Task 1.a Make the equation of motion for Joint 2 independent (of other joints). Justify your method.

$$\tau_{pendulum} = ml^2\ddot{\theta} - mglsin(\theta) \tag{1}$$

Task 1.b Transform the following independent joint control equation from time domain to Laplace domain:

$$u(t) = J\ddot{\theta}(t) + B\dot{\theta}(t) + D(t) \tag{2}$$

Laplace transform the equation(1) and we obtained as follows,

$$\begin{gathered} U = Js^2\theta + Bs\theta + D \\ U - D = Js^2\theta + Bs\theta \\ U - D = \theta(Js^2 + Bs) \\ \theta = \frac{U - D}{Js^2 + Bs} \\ \theta = \frac{U}{Js^2 + Bs} - \frac{D}{Js^2 + Bs} \\ \theta = HU - HD \end{gathered} \tag{3}$$

where $H = \frac{1}{Js^2+Bs}$

The terms in the equation we obtained in (a) that correspond to the terms J, B and D in the above equation would be

$J\ddot{\theta}(t)$ for inertial forces where $J = ml^2$, $m = m_2 + m_3$ and $l = L_2 + L_3$

$$J = (m_2 + m_3)(L_2 + L_3)^2$$

$$J = (0.2724 + 0.1406)(0.2221 + 0.1362)^2$$

$$J = 0.053$$

$B\dot{\theta}(t)$ for viscous friction and

$D(t)$ for gravitational forces where $D = mgl = (m_2 + m_3)(L_2 + L_3) * 9.81 = 1.45$

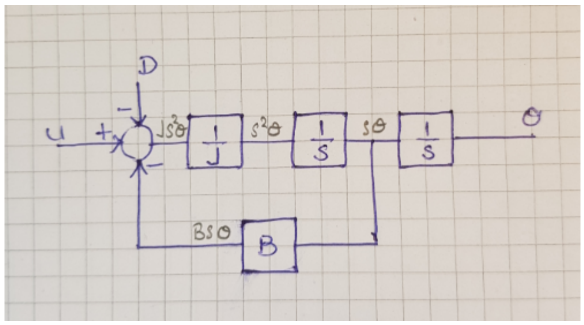
Task 1.c Draw a closed-loop block diagram for equation (2) in Laplace domain, using only simple blocks.

$$u(t) = J\ddot{\theta}(t) + B\dot{\theta}(t) + D(t)$$

$$U = Js^2\theta + Bs\theta + D$$

$$U - D - Bs\theta = Js^2\theta$$

So the transfer function block diagram can be describe as follows:



Add a PD-controller to the block diagram and derive the transfer function between the input $\theta_d(s)$ and the output $\theta(s)$.

The error is defined as $e(t) = \theta_d(t) - \theta(t)$.

Firstly, we would like to drive P-controller (K_p).

(The controller (K) use the error e(t) to calculate its output, called control effort, denote as u (U in block diagram)).

$$U(t) = K_p e(t)$$

where $e(t) = \theta_d(t) - \theta(t)$, and make Laplace transform.

$$U(s) = K_p E(s)$$

Then, we drive PD-controller (K_p and K_d).

$$u(t) = K_p e(t) + K_d \dot{e}(t)$$

$$u(t) = K_p(\theta_d - \theta) + K_d(\dot{\theta}_d - \dot{\theta})$$

Laplace transform:

$$U(s) = K_p(\theta_d - \theta) + K_d(s\theta_d - s\theta)$$

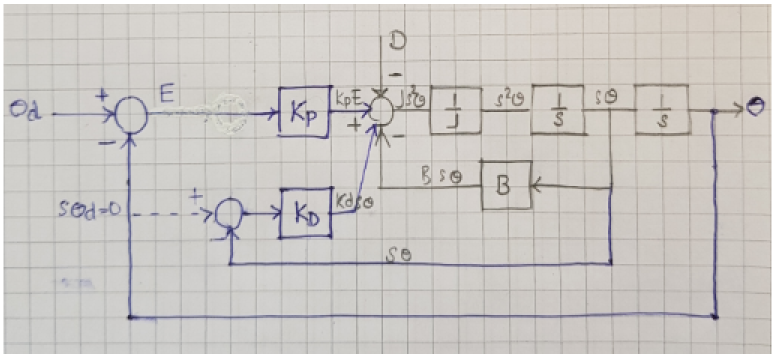
$$U(s) = K_p(\theta_d - \theta) + K_d s(\theta_d - \theta)$$

$$U(s) = (\theta_d - \theta)(K_p + K_d s)$$

From equation (3) we can make PD-transfer function as below:

$$\theta = \frac{(\theta_d - \theta)(K_p + K_d s) - D}{Js^2 + Bs} \tag{4}$$

The transfer function block diagram with Proportional Derivative (PD) Controller can be describe as follows:



Task 1.d With the PD-controller, the closed-loop system is now second order, and hence the step response is given by the closed-loop natural frequency ω and damping ratio ς.

We can derive the equation (4) as belows:

$$(Js^2 + Bs)\theta = (\theta_d - \theta)(K_p + K_d s) - D$$

$$Js^2\theta + Bs\theta = K_p\theta_d + K_d s\theta_d - K_p\theta - K_d s\theta - D$$

$$Js^2\theta + Bs\theta + K_d s\theta + K_p\theta = \theta_d(K_p + K_d s) - D$$

$$\theta(Js^2 + Bs + K_d s + K_p) = \theta_d(K_p + K_d s) - D$$

$$\theta = \frac{\theta_d(K_p + K_d s) - D}{Js^2 + Bs + K_d s + K_p}$$

$$\theta = \frac{\theta_d(K_p + K_d s) - D}{s^2 + \frac{B+K_d}{J}s + \frac{K_P}{J}}$$

The denominator is the characterististic polynomial and the roots of this determine the performance of the system.

$$s^2 + \frac{B + K_d}{J}s + \frac{K_P}{J} = 0$$

We could choose the values of K_P and K_d if we consider the closed-loop system as a damped second order system.

$$s^2 + 2\varsigma\omega s + \omega^2 = 0$$

The above two equation gives us K_P and K_d as belows:

$$2\varsigma\omega = \frac{B + K_d}{J}$$

$$K_d = 2J\varsigma\omega - B$$

$$\omega^2 = \frac{K_p}{J}$$

$$K_p = J\omega^2$$

Given the requirements of a natural frequency of 6 and a critically damped ($\varsigma = 1$) system, we can find values for K_P and K_d as belows:

$$K_d = 2 * 1 * 1 * 6 - 0.7$$

where $J = 0.053$, $\varsigma = 1$, $\omega = 6$ and $B = 0$

$$K_d = 0.636$$

$$K_p = 0.053 * 6^2$$

$$K_p = 1.908$$

where $J = 0.053$ and $\omega = 6$

Task 2.a A screenshot of Gazebo and rqt showing the functiong setup