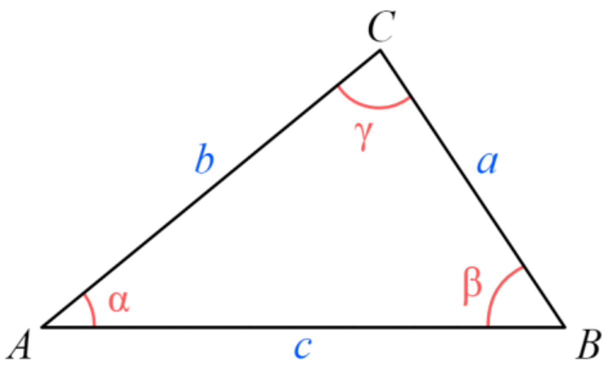
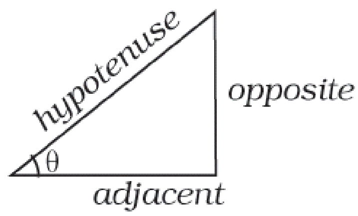
Rules & Formulas INF3480/INF4380

23. januar 2017 16:46



$$c^2 = a^2 + b^2 - 2ab\ \cos\gamma$$



$$\sin\theta = \frac{o}{h}$$

$$\cos\theta = \frac{a}{h}$$

$$\tan\theta = \frac{o}{a}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\text{radians} = \text{degrees} \times \frac{\pi}{180}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\sin^2 u = \frac{1-\cos(2u)}{2}$$

$$\cos^2 u = \frac{1+\cos(2u)}{2}$$

$$\tan^2 u = \frac{1-\cos(2u)}{1+\cos(2u)}$$

$$\sin(-u) = -\sin u \qquad \cos(-u) = \cos u$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \\ &= \frac{1-\tan^2\theta}{1+\tan^2\theta}\end{aligned}$$

Integral ↑ | Derivative ↓

$$\begin{array}{c}\sin\theta \\ \cos\theta \\ -\sin\theta \\ -\cos\theta \\ \sin\theta\end{array}$$

$$\sin(u+v) = \sin u\cos v + \cos u\sin v$$

$$\cos(u+v) = \cos u\cos v - \sin u\sin v$$

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u\tan v}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

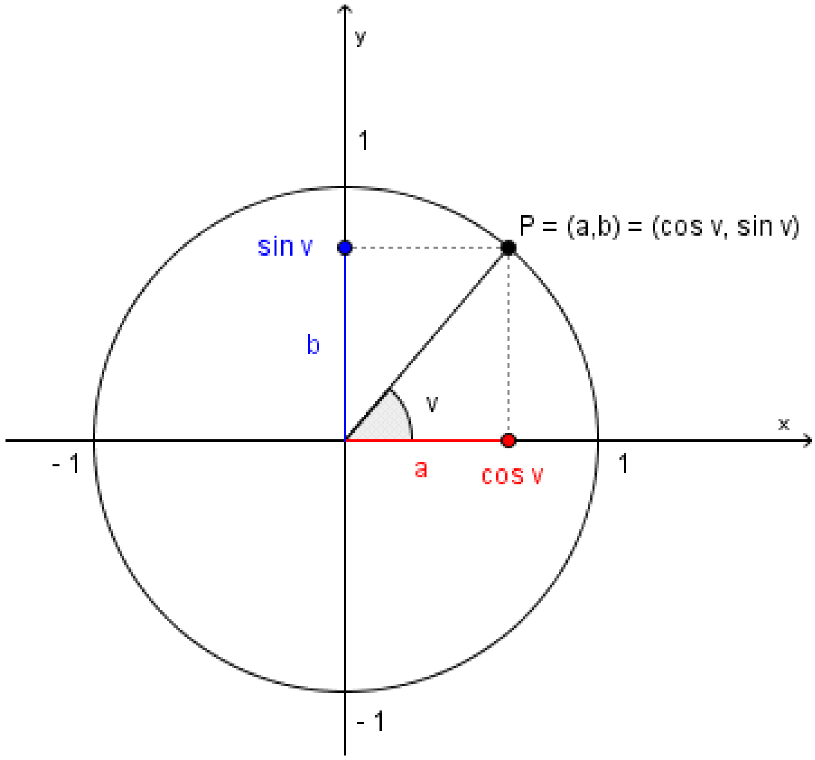
$$\sin(u-v) = \sin u\cos v - \cos u\sin v$$

$$\cos(u-v) = \cos u\cos v + \sin u\sin v$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u\tan v}$$

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta \\ &= \frac{2\tan\theta}{1+\tan^2\theta}\end{aligned}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$



$$f(x) = u * v$$
$$f'(x) = u' * v + u * v'$$

$$f(x) = g\big(u(x)\big)$$
$$f'(x) = g'(u) * u'(x)$$

$$\operatorname{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

Deg	0	30	45	60	90
Rad	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Sin	0	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
Cos	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	0
Tan	0	$\sqrt{3}^{-1}$	$\sqrt{3}^{0}$	$\sqrt{3}^{1}$	Not defined

$$\mathbf{A} = [a, b, c] \qquad \mathrm{B} = [d, e, f]$$

$$\begin{matrix} x & y & z & x & y \\ a & b & c & a & b \\ d & e & f & d & e \end{matrix}$$

$$\mathbf{A} \times \mathbf{B} = [(bf - ce), (cd - af), (ae - bd)]$$

Consider the matrices

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

Multiplying gives

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix}, \qquad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

Thus, $AB \neq BA$.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}$$

$$3 \times 2 \quad \times \quad 2 \times 5 \quad = \quad 3 \times 5$$

$$\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a(ei - hf) - d(bi - hc) + g(bf - ec)$$

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \times \underbrace{\begin{bmatrix} e & f \\ g & h \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}}_{C}$$

A, B and C are square metrices of size N x N
a, b, c and d are submatrices of A, of size N/2 x N/2
e, f, g and h are submatrices of B, of size N/2 x N/2

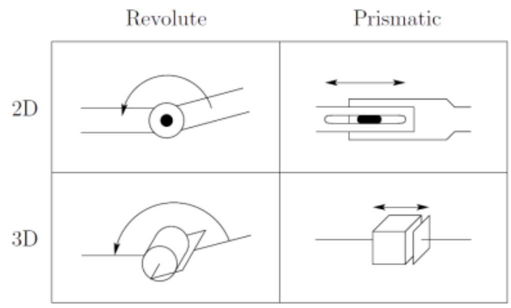
$$Rot_{x,\alpha}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\alpha & -S_\alpha \\ 0 & S_\alpha & C_\alpha \end{bmatrix}$$

$$Rot_{y,\beta}=\begin{bmatrix} C_\beta & 0 & S_\beta \\ 0 & 1 & 0 \\ -S_\beta & 0 & C_\beta \end{bmatrix}$$

$$Rot_{z,\gamma}=\begin{bmatrix} C_\gamma & -S_\gamma & 0 \\ S_\gamma & C_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_0^1 = (R_1^0)^T \qquad H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \qquad P^0 = H_1^0 H_2^1 P^2$$

$$(R_1^0)^T = (R_1^0)^{-1} \qquad H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix}$$



$$T_6^0 = T_1^0 T_2^1 T_3^2 T_4^3 T_5^4 T_6^5 = T_3^0 T_6^3$$

R_2^0 - Rotational matrix for coordinate system 2, relative to coordinate system 0.

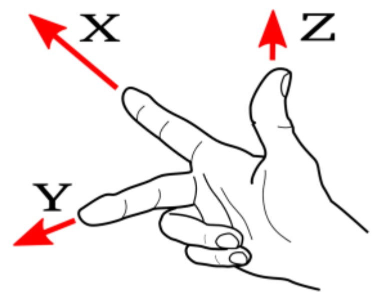
Consider three frames, o_0, o_1, and o_2 and corresponding rotation matrices $R_2{}^1$, and $R_1{}^0$

Let $d_2{}^1$ be the vector from the origin o_1 to o_2, $d_1{}^0$ from o_0 to o_1

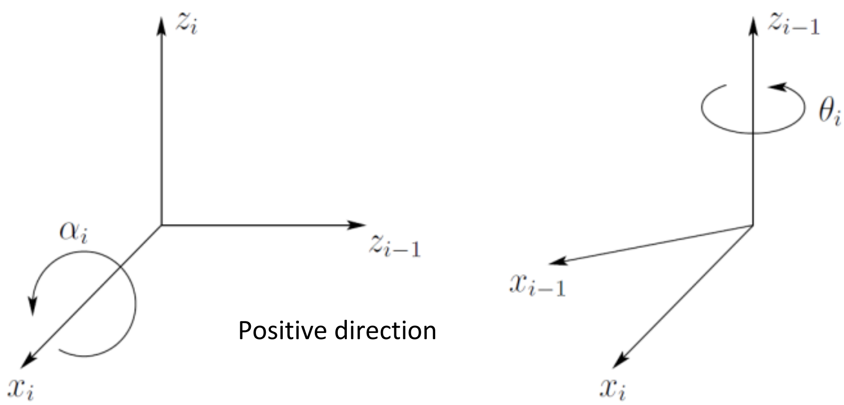
For a point p^2 attached to o_2, we can represent this vector in frames o_0 and o_1:

$$\begin{aligned} p^1 &= R_2^1 p^2 + d_2^1 \\ p^0 &= R_1^0 p^1 + d_1^0 \\ &= R_1^0 \left(R_2^1 p^2 + d_2^1 \right) + d_1^0 \\ &= R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0 \end{aligned}$$

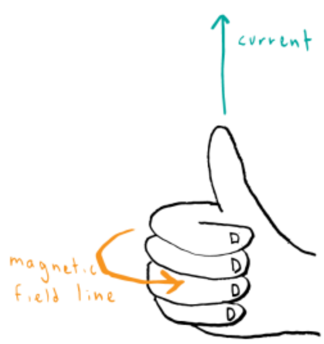
$$q = [q_1 \quad q_2 \quad \dots \quad q_n]^T$$



Positive direction will follow the alphabet



Positive direction



Denavit Hartenberg

$$A_i = \mathbf{Rot}_{z,\theta_i}\mathbf{Trans}_{z,d_i}\mathbf{Trans}_{x,a_i}\mathbf{Rot}_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Link parameters for 2-link planar manipulator

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

* variable

a_i: link length, distance between the z_0 and z_1 (along x_1)

α_i: link twist, angle between z_0 and z_1 (measured around x_1)

d_i: link offset, distance between o_0 and intersection of z_0 and x_1 (along z_0)

θ_i: joint angle, angle between x_0 and x_1 (measured around z_0)

Jacobian Matrix

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \left[\frac{dh(q)}{dq}\right]_{6\times n} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1}$$

Jacobian is a function of q, it is not a constant!

$$J = \left(\frac{dh(q)}{dq}\right)_{6\times n} = \begin{bmatrix} \frac{\partial h_1}{\partial q_1} & \frac{\partial h_1}{\partial q_2} & \cdots & \frac{\partial h_1}{\partial q_n} \\ \frac{\partial h_2}{\partial q_1} & \frac{\partial h_2}{\partial q_2} & \cdots & \frac{\partial h_2}{\partial q_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_6}{\partial q_1} & \frac{\partial h_6}{\partial q_2} & \cdots & \frac{\partial h_6}{\partial q_n} \end{bmatrix}_{6\times n}$$

The i^{th} column of J_v is given by:

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for } i \text{ revolute} \\ z_{i-1} & \text{for } i \text{ prismatic} \end{cases}$$

The i^{th} column of J_ω is given by:

$$J_{\omega_i} = \begin{cases} z_{i-1} & \text{for } i \text{ revolute} \\ 0 & \text{for } i \text{ prismatic} \end{cases}$$

$$\xi = J(q)\dot{q}$$

a configuration q is singular if and only if

$$\det J(q) = 0$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j$$

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

$$K = \frac{1}{2}mv^Tv + \frac{1}{2}\omega^T I\omega$$

Proportional Controller

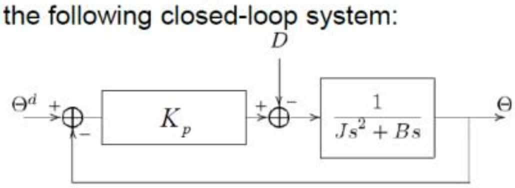
- Control law:

$$u(t) = K_p e(t)$$

  - Where $e(t) = \theta^d(t) - \theta(t)$
- in the Laplace domain:

$$U(s) = K_p E(t)$$

- This gives the following closed-loop system:



PID Controller

	Time Domain (t):	Frequency Domain (s):
	$u(t)$	$U(s)$
	$\parallel$	$\parallel$
P =	$K_p e(t)$	$K_P E(s)$
	+	+
I =	$K_i \int e(t)dt$	$\frac{K_I}{s} E(s)$
	+	+
D =	$K_d \dot{e}(t)$	$sK_d E(s)$

PD controller

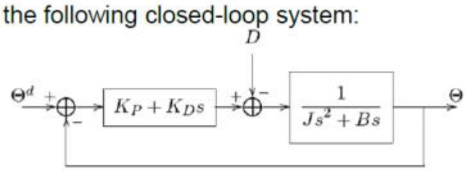
- Control law:

$$u(t) = K_p e(t) + K_d \dot{e}(t)$$

  - Where $e(t) = \theta^d(t) - \theta(t)$
- in the Laplace domain:

$$U(s) = (K_p + sK_d)E(t)$$

- This gives the following closed-loop system:



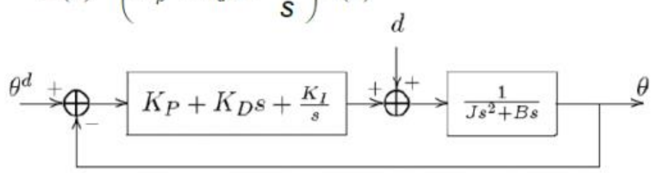
PID controller

- Control law:

$$u(t) = K_p e(t) + K_d \dot{e}(t) + K_i \int e(t)dt$$

- In the Laplace domain:

$$U(s) = \left(K_p + K_d s + \frac{K_i}{s}\right)E(s)$$



Time domain	Laplace domain
$x(t)$	$x(s) = L\{x(t)\} = \int_0^\infty e^{-st}x(t)dt$
$\dot{x}(t)$	$sx(s) - x(0)$
$\ddot{x}(t)$	$s^2x(s) - sx(0) - \dot{x}(0)$
Ct	$\frac{C}{s^2}$
step	$\frac{1}{s}$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$

Time domain	Laplace domain
$x(t - \alpha)H(t - \alpha)$	$e^{-\alpha s}x(s)$
$e^{-at}x(t)$	$x(s + a)$
$x(at)$	$\frac{1}{a}x\left(\frac{s}{a}\right)$
$C\delta(t)$	C

For example, if we have a 1DOF system described by:

$$\tau(t) = J\ddot{\theta}(t) + B\dot{\theta}(t)$$

We want the representation in the Laplace domain:

$$\tau(s) = s^2 J\theta(s) + sB\theta(s)$$
$$= s(sJ + B)\theta(s)$$