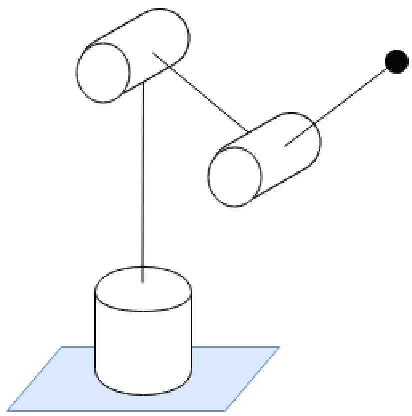
IN4140 Exam Solution (Spring 2019)

Task 1: Manipulator Classification (2%)

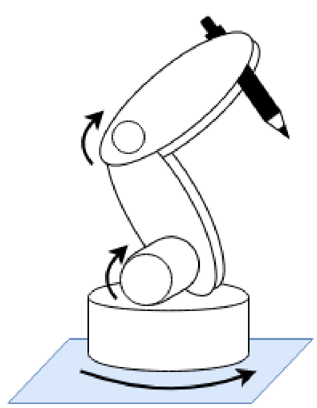
1.1 A manipulator of which type is displayed in the following image?



Select an alternative:
- Cartesian manipulator
- PPP manipulator
- Elbow-down manipulator
- **Articulated manipulator**

Maximum marks: 1

1.2 A manipulator of which type is displayed in the following image?



Select an alternative:
• SCARA manipulator
• Spherical manipulator
• **Anthropomorphic manipulator**
• Cylindrical manipulator

Maximum marks: 1

Task 2: Equation of motion (3%)

In control theory, motion of robotic systems can be described with the following equation:

$$J(q)\ddot{q} + C(q,\dot{q})\dot{q} + B\dot{q} + g(q) = \tau$$

2.1 Which of the alternatives describes the term J(q) most correctly?

Select an alternative:
• **J(q) is an inertia-related term**
• **J(q)** denotes inertial forces, among which are coriolis and centrifugal forces
• **J(q)** is the Jacobian matrix
• **J(q)** is the jamming coefficient

Maximum marks: 1.5

2.2 Which of the alternatives describes the term $\mathbf{C(q,\dot{q})\dot{q}}$ most correctly?

Select an alternative:
•$\mathbf{C(q,\dot{q})\dot{q}}$ describes colinear and centripetal forces
•$\mathbf{C(q,\dot{q})\dot{q}}$ is the viscous friction
•$\mathbf{C(q,\dot{q})\dot{q}}$ **denotes coriolis and centrifugal forces**
•$\mathbf{C(q,\dot{q})\dot{q}}$ is the control effort

Maximum marks: 1.5

Task 3: Robot Operating System (3%)

3.1 If you were to design a mobile robot, what kinds of nodes, in terms of functionality, would you use (for example, a Battery Indicator node may be one such node)?

Maximum marks: 3

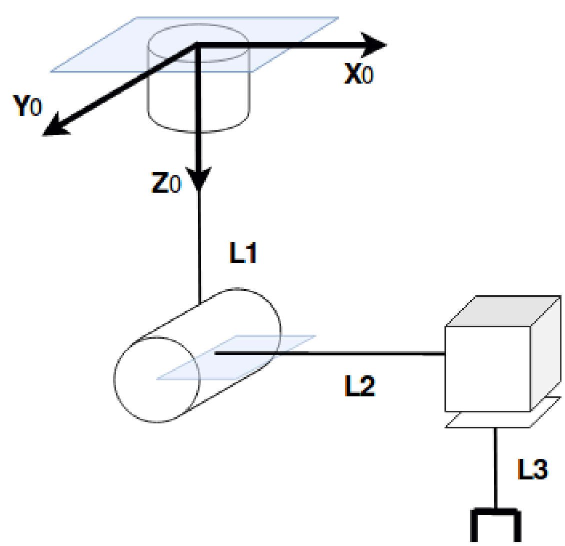
ANSWER GUIDELINES:

The nodes that can be used (not an exhaustive list):

- Camera/sensor node
- Mapping node (collision detection might be a part of it)
- Motion Planner node
- Trajectory Planner node
- Motion Controller node
- Wheel Encoders node

Justifying the choice of nodes or explicitly specifying their functionality is not really necessary, but might be helpful in certain cases. Some nodes may be considered more improtant than others: sensor nodes (to sense environment), path/trajectory planning/control nodes (to control movement), actuator/wheel/movement nodes (to actually move the robot).

Task 4: Kinematics (48%)



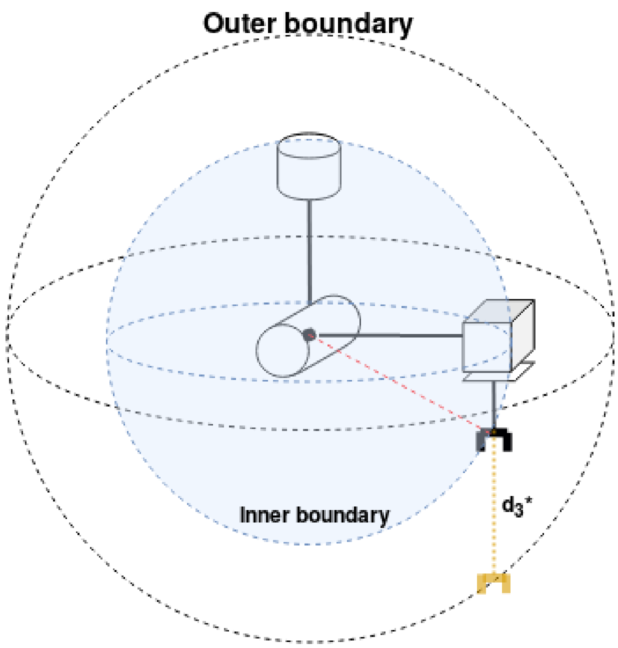
The image above shows the manipulator used in this task. The first revolute joint of the manipulator rotates about axis $\mathbf{Z_0}$. The second revolute joint rotates about the axis that is parallel to axis $\mathbf{Y_0}$. The prismatic joint moves along the axis that is parallel to axis $\mathbf{Z_0}$.

The axes $\mathbf{X_0}$, $\mathbf{Y_0}$ and $\mathbf{Z_0}$ are parallel to each other, and the lengths L1, L2 and L3 are fixed. The first revolute joint is considered to be in the base of the manipulator (at zero position).

a) 4% Draw manipulator workspace (include the outer and the inner boundaries).

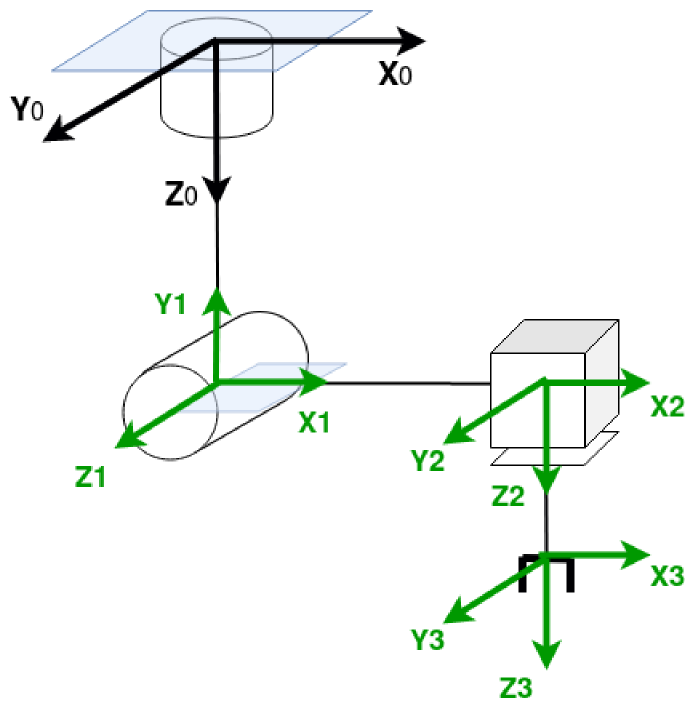
SOLUTION:

Assuming there are no mechanical constraints, the workspace of the manipulator is a sphere centered around the second joint. The inner boundary (the unreachable space) is also a sphere with the same origin, but a smaller radius (L_2, to be precise). An illustration:



b) 8% Assign coordinate frames to the manipulator from the above image in accordance with Denavit-Hartenberg convention. Mark the joint angles. Write the Denavit-Hartenberg parameters in a table.

SOLUTION:



Parameter Table:

i	**a**	α	**d**	θ
1	0	-90°	L_1	$\theta_1^\star$
2	L_2	90°	0	$\theta_2^\star$
3	0	0	L_3	0

The gripper frame conventions are not completely followed in the illustration, as they were not adequately taught in the lessons. Choosing to follow the conventions (by having θ_3 equal to $\pm$ 90°, for example) is most welcome, but may complicate the solution.

c) 8% Derive the forward kinematics of the manipulator from the base coordinate system to the tool coordinate system at the tip of the manipulator.

SOLUTION:

Canonical transformation:

$$\begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation matrices:

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & L_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & L_2c_2 \\ s_2 & 0 & -c_2 & L_2s_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

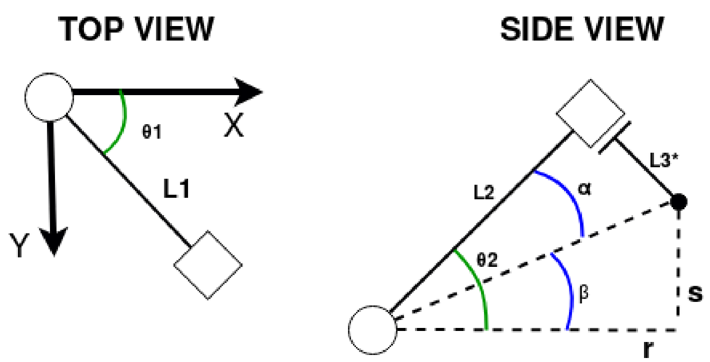
$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_3^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformation from base to tip:

$$H_t^B = A_1A_2A_3 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & c_1(L_2c_2 + L_3^*s_2) \\ s_1c_2 & c_1 & s_1s_2 & s_1(L_2c_2 + L_3^*s_2) \\ -s_2 & 0 & c_2 & L_1 - L_2s_2 + L_3^*c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d) 10% Derive the inverse kinematics of the manipulator.

GEOMETRIC SOLUTION:



Helper variables:

$$r^2 = x^2 + y^2$$

$$s = z - L_1$$

$$d^2 = r^2 + s^2$$

First joint variable:

$$\theta_1 = atan2(y/x)$$

Second joint variable:

$$\cos\alpha = \frac{L_2}{d}$$

$$\sin\alpha = \sqrt{1 - cos^2\alpha}$$

$$\alpha = atan2\left(\frac{\pm\sin\alpha}{\cos\alpha}\right)$$

$$\beta = atan2(r, s)$$
$$\theta_2 = \alpha + \beta$$

Third joint variable:

$$L_3^* = \sqrt{d^2 - L_2^2}$$
$$d_3 = L_3^* - L_3$$

e) 10% Derive the Jacobian matrix of the manipulator.

SOLUTION:
Linear part:

$$J_v = \begin{bmatrix} z_0^0 \times o_3^0 & z_1^0 \times (o_3^0 - o_3^1) & z_2^0 \end{bmatrix}$$

where

$$J_{v_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} c_1(L_2c_2 + L_3^*s_2) \\ s_1(L_2c_2 + L_3^*s_2) \\ L_1 - L_2s_2 + L_3^*c_2 \end{bmatrix} = \begin{bmatrix} -s_1(L_2c_2 + L_3^*s_2) \\ c_1(L_2c_2 + L_3^*s_2) \\ 0 \end{bmatrix}$$

$$J_{v_2} = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} c_1(L_2c_2 + L_3^*s_2) \\ s_1(L_2c_2 + L_3^*s_2) \\ -L_2s_2 + L_3^*c_2 \end{bmatrix} = \begin{bmatrix} -c_1(L_2s_2 - L_3^*c_2) \\ -s_1(L_2s_2 - L_3^*c_2) \\ -L_2c_2 - L_3^*s_2 \end{bmatrix} \leftarrow \text{shortened expression}$$

$$J_{v_3} = \begin{bmatrix} c_1s_2 \\ s_1s_2 \\ c_2 \end{bmatrix}$$

Angular part:

$$J_\omega = \begin{bmatrix} z_0^0 & z_1^0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The full Jacobian matrix:

$$J = \begin{bmatrix} -s_1(L_2c_2 + L_3^*s_2) & -c_1(L_2s_2 - L_3^*c_2) & c_1s_2 \\ c_1(L_2c_2 + L_3^*s_2) & -s_1(L_2s_2 - L_3^*c_2) & s_1s_2 \\ 0 & -L_2c_2 - L_3^*s_2 & c_2 \\ 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

f) 8% Find the singularities of the manipulator. Display the result geometrically.

SOLUTION:
Making a small change to J_v will make the task slightly easier to solve:

$$J_v = \begin{bmatrix} -s_1(L_2c_2 + L_3^*s_2) & -c_1(L_2s_2 - L_3^*c_2) & c_1s_2 \\ c_1(L_2c_2 + L_3^*s_2) & -s_1(L_2s_2 - L_3^*c_2) & s_1s_2 \\ 0 & -L_2c_2 - L_3^*s_2 & c_2 \end{bmatrix} = \begin{bmatrix} -s_1(L_2c_2 + L_3^*s_2) & -c_1(L_2s_2 - L_3^*c_2) & c_1s_2 \\ c_1(L_2c_2 + L_3^*s_2) & -s_1(L_2s_2 - L_3^*c_2) & s_1s_2 \\ 0 & -(L_2c_2 + L_3^*s_2) & c_2 \end{bmatrix}$$

Introducing two new variables:

$$L_2 c_2 + L_3^* s_2 = A$$

$$L_2 s_2 - L_3^* c_2 = B$$

Rewriting J_v:

$$J_v = \begin{bmatrix} -s_1 A & -c_1 B & c_1 s_2 \\ c_1 A & -s_1 B & s_1 s_2 \\ 0 & -A & c_2 \end{bmatrix}$$

We can calculate the arm singularities by solving the following equation:

$$det(J_v) = 0$$

$$det(J_v) = -s_1 A(-s_1 c_2 B + s_1 s_2 A) + c_1 B(c_1 c_2 A) + c_1 s_2(-c_1 A^2)$$

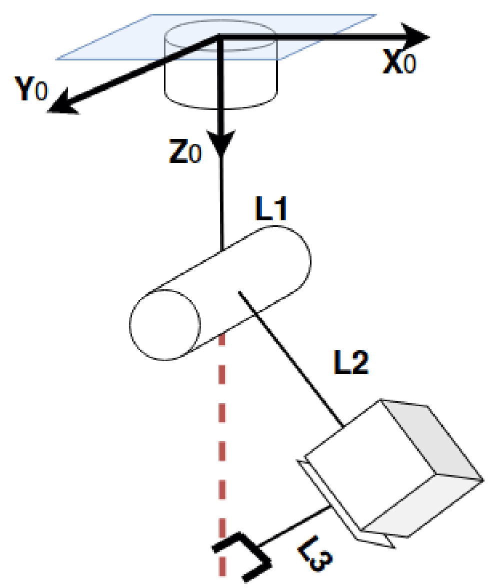
After simplifying:

$$det(J_v) = c_2 AB - s_2 A^2 = A(c_2 B - s_2 A)$$

Then, if $A(c_2 B - s_2 A) = 0$, there are two possible cases:

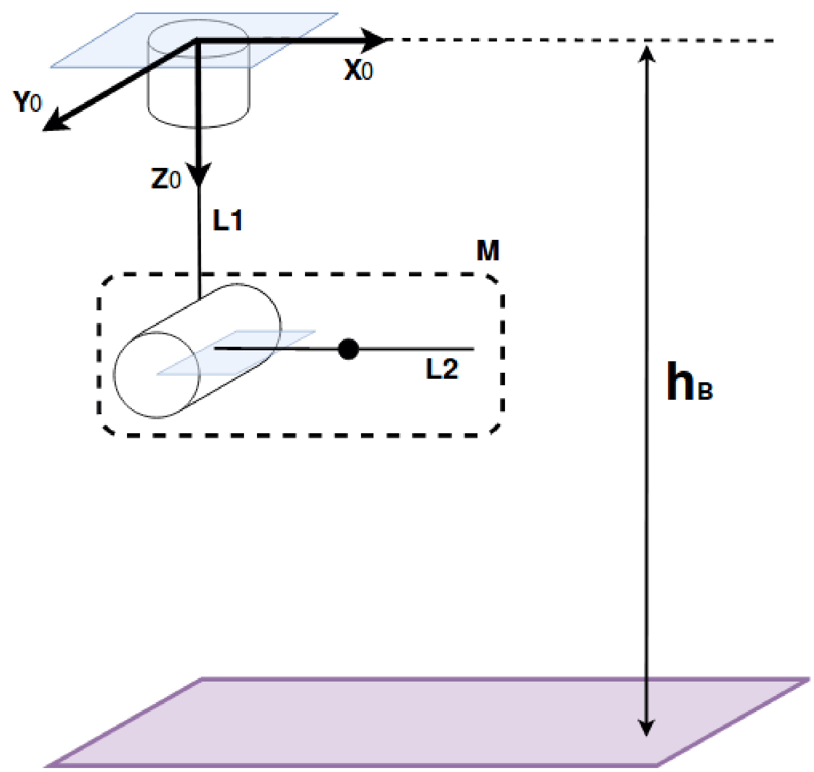
$$\begin{cases} A = 0 \\ c_2 B - s_2 B = 0 \end{cases} \Rightarrow \begin{cases} L_2 s_2 + L_3^* s_2 = 0 \\ \cancel{L_2 s_2 c_2} - L_3^* c_2^2 - \cancel{L_2 s_2 c_2} - L_3^* s_2^2 = 0 \end{cases}$$

The first case leads to all points along the axis z_0, resulting in infinite solutions, because any value of θ_1 can be chosen. Illustration:



The second case resolves to $L_3^* = 0$, which is rarely applicable. This case may be considered redundant.

Task 5: Dynamics (24%)



The image above shows a simplified version of the manipulator from the previous task. This manipulator has two degrees of freedom. Assume that the system has only one mass **M** - the mass of the second link (marked with the dashed line). Assume further that the mass is **uniformly distributed**, and that **its coordinate frame (in the initial position) is aligned in parallel with the frame of the second joint**. The distance between the ground level and the base of the manipulator is $\mathbf{h_B}$ (for this manipulator, **L1 + L2 ¡ hB**). Consider the inertia tensor to be as follows:

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

a) 2% Produce the symbolic expression of the system's potential energy due to the ground level.

SOLUTION:

$$P = Mgh$$

where

$$h = h_B - L_1 \pm \frac{L_2}{2} s_2$$

Total potential energy:

$$Mg(h_B - L_1 \pm \frac{L_2}{2}s_2)$$

Note: the actual sign in $\pm$ depends on configuration. Other expressions are also possible, depending on the geometric definition of θ_2.

b) 2% Assuming that L1 = 300mm, L2 = 200mm, h_B = 550mm, M = 500g, θ_2 = 30° and g = 10 m/s^2, numerically calculate the system's potential energy. State the units.

SOLUTION:

$$h = 0.55m - 0.3m + \frac{0.2m}{2} * 0.5 = 0.55m - 0.3m + 0.05m = 0.3m$$

$$P = 0.5kg * 10m/s^2 * 0.3m = 1.5J$$

Other valid answers: 1500mJ, 1,500,000μJ or similar, depending on configuration. For example, having "-" instead of "±" in the expression from (a) would give 1J as a result.

c) 12% Find the symbolic expression of the system's kinetic energy. You may use the following facts to assist your calculations:

$$R = R_1^0 R_{z,\theta_2} \qquad \omega^T R = (R^T \omega)^T$$

SOLUTION: The total kinetic energy of the system:

$$K = \frac{Mv^2}{2} + \frac{\omega^T R I R^T \omega}{2}$$

Translational part:

$$K_v = \frac{Mv^2}{2}$$
$$v = \mathcal{J}_v \dot{q}$$

where $\mathcal{J}$ is the Jacobian matrix of the manipulator. It can be obtained by adjusting the Jacobian matrix J of the three-link manipulator (from the previous task). Since it is the third joint that has been removed, the third column and all instances of L_3^* in J should be set to zero. Additionally, L_2 should be halved to represent the uniform mass distribution:

$$\mathcal{J} = \begin{bmatrix} -s_1(\frac{L_2}{2}c_2) & -c_1(\frac{L_2}{2}s_2) & 0 \\ c_1(\frac{L_2}{2}c_2) & -s_1(\frac{L_2}{2}s_2) & 0 \\ 0 & -\frac{L_2}{2}c_2 & 0 \\ 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{L_2}{2}s_1c_2 & -\frac{L_2}{2}c_1s_2 & 0 \\ \frac{L_2}{2}c_1c_2 & -\frac{L_2}{2}s_1s_2 & 0 \\ 0 & -\frac{L_2}{2}c_2 & 0 \\ 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Then,

$$v = \begin{bmatrix} -\frac{L_2}{2}s_1c_2 & -\frac{L_2}{2}c_1s_2 & 0 \\ \frac{L_2}{2}c_1c_2 & -\frac{L_2}{2}s_1s_2 & 0 \\ 0 & -\frac{L_2}{2}c_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{L}_3^* \end{bmatrix} = \begin{bmatrix} -\frac{L_2}{2}s_1c_2\dot{\theta}_1 - \frac{L_2}{2}c_1s_2\dot{\theta}_2 \\ \frac{L_2}{2}c_1c_2\dot{\theta}_1 - \frac{L_2}{2}s_1s_2\dot{\theta}_2 \\ -\frac{L_2}{2}c_2\dot{\theta}_2 \end{bmatrix}$$

and

$$v^2 = (-\frac{L_2}{2}s_1c_2\dot{\theta}_1 - \frac{L_2}{2}c_1s_2\dot{\theta}_2)^2 + (\frac{L_2}{2}c_1c_2\dot{\theta}_1 - \frac{L_2}{2}s_1s_2\dot{\theta}_2)^2 + (-\frac{L_2}{2}c_2\dot{\theta}_2)^2 =$$

$$\frac{L_2^2}{4}s_1^2c_2^2\dot{\theta}_1^{\,2} + \frac{L_2^2}{2}s_1c_1s_2c_2\dot{\theta}_1\dot{\theta}_2 + \frac{L_2^2}{4}c_1^2s_2^2\dot{\theta}_2^{\,2} + \frac{L_2^2}{4}c_1^2c_2^2\dot{\theta}_1^{\,2} - \frac{L_2^2}{2}s_1c_1s_2c_2\dot{\theta}_1\dot{\theta}_2 + \frac{L_2^2}{4}s_1^2s_2^2\dot{\theta}_2^{\,2} + \frac{L_2^2}{4}c_2^2\dot{\theta}_2^{\,2} =$$

$$\frac{L_2^2}{4}c_2^2\dot{\theta}_1^{\,2} + (\frac{L_2^2}{4}s_2^2\dot{\theta}_2^{\,2} + \frac{L_2^2}{4}c_2^2\dot{\theta}_2^{\,2}) = \frac{L_2^2}{4}(c_2^2\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2})$$

Finally, the translational kinetic energy of the system is

$$K_v = \frac{ML_2^2(c_2^2\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2})}{8}$$

<u>Rotational part:</u>

$$K_\omega = \frac{\omega^T R I R^T \omega}{2}$$

$$\omega = \mathcal{J}_\omega \dot{q} = \begin{bmatrix} 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{L}_3^* \end{bmatrix} = \begin{bmatrix} -s_1\dot{\theta}_2 \\ c_1\dot{\theta}_2 \\ \dot{\theta}_1 \end{bmatrix}$$

Using that $R = R_1^0 R_{z,\theta_2}$ (where R_1^0 is extracted from A_1 in forward kinematics),

$$R = R_1^0 R_{z,\theta_2} = \begin{bmatrix} c_1 & 0 & -s_1 \\ s_1 & 0 & c_1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1c_2 & -c_1s_2 & -s_1 \\ s_1c_2 & -s_1s_2 & c_1 \\ -s_2 & -c_2 & 0 \end{bmatrix}$$

and therefore

$$R^T\omega = \begin{bmatrix} c_1c_2 & s_1c_2 & -s_2 \\ -c_1s_2 & -s_1s_2 & -c_2 \\ -s_1 & -c_1 & 0 \end{bmatrix} \begin{bmatrix} -s_1\dot{\theta}_2 \\ c_1\dot{\theta}_2 \\ \dot{\theta}_1 \end{bmatrix} = \begin{bmatrix} \cancel{-s_1c_1c_2\dot{\theta}_2} + \cancel{s_1c_1c_2\dot{\theta}_2} - s_2\dot{\theta}_1 \\ \cancel{s_1c_1s_2\dot{\theta}_2} - \cancel{s_1c_1s_2\dot{\theta}_2} - c_2\dot{\theta}_1 \\ -s_1^2\dot{\theta}_2 - c_1^2\dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ -\dot{\theta}_2 \end{bmatrix}$$

Using the equality $\omega^T R = (R^T\omega)^T$,

$$\omega^T R = \begin{bmatrix} -s_2\dot{\theta}_1 & -c_2\dot{\theta}_1 & -\dot{\theta}_2 \end{bmatrix}$$

Then, the rotational kinetic energy of the system is

$$K_\omega = \frac{\begin{bmatrix} -s_2\dot{\theta}_1 & -c_2\dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ -\dot{\theta}_2 \end{bmatrix}}{2} =$$

$$\frac{\begin{bmatrix} -s_2\dot{\theta}_1 I_{xx} & -c_2\dot{\theta}_1 I_{yy} & -I_{zz}\dot{\theta}_2 \end{bmatrix} \begin{bmatrix} -s_2\dot{\theta}_1 \\ -c_2\dot{\theta}_1 \\ -\dot{\theta}_2 \end{bmatrix}}{2} =$$

$$\frac{s_2^2\dot{\theta}_1^{\,2} I_{xx} + c_2^2\dot{\theta}_1^{\,2} I_{yy} + \dot{\theta}_2^{\,2} I_{zz}}{2}$$

<u>Total kinetic energy:</u>

$$K = K_v + K_\omega = \frac{ML_2^2(c_2^2\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2})}{8} + \frac{s_2^2\dot{\theta}_1^{\,2} I_{xx} + c_2^2\dot{\theta}_1^{\,2} I_{yy} + \dot{\theta}_2^{\,2} I_{zz}}{2}$$

d) 8% For this task, assume that M is now a point mass located in the former mass center. Assemble the Lagrangian $\mathcal{L}$ accordingly. Then, derive the dynamics of the manipulator using the Euler-Lagrange formulation. Write the equations of motion in the following form:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau$$

SOLUTION:

Since **M** is now a point point mass, we can discard the influence of inertia (K_ω) on the system. The Lagrangian $\mathcal{L}$ is therefore as follows (*assuming that the potential energy expression has "+" instead of $\pm$, which depends on the choice of frame*):

$$\mathcal{L} = K_v - P$$

$$\mathcal{L} = \frac{ML_2^2(c_2^2\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2})}{8} - Mg(h_B - L_1 + \frac{L_2}{2}s_2)$$

<u>Unfactoring $\mathcal{L}$:</u>

$$\mathcal{L} = \frac{ML_2^2c_2^2\dot{\theta}_1^{\,2}}{8} + \frac{ML_2^2\dot{\theta}_2^{\,2}}{8} - Mgh_B + MgL_1 - \frac{MgL_2s_2}{2}$$

The torque:

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

<u>Torque on the first joint:</u>

$$\tau_1 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} - \frac{\partial \mathcal{L}}{\theta_1}$$

$$\tau_1 = \frac{d}{dt}\left(\frac{ML_2^2c_2^2\dot{\theta}_1}{4}\right) - 0 = \frac{ML_2^2c_2^2\ddot{\theta}_1}{4} - \frac{ML_2^2c_2s_2\dot{\theta}_1\dot{\theta}_2}{2}$$

Again, take note: $\frac{d}{dt}(c_2^2) = -2c_2s_2\dot{\theta}_2$ - this is a composite function with three layers (square function, cosine function, time-dependent function).

Torque on the second joint:

$$\tau_2 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} - \frac{\partial \mathcal{L}}{\theta_2}$$

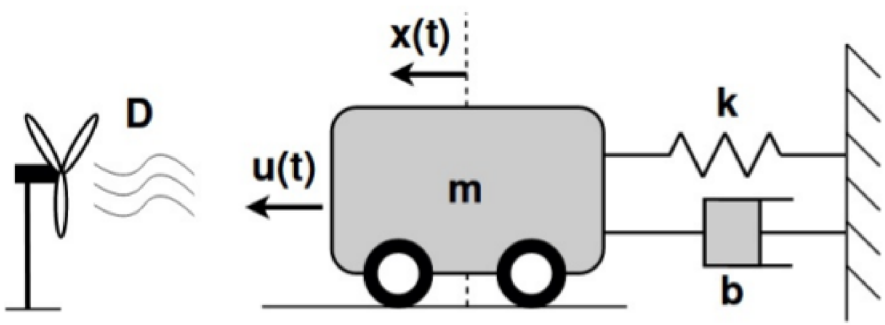
$$\tau_2 = \frac{ML_2^2\ddot{\theta}_2}{4} + \frac{ML_2^2 c_2 s_2 \dot{\theta}_1^{\,2}}{4} + \frac{MgL_2c_2}{2}$$

Rewriting τ:

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$

$$\tau = \begin{bmatrix} \frac{ML_2^2c_2^2}{4} & 0 \\ 0 & \frac{ML_2^2}{4} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -\frac{ML_2^2c_2s_2\dot{\theta}_2}{2} & 0 \\ \frac{ML_2^2c_2s_2\dot{\theta}_1}{4} & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{MgL_2c_2}{2} \end{bmatrix}$$

Task 6: Mass-spring-damper Control (20%)



The above image demonstrates a mass-spring-damper system, where m is the object mass, k is the spring constant, b is the damping coefficient, D is the system disturbance, x(t) is the mass displacement at time t, and u(t) is the input force. The system can be described with the following differential equation:

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) + D(t) = u(t)$$

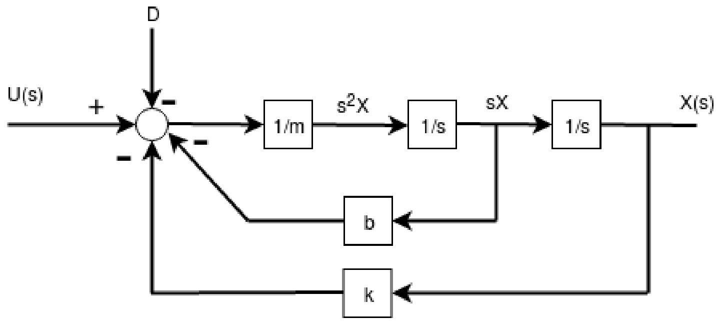
a) 5% Transform the above equation from time domain to Laplace domain. Draw the closed-loop block diagram of the system.

SOLUTION:

Laplace transform:

$$U(s) = ms^2X(s) + bsX(s) + kX(s) + D(s)$$

Block diagram of the system:



b) 5% Suppose that a PD compensator is added to the system, and $X_d(s)$ is the desired mass displacement. Transform the system equation in Laplace domain (that you obtained in (a)) so that it includes the PD compensator. Solve the resulting equation for $X(s)$.

SOLUTION:

The new equation with the PD compensator:

$$ms^2X + bsX + kX + D = K_p(X_d - X) - K_dsX$$

Solving for $X(s)$:

$$X(ms^2 + bs + k + K_p + K_ds) = K_pX_d - D$$

$$X = \frac{K_pX_d - D}{ms^2 + bs + k + K_p + K_ds}$$

c) 4% Assume that the system receives a step reference input $X_d(s) = \frac{\psi}{s}$ and constant disturbance $\frac{d}{s}$. Use the final value theorem below to determine the steady-state error:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

SOLUTION:

Mass displacement at $t \to \infty$:

$$lim_{t\to\infty}x(t) = \lim_{s\to 0} \frac{K_p\frac{\psi}{\cancel{s}}\cancel{s} - \frac{d}{\cancel{s}}\cancel{s}}{\cancel{ms^2} + \cancel{bs} + k + K_p + \cancel{K_ds}} = \frac{K_p\psi - d}{K_p + k} = \frac{K_p\psi}{K_p + k} - \frac{d}{K_p + k}$$

Steady-state error:

$$e(t) = X_d(t) - X(t)$$

$$lim_{t\to\infty}(X_d(t) - X(t)) = \lim_{s\to 0}\left(\cancel{s}\frac{\psi}{\cancel{s}} - sX(s)\right) = \psi - \left(\frac{K_p\psi}{K_p + k} + \frac{d}{K_p + k}\right)$$

d) 3% For which values of K_p and K_d is the system underdamped?

SOLUTION:

From the course book:

$$K_p = \omega^2 J$$

$$K_d = 2\zeta\omega J - B$$

In this case, m plays the role of J, while B should be replaced with b. Finally, for the system to be underdamped, ζ must less than 1. The result is:

$$K_p = \omega^2 m$$

$$K_d < 2\omega m - b$$

e) 3% If you needed to eliminate the steady-state error (while still suppressing oscillations), and could use a different compensator to do so, which compensator would you choose? Draw the closed-loop block diagram of the system with the new compensator.

SOLUTION:
To eliminate the steady-state error and not introduce oscialltions into the system, we need to choose a PID compensator.

Block diagram: