

# Functional Analysis Lecture Notes

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## Function analysis

- is the foundation of modern analysis
- grew up in the 1920's
- is the mathematical framework for quantum theory and many continuous models in engineering, economics, etc.
- combines analysis, topology, and algebra
- studies topological vector spaces and mappings between them
- generalizes linear algebra to infinite dimensions

## Topological Spaces

**Definition 1.** Let  $X$  be a set and  $\tau$  a collection of subsets of  $X$ .  $(X, \tau)$  is a *topological space* iff

1.  $\{X, \emptyset\} \in \tau$
2.  $U, V \in \tau \rightarrow U \cap V \in \tau$
3.  $\tau$  is closed under arbitrary unions, that is, if  $\forall i \in I : U_i \in \tau$  then  $\bigcup_{i \in I} U_i \in \tau$

$\tau$  is then called a *topology*, its elements are *open sets* and their complements are *closed sets*.

### Definition 2. Open Set

A set is called open if it is a neighbourhood of every point.

### Definition 3. Neighbourhood

A neighbourhood of set  $S$  is a set  $P$  that contains an open set  $U$  so  $S \subset U \subset P$ .

Topological spaces are the most general framework for 'doing analysis' (i.e. where notions like convergence, continuity, compactness, etc. are defined). The family of all topologies on  $X$  is partially ordered by inclusions if  $\tau_1, \tau_2$  are two topologies, we call  $\tau_1$  "weaker"/"coarser" than  $\tau_2$  iff  $\tau_1 \subseteq \tau_2$ . The "discrete topology"  $\tau_{discrete} := \{U \subseteq X\}$  is thus the strongest and the "trivial (or indiscrete) topology"  $\tau_{trivial} := \{\emptyset, X\}$  the weakest.

**Definition 4.** Let  $(X, \tau)$  be a topological space and  $A \subseteq X$ .

- The **closure** of  $A$ ,  $\overline{A}$ , is the smallest closed set including  $A$ . (i.e. the intersection of all closed sets containing  $A$ ).
- The **interior** of  $A$ ,  $\text{int}(A)$ , is the largest open set contained in  $A$ .
- The **boundary** of  $A$  is defined as  $\partial A := \overline{A} \setminus \text{int}(A) = \overline{A} \cap \overline{X \setminus A}$ .
- $A$  is called **dense** in  $X$  iff  $\overline{A} = X$ .
- $A$  is called **neighbourhood** of  $x \in X$  iff  $\exists U \in \tau : x \in U \subseteq A$ .
- $B \subseteq \tau$  is called a **base** for  $\tau$  iff  $\exists U \in \tau$  that is of the form  $U = \bigcup_{\alpha} V_{\alpha}$  for some family  $\{V_{\alpha} \subseteq B\}$ .
- $(X, \tau)$  is called **Hausdorff Space** iff  $\forall x, y \in X, \exists U_x, U_y \in \tau : x \in U_x \wedge y \in U_y \wedge (x \neq y \Rightarrow U_x \cap U_y = \emptyset)$ .
- $(X, \tau)$  is called **separable** iff it contains a countable, dense, subset  $C \subseteq X$ .

There are two important ways of constructing topologies:

1. **Metric Topologies:** If  $(X, d)$  is a metric space, then  $d$  induces a topology  $\tau$  via

$$U \in \tau \Leftrightarrow \forall x \in U \exists \epsilon > 0 : B_\epsilon(x) \subset U$$

where  $B_\epsilon := \{y \in X | d(x, y) < \epsilon\}$ .

- $\{B_\epsilon(x)\}_{\epsilon > 0, x \in X}$  is then a base for the topology.
- A topological space for which there is such a metric is called **metrizable**.
- Metrizable spaces are Hausdorff. Hence, non-Hausdorff topologies (such as the trivial topology if  $|X| \geq 2$  or the Zariski topology from algebraic geometry) are not metrizable.
- Most notions used in metric spaces have a topological generalization (see above). Important exceptions are **completeness**, **boundedness**, and **uniform continuity**.

2. **Weak topologies.** We need the notion of continuity for this.

**Definition 5.** Let  $(X_i, \tau_i)_{i \in \{1, 2\}}$  be topological spaces and  $f : X_1 \rightarrow X_2$ .  $f$  is called **continuous** iff the pre-image of any open set is open. It is called **open** iff the image of any open set is open and a **homeomorphism** iff it is an open, continuous bijection. Equivalently iff it is bijective and  $f$  as well as  $f^{-1}$  are continuous.

Note that for any function  $f : X_1 \rightarrow X_2$  between the sets, there are always topologies with respect to which  $f$  is continuous. E.g. if  $\tau_1$  is the discrete topology or if  $\tau_2$  is the trivial topology.

A so-called **weak topology** is now defined by requiring a family  $\bar{f}$  of functions from a set  $S$  into a topological space  $(X, \tau)$  to be continuous. In order to make this construction, unique, one takes the weakest topology on  $S$  for which this is the case.

A base for this topology is given by all finite intersections of sets of the form  $f^{-1}(u)$  where  $f \in \bar{f}$  and  $u \in \tau$ .

Three ways of constructing new topological spaces from old ones:

1. The **subspace** topology if a subset  $S$  of a topological space  $(X, \tau)$  is  $\tau_S := \{V \subseteq S | \exists U \in \tau : U \cap S = V\}$ . (Elements of  $\tau_S$  are sometimes called **relatively open**.)
2. The **product** topology of  $(X_1, \tau_1)$  and  $(X_2, \tau_2)$  is defined as

$$\{U \subseteq X_1 \times X_2 | \forall (x, y) \in U \text{ there are open neighbourhoods } U_x \in \tau_1, U_y \in \tau_2 : U_x \times U_y \subseteq U\}$$

3. the **quotient** topology of a quotient  $X \setminus \sim$  of  $(X, \tau)$  is defined as  $\{U \subseteq X \setminus \sim | q^{-1}(U) \in \tau\}$  where  $q : X \rightarrow X \setminus \sim : q(x) = q(y) \Leftrightarrow x \sim y$ .

**Definition 6.** Let  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$  be equipped with the usual (i.e. metric) topology. A  $\mathbb{K}$ -vector space  $X$  together with a topology on  $X$  is called **topological vector space** iff  $\cdot : \mathbb{K} \times X \rightarrow X$  and  $+$  :  $X \times X \rightarrow X$  are continuous with respect to the product topology.

We will later see that all Banach and Hilbert spaces are examples of topological vector spaces.

**Definition 7.** Let  $(X, \tau)$  be a topological space.

- A sequence  $(X_n)$  in  $X$  is said to **converge** to  $x_0 \in X$  iff for any neighbourhood  $U$  of  $x_0$  there is an  $m \in \mathbb{N}$  such that  $\forall n > m : x_n \in U$ .
- $A \subseteq X$  is called **sequentially compact** iff every sequence in  $A$  has a subsequence converging to an element in  $A$ .

- $A \subseteq X$  is called **compact** iff every open cover of  $A$  contains a finite subcover. That is,

$$(A \subseteq \cup_{i \in I} U_i \wedge \{U_i\}_{i \in I} \subseteq \tau \Rightarrow \exists \zeta \subseteq I : |\zeta| < \infty \wedge \cup_{i \in \zeta} U_i \supseteq A)$$

There is no general relation between compactness and sequential compactness. In metric spaces, however, they are equivalent.

**Corollary 1.** *If  $A$  is a closed subset of a compact topological space  $(X, \tau)$ , then  $A$  is compact.*

*Proof.* For any open cover  $\cup_{i \in I} U_i \supseteq A$ ,  $(\cup_{i \in I} U_i) \cup (X \setminus A) = X$  is an open cover of  $X$  with finite subcover  $\cup_{i \in \zeta} U_i \cup (X \setminus A) = X$ . Hence,  $\cup_{i \in \zeta} U_i \supseteq A$ .  $\square$

**Corollary 2.** *A compact set  $K$  in a Hausdorff space  $(X, \tau)$  is closed.*

*Proof.* We show that  $\forall x \in X \setminus K$  there is an open neighbourhood  $U \supseteq X \setminus K$ . Consequently  $X \setminus K$  is open. Due to the Hausdorff property :  $\forall y \in K$  there are disjoint open neighbourhoods  $U_y, V_y$  with  $x \in U_y$  and  $y \in V_y$  where  $\{V_y | y \in K\}$  is an open cover of  $K$  with finite subcover, say  $\{V_i\}_{i \in \zeta}$ . Then  $U_i := \cap_{i \in \zeta} U_i$  is an open neighbourhood of  $x$  disjoint from  $K$ .  $\square$

Products of compact spaces are again compact in the product topology (Tychonoff's Theorem). The following implies that taking quotients preserves compactness as well (as the quotient map is continuous).

**Theorem 1.** *If  $f : X_1 \rightarrow X_2$  is a continuous function between topological spaces  $(X_i, \tau_i)$  and  $K \subseteq X_1$  is compact, then  $f(K)$  is compact.*

*Proof.* Let  $\cup_{i \in I} V_i \supseteq f(K)$  be an open cover. Since  $V_i := f^{-1}(U_i)$  is open,  $\cup_{i \in I} V_i \supseteq K$  is an open cover containing a finite subcover  $\{V_i\}_{i \in \zeta}$ . Hence,

$$\begin{aligned} f(K) &\subseteq f(\cup_{i \in \zeta} V_i) = f(\cup_{i \in \zeta} f^{-1}(U_i)) \\ &= \cup_{i \in \zeta} f(f^{-1}(U_i)) \subseteq \cup_{i \in \zeta} U_i \end{aligned}$$

$\square$

Since a subset of  $\mathbb{R}$  is compact iff it is closed and bounded, we get:

**Corollary 3.** *If  $K$  is compact and  $f : K \rightarrow \mathbb{R}$  continuous, then  $f(K)$  has a minimum and maximum.*

## Metrisable Spaces

- A **metric space**  $(X, d)$  is a set  $X$  with a distance function  $d : X \times X \rightarrow [0, \infty)$  such that  $\forall x, y, z \in X$ 
  - (i)  $d(x, y) \geq 0$  with equality iff  $x = y$ .
  - (ii)  $d(x, y) = d(y, x)$
  - (iii)  $d(x, z) \leq d(x, y) + d(y, z)$
- A subset  $S$  of a metric space is **bounded** iff  $\exists r \in [0, \infty) : B_r(x) \supseteq S$  for some  $x$ .
- A metric space is called **complete** iff every Cauchy sequence converges
- A topological space  $(X, \tau)$  is called **metrizable** iff it is homeomorphic to a metric space (equivalently iff there is a metric such that  $\{B_{\epsilon(x)}\}_{\epsilon \geq 0, x \in X}$  is a base for  $\tau$ , i.e. the topology is **induced** by the metric) and **completely metrizable** iff it is homeomorphic to a complete metric space.
- In metric spaces: completeness  $\Leftrightarrow$  sequential compactness
- Every normed vector space becomes a metric space with  $d(x, y) := \|x - y\|$
- Separable completely metrizable topological spaces are called **Polish spaces**

- Since a subset of a metric space is closed iff it contains all limit points of a sequence, the topology of a metric space is completely determined by its converging sequences. In general, one has to consider so-called **nets**.

**Lemma 1.** *Let  $d_1, d_2$  be two metrics on  $X$ . They induce the same topology / are topologically equivalent if there are  $k_1, k_2 > 0$  such that  $\forall x, y \in X, k_1 d_2(x, y) \leq d_1(x, y) \leq k_2 d_2(x, y)$ .*

*Proof.* Define  $B_\epsilon^i(x) := \{y \in X | d_i(x, y) < \epsilon\}$ . We have to show that  $B_\epsilon^1(x)$  is open in the topology induced by  $d_2$ . By symmetry, the same holds for  $1 \leftrightarrow 2$ .  $\forall y \in B_\epsilon^1(x) \exists \delta > 0 : B_\delta^1(y) \subseteq B_\epsilon^1(x)$ . Hence,  $B_{\delta/k_1}^2(y) \subseteq B_\epsilon^1(x)$ . So for any  $y \in B_\epsilon^1(x)$  there is an open  $B^2$  neighbourhood inside  $B^1$ , which implies that  $B^1$  is open with respect to  $\tau_2$ .  $\square$

**Definition 8. Isometries.** Let  $(X_i, d_i)_{i \in \{1,2\}}$  be a metric space.

- $f : X_1 \rightarrow X_2$  is called **isometry** iff  $\forall x, y \in X_1 : d_1(x, y) = d_2(f(x), f(y))$ .
- The two metric spaces are called **isometric** iff there is a bijective isometry  $f : X_1 \rightarrow X_2$ .

Note that an isometry is automatically injective and that the inverse of a bijective isometry is again an isometry.

**Definition 9. Completion.** A complete metric space  $(Y, d)$  is a **completion** of the metric space  $(X, d)$  iff there exists an isometry  $f : X \rightarrow Y$  such that  $\overline{f(X)} = Y$ .

In practice, one often identifies  $X$  with  $f(X)$  and thus considers  $X$  as a dense subset of  $Y$ .

**Theorem 2. Existence of a completion** Every metric space  $(X, d)$  has a completion.

*Proof.* (Sketch) Define  $y := \{Cauchy - sequence in X\} \sim$  where  $(x_n) \sim (y_n) \Leftrightarrow \lim_{n \rightarrow \infty} d(x_n, y_n) = 0$  and  $\delta Y \times Y \rightarrow [0, \infty)$ .  $\delta([x], [y]) := \lim_{n \rightarrow \infty} d(x_n, y_n)$ .  $(Y, \delta)$  can be shown to be a complete metric space. Take  $f : X \rightarrow Y$  such that  $f(a) := (a, a, a, \dots)$ . This is an isometry, since  $\delta(f(a), f(b)) = \lim_{n \rightarrow \infty} d(a, b) = d(a, b)$ .

Moreover,  $\forall [x] \in Y$ , we can construct a sequence in  $f(X)$  that converges to  $[x]$ . In fact, for  $y_n := f(x_n)$  we obtain  $\delta(y_n, [x]) = \lim_{n \rightarrow \infty} d(y_n, x_n) = \lim_{n \rightarrow \infty} d(x_n, x_n) = 0$ . So  $\delta(y_n, [x]) \rightarrow 0$  for some  $n \rightarrow \infty$  since  $(x_n)$  is a Cauchy sequence.  $\square$

- The completion of a metric space is unique in the sense that any two completions are isometric.
- Since  $f(X)$  is dense in  $Y$ , the completion of a separable metric space is again separable.
- If there is a scalar product or norm, the completion preserves this structure.

**Example 1.** From the homework we know that  $(C([0, 1], \mathbb{K}), \|\cdot\|_\infty)$  with  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$  is complete.

**Theorem 3. Weierstrass Approximation Theorem.** The set  $X$  of polynomials on  $[0, 1]$  is dense in  $Y$  (with respect to the metric induced by  $\|\cdot\|_\infty$ ).

Consequently,  $Y$  is the completion of  $X$  with isometry  $f : X \rightarrow Y, f \mapsto f$ .

**Theorem 4.** If  $S \subseteq X$  is a subset of a complete metric space  $(X, d)$ , then  $(S, d)$  is complete iff  $S$  is closed.

*Proof.* ( $\Rightarrow$ ): Let  $(X_n)$  be a sequence in  $S$  that converges to  $x \in X$ .  $(X_n)$  is thus a Cauchy sequence and  $x \in S$  by completeness of  $S$ .

( $\Leftarrow$ ): Let  $(X_n)$  be a Cauchy sequence in  $S$ . By completeness of  $X$  it converges to some  $x \in X$ . Since  $S$  is closed,  $x \in S$ .  $\square$

- The ( $\Rightarrow$ ) part actually shows that every complete subspace of a (not necessarily complete) metric space is closed.
- The theorem enables us to prove **completeness of a space** by

- (i) Considering it as a subspace of a larger complete space
- (ii) Proving closedness

**Theorem 5. Baire's Category Theorem** *If  $(U_n)$  is a sequence of open, dense subsets of a complete metric space  $(X, d)$ , then the intersection  $U := \bigcap_{n \in \mathbb{N}} U_n$  is dense in  $X$*

*Proof.* We want to show that  $B_{\epsilon_0}(x_0) \cap U \neq \emptyset$  for any  $\epsilon_0 > 0, x_0 \in X$ . Define  $\overline{B}_\epsilon(x) := \{y \in X \mid d(x, y) \leq \epsilon\}$  and note that  $\overline{B}_\epsilon \subseteq B_{2\epsilon}(x)$ . Since  $U_n$  is open and dense,  $U_n \cap B_{\epsilon_0}(x_0)$  is open and non-empty. Consequently, there is  $x_1 \in X$  and  $\epsilon_1 > 0$  such that  $U_1 \cap B_{\epsilon_0}(x_0) \supseteq B_{2\epsilon_0}(x_0) \supseteq \overline{B}_{\epsilon_1}(x_1)$ . Now construct a sequence of points  $(x_n)$  in  $X$  and radii  $(\epsilon_n)$  such that  $\overline{B}_{\epsilon_n}(x_n) \subseteq U_n \cap B_{\epsilon_{n-1}}(x_{n-1})$  and  $\epsilon_n \in (0, 2^{-n}\epsilon_0]$ . By construction,  $\overline{B}_{\epsilon_n}(x_n) \subseteq \overline{B}_{\epsilon_{n-1}}(x_{n-1})$  and  $\epsilon_n \rightarrow 0$  for  $n \rightarrow \infty$ . In particular,  $d(x_n, x_m) \leq \epsilon_N = 2^{-N}\epsilon_0, \forall n, m > N$ . So,  $(x_n)$  is a Cauchy sequence and by completeness  $\lim_{n \rightarrow \infty} x_n = x_\infty \in \bigcup_{n \in \mathbb{N}} \overline{B}_{\epsilon_n}(x_n)$ . Since  $\forall n \in \mathbb{N} : x_\infty \in \overline{B}_{\epsilon_n}(x_n) \subseteq U_n$  we have  $x_\infty \in B_{\epsilon_0}(x_0) \cap U$ .  $\square$

A useful consequence is formulated in terms of **nowhere dense**.

**Definition 10.** A subset  $S$  of a topological space is called **nowhere dense** iff its closure has empty interior :  $\text{int}(\overline{S}) = \emptyset$ .

Note that  $S$  is nowhere dense iff its closure is.

**Corollary 4.** *A complete metric space is never the countable union of nowhere dense sets*

*Proof.* Assume  $X = \bigcup_{n \in \mathbb{N}} c_n$  where  $c_n$  is closed, that is,  $U_n := X \setminus c_n$  open. Then  $\bigcup_{n \in \mathbb{N}} U_n = \bigcup_{n \in \mathbb{N}} (X \setminus c_n) = X \setminus \bigcap_{n \in \mathbb{N}} c_n = \emptyset$ . By Baire's category theorem, at least one of the  $U_n$  is not dense. Then, however,  $\text{int}(c_n) = X \setminus \overline{U_n} \neq \emptyset$ .  $\square$

## Normed and Banach Spaces

**Definition 11.** Let  $X$  be a  $\mathbb{K}$ -vector space,  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ .

- $\|\cdot\| : X \mapsto [0, \infty)$  is called a **seminorm** iff  $\forall x, y \in X, \forall \alpha \in \mathbb{K}$ 
  - (i)  $\|x + y\| \leq \|x\| + \|y\|$
  - (ii)  $\|\alpha x\| = |\alpha| \|x\|$
- A seminorm is a **norm** iff  $\|x\| = 0 \Rightarrow x = 0$ .  $(X, \|\cdot\|)$  is then called a **normed space**.
- A **Banach Space** is a normed space that is complete
- The **linear span** of a subset  $S \subseteq X$  is defined as  $\text{span}(S) := \{\sum_{i=1}^n \alpha_i x_i \mid n \in \mathbb{N}, x_i \in S, \alpha_i \in \mathbb{K}\}$
- **Completeness** refers to the metric space with  $d(x, y) := \|x - y\|$ .
- A metric is induced by a norm iff  $\forall x, y, z \in X \forall \alpha \in \mathbb{K}$  :
  - (i)  $d(x + z, y + z) = d(x, y)$
  - (ii)  $d(\alpha x, \alpha y) = |\alpha| d(x, y)$

To see that this is sufficient for the existence of a norm, consider  $\|x\| = d(x, 0)$  and verify the necessary properties.

- Examples of **normed sequence spaces**

- (i)  $l_\infty := \{x \in \mathbb{K}^\mathbb{N} \mid \|x\|_\infty := \sup_n |x_n| < \infty\}$
- (ii)  $c_0 := \{x \in \mathbb{K}^\mathbb{N} \mid \lim_{n \rightarrow \infty} x_n = 0\}$  with  $\|\cdot\|_\infty$
- (iii)  $l_p := \{x \in \mathbb{K}^\mathbb{N} \mid \|x\|_p := (\sum_n |x_n|^p)^{1/p} < \infty\}, p \in [1, \infty)$

$$(iv) \ c_c := \{x \in \mathbb{K}^{\mathbb{N}} \mid |\{n \mid x_n \neq 0\}| < \infty\}$$

As sets, these satisfy  $c_c \subseteq l_p \subseteq l_q \subseteq c_0 \subseteq l_0$  for  $p \leq q < \infty$ .  $(l_\infty, \|\cdot\|_\infty)$  is a Banach space. Since  $c_0$  is a closed subset in  $l_\infty$ ,  $(c_0, \|\cdot\|_\infty)$  is also a Banach space. Also  $(l_p, \|\cdot\|_p)$  are Banach spaces.  $c_c$  is dense in  $l_p$  for  $p < \infty$  with respect to  $\|\cdot\|_p$  and  $c_c$  is dense in  $c_0$  with respect to  $\|\cdot\|_\infty$ . Since  $c_c \neq l_p$ , it is not a Banach space for any  $\|\cdot\|_p$ ,  $p \in [1, \infty]$ .

- $l_\infty$  can be generalized to arbitrary sets  $X$ :

$$l_\infty(X) := \left\{ f \in \mathbb{K}^X \mid \|f\|_\infty := \sup_{x \in X} |f(x)| < \infty \right\}.$$

$(l_\infty(X), \|\cdot\|_\infty)$  then turns out to be a Banach space.

- Examples of **normed function spaces**:

(1) **Continuous functions**: for any topological space  $(X, \tau)$  define,

$$(i) \ c_b(X) := \{f : X \rightarrow \mathbb{K} \mid f \text{ is continuous and } \|f\|_\infty := \sup_{x \in X} |f(x)| < \infty\}$$