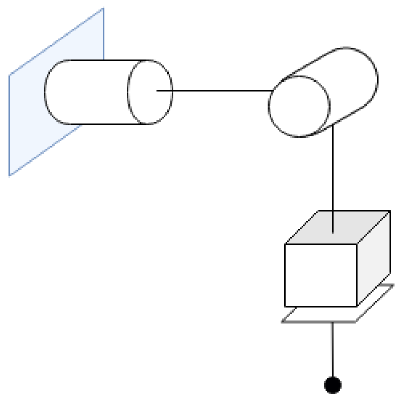
IN3140 Exam Solution (Spring 2019)

Task 1: Manipulator Classification (5%)

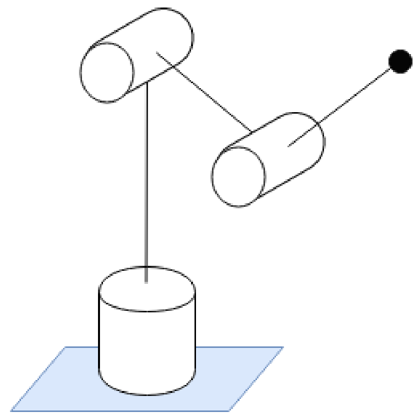
1.1 A manipulator of which type is displayed in the following image?



Select an alternative:
- Spherical wrist
- **Spherical manipulator**
- Articulated manipulator
- PPR manipulator

Maximum marks: 1.25

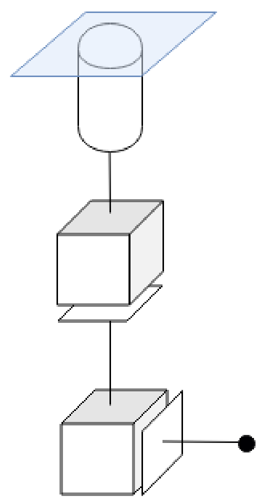
1.2 A manipulator of which type is displayed in the following image?



Select an alternative:
- Cartesian manipulator
- PPP manipulator
- Elbow-down manipulator
- **Articulated manipulator**

Maximum marks: 1.25

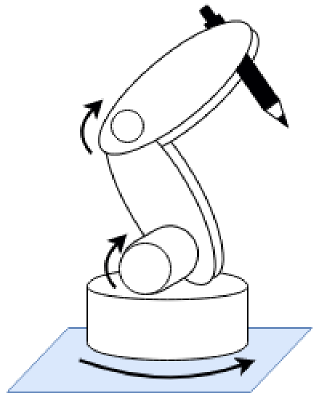
1.3 A manipulator of which type is displayed in the following image?



Select an alternative:
- PPR manipulator
- Spherical manipulator
- Articulated manipulator
- **Cylindrical manipulator**

Maximum marks: 1.25

1.4 A manipulator of which type is displayed in the following image?



Select an alternative:
- SCARA manipulator
- Myostatic manipulator
- **Anthropomorphic manipulator**
- Polar manipulator

Maximum marks: 1.25

Task 2: Equation of motion (5%)

In control theory, motion of robotic systems can be described with the following equation:

$$J(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = \tau$$

2.1 Which of the alternatives describes the term J(q) most correctly?

Select an alternative:
- **J(q) is an inertia-related term**
- **J(q)** denotes inertial forces, among which are coriolis and centrifugal forces
- **J(q)** is the Jacobian matrix
- **J(q)** is the jamming coefficient

Maximum marks: 1.5

2.2 Which of the alternatives describes the term $\mathbf{C(q, \dot{q})\dot{q}}$ most correctly?

Select an alternative:

•$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ describes colinear and centripetal forces
•$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ is the viscous friction
•$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ **denotes coriolis and centrifugal forces**
•$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}$ is the control effort

Maximum marks: 1.5

2.3 When attempting to independently control only one manipulator joint, we classify some of the terms in the equation as disturbance D. If we were to introduce this change, what would the new equation look like?

Select an alternative:
•$J\ddot{q} + C\dot{q} + B\dot{q} + D = \tau$
•$J(q)\ddot{q} + B(q)\dot{q} + D = \tau$
•$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$
•$\mathbf{J}\ddot{\mathbf{q}} + \mathbf{B}\dot{\mathbf{q}} + \mathbf{D} = \tau$

Maximum marks: 2

Task 3: Robot Operating System (5%)

3.1 Which of the following statements about ROS concepts is true?

Select an alternative:
•ROS Messages are processes that perform computation
•**ROS Topics may have multiple concurrent publishers and subscribers**
•ROS Melodic has name registration as its main function - nodes cannot find each other without it
•ROS Services employ request/deny communication model

Maximum marks: 1.5

3.2 If you were to design a mobile robot, what kinds of nodes, in terms of functionality, would you use (for example, a Battery Indicator node may be one such node)?

Maximum marks: 3.5

ANSWER GUIDELINES:
The nodes that can be used (not an exhaustive list):
- Camera/sensor node
- Mapping node (collision detection might be a part of it)
- Motion Planner node
- Trajectory Planner node
- Motion Controller node
- Wheel Encoders node

Justifying the choice of nodes or explicitly specifying their functionality is not really necessary, but might be helpful in certain cases. Some nodes may be considered more improtant than others: sensor nodes (to sense environment), path/trajectory planning/control nodes (to control movement), actuator/wheel/movement nodes (to actually move).

Task 4: Evolutionary Robotics (5%)

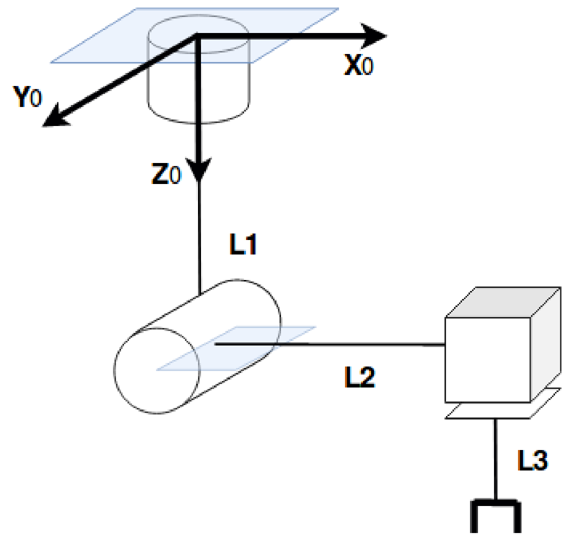
4.1 Briefly explain how an evolutionary algorithm works, outlining the different stages. Which problems within traditional robotics can be efficiently solved by evolutionary algorithms?

Maximum marks: 5

ANSWER GUIDELINES:
You are expected to write a short explanation that includes the following stages:
- Initialization of a random population
- Evaluation of individuals
- Termination criterion check
- Creation of new population from fit individuals (if termination criterion is not reached)
- Verification and application of solutions

Evolutionary algorithms are applicable for adaptation, optimization and design exploration. Specific problem examples and unique insights are welcome.

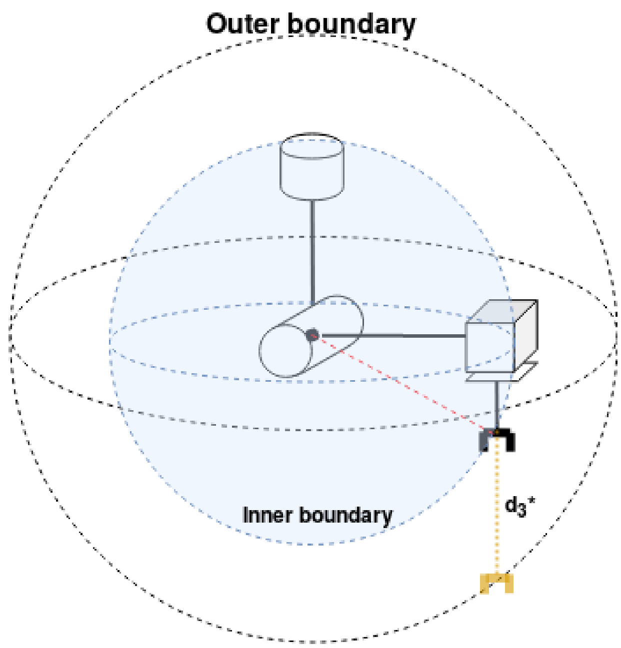
Task 5: Kinematics (40%)



The image above shows the manipulator used in this task. The first revolute joint of the manipulator rotates about axis $\mathbf{Z_0}$. The second revolute joint rotates about the axis that is parallel to axis $\mathbf{Y_0}$. The prismatic joint moves along the axis that is parallel to axis $\mathbf{Z_0}$. The axes $\mathbf{X_0}$, $\mathbf{Y_0}$ and $\mathbf{Z_0}$ are parallel to each other, and the lengths L1, L2 and L3 are fixed. The first revolute joint is considered to be in the base of the manipulator (at zero position).

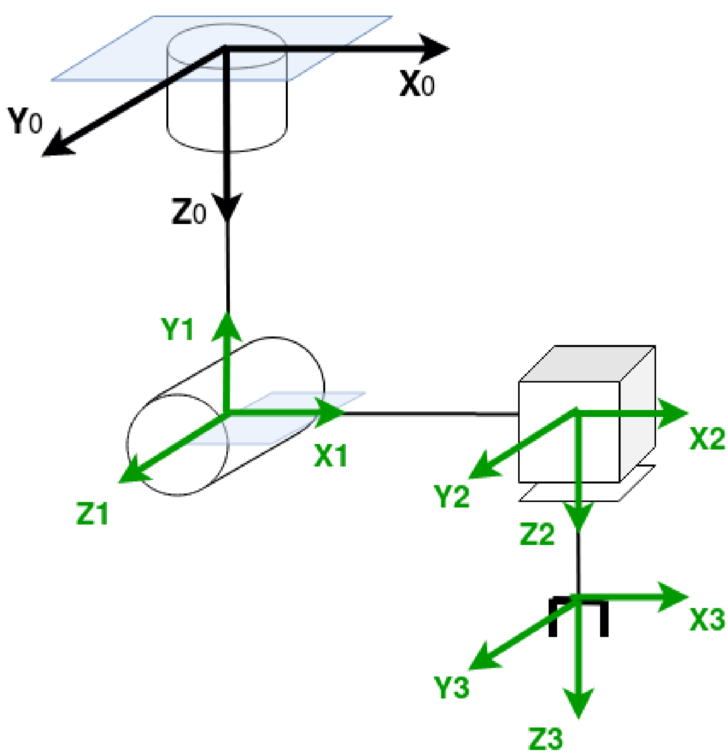
a) 4% Draw manipulator workspace (include the outer and the inner boundaries).

SOLUTION:
Assuming there are no mechanical constraints, the workspace of the manipulator is a sphere centered around the second joint. The inner boundary (the unreachable space) is also a sphere with the same origin, but a smaller radius. The inner boundary is reached at the minimal extension of the third link ($d^*_{3_{min}}$), while the outer boundary is reached at the maximal extension ($d^*_{3_{max}}$). A possible illustration:



b) 8% Assign coordinate frames to the manipulator from the above image in accordance with Denavit-Hartenberg convention. Mark the joint angles. Write the Denavit-Hartenberg parameters in a table.

SOLUTION:



Parameter Table:

i	**a**	α	**d**	θ
1	0	-90°	L_1	$\theta_1^\star$
2	L_2	90°	0	$\theta_2^\star$
3	0	0	L_3	0

The gripper frame conventions are not completely followed in the illustration, as they were not adequately taught in the lessons. Choosing to follow the conventions (by having θ_3 equal to $\pm$ 90°, for example) is most welcome, but may complicate the solution.

c) 8% Derive the forward kinematics of the manipulator from the base coordinate system to the tool coordinate system at the tip of the manipulator.

SOLUTION:

Canonical transformation:

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation matrices:

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & L_2c_2 \\ s_2 & 0 & -c_2 & L_2s_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

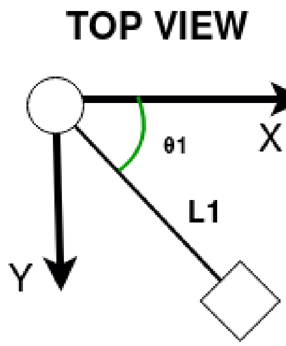
$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_3^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

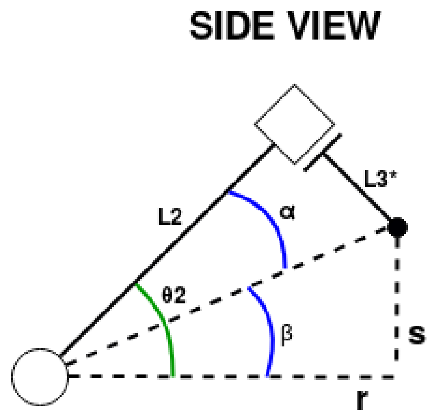
Transformation from base to tip:

$$H_t^B = A_1A_2A_3 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1(L_2c_2 + L_3^*s_2) \\ s_1c_2 & c_1 & s_1s_2 & s_1(L_2c_2 + L_3^*s_2) \\ -s_2 & 0 & c_2 & L_1 - L_2s_2 + L_3^*c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d) 10% Derive the inverse kinematics of the manipulator.

GEOMETRIC SOLUTION:





Helper variables:

$$r^2 = x^2 + y^2$$
$$s = z - L_1$$
$$d^2 = r^2 + s^2$$

First joint variable:

$$\theta_1 = atan2(y/x)$$

Second joint variable:

$$\cos\alpha = \frac{L_2}{d}$$
$$\sin\alpha = \sqrt{1 - cos^2\alpha}$$
$$\alpha = atan2\left(\frac{\pm\sin\alpha}{\cos\alpha}\right)$$
$$\beta = atan2(r, s)$$
$$\theta_2 = \alpha + \beta$$

Third joint variable:

$$L_3^* = \sqrt{d^2 - L_2^2}$$
$$d_3 = L_3^* - L_3$$

e) 10% Derive the Jacobian matrix of the manipulator.

SOLUTION:
Linear part:

$$J_v = \begin{bmatrix} z_0^0 \times o_3^0 & z_1^0 \times (o_3^0 - o_3^1) & z_2^0 \end{bmatrix}$$

where

$$J_{v_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} c_1(L_2c_2 + L_3^*s_2) \\ s_1(L_2c_2 + L_3^*s_2) \\ L_1 - L_2s_2 + L_3^*c_2 \end{bmatrix} = \begin{bmatrix} -s_1(L_2c_2 + L_3^*s_2) \\ c_1(L_2c_2 + L_3^*s_2) \\ 0 \end{bmatrix}$$

$$J_{v_2} = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} c_1(L_2c_2 + L_3^*s_2) \\ s_1(L_2c_2 + L_3^*s_2) \\ -L_2s_2 + L_3^*c_2 \end{bmatrix} = \begin{bmatrix} -c_1(L_2s_2 - L_3^*c_2) \\ -s_1(L_2s_2 - L_3^*c_2) \\ -L_2c_2 - L_3^*s_2 \end{bmatrix} \leftarrow \text{shortened expression}$$

$$J_{v_3} = \begin{bmatrix} c_1s_2 \\ s_1s_2 \\ c_2 \end{bmatrix}$$

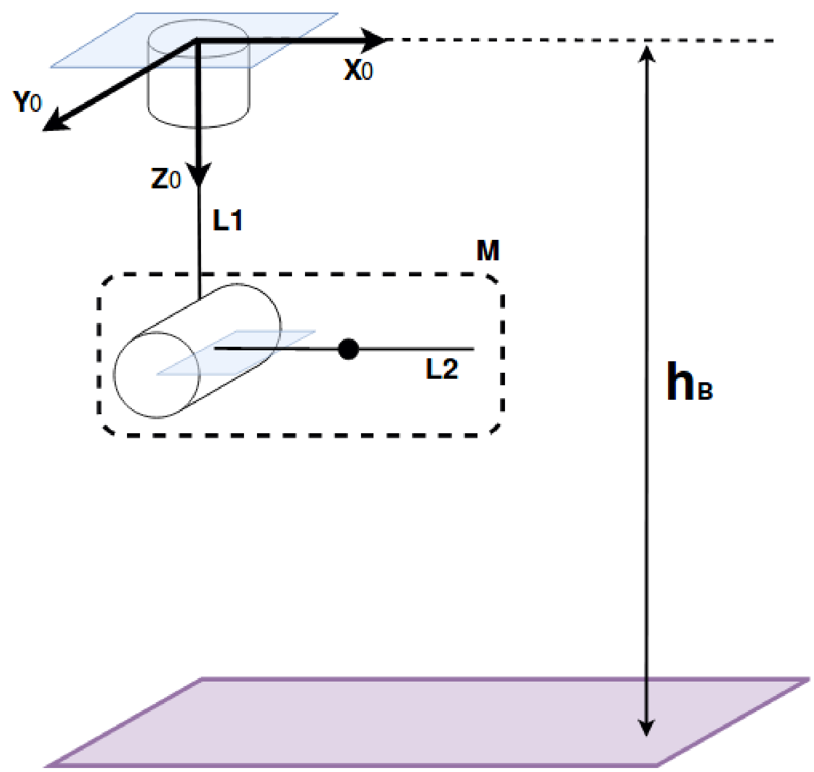
Angular part:

$$J_\omega = \begin{bmatrix} z_0^0 & z_1^0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The full Jacobian matrix:

$$J = \begin{bmatrix} -s_1(L_2c_2 + L_3^*s_2) & -c_1(L_2s_2 - L_3^*c_2) & c_1s_2 \\ c_1(L_2c_2 + L_3^*s_2) & -s_1(L_2s_2 - L_3^*c_2) & s_1s_2 \\ 0 & -L_2c_2 - L_3^*s_2 & c_2 \\ 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Task 6: Dynamics (24%)



The image above shows a simplified version of the manipulator from the previous task. This manipulator has two degrees of freedom. Assume that the system has only one mass **M** - the mass of the second link (marked with the dashed line). Assume further that the mass is **uniformly distributed**, and that **its coordinate frame (in the initial position) is aligned in parallel with the frame of the second joint**. The distance between the ground level and the base of the manipulator is $\mathbf{h_B}$ (for this manipulator, **L1 + L2 ¡ hB**). Consider the inertia tensor to be as follows:

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

a) 2% Produce the symbolic expression of the system's potential energy due to the ground level.

SOLUTION:

$$P = Mgh$$

where

$$h = h_B - L_1 \pm \frac{L_2}{2} s_2$$

Total potential energy:

$$Mg(h_B - L_1 \pm \frac{L_2}{2} s_2)$$

Note: the actual sign in ± depends on configuration. Other expressions are also possible, depending on the geometric definition of θ_2.

b) 2% Assuming that L1 = 300mm, L2 = 200mm, h_B = 550mm, M = 500g, θ_2 = 30° and g = 10 m/s^2, numerically calculate the system's potential energy. State the units.

SOLUTION:

$$h = 0.55m - 0.3m + \frac{0.2m}{2} * 0.5 = 0.55m - 0.3m + 0.05m = 0.3m$$

$$P = 0.5kg * 10m/s^2 * 0.3m = 1.5J$$

Other valid answers: 1500mJ, 1,500,000μJ or similar, depending on configuration. For example, having "-" instead of "±" in the expression from (a) would give 1J as a result.

c) 12% Find the symbolic expression of the system's kinetic energy. You may use the following facts to assist your calculations:

$$R = R_1^0 R_{z,\theta_2} \qquad \omega^T R = (R^T \omega)^T$$

SOLUTION: The total kinetic energy of the system:

$$K = \frac{Mv^2}{2} + \frac{\omega^T R I R^T \omega}{2}$$

Translational part:

$$K_v = \frac{Mv^2}{2}$$
$$v = \mathcal{J}_v \dot{q}$$

where $\mathcal{J}$ is the Jacobian matrix of the manipulator. It can be obtained by adjusting the Jacobian matrix J of the three-link manipulator (from the previous task). Since it is the third joint that has been removed, the third column and all instances of L_3^* in J should be set to zero. Additionally, L_2 should be halved to represent the uniform mass distribution:

$$\mathcal{J} = \begin{bmatrix} -s_1(\frac{L_2}{2}c_2) & -c_1(\frac{L_2}{2}s_2) & 0 \\ c_1(\frac{L_2}{2}c_2) & -s_1(\frac{L_2}{2}s_2) & 0 \\ 0 & -\frac{L_2}{2}c_2 & 0 \\ 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{L_2}{2}s_1c_2 & -\frac{L_2}{2}c_1s_2 & 0 \\ \frac{L_2}{2}c_1c_2 & -\frac{L_2}{2}s_1s_2 & 0 \\ 0 & -\frac{L_2}{2}c_2 & 0 \\ 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then,

$$v = \begin{bmatrix} -\frac{L_2}{2}s_1c_2 & -\frac{L_2}{2}c_1s_2 & 0 \\ \frac{L_2}{2}c_1c_2 & -\frac{L_2}{2}s_1s_2 & 0 \\ 0 & -\frac{L_2}{2}c_2 & 0 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot L_3^* \end{bmatrix} = \begin{bmatrix} -\frac{L_2}{2}s_1c_2\dot\theta_1 - \frac{L_2}{2}c_1s_2\dot\theta_2 \\ \frac{L_2}{2}c_1c_2\dot\theta_1 - \frac{L_2}{2}s_1s_2\dot\theta_2 \\ -\frac{L_2}{2}c_2\dot\theta_2 \end{bmatrix}$$

and

$$v^2 = (-\frac{L_2}{2}s_1c_2\dot\theta_1 - \frac{L_2}{2}c_1s_2\dot\theta_2)^2 + (\frac{L_2}{2}c_1c_2\dot\theta_1 - \frac{L_2}{2}s_1s_2\dot\theta_2)^2 + (-\frac{L_2}{2}c_2\dot\theta_2)^2 =$$

$$\underline{\frac{L_2^2}{4}s_1^2c_2^2\dot\theta_1^{\,2}} + \frac{L_2^2}{2}s_1c_1s_2c_2\dot\theta_1\dot\theta_2 + \frac{L_2^2}{4}c_1^2s_2^2\dot\theta_2^{\,2} + \underline{\frac{L_2^2}{4}c_1^2c_2^2\dot\theta_1^{\,2}} - \frac{L_2^2}{2}s_1c_1s_2c_2\dot\theta_1\dot\theta_2 + \frac{L_2^2}{4}s_1^2s_2^2\dot\theta_2^{\,2} + \frac{L_2^2}{4}c_2^2\dot\theta_2^{\,2} =$$

$$\underline{\frac{L_2^2}{4}c_2^2\dot\theta_1^{\,2}} + (\frac{L_2^2}{4}s_2^2\dot\theta_2^{\,2} + \frac{L_2^2}{4}c_2^2\dot\theta_2^{\,2}) = \frac{L_2^2}{4}(c_2^2\dot\theta_1^{\,2} + \dot\theta_2^{\,2})$$

Finally, the translational kinetic energy of the system is

$$K_v = \frac{ML_2^2(c_2^2\dot\theta_1^{\,2} + \dot\theta_2^{\,2})}{8}$$

<u>Rotational part:</u>

$$K_\omega = \frac{\omega^T R I R^T \omega}{2}$$

$$\omega = \mathcal{J}_\omega \dot q = \begin{bmatrix} 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot L_3^* \end{bmatrix} = \begin{bmatrix} -s_1\dot\theta_2 \\ c_1\dot\theta_2 \\ \dot\theta_1 \end{bmatrix}$$

Using that $R = R_1^0 R_{z,\theta_2}$ (where R_1^0 is extracted from A_1 in forward kinematics),

$$R = R_1^0 R_{z,\theta_2} = \begin{bmatrix} c_1 & 0 & -s_1 \\ s_1 & 0 & c_1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 \\ s_1c_2 & -s_1s_2 & c_1 \\ -s_2 & -c_2 & 0 \end{bmatrix}$$

and therefore

$$R^T\omega = \begin{bmatrix} c_1c_2 & s_1c_2 & -s_2 \\ -c_1s_2 & -s_1s_2 & -c_2 \\ -s_1 & -c_1 & 0 \end{bmatrix} \begin{bmatrix} -s_1\dot\theta_2 \\ c_1\dot\theta_2 \\ \dot\theta_1 \end{bmatrix} = \begin{bmatrix} \cancel{-s_1c_1c_2\dot\theta_2} + \cancel{s_1c_1c_2\dot\theta_2} - s_2\dot\theta_1 \\ \cancel{s_1c_1s_2\dot\theta_2} - \cancel{s_1c_1s_2\dot\theta_2} - c_2\dot\theta_1 \\ -s_1^2\dot\theta_2 - c_1^2\dot\theta_2 \end{bmatrix} = \begin{bmatrix} -s_2\dot\theta_1 \\ -c_2\dot\theta_1 \\ -\dot\theta_2 \end{bmatrix}$$

Using the equality $\omega^T R = (R^T\omega)^T$,

$$\omega^T R = \begin{bmatrix} -s_2\dot\theta_1 & -c_2\dot\theta_1 & -\dot\theta_2 \end{bmatrix}$$

Then, the rotational kinetic energy of the system is

$$K_\omega = \frac{\begin{bmatrix} -s_2\dot\theta_1 & -c_2\dot\theta_1 & \dot\theta_2 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} -s_2\dot\theta_1 \\ -c_2\dot\theta_1 \\ -\dot\theta_2 \end{bmatrix}}{2} =$$

$$\frac{\begin{bmatrix}-s_2\dot{\theta}_1 I_{xx} & -c_2\dot{\theta}_1 I_{yy} & -I_{zz}\dot{\theta}_2\end{bmatrix}\begin{bmatrix}-s_2\dot{\theta}_1\\ -c_2\dot{\theta}_1\\ -\dot{\theta}_2\end{bmatrix}}{2} =$$

$$\frac{s_2^2\dot{\theta}_1^{\,2} I_{xx} + c_2^2\dot{\theta}_1^{\,2} I_{yy} + \dot{\theta}_2^{\,2} I_{zz}}{2}$$

Total kinetic energy:

$$K = K_v + K_\omega = \frac{ML_2^2(c_2^2\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2})}{8} + \frac{s_2^2\dot{\theta}_1^{\,2} I_{xx} + c_2^2\dot{\theta}_1^{\,2} I_{yy} + \dot{\theta}_2^{\,2} I_{zz}}{2}$$

d) 8% For this task, assume that M is now a point mass located in the former mass center. Assemble the Lagrangian $\mathcal{L}$ accordingly. Then, derive the dynamics of the manipulator using the Euler-Lagrange formulation. Write the equations of motion in the following form:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

SOLUTION:

Since **M** is now a point point mass, we can discard the influence of inertia (K_ω) on the system. The Lagrangian $\mathcal{L}$ is therefore as follows (*assuming that the potential energy expression has "+" instead of* $\pm$*, which depends on the choice of frame*):

$$\mathcal{L} = K_v - P$$

$$\mathcal{L} = \frac{ML_2^2(c_2^2\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2})}{8} - Mg(h_B - L_1 + \frac{L_2}{2}s_2)$$

Unfactoring $\mathcal{L}$:

$$\mathcal{L} = \frac{ML_2^2c_2^2\dot{\theta}_1^{\,2}}{8} + \frac{ML_2^2\dot{\theta}_2^{\,2}}{8} - Mgh_B + MgL_1 - \frac{MgL_2s_2}{2}$$

The torque:

$$\tau = \begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}$$

Torque on the first joint:

$$\tau_1 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} - \frac{\partial\mathcal{L}}{\theta_1}$$

$$\tau_1 = \frac{d}{dt}\left(\frac{ML_2^2c_2^2\dot{\theta}_1}{4}\right) - 0 = \frac{ML_2^2c_2^2\ddot{\theta}_1}{4} - \frac{ML_2^2c_2s_2\dot{\theta}_1\dot{\theta}_2}{2}$$

Again, take note: $\frac{d}{dt}(c_2^2) = -2c_2s_2\dot{\theta}_2$ - this is a composite function with three layers (square function, cosine function, time-dependent function).

Torque on the second joint:

$$\tau_2 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} - \frac{\partial \mathcal{L}}{\theta_2}$$

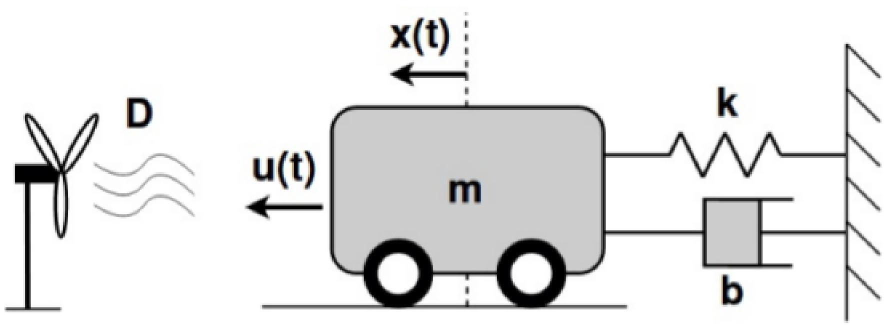
$$\tau_2 = \frac{ML_2^2\ddot{\theta}_2}{4} + \frac{ML_2^2 c_2 s_2 \dot{\theta}_1^{\,2}}{4} + \frac{MgL_2c_2}{2}$$

Rewriting τ:

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$

$$\tau = \begin{bmatrix} \frac{ML_2^2c_2^2}{4} & 0 \\ 0 & \frac{ML_2^2}{4} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -\frac{ML_2^2c_2s_2\dot{\theta}_2}{2} & 0 \\ \frac{ML_2^2c_2s_2\dot{\theta}_1}{4} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{MgL_2c_2}{2} \end{bmatrix}$$

Task 7: Mass-spring-damper Control (16%)



The above image demonstrates a mass-spring-damper system, where m is the object mass, k is the spring constant, b is the damping coefficient, D is the system disturbance, x(t) is the mass displacement at time t, and u(t) is the input force. The system can be described with the following differential equation:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) + D(t) = u(t)$$

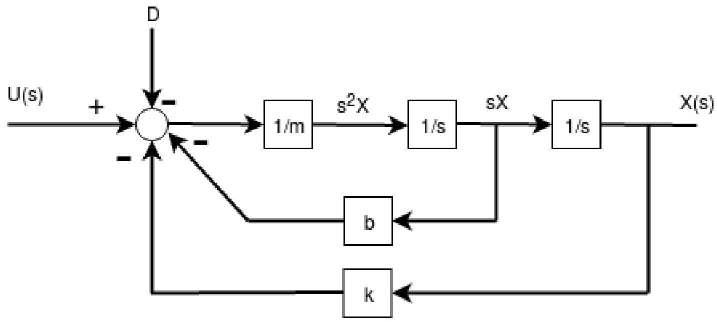
a) 5% Transform the above equation from time domain to Laplace domain. Draw the closed-loop block diagram of the system.

SOLUTION:

Laplace transform:

$$U(s) = ms^2X(s) + bsX(s) + kX(s) + D(s)$$

Block diagram of the system:



b) 6% Suppose that a PD compensator is added to the system, and $X_d(s)$ is the desired mass displacement. Transform the system equation in Laplace domain (that you obtained in (a)) so that it includes the PD compensator. Solve the resulting equation for $X(s)$.

SOLUTION:

The new equation with the PD compensator:

$$ms^2X + bsX + kX + D = K_p(X_d - X) - K_d sX$$

Solving for $X(s)$:

$$X(ms^2 + bs + k + K_p + K_d s) = K_p X_d - D$$

$$X = \frac{K_p X_d - D}{ms^2 + bs + k + K_p + K_d s}$$

c) 5% Assume that the system receives a step reference input $X_d(s) = \frac{\psi}{s}$ and constant disturbance $\frac{d}{s}$. Use the final value theorem below to determine the steady-state error:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

If you needed to eliminate the steady-state error and could use a different compensator to do so, which compensator would you choose?

SOLUTION:

Mass displacement at $t \to \infty$:

$$lim_{t\to\infty} x(t) = \lim_{s\to 0} \frac{K_p \frac{\psi}{\cancel{s}}\cancel{s} - \frac{d}{\cancel{s}}\cancel{s}}{\cancel{ms^2} + \cancel{bs} + k + K_p + \cancel{K_d s}} = \frac{K_p\psi - d}{K_p + k} = \frac{K_p\psi}{K_p + k} - \frac{d}{K_p + k}$$

Steady-state error:

$$e(t) = X_d(t) - X(t)$$

$$lim_{t\to\infty}(X_d(t) - X(t)) = \lim_{s\to 0}\left(\cancel{s}\frac{\psi}{\cancel{s}} - sX(s)\right) = \psi - \left(\frac{K_p\psi}{K_p + k} + \frac{d}{K_p + k}\right)$$

There two compensator types that pass the requirement of having no steady-state error: PI and PID.