

Functional Analysis Lecture Notes

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Function analysis

- is the foundation of modern analysis
- grew up in the 1920's
- is the mathematical framework for quantum theory and many continuous models in engineering, economics, etc.
- combines analysis, topology, and algebra
- studies topological vector spaces and mappings between them
- generalizes linear algebra to infinite dimensions

Topological Spaces

Definition 1. Let X be a set and τ a collection of subsets of X . (X, τ) is a *topological space* iff

1. $\{X, \emptyset\} \in \tau$
2. $U, V \in \tau \rightarrow U \cap V \in \tau$
3. τ is closed under arbitrary unions, that is, if $\forall i \in I : U_i \in \tau$ then $\bigcup_{i \in I} U_i \in \tau$

τ is then called a *topology*, its elements are *open sets* and their complements are *closed sets*.

Definition 2. Open Set

A set is called open if it is a neighbourhood of every point.

Definition 3. Neighbourhood

A neighbourhood of set S is a set P that contains an open set U so $S \subset U \subset P$.

Topological spaces are the most general framework for 'doing analysis' (i.e. where notions like convergence, continuity, compactness, etc. are defined). The family of all topologies on X is partially ordered by inclusions if τ_1, τ_2 are two topologies, we call τ_1 "weaker"/"coarser" than τ_2 iff $\tau_1 \subseteq \tau_2$. The "discrete topology" $\tau_{discrete} := \{U \subseteq X\}$ is thus the strongest and the "trivial (or indiscrete) topology" $\tau_{trivial} := \{\emptyset, X\}$ the weakest.

Definition 4. Let (X, τ) be a topological space and $A \subseteq X$.

- The **closure** of A , \overline{A} , is the smallest closed set including A . (i.e. the intersection of all closed sets containing A).
- The **interior** of A , $\text{int}(A)$, is the largest open set contained in A .
- The **boundary** of A is defined as $\partial A := \overline{A} \setminus \text{int}(A) = \overline{A} \cap \overline{X \setminus A}$.
- A is called **dense** in X iff $\overline{A} = X$.
- A is called **neighbourhood** of $x \in X$ iff $\exists U \in \tau : x \in U \subseteq A$.
- $B \subseteq \tau$ is called a **base** for τ iff $\exists U \in \tau$ that is of the form $U = \bigcup_{\alpha} V_{\alpha}$ for some family $\{V_{\alpha} \subseteq B\}$.
- (X, τ) is called **Hausdorff Space** iff $\forall x, y \in X, \exists U_x, U_y \in \tau : x \in U_x \wedge y \in U_y \wedge (X \neq y \Rightarrow U_x \cap U_y = \emptyset)$.
- (X, τ) is called **separable** iff it contains a countable, dense, subset $C \subseteq X = \tau$.

There are two important ways of constructing topologies:

1. **Metric Topologies:** If (X, d) is a metric space, then d induces a topology τ via

$$U \in \tau \Leftrightarrow \forall x \in U \exists \epsilon > 0 : B_\epsilon(x) \subset U$$

where $B_\epsilon := \{y \in X | d(x, y) < \epsilon\}$.

- $\{B_\epsilon(x)\}_{\epsilon > 0, x \in X}$ is then a base for the topology.
- A topological space for which there is such a metric is called **metrizable**.
- Metrizable spaces are Hausdorff. Hence, non-Hausdorff topologies (such as the trivial topology if $|X| \geq 2$ or the Zariski topology from algebraic geometry) are not metrizable.
- Most notions used in metric spaces have a topological generalization (see above). Important exceptions are **completeness**, **boundedness**, and **uniform continuity**.

2. **Weak topologies.** We need the notion of continuity for this.

Definition 5. Let $(X_i, \tau_i)_{i \in \{1, 2\}}$ be topological spaces and $f : X_1 \rightarrow X_2$. f is called **continuous** iff the pre-image of any open set is open. It is called **open** iff the image of any open set is open and a **homeomorphism** iff it is an open, continuous bijection. Equivalently iff it is bijective and f as well as f^{-1} are continuous.

Note that for any function $f : X_1 \rightarrow X_2$ between the sets, there are always topologies with respect to which f is continuous. E.g. if τ_1 is the discrete topology or if τ_2 is the trivial topology.

A so-called **weak topology** is now defined by requiring a family \bar{f} of functions from a set S into a topological space (X, τ) to be continuous. In order to make this construction, unique, one takes the weakest topology on S for which this is the case.

A base for this topology is given by all finite intersections of sets of the form $f^{-1}(u)$ where $f \in \bar{f}$ and $u \in \tau$.

Three ways of constructing new topological spaces from old ones:

1. The **subspace** topology if a subset S of a topological space (X, τ) is $\tau_S := \{V \subseteq S | \exists U \in \tau : U \cap S = V\}$. (Elements of τ_S are sometimes called **relatively open**.)
2. The **product** topology of (X_1, τ_1) and (X_2, τ_2) is defined as

$$\{U \subseteq X_1 \times X_2 | \forall (x, y) \in U \text{ there are open neighbourhoods } U_x \in \tau_1, U_y \in \tau_2 : U_x \times U_y \subseteq U\}$$

3. the **quotient** topology of a quotient $X \setminus \sim$ of (X, τ) is defined as $\{U \subseteq X \setminus \sim | q^{-1}(U) \in \tau\}$ where $q : X \rightarrow X \setminus \sim : q(x) = q(y) \Leftrightarrow x \sim y$.

Definition 6. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ be equipped with the usual (i.e. metric) topology. A \mathbb{K} -vector space X together with a topology on X is called **topological vector space** iff $\cdot : \mathbb{K} \times X \rightarrow X$ and $+$: $X \times X \rightarrow X$ are continuous with respect to the product topology.

We will later see that all Banach and Hilbert spaces are examples of topological vector spaces.

Definition 7. Let (X, τ) be a topological space.

- A sequence (X_n) in X is said to **converge** to $x_0 \in X$ iff for any neighbourhood U of x_0 there is an $m \in \mathbb{N}$ such that $\forall n > m : x_n \in U$.
- $A \subseteq X$ is called **sequentially compact** iff every sequence in A has a subsequence converging to an element in A .

- $A \subseteq X$ is called **compact** iff every open cover of A contains a finite subcover. That is,

$$(A \subseteq \cup_{i \in I} U_i \wedge \{U_i\}_{i \in I} \subseteq \tau \Rightarrow \exists \zeta \subseteq I : |\zeta| < \infty \wedge \cup_{i \in \zeta} U_i \supseteq A)$$

There is no general relation between compactness and sequential compactness. In metric spaces, however, they are equivalent.

Corollary 1. *If A is a closed subset of a compact topological space (X, τ) , then A is compact.*

Proof. For any open cover $\cup_{i \in I} U_i \supseteq A$, $(\cup_{i \in I} U_i) \cup (X \setminus A) = X$ is an open cover of X with finite subcover $\cup_{i \in \zeta} U_i \cup (X \setminus A) = X$. Hence, $\cup_{i \in \zeta} U_i \supseteq A$. \square

Corollary 2. *A compact set K in a Hausdorff space (X, τ) is closed.*

Proof. We show that $\forall x \in X \setminus K$ there is an open neighbourhood $U \supseteq X \setminus K$. Consequently $X \setminus K$ is open. Due to the Hausdorff property : $\forall y \in K$ there are disjoint open neighbourhoods U_y, V_y with $x \in U_y$ and $y \in V_y$ where $\{V_y | y \in K\}$ is an open cover of K with finite subcover, say $\{V_i\}_{i \in \zeta}$. Then $U_i := \cap_{i \in \zeta} U_i$ is an open neighbourhood of x disjoint from K . \square

Products of compact spaces are again compact in the product topology (Tychonoff's Theorem). The following implies that taking quotients preserves compactness as well (as the quotient map is continuous).

Theorem 1. *If $f : X_1 \rightarrow X_2$ is a continuous function between topological spaces (X_i, τ_i) and $K \subseteq X_1$ is compact, then $f(K)$ is compact.*

Proof. Let $\cup_{i \in I} V_i \supseteq f(K)$ be an open cover. Since $V_i := f^{-1}(U_i)$ is open, $\cup_{i \in I} V_i \supseteq K$ is an open cover containing a finite subcover $\{V_i\}_{i \in \zeta}$. Hence,

$$\begin{aligned} f(K) &\subseteq f(\cup_{i \in \zeta} V_i) = f(\cup_{i \in \zeta} f^{-1}(U_i)) \\ &= \cup_{i \in \zeta} f(f^{-1}(U_i)) \subseteq \cup_{i \in \zeta} U_i \end{aligned}$$

\square

Since a subset of \mathbb{R} is compact iff it is closed and bounded, we get:

Corollary 3. *If K is compact and $f : K \rightarrow \mathbb{R}$ continuous, then $f(K)$ has a minimum and maximum.*

Metrisable Spaces

- A **metric space** (X, d) is a set X with a distance function $d : X \times X \rightarrow [0, \infty)$ such that $\forall x, y, z \in X$
 - (i) $d(x, y) \geq 0$ with equality iff $x = y$.
 - (ii) $d(x, y) = d(y, x)$
 - (iii) $d(x, z) \leq d(x, y) + d(y, z)$
- A subset S of a metric space is **bounded** iff $\exists r \in [0, \infty) : B_r(x) \supseteq S$ for some x .
- A metric space is called **complete** iff every Cauchy sequence converges
- A topological space (X, τ) is called **metrizable** iff it is homeomorphic to a metric space (equivalently iff there is a metric such that $\{B_{\epsilon(x)}\}_{\epsilon \geq 0, x \in X}$ is a base for τ , i.e. the topology is **induced** by the metric) and **completely metrizable** iff it is homeomorphic to a complete metric space.
- In metric spaces: completeness \Leftrightarrow sequential compactness
- Every normed vector space becomes a metric space with $d(x, y) := \|x - y\|$
- Separable completely metrizable topological spaces are called **Polish spaces**

- Since a subset of a metric space is closed iff it contains all limit points of a sequence, the topology of a metric space is completely determined by its converging sequences. In general, one has to consider so-called **nets**.

Lemma 1. *Let d_1, d_2 be two metrics on X . They induce the same topology / are topologically equivalent if there are $k_1, k_2 > 0$ such that $\forall x, y \in X, k_1 d_2(x, y) \leq d_1(x, y) \leq k_2 d_2(x, y)$.*

Proof. Define $B_\epsilon^i(x) := \{y \in X \mid d_i(x, y) < \epsilon\}$. We have to show that $B_\epsilon^1(x)$ is open in the topology induced by d_2 . By symmetry, the same holds for $1 \leftrightarrow 2$. $\forall y \in B_\epsilon^1(x) \exists \delta > 0 : B_\delta^2(y) \subseteq B_\epsilon^1(x)$. Hence, $B_{\delta/k_1}^2(y) \subseteq B_\epsilon^1(x)$. So for any $y \in B_\epsilon^1(x)$ there is an open B^2 neighbourhood inside B^1 , which implies that B^1 is open with respect to τ_2 . \square

Definition 8. Isometries. Let $(X_i, d_i)_{i \in \{1,2\}}$ be a metric space.

- $f : X_1 \rightarrow X_2$ is called **isometry** iff $\forall x, y \in X_1 : d_1(x, y) = d_2(f(x), f(y))$.
- The two metric spaces are called **isometric** iff there is a bijective isometry $f : X_1 \rightarrow X_2$.

Note that an isometry is automatically injective and that the inverse of a bijective isometry is again an isometry.

Definition 9. Completion. A complete metric space (Y, d) is a **completion** of the metric space (X, d) iff there exists an isometry $f : X \rightarrow Y$ such that $f(X) = Y$.

In practice, one often identifies X with $f(X)$ and thus considers X as a dense subset of Y .

Theorem 2. Existence of a completion Every metric space (X, d) has a completion.

Proof. (Sketch) Define $y := \{\text{Cauchy-sequence in } X\} \sim$ where $(x_n) \sim (y_n) \Leftrightarrow \lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ and $\delta Y \times Y \rightarrow [0, \infty)$. $\delta([x], [y]) := \lim_{n \rightarrow \infty} d(x_n, y_n)$. (Y, δ) can be shown to be a complete metric space. Take $f : X \rightarrow Y$ such that $f(a) := (a, a, a, \dots)$. This is an isometry, since $\delta(f(a), f(b)) = \lim_{n \rightarrow \infty} d(a, b) = d(a, b)$.

Moreover, $\forall [x] \in Y$, we can construct a sequence in $f(X)$ that converges to $[x]$. In fact, for $y_n := f(x_n)$ we obtain $\delta(y_n, [x]) = \lim_{n \rightarrow \infty} d(y_n, x_n) = \lim_{n \rightarrow \infty} d(x_n, x_n) = 0$. So $\delta(y_n, [x]) \rightarrow 0$ for some $n \rightarrow \infty$ since (x_n) is a Cauchy sequence. \square

- The completion of a metric space is unique in the sense that any two completions are isometric.
- Since $f(X)$ is dense in Y , the completion of a separable metric space is again separable.
- If there is a scalar product or norm, the completion preserves this structure.

Example 1. From the homework we know that $(C([0, 1], \mathbb{K}), \|\cdot\|_\infty)$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is complete.

Theorem 3. Weierstrass Approximation Theorem. The set X of polynomials on $[0, 1]$ is dense in Y (with respect to the metric induced by $\|\cdot\|_\infty$).

Consequently, Y is the completion of X with isometry $f : X \rightarrow Y, f \mapsto f$.

Theorem 4. If $S \subseteq X$ is a subset of a complete metric space (X, d) , then (S, d) is complete iff S is closed.

Proof. (\Rightarrow) : Let (X_n) be a sequence in S that converges to $x \in X$. (X_n) is thus a Cauchy sequence and $x \in S$ by completeness of S .

(\Leftarrow) : Let (X_n) be a Cauchy sequence in S . By completeness of X it converges to some $x \in X$. Since S is closed, $x \in S$. \square

- The (\Rightarrow) part actually shows that every complete subspace of a (not necessarily complete) metric space is closed.
- The theorem enables us to prove **completeness of a space** by
 - (i) Considering it as a subspace of a larger complete space
 - (ii) Proving closedness