QUANTUM ERROR CORRECTION

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OVERVIEW

1. What is Classical Error Correction?
2. What is Quantum Error Correction?
3. Quantum Errors: Causes and Effects
4. Error Correcting Codes
5. Stabilizer Formalism
6. Surface Code

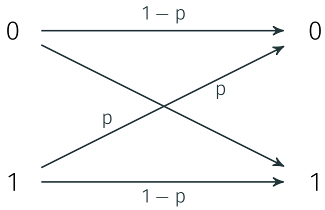
WHAT IS CLASSICAL ERROR CORRECTION?

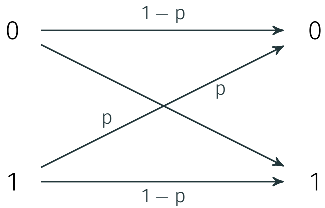
- Noise is the main opponent of information processing systems, it causes information to become erroneous

- Error correction is the act of preventing or fixing errors in information caused by noise

WHAT IS CLASSICAL ERROR CORRECTION?

- A simple strategy for protecting messages: encode redundant information
- Example: sending a classical bit over a channel
    - Errors cause the bit flips
    - Assume probability of error is $p > 0$ and $1 - p$ error free





Protocol

- Encode: $0 \to 000$, $1 \to 111$
- Send all three bits through the symmetric binary channel
- Receiver uses majority voting to decode the message

Analysis

- The probability of error is $3p^2 - 2p^3 < p$ if $p < 0.5$.

- Quantum error correction is the act of protecting quantum states against the effects of noise
- Quantum error correcting codes are based on similar principles of classical codes but differ because of
  - No cloning theorem
  - Measurement destroys quantum information
  - Bit flip and phase errors: $|0\rangle \leftrightarrow |1\rangle$ & $|1\rangle \leftrightarrow -|1\rangle$
  - Errors are continuous: e.g. $|0\rangle \rightarrow e^{i\epsilon\sigma_x}|0\rangle$

  where $\sigma_x \equiv$ Pauli X operation

We investigate two simple algorithms and how common errors affect their outcomes

- $|\psi\rangle_{\text{final}} = \prod_{i=1}^{N} I_i |0\rangle = |0\rangle$
- $|\psi\rangle_{\text{final}} = HIH|0\rangle = HI\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = H\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |0\rangle$

For:

$$I_i \equiv \text{Identity operation for all } i = 1, ..., N$$
$$H \equiv \text{Hadamard operation}$$

1) Coherent quantum errors: Gates incorrectly applied

- The system is assumed to be governed by Hamiltonian H' when it is governed by H
- For $\prod_{i=1}^{N} I_i |0\rangle$, I_i could instead, for example, rotate the qubit state by a small angle around the X-axis of the Bloch sphere
- $|\psi\rangle_{\text{final}} = \prod_{i=1}^{N} e^{i\epsilon\sigma_x} |0\rangle$ = $\cos(N\epsilon)|0\rangle + i\sin(N\epsilon)|1\rangle$
- Measuring $|\psi\rangle_{\text{final}}$ in $|0\rangle, |1\rangle$ basis
  - $P(|0\rangle) = \cos^2(N\epsilon) \approx 1 - (N\epsilon)^2$
  - $P(|1\rangle) = \sin^2(N\epsilon) \approx (N\epsilon)^2$

2) Environmental Decoherence

- Consider a simple two level environment
  - Basis states: $|e_0\rangle$, $|e_1\rangle$; $\langle e_i|e_j\rangle = \delta_{ij}$; $|e_0\rangle\langle e_0| + |e_1\rangle\langle e_1| = I$
  - Assume when the qubit is in $|1\rangle$ coupling with the environment flips the environment state, but does nothing when the qubit it in $|0\rangle$
  - Assume interactions with the environment occur only during "wait stage", that is, during the I operation
  - Assume the environment is initialized to $|E\rangle = |e_0\rangle$

$$\begin{aligned} HIH|0\rangle|E\rangle &= HI\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|e_0\rangle \\ &= H\frac{1}{\sqrt{2}}(|0\rangle|e_0\rangle + |1\rangle|e_1\rangle) \\ &= \frac{1}{2}(|0\rangle + |1\rangle)|e_0\rangle + \frac{1}{2}(|0\rangle - |1\rangle)|e_1\rangle \end{aligned}$$

In density matrix formulation for HIH $|0\rangle\,|E\rangle$:

$$\begin{aligned}\rho = &\frac{1}{4}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|)\,|e_0\rangle\langle e_0| \\ &+ \frac{1}{4}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|)\,|e_1\rangle\langle e_1| \\ &+ \frac{1}{4}(|0\rangle\langle 0| - |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)\,|e_0\rangle\langle e_1| \\ &+ \frac{1}{4}(|0\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 0| - |1\rangle\langle 1|)\,|e_1\rangle\langle e_0|\end{aligned}$$

Tracing out the environment,

$$\mathrm{Tr}_E(\rho) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

- With this simple scenario, there is a 50/50 chance of measuring $|0\rangle$ or $|1\rangle$

3) Loss, leakage, measurement and Initialization
Other types of errors are

- Measurement errors
- Loss errors
- Leakage errors
- Initialization errors

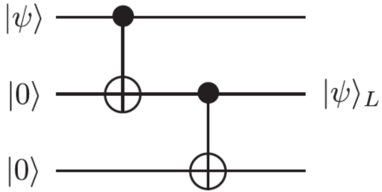
"We have learned that it is possible to fight entanglement with entanglement." - John Preskill

- Error correcting codes (ECC) are encoding and decoding schemes that allow errors in information to be detected and located

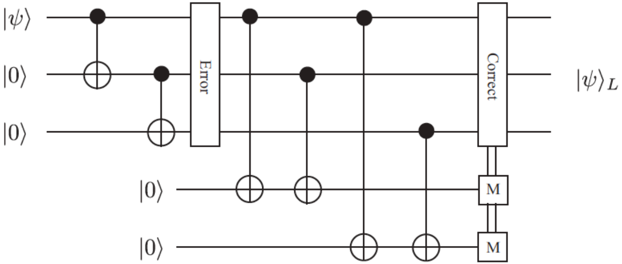
- E.g. 3-qubit bit flip code
- Similar to the classical code, the 3-qubit code encodes $|0\rangle$ and $|1\rangle$ using logical qubits

$$|0\rangle_L = |000\rangle, |1\rangle_L = |111\rangle$$

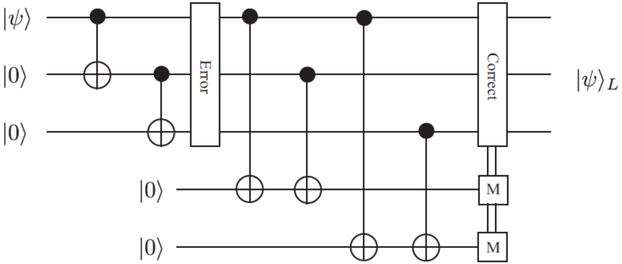
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow |\psi\rangle_L \equiv \alpha|0\rangle_L + \beta|1\rangle_L$$



- The 3-qubit code can correct up to one bit flip error
- We introduce two ancilla qubits to the system to extract "syndrome" information



- With probability p for a bit flip error, the probability ≤ 1 bit flips occurs is $(1-p)^3 + 3p(1-p)^2 = 1 - 3p^2 + 2p^3$
- The probability of error is $3p^2 - 2p^3$, the same as the classical case, which is an improvement to no code for $p < 0.5$.



E.g. $|\psi\rangle_\text{L} = |000\rangle$

$$|000\rangle\,|00\rangle \xrightarrow{\text{error}} |100\rangle\,|00\rangle \xrightarrow{\text{CNOTs}} |100\rangle\,|11\rangle \xrightarrow{\text{correction}} |000\rangle$$

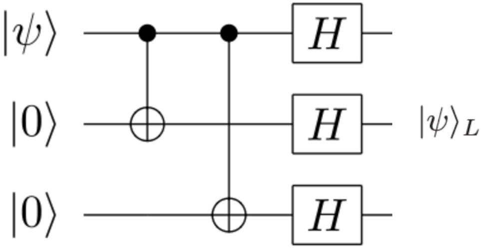
$$|010\rangle\,|00\rangle \xrightarrow{\text{CNOTs}} |100\rangle\,|10\rangle \xrightarrow{\text{correction}} |000\rangle$$

$$|001\rangle\,|00\rangle \xrightarrow{\text{CNOTs}} |100\rangle\,|01\rangle \xrightarrow{\text{correction}} |000\rangle$$

$$|101\rangle\,|00\rangle \xrightarrow{\text{CNOTs}} |101\rangle\,|10\rangle \xrightarrow{\text{correction}} |111\rangle$$

3-qubit phase flip code

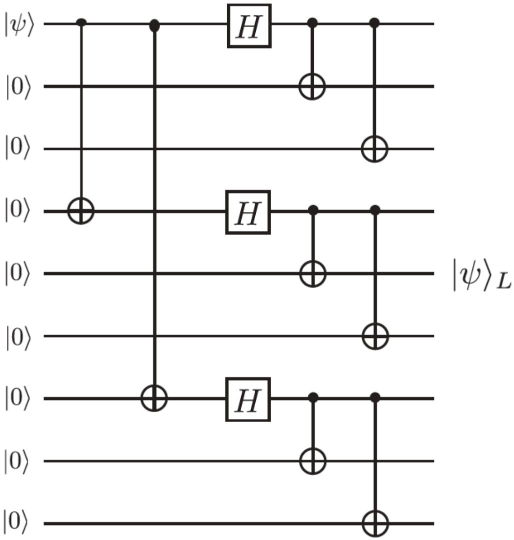
- Similar to the bit-flip encoding, but we apply Hadamard gates to switch the basis
- Can use the same strategy to detect phase flips in this new basis



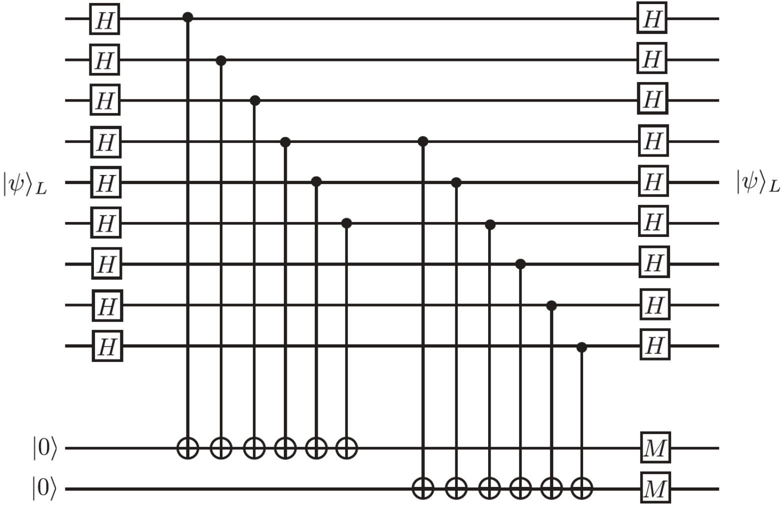
The Shor 9-qubit code

- Can correct one bit and one phase flip error (or one of both)
- $|0\rangle_L = \frac{1}{\sqrt{8}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$
- $|1\rangle_L = \frac{1}{\sqrt{8}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$

The Shor 9-qubit code



ERROR CORRECTING CODES



STABILIZER FORMALISM

- Using vector notation for to represent logical qubits is inefficient
- Stabilizer states use operators to represent states rather than vectors (Heisenberg representation)
- A state $|\psi\rangle$ is represented or "stabilized" by an operator A if A $|\psi\rangle = |\psi\rangle$ (i.e $|\psi\rangle$ is an eigenvector of A with eigenvalue 1)
- E.g. $\sigma_z |0\rangle = |0\rangle$
- Stabilizer codes are a collection of stabilizer states used to form logical qubits

- The Pauli group

$$\mathcal{P} = \{\pm \mathrm{I}, \pm i\mathrm{I}, \pm\sigma_x, \pm i\sigma_x, \pm\sigma_z, \pm i\sigma_z, \pm\sigma_y, \pm i\sigma_y\}$$

  can be used to represent N qubit states
- Formal definition of a stabilizer for N qubits:

  Def: Let S be a subgroup of $\mathcal{P}^{\otimes N}$. Define V_S to be the set of N qubit states which are stabilized by every element in S. V_S is called the vector space stabilized by S, and S is the stabilizer for V_S

- V_S can be found via the intersection of each subspace spanned by the operators in S.
- Notation: E.g. N=3, a stabilizer is represented as $\sigma_{z_1}\sigma_{z_2} \equiv \sigma_{z_1} \otimes \sigma_{z_2} \otimes \mathrm{I}_3$

- E.g. Stabilizer for the 3-qubit bit flip code. $N = 3$ and $S = \{I, \sigma_{z_1}\sigma_{z_2}, \sigma_{z_2}\sigma_{z_3}, \sigma_{z_1}\sigma_{z_3}\}$
    - $\sigma_{z_1}\sigma_{z_2}$ stabilizes $|000\rangle, |001\rangle, |110\rangle, |111\rangle$
    - $\sigma_{z_2}\sigma_{z_3}$ stabilizes $|000\rangle, |100\rangle, |011\rangle, |111\rangle$
    - $\sigma_{z_1}\sigma_{z_3}$ stabilizes $|000\rangle, |010\rangle, |101\rangle, |111\rangle$
    - $V_S = \{|000\rangle, |111\rangle\}$, which are the logical qubits
- $S = \langle \sigma_{z_1}\sigma_{z_2}, \sigma_{z_2}\sigma_{z_3} \rangle$, where $\langle \cdot \rangle$ represents every product of the elements called the generator
- E.g. Stabilizer of Shor's code:

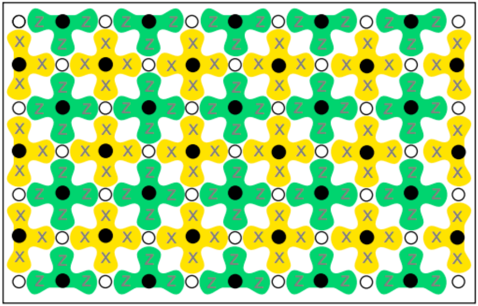
$$S = \langle \sigma_{z_1}\sigma_{z_2}, \sigma_{z_1}\sigma_{z_3}, \sigma_{z_4}\sigma_{z_5}, \sigma_{z_4}\sigma_{z_6}, \sigma_{z_7}\sigma_{z_8}, \sigma_{z_7}\sigma_{z_9},$$
$$\sigma_{x_1}\sigma_{x_2}\sigma_{x_3}\sigma_{x_4}\sigma_{x_5}\sigma_{x_6}, \sigma_{x_1}\sigma_{x_2}\sigma_{x_3}\sigma_{x_7}\sigma_{x_8}\sigma_{x_9} \rangle$$

- Benefits to using the stabilizer code are seen when looking at error detection and correction

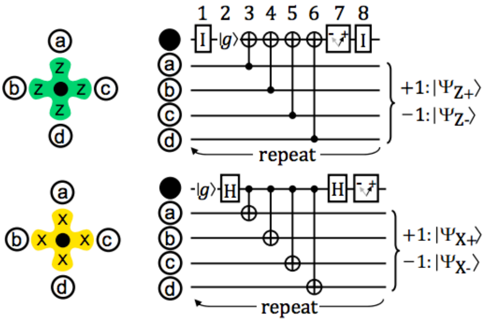
- E.g. 2-qubit surface code

$\sigma_{z_1}\sigma_{z_2}$	$\sigma_{x_1}\sigma_{x_2}$	$\|\psi\rangle$
+1	+1	$(\|00\rangle + \|11\rangle)/\sqrt{2}$
+1	-1	$(\|00\rangle - \|11\rangle)/\sqrt{2}$
-1	+1	$(\|01\rangle + \|10\rangle)/\sqrt{2}$
-1	-1	$(\|01\rangle - \|10\rangle)/\sqrt{2}$

$|\psi\rangle$ is initialized to $(|00\rangle + |11\rangle)/\sqrt{2}$ and a σ_{x_1} error (i.e. a bit flip on the first qubit) occurs, the state will change to $(|01\rangle + |10\rangle)/\sqrt{2}$ and the eigenvalue pair on the next stabilizer measurement $\sigma_{z_1}\sigma_{z_2}$ is $(-1, +1)$

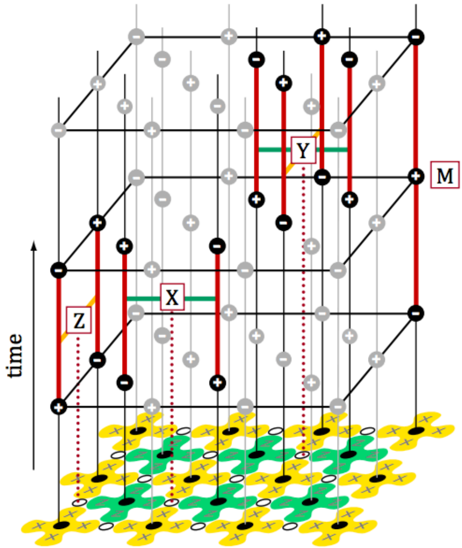


- Data qubit (white) or measurement qubit (black)
- Measure-Z (green) and measure-X (yellow)
- Each data qubit is coupled to two measure-Z and two measure-X qubits
- Measure-Z: $\sigma_{Z_a}\sigma_{Z_b}\sigma_{Z_c}\sigma_{Z_d}$, measure-X: $\sigma_{X_a}\sigma_{X_b}\sigma_{X_c}\sigma_{X_d}$



Eigenvalue	$\hat{Z}_a\hat{Z}_b\hat{Z}_c\hat{Z}_d$	$\hat{X}_a\hat{X}_b\hat{X}_c\hat{X}_d$
$+1$	$\|gggg\rangle$	$\|++++\rangle$
	$\|ggee\rangle$	$\|++--\rangle$
	$\|geeg\rangle$	$\|+--+\rangle$
	$\|eegg\rangle$	$\|--++\rangle$
	$\|egge\rangle$	$\|-++-\rangle$
	$\|gege\rangle$	$\|+-+-\rangle$
	$\|egeg\rangle$	$\|-+-+\rangle$
	$\|eeee\rangle$	$\|----\rangle$
-1	$\|ggge\rangle$	$\|+++-\rangle$
	$\|ggeg\rangle$	$\|++-+\rangle$
	$\|gegg\rangle$	$\|+-++\rangle$
	$\|eggg\rangle$	$\|-+++\rangle$
	$\|geee\rangle$	$\|+---\rangle$
	$\|egee\rangle$	$\|-+--\rangle$
	$\|eege\rangle$	$\|--+-\rangle$
	$\|eeeg\rangle$	$\|---+\rangle$

SURFACE CODE



THANKS!

Questions

1. Why does Shor's 9-qubit gate need 8 logical qubits instead of 4 (i.e. 4 for bit flip errors and 4 for phase flip errors instead of 2 and 2)?
2. Can you find a stabilizer state representation for the $|\mathrm{GHZ}\rangle_3 = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$?
3. Why in the examples of stabilizer codes shown does σ_y never appear?
4. What are some flaws for the two qubit surface code example?

1. Simon Devitt, William Munro, Kae Nemoto
   Quantum Error Correction for Beginners
   arXiv:0905.2794v4
2. Austin G. Fowler, Matteo Mariantoni, John M. Martinis, Andrew N. Cleland
   Surface codes: Towards practical large-scale quantum computation
   arXiv:1208.0928v2
3. Michael Nielsen, Isaac Chuang
   Quantum Computation and Quantum Information