TECHNISCHE UNIVERSITÄT MÜNCHEN

Zentrum Mathematik

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http://www-m16.ma.tum.de/Allgemeines/CompPlasmaPhys16

Exercise sheet 3 (May 10 and May 17, 2016)

1. **Constant-coefficient advection.**
For $x \in [0, L]$ and $t \in [0, T]$, consider the advection problem

$$\begin{cases} \dfrac{\partial}{\partial t} u(t, x) + a \dfrac{\partial}{\partial x} u(t, x) = 0\,, \qquad a \in \mathbb{R}\,, \\ u(0, x) = u_0(x) = \dfrac{1}{(2\pi\sigma^2)^{1/2}}\, e^{-(x-L/2)^2/(2\sigma^2)}\,, \end{cases} \tag{1}$$

where u is L-periodic. Denoting by $u_j^n := u(t_n, x_j)$ on a uniform space-time grid with parameters Δt and h, implement the following solvers for problem (1), for arbitrary values of L, T, a and σ:

(a) an explicit Euler upwind scheme,

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{h}\Big[a_-(u_{j+1}^n - u_j^n) + a_+(u_j^n - u_{j-1}^n)\Big]\,, \quad a_- = \min(0, a)\,,\ a_+ = \max(0, a)\,.$$

(b) a Lax-Wendroff scheme,

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2h}(u_{j+1}^n - u_{j-1}^n) + \frac{a^2 \Delta t^2}{2h^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)\,.$$

(c) a spectral scheme (solve analytically in Fourier space),

$$\hat{u}_k^{n+1} = \exp\left(\frac{-2\pi i}{L} k\, a\, \Delta t\right) \hat{u}_k^n\,, \quad k = -N/2, \dots, N/2 - 1\,,$$

where $\hat{u}_k = P_{kj}^* u_j$ is the k-th discrete Fourier mode.

Solve (1) with these schemes for $L = 1$, $T = 1.5$, $a = \pm 2$ and $\sigma = 0.1$. Check the conservation of total mass, maximum of u and of the L^2-norm. Make sure the CFL condition is satisfied for stability in the schemes a) and b). Use different spatial discretisations in order to perform convergence tests, by comparing to the exact solution $u_e(t, x) = u_0(x - at)$. Visualise your results and make a video of the evolution of u.

2. **1D-1V Vlasov-Poisson.**
We shall apply our knowledge gained in the previous exercises for solving the Vlasov-Poisson system for electrons,

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \dfrac{\partial f}{\partial x} - E \dfrac{\partial f}{\partial v} = 0\,, \\ -\dfrac{\partial^2 \phi}{\partial x^2} = 1 - \displaystyle\int f\, dv\,, \qquad E = -\dfrac{\partial \phi}{\partial x}\,. \end{cases} \tag{2}$$

Here, $f > 0$ is a function of $t \in [0, T]$, $x \in [0, L_x]$ and $v \in [v_{min}, v_{max}]$, $v_{min} < 0 < v_{max}$, assumed L_x-periodic and $(v_{max} - v_{min})$-periodic, respectively, and $\phi = \phi(t, x)$ is L_x-periodic.

(a) Using the (periodic) Poisson solver from Exercise sheet 2, implement a solver for (2) via a splitting method for the Vlasov equation, i.e. at each time step solve sequentially the two problems (A) and (B), for $t \in [t_n, t_{n+1}]$,

$$(A): \quad \frac{\partial f^*}{\partial t} - E\,\frac{\partial f^*}{\partial v} = 0\,, \qquad\qquad (B): \quad \frac{\partial f}{\partial t} + v\,\frac{\partial f}{\partial x} = 0\,,$$

$$f^*(t_n) = f(t_n)\,, \qquad\qquad f(t_n) = f^*(t_{n+1})\,.$$

(b) Show that (A) is a constant-coefficient advection problem. Determine the splitting error we commit by computing, formally, $f^{n+1} = e^B e^A f^n$ instead of $f^{n+1} = e^{(A+B)} f^n$, A and B denoting the respective advection operators. Implement the most accurate solver from exercise 1 for both splitting steps.

(c) Solve system (2) for $T = 30$, $L_x = 12$, $v_{max} = -v_{min} = 5$ and the initial condition

$$f(0, x, v) = f_0(x, v) = [1 + 0.01\cos(2\pi x/L_x)]\,\frac{1}{(2\pi)^{1/2}}\,e^{-v^2/2}\,.$$

Choose an appropriate time step. Check for conservation of total mass, momentum, total energy and L^2-norm. Make a video of the evolution of f in the (x, v)-plane. Plot the field energy $\int |E|^2/2\,dx$ as a function of time. Does your result change when changing phase space resolution (more grid points).

(d) Replace your Poisson solver by a spectral solver for $\partial_x E = 1 - \int f\,dv$ and repeat the numerical experiment.