Category Theory Seminar

Monoidal Categories in Physics

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1 Introduction

In this report, we will investigate how physical systems and interactions between them has the structure of a category and will explore a concrete physical example using ideas from quantum theory. We will see that when we consider the joining of two systems via a tensor product, in order to model the composite system, we will need a new construct from category theory, namely the symmetric monoidal category. We will define these categories and see how physics does indeed fit this model.

2 Categories in Physics

Physical interactions between physics systems can be modeled with category theory. We can see this through a simple example. For some physical system A, and an operation that can be performed on A, $f : A \to B$, to obtain a resulting system B, denoted,

$$A \xrightarrow{\quad f \quad} B,$$

where the arrow represents time. We can perform a second operation, $g : B \to C$, on B to get a third system of type C,

$$B \xrightarrow{\quad g \quad} C.$$

We can compose g and f to get the composition of operations $g \circ f$. If we introduce a third operation h that acts on C, it is clear that associativity holds, that is $(h \circ g) \circ f = h \circ (g \circ f)$. There are operations that "do nothing" to the system,

$$A \xrightarrow{\quad 1_A \quad} A,$$

which behaves as an identity morphism. It is also clear that if we have two "identity" operations for two systems A and B, then,

$$1_B \circ f = f = f \circ 1_A,$$

and this structure is exactly a category.

Example. A more concrete example can be seen in quantum theory, where the systems are generally composed of objects called qubits whose states are represented by state vectors in a complex Hilbert space, $|\psi\rangle \in \mathcal{H}$. The state of the qubit is what is of interest in quantum theory, and not what the qubit is in reality, so qubits can be for example electrons or photons, but both of these contain the same type of "quantumness" and so are generalized to be qubits. The operations that are performed on qubits just affect the state vector. The operators can therefore be represented by linear operators (i.e. the elements in $\mathcal{L}(\mathcal{H})$). So it is possible to transform qubit states from one to another via these operators, for example $R : |\psi\rangle \to |\psi'\rangle$, or using the style above,

$$|\psi\rangle \xrightarrow{\quad R \quad} |\psi'\rangle.$$

In physics, it is useful to consider multiple systems composed as one composite system. It is possible to consider multiple systems as a single object via tensor products. When multiple systems are considered as a whole, compound operations need to be considered as well. For example, with systems A, B, C and D and operators $R : A \to C$ and $S : B \to D$ we have,

$$A \otimes B \xrightarrow{R \otimes S} C \otimes D.$$

To model this with category theory, we need a particular construct the comes from the 2-dimensional variant of categories called monodial categories, which are a type of product category.

3 Monoidal Categories

Definition. A **symmetric monoidal category** is a category $\mathcal{C}$ with a unit object I, equipped with bifunctorial functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called a symmetric monoidal tensor, which is an assignment for both pairs of objects and pairs of morphisms. Formally, for object A, B, C and D and morphisms f and g, the monoidal tensor should assign the tensored pairs,

$$\begin{array}{rcl} (A, B) & \mapsto & A \otimes B \\ (f : A \to B, g : C \to D) & \mapsto & f \otimes g : A \otimes C \longrightarrow B \otimes D \end{array}$$

to their tensor product. The monoidal category comes together with left and right unital natural isomorphisms, a symmetry isomorphism, and an associativity isomorphism, which are as follows,

$$\begin{aligned}
\lambda_A &: I \otimes A \longrightarrow A \\
\rho_A &: A \otimes I \longrightarrow A \\
\beta_{A,B} &: A \otimes B \longrightarrow B \otimes A \\
\alpha_{A,B,C} &: (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)
\end{aligned}$$

We demand $\alpha_{A,B,C}$ allows the following diagram to commute,

$$\begin{tikzcd}
 & ((A \otimes B) \otimes C) \otimes D \arrow[dl, "\alpha_{A,B,C} \otimes 1_D"'] \arrow[dr, "\alpha_{A \otimes B,C,D}"] & \\
(A \otimes (B \otimes C)) \otimes D \arrow[d, "\alpha_{A,B \otimes C,D}"'] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A,B,C \otimes D}"] \\
A \otimes ((B \otimes C) \otimes D) \arrow[rr, "1_A \otimes \alpha_{B,C,D}"] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

which is called the pentagon identity. We demand further that $\alpha_{A,B,C}$ with the unital isomorphisms obey the triangle identity,

$$\begin{tikzcd}
(A \otimes 1) \otimes B \arrow[rr, "\alpha_{A,1,B}"] \arrow[dr, "\rho_A \otimes 1_B"'] & & A \otimes (1 \otimes B) \arrow[dl, "1_A \otimes \lambda_B"] \\
 & A \otimes B &
\end{tikzcd}$$

It is also necessary that $\beta_{A,B}$ and $\alpha_{A,B,C}$ obey the hexagon identity,

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\alpha_{A,B,C}"] \arrow[d, "\beta_{A,B} \otimes 1_C"'] & A \otimes (B \otimes C) \arrow[r, "\beta_{A,B \otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[r, "1_B \otimes \beta_{A,C}"'] & B \otimes (C \otimes A)
\end{tikzcd}$$

Finally, $\beta_{A,B}$ should be such that,

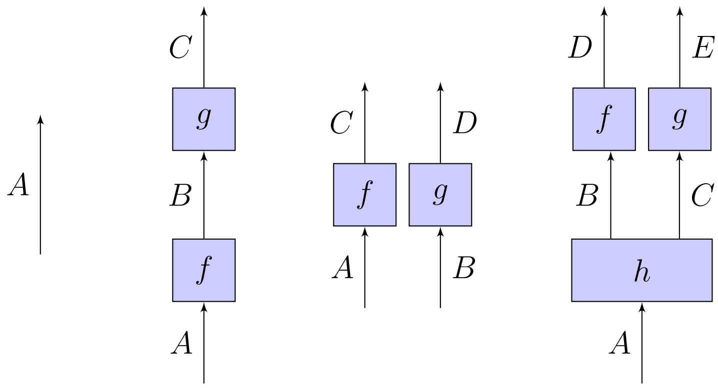
$$\beta_{B,A} \circ \beta_{A,B} = 1_{A \otimes B}.$$

The difference between strict monoidal categories and symmetric monoidal categories is that for two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, $A \otimes B$ is naturally isomorphic to $B \otimes A$, where this does not hold in a strict monoidal category.

4 Symmetric Monoids in Physics

We have seen in the second section a simple case of how category theory applies to a quantum system and how operators can change the state vector a qubit. What we will see in this section is how the monoidal structure is used to represent the notion of representing two distinct qubits as one, which is the tensor product of their state vectors. We will also explore how symmetric monoids model some interesting quantum properties.

In this section physical processes, which are the morphisms, will be represented by square boxes and the systems will evolve on the directed arrows, representing time. Tensored systems will appear side by side. Some examples are as follows,



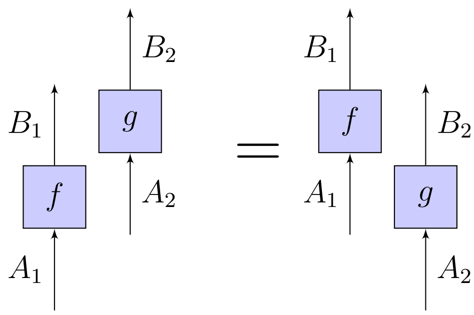
These represent,

- $1_A : A \to A$
- $f : A \to B, g : B \to C, g \circ f : A \to C$
- $f : A \to C, g : B \to D, f \otimes g : A \otimes B \to C \otimes D$
- $f : B \to D, g : C \to E, f \otimes g : B \otimes C \to D \otimes E, h : A \to B \otimes C$

Categorically, bifunctorality of the tensor product $\otimes$ means that for objects A_1, A_2, B_1, and B_2 with the morphisms $f : A_1 \to B_1$ and $g : A_2 \to B_2$, the following diagram commutes,

$$\begin{tikzcd}
A_1 \otimes A_2 \arrow[r, "f \otimes 1_{A_2}"] \arrow[d, "1_{A_1} \otimes g"'] & B_1 \otimes A_2 \arrow[d, "1_{B_1} \otimes g"] \\
A_1 \otimes B_2 \arrow[r, "f \otimes 1_{B_2}"'] & B_1 \otimes B_2
\end{tikzcd}$$

In the style of the physical diagrams, bifunctorality is represented as,

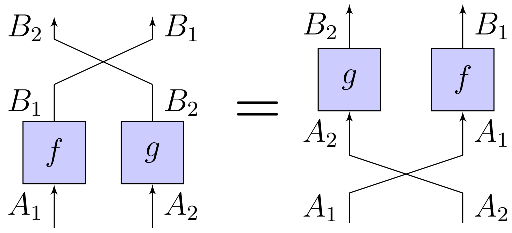


which says that if we perform f on system A_1, and then perform g on A_2, it is the same as first applying g on A_2 and then f on A_1. Both result in the system $B_1 \otimes B_2$.

For symmetry, we have that the following diagram commutes,

$$\begin{tikzcd}
A_1 \otimes A_2 \arrow[r, "f \otimes g"] \arrow[d, "\beta_{A_1,A_2}"'] & B_1 \otimes B_2 \arrow[d, "\beta_{B_1,B_2}"] \\
A_2 \otimes A_1 \arrow[r, "g \otimes f"'] & B_2 \otimes B_1
\end{tikzcd}$$

Representing this as a physical diagram we have,



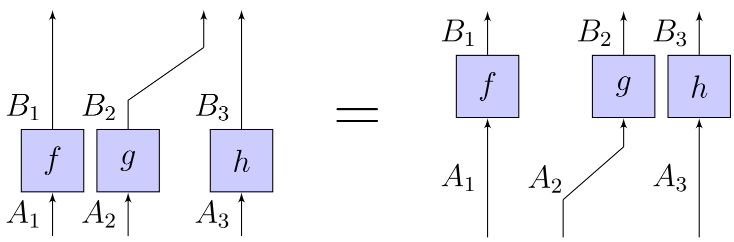
which implies that if we swap the location of the systems, then we must also swap the location of the operations that are being performed on each respective system.

For example, in an experiment working with qubits, it is often the case that a laser beam is shone on the qubit to manipulate its state. If there are two qubits $|\psi\rangle$ and $|\phi\rangle$, and for qubit $|\psi\rangle$ a laser frequency with $1GHz$ is used and for the second qubit a laser of frequency $2GHz$ is used. If we switch the location of the qubits, to perform the same action, we have to also move the lasers. This seems quite obvious, but must be enforced nonetheless.

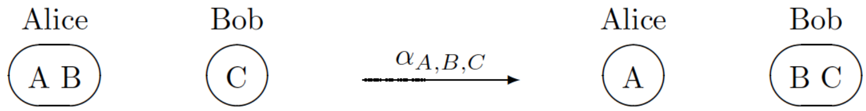
For associativity, with the addition of systems A_3 and B_3 with morphism $h : A_3 \to B_3$, we have that the following diagram commutes,

$$\begin{tikzcd}
(A_1 \otimes A_2) \otimes A_3 \arrow[r, "(f \otimes g) \otimes h"] \arrow[d, "\alpha_{A_1,A_2,A_3}"'] & (B_1 \otimes B_2) \otimes B_3 \arrow[d, "\alpha_{B_1,B_2,B_3}"] \\
A_1 \otimes (A_2 \otimes A_3) \arrow[r, "f \otimes (g \otimes h)"'] & B_1 \otimes (B_2 \otimes B_3)
\end{tikzcd}$$

which can be seen using the physical representation as,



Technically, nothing between the left and right side of the equation is different. Associativity in a physical sense is more an idea of grouping systems rather than a having a physical meaning. Associativity is important when it comes to composite systems that are unable to interact for some reason. An example is in quantum information theory where two parties, Alice and Bob, would like to communicate with each other. The act of communicating can be seen as Alice preparing her qubit in some particular state $|\psi\rangle$ representing a message she would like to communicate with Bob. She (physically) sends her qubit over some quantum channel over to Bob and he is able to measure this qubit to decode the message. This can be depicted as,



where Alice is sending Bob qubit B via the associativity of $\alpha_{A,B,C}$. Qubit B is in the same state on the left and right side (until Bob measures it), but with associativity it is clear which party is in possession of which qubits.

Lastly we examine what the unit object and the left and right unital transformations represent in physics. The unit object I in regards to category theory is similar to the other objects in the category with the special requirement that $A \otimes I \simeq A \simeq I \otimes A$. It follows that the following diagrams commutes,

$$\begin{tikzcd}
A \otimes I \arrow[r, "f \otimes 1_I"] \arrow[d, "\rho_A"'] & B \otimes I \arrow[d, "\rho_B"] & I \otimes A \arrow[r, "1_I \otimes f"] \arrow[d, "\lambda_A"'] & I \otimes B \arrow[d, "\lambda_B"] \\
A \arrow[r, "f"'] & B & A \arrow[r, "f"'] & B
\end{tikzcd}$$

The unit object intuitively plays the role that it does not affect another object when joined by the tensor. In nature, a system that does not interact with its surrounding systems is quite a special system in sense that it is no system at all. The unit system is the absence of a system, and so introducing nothing to another system should not have any effect on it, that is, the isomorphic behavior of the tensor between a system and a unit object, is preserved.

5 Conclusion

In summary, we have seen that a symmetric monoidal category can model some simple properties of physics. Although physical interactions can behave in a much more complicated manner than a simple change of system state, using the idea of a monoidal category can help visualize physical interactions in a new way. The information presented can be extended further to mimic even more complicated ideas such as quantum entanglement, and can be used to prove theorems like the no-cloning and no-deleting theorem via the commutativity (or lack thereof) of diagrams. Overall, it is a good tool for physicists to keep in mind, since it can simplify concepts to be understood more deeply.

References

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[3] Urs Schreiber - https://ncatlab.org/nlab/show/symmetric+monoidal+category

[4] John Baez Blog https://golem.ph.utexas.edu/category/2006/08/quantum_computation_and_symmet.html