Exercise 4

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Problem 13

Let $\mathcal{A}$ be a commutative Banach algebra, then the following are equivalent:

1. $\mathcal{A}$ is semi-simple
2. $\text{Rad}(\mathcal{A}) = \{0\}$
3. $\Delta\mathcal{A}$ separates points on $\mathcal{A}$

Proof. Assume $\mathcal{A}$ is a commutative Banach Algebra.

$(1) \Rightarrow (2)$: Assume that $\mathcal{A}$ is semi-simple, that is, $\hat{} : \mathcal{A} \to \hat{\mathcal{A}}$ is injective. Assume that $\text{Rad}(\mathcal{A}) \neq \{0\}$ and therefore $\exists x \in \text{Rad}(\mathcal{A})$, such that $x \neq 0$, since $\text{Rad}(\mathcal{A}) \neq \emptyset$. Since $x \in \text{Rad}(\mathcal{A})$, by definition, we have that $\forall \varphi \in \Delta\mathcal{A}, \varphi(x) = 0$ and thus, $\forall \varphi \in \Delta\mathcal{A}, \hat{x}(\varphi) = \varphi(x) = 0 = \varphi(0) = \hat{0}(\varphi) \Rightarrow \hat{x} = \hat{0}$, but, by assumption, $\hat{} : \mathcal{A} \to \hat{\mathcal{A}}$ is injective, and therefore $x = 0$ which is a contradiction.

$(1) \Rightarrow (3)$ Assume $\hat{} : \mathcal{A} \to \hat{\mathcal{A}}$ is injective and let $x, y \in \mathcal{A}, x \neq y$. By assumption of injectivity, $\hat{x} \neq \hat{y}$, therefore, $\exists \varphi \in \Delta\mathcal{A}$ such that $\hat{x}(\varphi) \neq \hat{y}(\varphi) \Rightarrow \varphi(x) \neq \varphi(y)$. Thus $\exists \varphi \in \Delta\mathcal{A}$ that separates x from y. Since x and y were arbitrary, $\Delta\mathcal{A}$ separates points on $\mathcal{A}$.

$(3) \Rightarrow (1)$ Assume $\Delta\mathcal{A}$ separates points on $\mathcal{A}$ and let $x, y \in \mathcal{A}$ such that $\hat{x} = \hat{y}$, where $\hat{} : \mathcal{A} \to \hat{\mathcal{A}}$. Then $\hat{x} = \hat{y} \Rightarrow \forall \varphi \in \Delta\mathcal{A}, \hat{x}(\varphi) = \hat{y}(\varphi) \Rightarrow \varphi(x) = \varphi(y)$. From the assumption that $\Delta\mathcal{A}$ separates points, this can only hold if $x = y$ and therefore $\hat{} : \mathcal{A} \to \hat{\mathcal{A}}$ is injective.

$(2) \Rightarrow (1)$ Assume $\text{Rad}(\mathcal{A}) = \{0\}$, then by definition, $\forall \varphi \in \Delta\mathcal{A}, \varphi(0) = 0 = \hat{0}(\varphi)$. Since 0 is the only element of $\text{Rad}(\mathcal{A})$, $\ker(\hat{}) = \{0\}$, and since $\hat{} : \mathcal{A} \to \hat{\mathcal{A}}$ is linear, it is injective.

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