Suggested Solutions Inf3480 Exam 2013

June 18, 2013

Exercise 1

a) When a robot is in a singularity the manipulator looses one or more degrees of freedom. This can be found using the Jacobian. When the Jacobian in not of full rank (dim J ¿ rank J), there is a singularity. Equally one can find singularities by finding when the determinant of J equals zero.

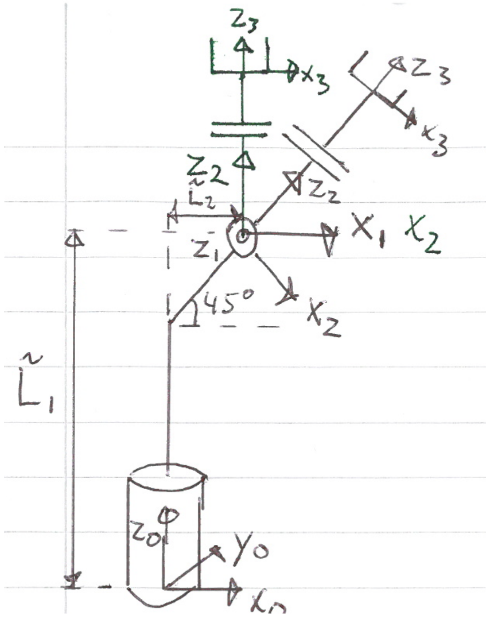
b) Constrained cv. unconstrained environment
Mobile robots need external sensing to determine position
Manipulators may not need to percieve the worlds around them

c) The difference between simulations and the real world.

d) P-Proportional - Minimizes the error
I-Integral - Removes steady state error
D-Derivative - Decrease oscillations

Exercise 2

a) To parameter solutions are found. One where the tool has a 45° angle to the base and one where the tool points upwards. The last solution is given in green in the figure. The coordinate frames 0 and 1 are the same for the two solutions.



	a_i	d_i	α_i	θ_i
1	$\tilde{L}_2$	$\tilde{L}_1$	90°	θ_1^*
2	0	0	−90°	$\theta_2^* - 45°$
3	0	L_3^*	0	0
1	$\tilde{L}_2$	$\tilde{L}_1$	90°	θ_1^*
2	0	0	−90°	θ_2^*
3	0	L_3^*	0	0

$$\tilde{L}_1 = L_1 + \frac{1}{2}\sqrt{2}L_2$$
$$\tilde{L}_2 = \frac{1}{2}\sqrt{2}L_2$$

We will use the solution without the 45° offset in the rest of this exercise.

b)

$$\boldsymbol{T}_1^0 = \begin{bmatrix} c_1 & 0 & s_1 & \tilde{L}_2 c_1 \\ s_1 & 0 & -c_1 & \tilde{L}_2 s_1 \\ 0 & 1 & 0 & \tilde{L}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{T}_2^1 = \begin{bmatrix} c_2 & 0 & -s_2 & 0 \\ s_2 & 0 & c_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{T}_3^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{T}_1^0 \boldsymbol{T}_2^1 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & \tilde{L}_2 c_1 \\ s_1 c_2 & -c_1 & -s_1 s_2 & \tilde{L}_2 s_1 \\ s_2 & 0 & c_2 & \tilde{L}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{T}_2^0 \boldsymbol{T}_3^2 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & -c_1 s_2 L_3 + \tilde{L}_2 c_1 \\ s_1 c_2 & -c_1 & -s_1 s_2 & -s_1 s_2 L_3 + \tilde{L}_2 s_1 \\ s_2 & 0 & c_2 & c_2 L_3 + \tilde{L}_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c)

$$\begin{aligned} \boldsymbol{J}_{v_1} &= z_0 \times o_3 & \qquad \boldsymbol{J}_{\omega_1} &= z_0 \\ \boldsymbol{J}_{v_2} &= z_1 \times (o_3 - o_1) & \qquad \boldsymbol{J}_{\omega_2} &= z_1 \\ \boldsymbol{J}_{v_3} &= z_2 & \qquad \boldsymbol{J}_{\omega_3} &= 0 \end{aligned}$$

$$z_0 \times o_3 = \begin{bmatrix} s_1 s_2 L_3 - \tilde{L}_2 s_1 \\ -c_1 s_2 L_3 + \tilde{L}_2 c_1 \\ 0 \end{bmatrix}$$

$$o_3 - o_1 = \begin{bmatrix} -c_1 s_2 L_3 \\ -s_1 s_2 L_3 \\ c_2 L_3 \end{bmatrix} \qquad z_1 \times (o_3 - o_1) = \begin{bmatrix} -c_1 c_2 L_3 \\ -s_1 c_2 L_3 \\ -s_2 L_3 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} s_1 s_2 L_3 - \tilde{L}_2 s_1 & -c_1 c_2 L_3 & -c_1 s_2 \\ -c_1 s_2 L_3 + \tilde{L}_2 c_1 & -s_1 c_2 L_3 & -s_1 s_2 \\ 0 & -s_2 L_3 & c_2 \\ 0 & s_1 & 0 \\ 0 & -c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

d) There are two ways to solve the inverse kinematics, either *analytically* or *geometrically.* First we solve the equations analytically using the equations

$$\begin{aligned} p_x &= -c_1 s_2 L_3 + \tilde{L}_2 c_1 \\ p_y &= -s_1 s_2 L_3 + \tilde{L}_2 s_1 \\ p_z &= c_2 L_3 + \tilde{L}_1 \end{aligned}$$

Dividing the second equation on the first equation yields

$$\begin{aligned} \frac{p_y}{p_x} &= \frac{-s_1 s_2 L_3 + \tilde{L}_2 s_1}{-c_1 s_2 L_3 + \tilde{L}_2 c_1} \\ &= \frac{s_1}{c_1} \frac{-s_2 L_3 + \tilde{L}_2}{-s_2 L_3 + \tilde{L}_2} \\ &= \tan \theta_1 \end{aligned}$$

Which yields the solutions

$$\theta_1 = \operatorname{atan2}(p_y, p_x) \qquad\qquad \theta_1 = \pi + \operatorname{atan2}(p_y, p_x) \tag{1}$$

Now we square the two first equations

$$\begin{aligned} p_x^2 &= -c_1^2 s_2^2 L_3^2 + \tilde{L}_2^2 c_1^2 - 2c_1^2 s_2 \tilde{L}_2 L_3 \\ p_y^2 &= -s_1^2 s_2^2 L_3^2 + \tilde{L}_2^2 s_1^2 - 2s_1^2 s_2 \tilde{L}_2 L_3 \end{aligned}$$

The we sum the together

$$p_x^2 + p_y^2 = s_2^2 L_3^2 + \tilde{L}_2^2 - 2s_2 \tilde{L}_2 L_3$$

The right hand side can be transformed using the relation $a^2 + b^2 - 2ab = (a-b)^2$ in two different ways

$$p_x^2 + p_y^2 = \left(s_2 L_3 - \tilde{L}_2\right)^2 \qquad\qquad p_x^2 + p_y^2 = \left(-s_2 L_3 + \tilde{L}_2\right)^2$$

Taking the square root yields

$$\sqrt{p_x^2 + p_y^2} = p_{xy} = s_2 L_3 - \tilde{L}_2 \qquad\qquad \sqrt{p_x^2 + p_y^2} = p_{xy} = -s_2 L_3 + \tilde{L}_2$$

Dividing these two equations on $p_z = c_2 L_3 + \tilde{L}_1$ yields

$$\begin{aligned} \frac{p_{xy} + \tilde{L}_2}{p_z - \tilde{L}_1} &= \frac{s_2 L_3}{c_2 L_3} \\ &= \frac{s_2}{c_2} \end{aligned} \qquad\qquad \begin{aligned} \frac{p_{xy} - \tilde{L}_2}{p_z - \tilde{L}_1} &= -\frac{s_2 L_3}{c_2 L_3} \\ &= -\frac{s_2}{c_2} \end{aligned}$$

This yields three solution (the minus sign on the right hand side of the right equation yields two solutions, because it could go both above and below the division line)

$$\begin{aligned} \theta_2 &= \operatorname{atan2}(p_{xy} + \tilde{L}_2, p_z - \tilde{L}_1) \\ \theta_2 &= \operatorname{atan2}(-(p_{xy} - \tilde{L}_2), p_z - \tilde{L}_1) \\ \theta_2 &= \operatorname{atan2}(p_{xy} - \tilde{L}_2, -(p_z - \tilde{L}_1)) \end{aligned}$$

Because we have used the square root we have to check that the solution is correct. We could en up with negative values when we want positive ones because we have squared and then used squared root.

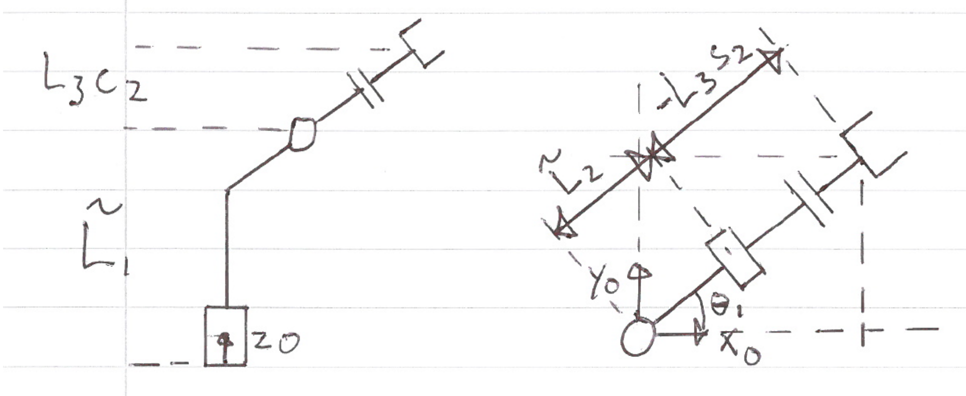
The last unknown can be found using the equation

$$p_z = c_2 L_3 + \tilde{L}_1$$

This yields

$$L_3 = \frac{p_z - \tilde{L}_1}{c_2}$$

For the geometric solution we we draw the robot



From the drawing we can see that θ_1 is only dependent of p_x and p_y. We get the solution

$$\theta_1 = \operatorname{atan2}(p_y, p_x) \qquad\qquad \theta_1 = \pi + \operatorname{atan2}(p_y, p_x) \tag{2}$$

For the two other unknowns we get the equations

$$\begin{aligned} p_{xy} &= \tilde{L}_2 - L_3 s_2 \\ p_z &= \tilde{L}_1 + L_3 c_2 \end{aligned}$$

The minus sign is a result of how we defined positive values of θ_2. Here we use the same procedure as in the analytic case and find

$$\begin{aligned} \theta_2 &= \operatorname{atan2}(-(p_{xy} - \tilde{L}_2), p_z - \tilde{L}_1) \\ \theta_2 &= \operatorname{atan2}(p_{xy} - \tilde{L}_2, -(p_z - \tilde{L}_1)) \\ L_3 &= \frac{p_z - \tilde{L}_1}{c_2} \end{aligned}$$

e) Four possible solutions are given in the table below. Only one of them is required to answer the exercise. L_3 should be positive.

θ_1 [°]	26.57	26.57	206.57	206.57
θ_2 [°]	-114.47	65.54	107.21	287.21
L_3 $[m]$	0.498	-0.498	0.696	-0.696
q_3 $[m]$	0.398	-0.598	0.596	-0.796

f) Torque in the x-direction (τ_x) can not be exerted.

$$\boldsymbol{\tau} = \begin{bmatrix} 0 & s_2 L_3 + \tilde{L}_2 & 0 & 0 & 0 & 1 \\ -c_2 L_3 & 0 & -s_2 L_3 & 0 & -1 & 0 \\ s_2 & 0 & c_2 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{F}$$

Exercise 3

a) First we find the velocity of the mass

$$\begin{aligned} v_x &= c_1 L_2 \dot{\theta}_1 + s_1 \dot{L}_2 \\ v_y &= -s_1 L_2 \dot{\theta}_1 + c_1 \dot{L}_2 \end{aligned}$$

Then we compute the squared velocity

$$\begin{aligned} v_x^2 &= c_1^2 L_2^2 \dot{\theta}_1^2 + s_1^2 \dot{L}_2^2 + c_1 s_1 L_2 \dot{\theta}_1 L_2 \\ v_y^2 &= -s_1^2 L_2^2 \dot{\theta}_1^2 + c_1^2 \dot{L}_2^2 - c_1 s_1 L_2 \dot{\theta}_1 L_2 \\ \boldsymbol{v}^2 &= L_2^2 \dot{\theta}_1^2 + \dot{L}_2^2 \end{aligned}$$

The we find the height

$$h = L_1 + c_1 L_2$$

Inserting this into the Lagrangian yields

$$\begin{aligned} \mathcal{L} &= \mathcal{K} - \mathcal{P} \\ &= \frac{1}{2} m \boldsymbol{v}^2 - mgh \\ &= \frac{1}{2} m \left(L_2^2 \dot{\theta}_1^2 + \dot{L}_2^2 \right) - mg \left(L_1 + c_1 L_2 \right) \end{aligned}$$

b) First we find the intermediate results

$$\frac{\delta \mathcal{L}}{\delta \dot{\theta}_1} = mL_2^2\dot{\theta}_1 \qquad \frac{\delta \mathcal{L}}{\delta \dot{L}_2} = m\dot{L}_2$$

$$\frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \dot{\theta}_1} = mL_2^2\ddot{\theta}_1 + 2mL_2\dot{L}_2\dot{\theta}_1 \qquad \frac{d}{dt}\frac{\delta \mathcal{L}}{\delta \dot{L}_2} = m\ddot{L}_2$$

$$\frac{\delta \mathcal{L}}{\delta \theta_1} = mgs_1L_2 \qquad \frac{\delta \mathcal{L}}{\delta L_2} = mL_2\dot{\theta}_1^2 + mgc_1$$

This yields the two motion equations

$$\begin{aligned} mL_2^2\ddot{\theta}_1 + 2mL_2\dot{L}_2\dot{\theta}_1 - mgs_1L_2 &= \tau_1 \\ m\ddot{L}_2 - mL_2\dot{\theta}_1^2 - mgc_1 &= F_2 \end{aligned}$$

c) Inserting $\dot{L}_2 = 0$ and $s_1 = \theta_1$ into the first equation of motion yields

$$mL_2^2\ddot{\theta}_1 - mgL_2\theta_1 = \tau_1 \tag{3}$$

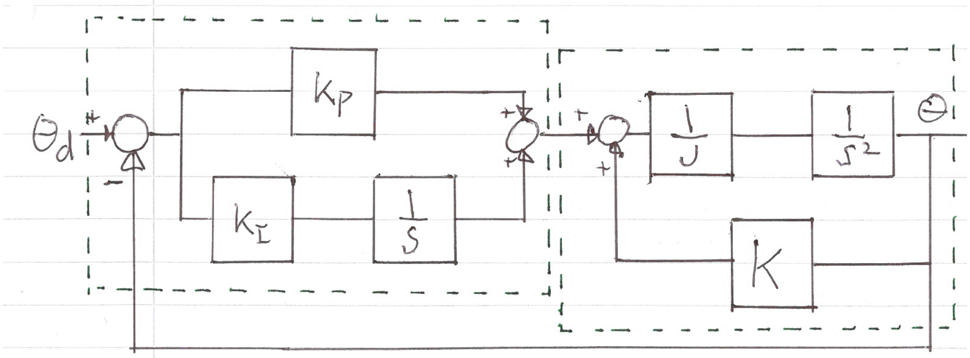
This yields the following parameters

$$\begin{aligned} J &= mL_2^2 \\ b &= 0 \\ k &= mgL_2 \end{aligned}$$

Transforming the equation into the Laplace domain

$$J\theta(s)s^2 + k\theta(s) = \tau(s)$$

d) The block diagram is given below



The left part is the controller and the right part is the system.