Functional Analysis Lecture Notes

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Function analysis

- is the foundation of modern analysis
- grew up in the 1920's
- is the mathematical framework for quantum theory and many continuous models in engineering, economics, etc.
- combines analysis, topology, and algebra
- studies topological vector spaces and mappings between them
- generalizes linear algebra to infinite dimensions

Topological Spaces

Definition 1. Let X be a set and τ a collection of subsets of X. (X, τ) is a *topological space* iff

1. {X, $\varnothing$} $\in \tau$
2. U, V $\in \tau \to$ U $\cap$ V $\in \tau$
3. τ is closed under arbitrary unions, that is, if $\forall i \in I : U_i \in \tau$ then $\bigcup_{i \in I} U_i \in \tau$

τ is then called a *topology*, its elements are *open sets* and their complements are *closed sets*.

Definition 2. Open Set
A set is called open if it is a neighbourhood of every point.

Definition 3. Neighbourhood
A neighbourhood of set S is a set P that contains an open set U so $S \subset U \subset P$.

Topological spaces are the most general framework for 'doing analysis' (i.e. where notions like convergence, continuity, compactness, etc. are defined). The family of all topologies on X is partially ordered by inclusions if τ_1, τ_2 are two topologies, we call τ_1 "weaker"/"courser" than τ_2 iff $\tau_1 \subseteq \tau_2$. The "discrete topology" $\tau_{discrete} \coloneqq \{U \subseteq X\}$ is thus the strongest and the "trivial (or indiscrete) topology" $\tau_{trivial} \coloneqq \{\varnothing, X\}$ the weakest.

Definition 4. Let (X, τ) be a topological space and $A \subseteq X$.

- The **closure** of A, $\overline{A}$, is the smallest closed set including A. (i.e. the intersection of all closed sets containing A).
- The **interior** of A, $\mathrm{int}(A)$, is the largest open set contained in A.
- The **boundary** of A is defined as $\partial A \coloneqq \overline{A} \setminus \mathrm{int}(A) = \overline{A} \cap \overline{[XA]}$.
- A is called **dense** in X iff $\overline{A} = X$.
- A is called **neighbourhood** of $x \in X$ iff $\exists U \in \tau : x \in U \subseteq A$.
- $B \subseteq \tau$ is called a **base** for τ iff $\exists U \in \tau$ that is of the form $U = \cup_\alpha V_\alpha$ for some family $\{V_\alpha \subseteq B$.
- (X, tau) is called **Hausdorff Space** iff $\forall x, y \in X, \exists U_x, U_y \in \tau : x \in U_x \wedge y \in U_y \wedge (X \neq y \Rightarrow U_x \cap U_y = \varnothing$.
- (X, τ) is called **seperable** iff it contains a countable, dense, subset $C \subseteq X = \tau$.

There are two important ways of constructing topologies:

1. **Metric Topologies**: If (X, d) is a metric space, then d induces a topology τ via

$$U \in \tau \Leftrightarrow \forall x \in U \exists \epsilon > 0 : B_\epsilon(x) \subset U$$

   where $B_\epsilon := \{y \in X | d(x, y) < \epsilon\}$.

   - $\{B_\epsilon(x)\}_{\epsilon>0, x\in X}$ is then a base for the topology.
   - A topological space for which there is such a metric is called **metrizable**.
   - Metrizable spaces are Hansdorff. Hence, non-Hansdorff topologies (such as the trival topology if $|X| \geq 2$ or the Zariski topology from algebraic geometry) are not metrizable.
   - Most notions used in metric spaces have a topological generalization (see above). Important exceptions are **completeness**, **boundedness**, and **uniform continuity**.

2. **Weak topologies**. We need the notion of continuity for this.

   Definition 5. Let $(X_i, \tau_i)_{(i\in\{1,2\})}$ be topological spaces and $f : X_1 \to X_2$. f is called **continuous** iff the pre-image of any open set is open. It is called **open** iff the image of any open set is open and a **homeomorphism** iff it is an open, continuous bijection. Equivalently iff it is bijective and f as well as f^{-1} are continuous.

Note that for any function $f : X_1 \to X_2$ between the sets, there are always topologies with respect to which f is continuous. E.g. if tau_1 is the discrete topology or if τ_2 is the trivial topology.

A so-called **weak topology** is now defined by requiring a family $\overline{f}$ of functions from a set S into a topological space (X, τ) to be continuous. In order to make this construction, unique, one takes the weakest topology on S for which this is the case.

A base for this topology is given by all finite intersections of sets of the form $f^{-1}(u)$ where $f \in \overline{f}$ and $u \in \tau$.

Three ways of constructing new topological spaces from old ones:

1. The **subspace** topology if a subset S of a topological space (X, τ) is $\tau_S := \{V \subseteq S | \exists U \in \tau : U \cap S = V\}$. (Elements of τ_S are sometimes called **relatively open**.
2. The **product** topology of (X_1, τ_1) and (X_2, τ_2) is defined as

$$\{U \subseteq X_1 \times X_2 | \forall (x, y) \in U \text{ there are open neighbourhoods } U_x \in \tau_1, U_y \in \tau_2 : U_x \times U_y \subseteq U\}$$

3. the **quotient** topology of a quotient $X \backslash \sim$ of (X, τ) is defined as $\{U \subseteq X \backslash \sim | q^{-1}(U) \in \tau\}$ where $q : X \to X \backslash \sim : q(x) = q(y) \Leftrightarrow x \sim y$.

Definition 6. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be equipped with the usual (i.e. metric) topology. A $\mathbb{K}$-vector space X together with a topology on X is called **topological vector space** iff $\cdot : \mathbb{K} \times X \to X$ and $+ : X \times X \to X$ are continuous with respect to the product topology.

We will later see that all Banach and Hilbert spaces are examples of topological vector spaces.

Definition 7. Let (X, τ) be a topological space.

- A sequence (X_n) in X is said to **converge** to $x_0 \in X$ iff for any neighbourhood U of x_0 there is an $m \in \mathbb{N}$ such that $\forall n > m : x_n \in U$.
- $A \subseteq X$ is called **sequentially compact** iff every sequence in A has a subsequence converging to an element in A.

- $A \subseteq X$ is called **compact** iff every open cover of A contains a finite subcover. That is,

$$(A \subseteq \cup_{i \in I} U_i \wedge \{U_i\}_{i \in I} \subseteq \tau \Rightarrow \exists \zeta \subseteq I : |\zeta| < \infty \wedge \cup_{i \in \zeta} U_i \supseteq A)$$

There is no general relation between compactness and sequential compactness. In metric spaces, however, they are equivalent.

Corollary 1. *If A is a closed subset of a compact topological space (X, τ), then A is compact.*

Proof. For any open over $\cup_{i \in I} U_i \supseteq A, (\cup_{i \in I} U_i) \cup (X \setminus A) = X$ is an open cover of X with finite subcover $\cup_{i \in \zeta} U_i \cup (X \setminus A) = X$. Hence, $\cup_{i \in \zeta} U_i \supseteq A$. ☐

Corollary 2. *A compact set K in a Hansdorff space (X, τ) is closed.*

Proof. We show that $\forall x \in X \setminus K$ there is an open neighbourhood $U \supseteq X \setminus K$. Consequently $X \setminus K$ is open. Due to the Hansdorff property : $\forall y \in K$ there are disjoint open neighbourhoods U_y, V_y with $x \in U_y$ and $y \in V_y$ where $\{V_y | y \in K\}$ is an open cover of K with finite subcover, say $\{V_i\}_{i \in \zeta}$. Then $U_i \coloneqq \cap_{i \in \zeta} U_i$ is an open neighbourhood of x disjoint from K. ☐

Products of compact spaces are again compact in the product topology (Tychonoff's Theorem). The following implies that taking quotients preserves compactness as well (as the quotient map is continuous).

Theorem 1. *If $f : X_1 \to X_2$ is a continuous function between topological spaces (X_i, τ_i) and $K \subseteq X_1$ is compact, then $f(K)$ is compact.*

Proof. Let $\cup_{i \in I} \supseteq f(K)$ be an open cover. Since $V_i \coloneqq f^{-1}(U_i)$ is open, $\cup_{i \in I} V_i \supseteq K$ is an open cover containing a finite subcover $\{V_i\}_{i \in \zeta}$. Hence,

$$\begin{aligned} f(K) \subseteq f\left(\cup_{i \in \zeta} V_i\right) &= f\left(\cup_{i \in \zeta} f^{-1}(U_i)\right) \\ &= \cup_{i \in \zeta} f\left(f^{-1}(U_i)\right) \subseteq \cup_{i \in \zeta} U_i \end{aligned}$$

☐

Since a subset of $\mathbb{R}$ is compact iff it is closed and bounded, we get:

Corollary 3. *If K is compact and $f : K \to \mathbb{R}$ continuous, then $f(K)$ has a minimum and maximum.*

Metrizable Spaces

- A **metric space** (X, d) is a set X with a distance function $d : X \times X \to [0, \infty)$ such that $\forall x, y, z \in X$
  (i) $d(x, y) \geq 0$ with equality iff $x = y$.
  (ii) $d(x, y) = d(y, x)$
  (iii) $d(x, z) \leq d(x, y) + d(y, z)$
- A subset S of a metric space is **bounded** iff $\exists r \in [0, \infty) : B_r(x) \supseteq S$ for some x.
- A metric space is called **complete** iff every Cauchy sequence converges
- A topological space (X, τ) is called **metrizable** iff it is homeomorphic to a metric space (equivalently iff there is a metric such that $\{B_{\epsilon(x)}\}_{\epsilon \geq 0, x \in X}$ is a base for τ, i.e. the topology is **induced** by the metric) and **completely metrizable** iff it is homeomorphic to a complete metric space.
- In metric spaces: completeness $\Leftrightarrow$ sequential compactness
- Every normal vector space becomes a metric space with $d(x, y) \coloneqq \|x - y\|$
- Separable completely metrizable topological spaces are called **Polish spaces**

- Since a subset of a metric space is closed iff it contains all limit points of a sequence, the topology of a metric space is completely determined by its converging sequences. In general, one has to consider so-called **nets**.

Lemma 1. *Let d_1, d_2 be two metrics on X. They induce the same topology / are topologically equivalent if there are $k_1, k_2 > 0$ such that $\forall x, y \in X$, $k_1 d_2(x, y) \leq d_1(x, y) \leq k_2 d_2(x, y)$.*

Proof. Define $B^i_\epsilon(x) :- \{y \in X | d_i(x, y) < \epsilon\}$. We have to show that $B^1_\epsilon(x)$ is open in the topology induced by d_2. By symmetry, the same holds for $1 \leftrightarrow 2$. $\forall y \in B^1_\epsilon(x) \exists \delta > 0 : B^1_\delta(y) \subseteq B^1_\epsilon(x)$. Hence, $B^2_{\delta/k_1}(y) \subseteq B^1_\epsilon(x)$. So for any $y \in B^1_\epsilon(x)$ there is an open B^2 neighbourhood inside B^1, which implies that B^1 is open with respect to τ_2. □

Definition 8. **Isometries**. Let $(X_i, d_i)_{i\in\{1,2\}}$ be a metric space.

- $f : X_1 \to X_2$ is called **isometry** iff $\forall x, y \in X_1 : d_1(x, y) = d_2(f(x), f(y))$.
- The two metric spaces are called **isometric** iff there is a bijective isometry $f : X_1 \to X_2$.

Note that an isometry is automatically injective and that the inverse of a bijective isometry is again an isometry.

Definition 9. **Completion**. A complete metric space $(y, delta)$ is a **completion** of the metric space (X, d) iff there exists an isometry $f : X \to Y$ such that $\overline{f(X)} = Y$.

In practice, one often identifies X with $f(X)$ and thus considers X as a dense subset of Y.

Theorem 2. ***Existence of a completion*** *Every metric space (X, d) has a completion.*

Proof. (Sketch) Define $y := \{Cauchy - sequenceinX\} \backslash \sim$ where $(x_n) \sim (y_n) \Leftrightarrow \lim_{n\to\infty} d(x_n, y_n) = 0$ and $\delta Y \times Y \to [0, \infty)$. $\delta([x], [y]) := \lim_{n\to\infty} d(x_n, y_n)$. (Y, δ) can be shown to be a complete metric space. Take $f : X \to Y$ such that $f(a) := (a, a, a, ...)$. This is an isometry, since $\delta(f(a), f(b)) = \lim_{n\to\infty} d(a, b) = d(a, b)$.

Moreover, $\forall [x] \in Y$, we can construct a sequence in $f(x)$ that convergest to $[x]$. In fact, for $y_n := f(x_n)$ we obtain $\delta(y_n, [x]) = \lim_{n\to\infty} d(y_{n,m}, x_m) = \lim_{n\to\infty} d(x_n, x_m)$. So $\delta(y_n, [x]) \to 0$ for some $n \to \infty$ since (x_n) is a Cauchy sequence. □

- The completion of a metric space is unique in the sense that any two completions are isometric.
- Since $f(x)$ is dense in Y, the completion of a separable metric space is again separable.
- If there is a scalar product or norm, the completion preserves this structure.

Example 1. *From the homework we know that $(C([0, 1], \mathbb{K}, \| \cdot \|_\infty)$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is complete.*

Theorem 3. ***Weierstrass Approximation Theorem****. The set X of polynomials on $[0, 1]$ is dense in Y (with respect to the metric induced by $\| \cdot \|_\infty$).*

Consequently, Y is the completion of X with isometry $f : X \to Y, f \mapsto f$.

Theorem 4. *If $S \subseteq X$ is a subset of a complete metric space (X, d), then (S, d) is compelte iff S is closed.*

Proof. $(\Rightarrow)$: Let (X_n) be a sequence in S that converges to $x \in X$. (X_n) is thus a Cauchy sequence and $x \in S$ by completeness of S.

$(\Leftarrow)$: Let (X_n) be a Cauchy sequence in S. By completeness of X it converges to some $x \in X$. Since S is closed, $x \in S$. □

- The $(\Rightarrow)$ part actually shows that every complete subspace of a (not necessarily complete) metric space is closed.
- The theorem enables us to prove **completeness of a space** by

(i) Considering it as a subspace of a larger complete space

(ii) Proving closedness

Theorem 5. ***Baire's Category Theorem*** *If (U_n) is a sequence of open, dense subsets of a complete metric space (X, d), then the intersection $U \coloneqq \cup_{n in \mathbb{N}}$ is dense in X*

Proof. We want to show that $B_{\epsilon_0}(x_0) \cup U \neq \varnothing$ for any $\epsilon_0 > 0, x_0 \in X$. Define $\overline{B}_\epsilon(x) \coloneqq \{y \in X | d(x, y) \leq \epsilon\}$ and note that $\overline{B}_\epsilon \subseteq B_{2\epsilon}(x)$. Since U_n is open and dense, $U_n \cup B_{\epsilon_0}(x_0)$ is open and non-empty. Consequently, there is $x_1 \in X$ and $\epsilon_1 > 0$ such that $U_1 \cup B_{\epsilon_0}(x_0) \supseteq B_{2\epsilon_0}(x_0) \supseteq \overline{B}_{\epsilon_1}(x_1)$. Now construct a sequence of points (x_n) in X and radii (ϵ_n) such that $\overline{B}_{\epsilon_n}(x_n) \subseteq U_n \cup B_{\epsilon_{n-1}}(x_{n-1})$ and $\epsilon_n \in (0, 2^{-n}\epsilon_0]$. By construction, $\overline{B}_{\epsilon_n}(x_n) \subseteq \overline{B}_{n-1}(x_{n-1})$ and $\epsilon_n \to 0$ for $n \to \infty$. In particular, $d(x_n, x_m) \leq \epsilon_N = 2^{-N}\epsilon_0, \forall n, m > N$. So, (x_n) is a Cauchy sequence and by completeness $\lim_{n \to 0} x_n = x_\infty \in \cup_{n \in \mathbb{N}} \overline{B}_{\epsilon_n}(x_n)$. Since $\forall n \in \mathbb{N} : x_\infty \in \overline{B}_{\epsilon_n}(x_n) \subseteq U_n$ we have $x_\infty \in B_{\epsilon_0}(x_0) \cup U$. ☐

A useful consequence is formulated in terms of **nowhere dense**.

Definition 10. A subset S of a topological space is called **nowhere dense** iff its closure has empty interior : $\mathrm{int}(\overline{S} = \varnothing$.

Note that S is nowhere dense iff its closure is.

Corollary 4. *A complete metric space is never the countable union of nowhere dense sets*

Proof. Assume $X = \bigcup_{n \in \mathbb{N}} c_n$ where c_n is closed, that is, $U_n \coloneqq X \setminus c_n$ open. Then $\bigcup_{n \in \mathbb{N}} U_n = \bigcup_n (X \setminus c_n) = X \setminus \bigcup_n c_n = \varnothing$. By Baire's category theorem, at least one of the $U_n$is not dense. Then, however, $\mathrm{int}(c_n) = X \setminus \overline{U_n} \neq \varnothing$. ☐

Normed and Banach Spaces

Definition 11. Let X be a $\mathbb{K}$-vector space, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

- $\| \cdot \| : X \mapsto [0, \infty)$ is called a **seminorm** iff $\forall x, y \in X, \forall \alpha \in \mathbb{K}$
  (i) $\|x + y\| \leq \|x\| + \|y\|$
  (ii) $\|\alpha x\| = |\alpha| \|x\|$
- A seminorm is a **norm** iff $\|x\| = 0 \Rightarrow x = 0$. $(X, \|\cdot\|)$ is then called a **normed space**.
- A **Banach Space** is a normed space that is complete
- The **linear span** of a subset $S \subseteq X$ is defined as $\mathrm{span}(S) \coloneqq \{\sum_{i=1}^n \alpha_i x_i | n \in \mathbb{N} x_i \in S, \alpha_i \in \mathbb{R}\}$
- **Completeness** refers to the metric space with $d(x, y) \coloneqq \|x - y\|$.
- A metric is induced by a norm iff $\forall x, y, z \in X \forall \alpha \in \mathbb{K}$:
  (i) $d(x + z, y + z) = d(x, y)$
  (ii) $d(\alpha x, \alpha y) = |\alpha| d(x, y)$

  To see that this is sufficient for the existence of a norm, consider $\|x\| = d(x, 0)$ and verify the necessary properties.
- Examples of **normed sequence spaces**
  (i) $l_\infty \coloneqq \left\{x \in \mathbb{K}^\mathbb{N} \mid \|x\|_\infty \coloneqq \sup_n |x_n| \leq \infty\right\}$
  (ii) $c_0 \coloneqq \left\{x \in \mathbb{K}^\mathbb{N} \mid \lim_{n \to \infty} x_n = 0\right\}$ with $\|\cdot\|_\infty$
  (iii) $l_p \coloneqq \left\{x \in \mathbb{K}^\mathbb{N} \mid \|x\|_p \coloneqq \left(\sum_n |x_n|\right)^{1/p} < \infty\right\}, p \in [1, \infty)$

(iv) $c_c \coloneqq \left\{x \in \mathbb{K}^{\mathbb{N}} \mid |\{n \mid x_n \neq 0\}| < \infty\right\}$

As sets, these satisfy $c_c \subseteq l_p \subseteq l_q \subseteq c_0 \subseteq l_0$ for $p \leq q < \infty$. $(l_\infty, \|\cdot\|_\infty)$ is a Banach space. Since c_0 is a closed subset in l_∞, $(c_0, \|\cdot\|_\infty)$ is also a Banach space. Also $\left(l_p, \|\cdot\|_p\right)$ are Banach spaces. c_c is dense in l_p for $p < \infty$ with respect to $\|\cdot\|_p$ and c_c is dense in c_0 with respect to $\|\cdot\|_\infty$. Since $c_c \neq l_p$, it is not a Banach space for any $\|\cdot\|_p$, $p \in [1, \infty]$.

- l_∞ can be generalized to arbitrary sets X:

$$l_\infty(X) \coloneqq \left\{f \in \mathbb{K}^{\mathbb{N}} \mid \|f\|_\infty \coloneqq \sup_{x \in X} |f(x)| < \infty\right\}.$$

$(l_\infty(x), \|\cdot\|_\infty)$ then turns out to be a Banach space.

- Examples of **normed function spaces**:

(1) **Continuous functions**: for any topological space (X, τ) define,

(i) $c_b(x) \coloneqq \{f : X \mapsto \mathbb{K} \mid f \text{ is continuous and } \|f\|_\infty \coloneqq \sup_{x \in X} |f(x)| < \infty\}$ (bounded)

(ii) $c_0(x) \coloneqq \{f : X \mapsto \mathbb{K} \mid f \text{ is continuous} \wedge \forall \epsilon > 0 : \{x \in X \mid |f(x)| \geq \epsilon\}\}$ is compact (vanishing at infinity)

(iii) $c_c(x) \coloneqq \left\{f : X \mapsto \mathbb{K} \mid f \text{ is continuous} \wedge \sup(f) \coloneqq \overline{\{x \in X \mid f(x) \neq 0\}}\right\}$ is compact (compact support)

Clearly, $c_c(x) \subseteq c_0(x) \subseteq c_b(x)$ with equality for compact X. In that case we write $C(x)$. With respect to $\|\cdot\|_\infty$ the space $c_b(x)$ is a closed subspace of $c_\infty(x)$ (recall that uniform limits of continuous bounded functions are still continuous and bounded). So $(c_b(x), \|\cdot\|_\infty)$ is a Banach space. Similarly, $c_0 x$ is closed in $c_b(x)$ with respect to $\|\cdot\|_\infty$.

(2) L_p**-spaces**: Let (X, Σ, μ) be a measure space and $p \in [1, \infty]$. $L_p(x)$ denotes the set of equivalent classes of measure functions $f : X \mapsto \mathbb{C}$ for which $\|f\|_p \coloneqq \left(\int_x |f(x)|^p d\mu(x)\right)^{1/p} < \infty$. Here, $f \sim g$ iff they differ only on a set of measure zero. Similarly, $L_\infty(x)$ is the set of equivalence classes of measure functions $f : X \mapsto \mathbb{C}$ for which the **essential supremum** is finite, that is,

$$\|f\|_\infty \coloneqq \inf\{\mu \in \mathbb{R} \mid \mu\{x \mid |f(x)| > \mu\} = 0\} < \infty$$

Properties

- $\left(L_p(x), \|\cdot\|_p\right)$ is a Banach space $\forall p \in [1, \infty]$.
- Hölders inequality: If $p \in [0, \infty], \frac{1}{p} + \frac{1}{q} = 1, f \in L_p(x), g \in L_q(x)$, then $fg \in L_1(x)$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$
- For $(X, \Sigma, \mu) = \left(\mathbb{N}, 2^{\mathbb{N}}, \text{counting measure}\right)$ we obtain $L_p(x) = L_p$.
- If $p < q$, then $\|f\|_p \leq \|f\|_q \mu(x)^{\frac{q-p}{pq}}$, so $\mu(x) < \infty \Rightarrow L_p \supseteq L_q$. (but $L_p \subseteq L_q$).

(3) **Spaces of Differentiable functions**: The space $c^n([a, b], \mathbb{K})$ of $n \in \mathbb{N}$ times continuously differentiable functions is a subspace of $c([a, b])$, but not closed with respect to $\|\cdot\|_\infty$ (e.g. $\sqrt{x^2 + \frac{1}{n}}\ \ n \to \infty\ \ |x|$). It becomes a Banach space when equipped with $\|f\| \coloneqq \sum_{k=0}^n \left\|f^{(k)}\right\|_\infty$. (Note that now for the above example $\|f_n - f_m\| \not\to 0$).

(4) **Spaces of Holomorphic Functions**: For $mathbbD \coloneqq \{z \in \mathbb{C} \mid |z| < 1\}$ the **Hardy space** H_∞ is the space of all bounded holomorphic functions $f : \mathbb{D} \to \mathbb{C}$. This is a closed subspace of $(C_b(\mathbb{D}), \|\cdot\|_\infty)$ since uniform limits of holomorphic functions are holomorphic. So $(H_\infty, \|\cdot\|_\infty)$ is a Banach space.

Theorem 6. *A normed space $(X, \|\cdot\|)$ is complete iff every absolutely convergent series converges (i.e. $\sum_{k=1}^\infty \|x_k\| < \infty \to \exists x_\infty \in X : \|\sum_{k=1}^n x_k - x_\infty\| \to 0$).*

Proof. Assume $(X, \|\cdot\|)$ is complete and $\sum_{k=1}^{\infty} < \infty$. Define $S_n := \sum_{k=1}^{n} x_k$. With $n > m : \|S_n - S_m\| = \left\|\sum_{k=m+1}^{n} x_k\right\| \leq \sum_{k=m+1}^{n} \|x_k\| \leq \sum_{k=m+1}^{\infty} \|x_k\| \to 0$. Hence, (S_n) is a Cauchy sequence and thus convergent.

In the other direction, consider a Cauchy sequence (x_n). By taking a subsequence we can, without loss of generality, ensure that this converges fast in the sense that $\|x_n - x_m\| < 2^{-n}$ for $n \leq m$. With $x_0 := 0$ and $y_n := x_k - x_{k-1}$ we obtain $x_n = \sum_{k=1}^{n} y_k$ and $\|y_k\| < 2^{1-k}$. So $\sum_k y_k$ is absolutely convergent, and by assumption $x_n = \sum_{k=1}^{n} y_k \to x_\infty \in X$. $\square$

Remark 1. In a Banach space every absolutely convergent series converges unconditionally in the sense that $\sum_k x_k = \sum_{\pi(k)}$ for every permutation $\pi : \mathbb{N} \to \mathbb{N}$. The converse holds (unconditional $\Rightarrow$ absolute convergence) iff the Banach space is finite dimensional.

Corollary 5. *$(L_\infty, \|\cdot\|)$ is complete.*

Proof. Suppose $x_n \in L_\infty$ and $\sum_{n=0}^{\infty} \|x_n\|_\infty < \infty$. Then $\|\sum_{n=0}^{\infty}\|_\infty \leq \sum n = 0^\infty \|x_n\|_\infty < \infty$, so $x \in L_\infty$. Moreover, $\sum_{n=1}^{N} x_n \quad N \to \infty \quad x$ since $\left\|x - \sum_{n=1}^{N} x_n\right\|_\infty = \|\sum_{x_n}\| \leq \sum_{n=N+1}^{\infty} \|x_n\| \to 0 \quad N \to \infty$ $\square$

Theorem 7. ***(Mazur-Ulam)*** *Every surjective isometry $f : X \to Y$ between real normed spaces is affine (i.e., $\tilde{f}(x) := f(x) - f(0)$ is linear).*

Proof. It is sufficient to prove $f(\frac{x+y}{2}) = \frac{1}{2}(f(x) + f(y))$ for any $x, y \in X$. The rest is done by iteration and continuity. Fix x, y and define $\Delta(f) := \left\|f(\frac{x+y}{2} - \frac{f(x)+f(y)}{2}\right\|$. Δ is then uniformly bounded over all isometries in the sense that

$$\Delta(f) \leq \frac{1}{2}\left\|f(\frac{x+y}{2}) - f(x)\right\| + \frac{1}{2}\left\|f(\frac{x+y}{2}) - f(y)\right\| = \frac{\|x-y\|}{2}$$

For any bijective isometry f we can, however, construct another bijective isometry f' such that $\Delta(f') = z\Delta(f)$. Iterating this, it eventually contradicts the bound unless $\Delta(f) = 0$. f' is constructed ad follows:

$$f' := f^{-1} \circ \psi \circ f \text{ with } \psi(z) := f(x) + f(y) - z,$$

a reflection such that $f'(x) = y, f'(y) = x, \psi = \psi^{-1}$ is an isometry. Then

$$\begin{aligned}
\Delta(f') &= \left\|f^{-1}\left(f(x) - f(y) - f\left(\frac{x+y}{2}\right)\right) - \frac{x+y}{2}\right\| \\
&= \left\|f(x) - f(y) - f\left(\frac{x+y}{2}\right) - f\left(\frac{x+y}{2}\right)\right\| \\
&= 2\Delta(f)
\end{aligned}$$

$\square$

The assumptions that f is surjective or that the spaces are over $\mathbb{R}$ cannot be dropped. However, if y is so-called 'strictly convex' (i.e. if its unit ball is strictly convex), then surjectiveity is not required. Our $\mathbb{C}$ linearity has to be replied by linear or conjugate-linear (which is the case for complex conjugation).