Banach Algebras - Homework 8 Solutions

27) a) Consider $(\mathcal{B}(H), SOT)$ and $\mathcal{B}(H)\times\mathcal{B}(H)$ with the product topology. Suppose $(T_\alpha)_{\alpha\in I}, (S_\beta)_{\beta\in J} \subset \mathcal{B}(H)$ are nets with $T_\alpha \to T, S_\beta \to S$. We then obtain

$$\| \left((T_\alpha + S_\beta) - (T+S)\right) x\| \leq \|(T_\alpha - T)x\| + \|(S_\beta - S)x\| \to 0.$$

By exercise 23.b, we see

$$+ : \mathcal{B}(H) \times \mathcal{B}(H) \to \mathcal{B}(H), \quad (T,S) \mapsto T+S$$

is SOT-continuous.

Analogously $\mathbb{C} \times \mathcal{B}(H) \to \mathcal{B}(H), (\alpha, T) \mapsto \alpha T$ is SOT-continuous. Using 23.a, we prove the WOT-continuity of these maps analogously.

Claim: $(\mathcal{B}(H), WOT), (\mathcal{B}(H), SOT)$ are locally convex vector spaces.

> **Reminder**: Let (V,τ) be a topological vector space. Then $(V\tau)$ is a locally convex topological vector space iff there is a family of semi-normls $\mathcal{P}$ on V such that $(V,\mathcal{P})$ is a semi-normed space and $\tau = \tau_\mathcal{P}$
>
> SOT is the initial topology on $\mathcal{B}(H)$ with respect to
>
> $$\mathcal{P}_H = \{p_x \mid x \in H\},\ p_x(T) = \|Tx\|$$
>
> since,
>
> $$p_x(T) = \|Tx\| = 0, \forall x \in H \Leftrightarrow T = 0$$
>
> $\mathcal{P}_H$ is a family of semi-norms. Hence $(\mathcal{B}(H), SOT)$ is a locally convex vector space.
>
> WOT is the initial topology on $\mathcal{B}(H)$ with respect to
>
> $$\mathcal{P}_H = \{p_{xy} \mid x,y \in H\},\ p_{xy}(T) = |\langle Tx, y\rangle|$$
>
> Since
>
> $$p_{xy}(T) = |\langle Tx,y\rangle| = 0 \forall x,y \in H \Leftrightarrow \langle Tx,y\rangle = 0 \forall x,y \in H \Leftrightarrow T = 0$$
>
> we see $(\mathcal{B}(H), WOT)$ is a locally convex vector space.

b)For $S \in \mathcal{B}(H)$ and $(T_i)_{i\in I} \subset \mathcal{B}(H)$ with $T_i \to T$, we get

$$|\langle ST_i x, y\rangle - \langle STx, y\rangle| = |\langle S(T-T_i)x, y\rangle| \to 0$$

since $\langle\cdot,\cdot\rangle$ is continuous and $S \in B(H)$ Thus, $\mathcal{B}(H) \to \mathcal{B}(H), T \mapsto ST$ is WOT-continuous. Moreover,

$$|\langle T_i x, y\rangle - \langle Tx, y\rangle| = |\langle x, (T^* - T_i^*)y\rangle| = |\langle (T^* - T_i^*)y, x\rangle| \to 0$$

that is, $\mathcal{B}(H) \to \mathcal{B}(H), T \mapsto T^*$ is WOT-continuous. We have

$$\|S(T-T_i)x\| \leq \|S\|\|(T-T_i)x\| \to 0$$

that is, $\mathcal{B}(H) \to \mathcal{B}(H), T \mapsto ST$ is also SOT-continuous.

c) Let $(T_\alpha)_{\alpha\in I} \subset \mathcal{B}(H), (S_\beta)_{\beta_{\beta\in J}} \subset \mathcal{B}(H)_n, T_\alpha \to T, S_\beta \to S$, Then,

$$\|S_\beta T_\alpha x - STx\| \leq \|S_\beta\|\|(T_\alpha - T)x\| + \|(S_\beta - S)Tx\| \leq n\|(T_\alpha - T)x\| + \|(S_\beta - S)Tx\| \to 0$$

Hence, $\mathcal{B}(H)_n \times \mathcal{B}(H) \to \mathcal{B}(H)(S,T) \mapsto ST$ is SOT-continuous.

28)Let H be an infinite dimensional Hilbert space, $(y_n)_{n\geq 1} \subset H$ an orthonormal sequence, and

$$V_n : H \to H, \;\; V_n x = \langle x, y_n\rangle y_1$$

We have for $x, y \in H$,

$$\langle V_n x, y\rangle = \langle\langle x, y_n\rangle y_1, y\rangle = \langle x, y_n\rangle\langle y, y_1\rangle = \langle x, \langle y, y_1\rangle y_n\rangle = \langle x, V_n^* y\rangle$$

where $V_n^* y = \langle y, y_1\rangle y_n$. Further, $\|V_n x\| = |\langle x, y_n\rangle|\|y_1\| = |\langle x, y_n\rangle|$. Since $(y_n)_{n\geq 1}$ is an orthonormal sequence, we get by Bessel's inequality,

$$\sum_{n\geq 1} |\langle x, y_n\rangle|^2 \leq \|x\|^2$$

Hence, $|\langle x, y_n\rangle| \to 0, n \to \infty$. This shows $V_n \to 0$ in SOT. But,

$$\|V_n^* x\| = |\langle x, y_1\rangle|\|y_n\| = |\langle x, y_1\rangle| \nrightarrow 0$$

Hence $\mathcal{B}(H) \to \mathcal{B}(H), T \mapsto T^*$ can not be SOT-continuous.

29) Let H be an infinite dimensional Hilbert space such that $\{0\} = F \subset H$ is a subspace with $\dim(F) = n$.

$$A_F = n(I - P_f), \;\; V_F x = \frac{1}{n}\sum_{j=1}^{n}\langle x, x_j\rangle y_j$$

with, $\{x_1, ..., x_n, y_1, ..., y_n\}$ orthonormal and $F = span\{x_j \mid j = 1, ..., n\}$

1. If $F_1 \leq F_2$, then $F_2^\perp \leq F_1^\perp$. Hence $\dim(F^\perp) \to 0$ as $F \nearrow H$. Since $A_F x = n(I - P_F)x \in F^\perp$ we get $\|A_F x\| \to 0$.
2. $\|V_F x\| = \frac{1}{n}\|\sum_{j=1}^n \langle x, x_j\rangle y_j\| = \frac{1}{n}\sum_{j=1}^n |\langle x, x_j\rangle|^2\|y_j\|^2 = \frac{1}{n}\sum_{j=1}^n |\langle x, x_j\rangle|^2 \leq \frac{1}{n}\|x\| \to 0$
3. We have for $x \in H$,

$$\begin{aligned} A_F V_F x &= n(V_F x - P_F V_F x) \\ &= n\left(\frac{1}{n}\sum_{j=1}^{n}\langle x, x_j\rangle y_j - \frac{1}{n}\sum_{j=1}^{n}\langle x, x_j\rangle \underbrace{P_F y_j}_{=0}\right) \\ &= \sum_{j=1}^{n}\langle x, x_j\rangle y_j \end{aligned}$$

Hence $\|A_F V_F x\| \nrightarrow 0$. This demonstrates that $\mathcal{B}(H) \times \mathcal{B}(H) \to \mathcal{B}(H), (S,T) \mapsto ST$ is not SOT-continuous.

Remark: Denote by $N \subset \mathcal{B}(H)$ the set of normal operators. We have seen that $\mathcal{B}(H) \to \mathcal{B}(H), T \mapsto T^*$ is not SOT-continuous, but this map is SOT-continuous on N. Note that if $T \in N$, then $\|Tx\| = \|T^*x\|, \forall x \in H$. Indeed,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle = \langle, TT^*x\rangle = \langle T^*x, T^*x\rangle = \|T^*x\|$$

We easily see the converse is true as well, that is, if $\|Tx\| = \|T^*x\|, \forall x \in H$ then $T^*T = TT^*$.

Now,

$$
\begin{aligned}
\|T_i^* x - Tx\|^2 &= \|T_i^* x\|^2 + \|T^* x\| - 2\Re\langle T_i^* x, T_i^* x\rangle \\
&\vdots \\
&\leq (\|T_i x\|^2 - \|Tx\|^2)(\|Tx\|^2 + \|Tx\|^2) - 2\|x\|\|(T_i - T)T^* x\| \\
&\leq \|(T_i - T)x(\|(\|T_i - T)x\| + 2\|Tx\| + 2\|x\|\|(T_i - T)T^* x\|)
\end{aligned}
$$

and from this the assertion follows.

30) Let H be a Hilbert space, $P \subset \mathcal{B}(H)$ the set of projections. We have to show that $id : (P, \tau_{SOT}) \to (P, \tau_{WOT})$ is a homeomorphism. Since $\tau_{WOT} \subseteq \tau_{SOT}$ we have the continuity of id. It remains to show that id^{-1} is continuous. Let $E_0 \in P, u \in H, \|u\| = 1$ and $\epsilon > 0$.

Claim: There are $x_1, ..., x_n, y_1, ..., y_n \in H$ such that if $E \in P$ with $\|(E - E_0)u\| < \epsilon$, then $\exists \delta > 0$, $|\langle (E - E_0)x_j, y_j\rangle| < \delta, j = 1, ..., n$.

Proof: Suppose $E \in P$ satisfies

$$
|\langle (E - E_0)u, u\rangle| < \frac{\epsilon^2}{4} \text{ and } |\langle (E - E_0)u, E_0 u\rangle| < \frac{\epsilon^2}{4}
$$

Then,

$$
\begin{aligned}
\|(E - E_0)u\|^2 &= \langle Eu, u\rangle + \langle E_0 u, u\rangle - 2\Re\langle Eu, E_0 u\rangle \\
&= \langle (E - E_0)u, u\rangle + 2\langle E_0 u, E_0 u\rangle - 2\Re\langle Eu, E_0 u\rangle \\
&\leq |\langle (E - E_0)u, u\rangle| + 2|\langle (E_0 - E)u, E_0 u\rangle| \\
&< \epsilon^2
\end{aligned}
$$

This shows that id^{-1} is continuous as well. Hence id is a homeomorphism.

31) Let $\mathfrak{U} \subset \mathcal{B}(H)$ be the set of unitary operators. Again, we have to show that $id : (\mathfrak{U}, \tau_{SOT}) \to (\mathfrak{U}, \tau_{WOT})$ is a homeomorphism. It suffices to show that id^{-1} is continuous. Let $U_0 \in \mathfrak{U}, u \in H, \|u\| = 1, \epsilon > 0$. Suppose that $U \in \mathfrak{U}$, then

$$
\begin{aligned}
\|(U - U_0)u\|^2 &= 2\langle u, u\rangle - 2\Re\langle Uu, U_0 u\rangle \\
&= 2\Re\langle (U - U_0)u, U_0 u\rangle \\
&\leq 2|\langle (U - U_0)u, U_0 u\rangle| \\
&< \epsilon^2
\end{aligned}
$$

as long as $|\langle (U - U_0)u, U_0 u\rangle| < \epsilon^2/2$. With this, the assertion follows.