Banach Algebras - Homework 7 Solutions

25)
(1) For the orthogonal projection $E : H \to Y$ we have, $Ex = E(y + y^{\perp}) = y$. Using this, we obtain,

$$\|Ex\|^2 = \|y\|^2 \leq \|y\|^2 + \|y^{\perp}\|^2 = \|x\|^2 \Rightarrow \|E\| \leq 1.$$

Now, for $x \in Y$, we have $\|Ex\| = \|x\|$, hence $\|E\| = 1$. Moreover,

$$\langle Ex, x\rangle = \langle y, y + y^{\perp}\rangle = \langle y, y\rangle = \|y\|^2 \geq 0 \Rightarrow E \geq 0.$$

(2) Follows from (1).

(3) Let $E \in \mathcal{B}(H)$ with $E^2 = E = E^*$. We have that $\ker(E) = \mathrm{Rg}(E^*)^{\perp} = \mathrm{Rg}(E)^{\perp}$. Since $\mathrm{Rg}(E)^{\perp}$ is closed, the statement follows.

(4) Let E_1, E_2 be orthogonal projections onto closed subspaces $Y_1, Y_2 \subset H$.

$(i) \Rightarrow (ii)$: Let $Y_1 \subseteq Y_2, x = y_1 + y_1^{\perp}$. Then $E_1 x = y_1$ and since $E_2 y_2 = y_2$ for every $y_2 \in Y_2$, $E_2 E_1 x = y_1 = E_1 x$

$(ii) \Rightarrow (iii)$: Since $E_1^2 = E_1$, we get from $E_2 E_1 = E_1$ the equation $E_1 E_2 E_1 = E_1^2$ or equivalently $(E_1 E_1 - E_1)E_1 = 0$, and thus $E_1 E_2 = E_1$.

$(iii) \Rightarrow (iv)$: For $x \in H$, we have $\|E_1 x\| = \|E_1 E_2 x\| \leq \|E_2 x\|$, using the fact that $E_1 \in \mathcal{B}(H)$ and $\|E_1\| \leq 1$.

$(iv) \Rightarrow (v)$: We have to show that $\langle (E_2 - E_1)x, x\rangle \geq 0, \forall x \in H$.

$$\langle E_1 x, x\rangle = \langle E_1 x, E_1 x\rangle = \|E_1 x\|^2 \leq \|E_2 x\|^2 = \langle E_2 x, E_2 x\rangle = \langle E_2 x, x\rangle.$$

Hence $0 \leq \langle (E_2 - E_1)x, x\rangle$.

$(v) \Rightarrow (i)$: Let $y \in Y_1$, then $(E_2 - E_1)y = E_2 y - y$ and

$$\|y\|^2 = \langle E_1 y, y\rangle \leq \langle E_2 y, y\rangle = \|E_2 y\|^2 \leq \|y\|^2,$$

from which it follows that $\|E_2 y\|^2 = \|y\|^2$, and by (2), $y \in Y_2$.

(5) If $\{E_\alpha\}$ is a family of projections, it follows from (4) that $\check{E} = \bigvee E_\alpha$, $\hat{E} = \bigwedge E_\alpha$ are the projections onto $\bigvee Y_\alpha$ and $\bigwedge Y_\alpha$ respectively, where Y_α is the closed subspace with $E_\alpha Y_\alpha = Y_\alpha$.

(6) If $E_1 \leq E_2$, then $I = E_2 \leq (I - E_1)$. Hence $\bigwedge(I - E_\alpha) = I - \bigvee E_\alpha$ and $\bigvee(I - E_\alpha) = I - \bigwedge E_\alpha$.

(7) By assumption $\{E_\alpha(H)\}_\alpha$ is an increasing net of closed subspaces of H. Moreover, $\bigcup\limits_\alpha E_\alpha(H)$ is a subspace of H with $\mathrm{cl}(\bigcup\limits_\alpha E_\alpha(H)) = E(H)$. Now let $x \in H, \epsilon > 0$. Since $Ex \in E(H)$, there is $y \in E_\alpha(H)$ for some α with $\|Ex - y\| < \epsilon$. If $\alpha \leq \beta$, we get $E_\alpha \leq E_\beta \leq E$, and so $y \in E_\alpha(H) \subseteq E_\beta(H) \subseteq E(H)$ which implies

$$\|Ex - E_\alpha x\| = \|E(Ex - y) - E_\beta(Ex - y)\| \leq \|E - E_\beta\| \|Ex - y\| < \epsilon$$

(8) Working with $I - E_\alpha$, the result follows from (7).

(9) If the index set A is finite, then $E = \sum\limits_{\alpha \in A} E_\alpha$ is clear. If A is an infinite set, let $\mathcal{F} = \{F \subset A \mid |F| < \infty\}$. For $F \in \mathcal{F}$, let $G_F = \sum\limits_{\alpha \in F} E_\alpha$, then by assumption on the net $\{E_\alpha\}_{\alpha \in A}$ and (7), we see $G_F = \bigvee\limits_{\alpha \in F} E_\alpha$. Hence $(G_F)_{F \in \mathcal{F}}$ is an increasing net of projections and

$$\bigvee_{F \in \mathcal{F}} G_F = \bigvee_{F \in \mathcal{F}} \{ \bigvee_{\alpha \in F} E_\alpha \} = \bigvee_{\alpha \in A} E_\alpha = E$$

and by (7) the assertion follows.

26)
(i) Let $\mathcal{A} = \{M_\varphi \mid \varphi \in L^\infty(\mu)\}$.
Commutativity: Let $M_\varphi, M_\psi \in \mathcal{A}$, then $M_\varphi M_\psi f = \varphi \psi f = \psi \varphi f = M_\psi M_\varphi f$.
C* property: We have to show that $\|M_\varphi^* M_\varphi\| = \|M_\varphi\|^2$. This follows from example 1.6 (i).

(ii) Let $S \subset X$ be any measurable set and χ_S its characteristic function. We have, for any $f \in L^2(\mu)$,

$$M_{\chi_S} M_{\chi_S} f = \chi_S \chi_S f = \chi_S f = M_{\chi_S} f.$$

Hence $M_{\chi_S}^2 = M_{\chi_S}$. Moreover, $M_{\chi_S}^* f = M_{\overline{\chi_S}} f = M_{\chi_S} f$, that is, $M_{\chi_S}^* = M_{\chi_S}$. By 25)(3) we get that M_{χ_S} is a projection.

(iii) Let $\{S_n\}_{n \geq 1}$ be a sequence of subsets of Ω which $S_n \cap S_m = \emptyset, n \neq m$, $\bigcup\limits_{n \geq 1} S_n = X$. By (ii), $M_{\chi_{S_n}}$ are projections and by 25)(4) we get

$$\|f - \sum_{j=1}^{n} M_{\chi_{S_j}} f\| \to 0, \;\; n \to \infty.$$