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**Control theory**

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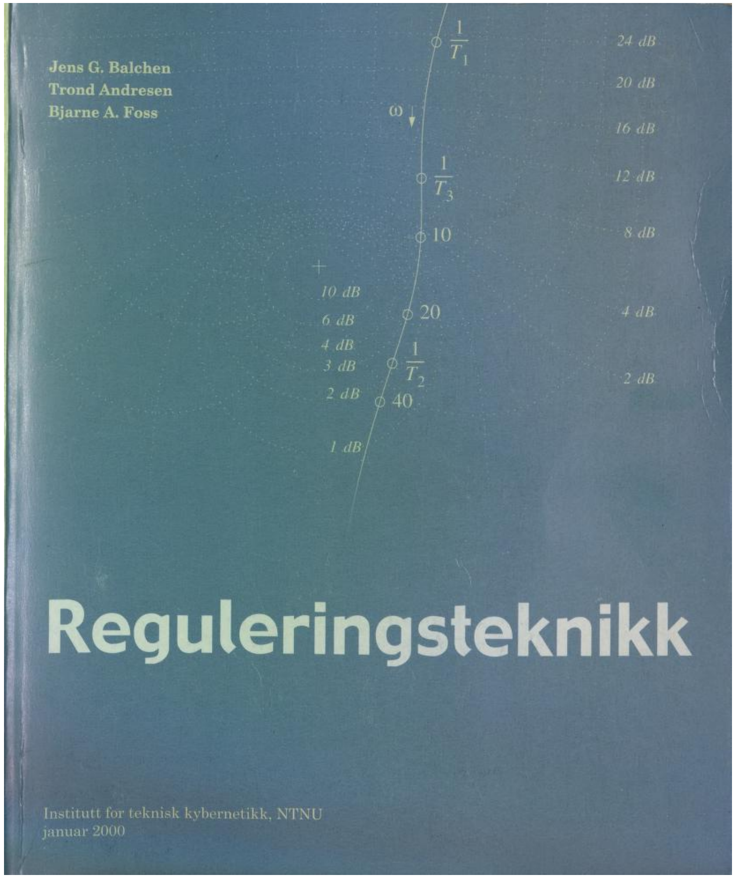
# Lecture overview

- 1. Laplace transform
  - 1.1 Transfer functions and block diagrams
  - 1.2 Roots and zeros
  - 1.3 Root locus plots
- 2. Frequency analysis
  - Bode plots
- 3. State space systems
- 4. Feedback systems
- 5. Stability
  - 5.1 Frequency domain
  - 5.2 State space systems
  - 5.3 Non-linear systems

# Additional litterature

- *Reguleringsteknikk* Balchen et. al.
- Wikibook on Control Systems wikibooks.org/wiki/Control_Systems



# 1. Laplace transform

- A tool to solve linear higher order differential equations
- Example:

Mass-spring-damper system

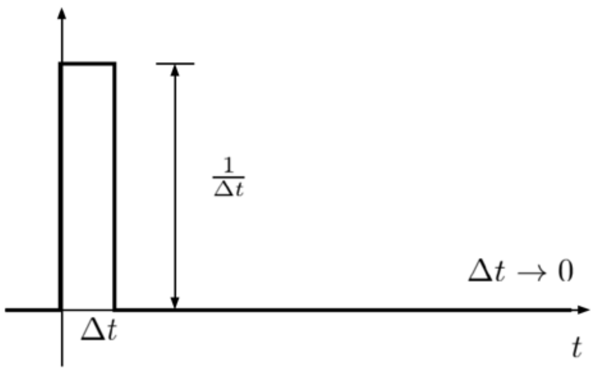
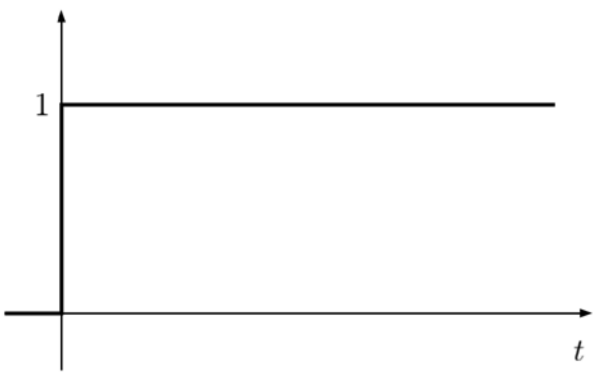
$$m\ddot{x} + f\dot{x} + kx = u$$

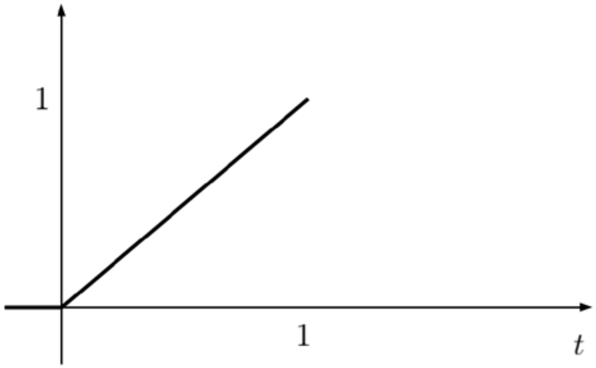
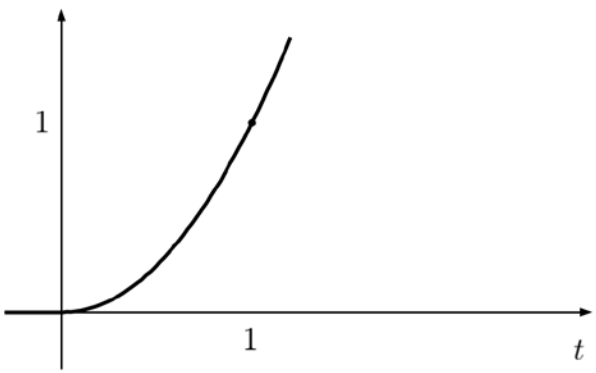
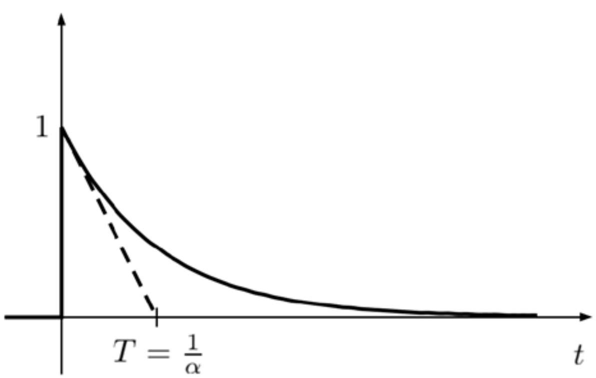
Transforms to

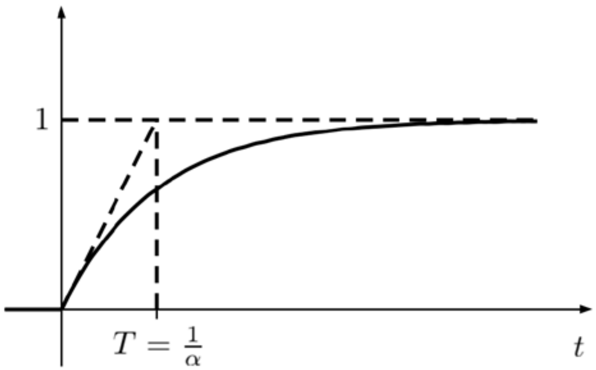
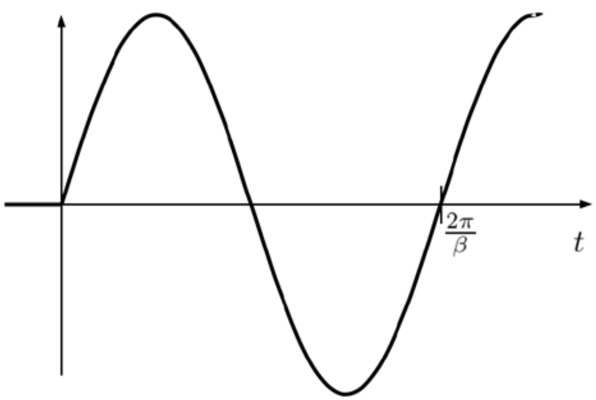
$$mxs^2 + fxs + kx = u$$

This system is now a linear equation

# Common Laplace functions

| | | |
|---|---|---|
|   | $\delta(t) = \lim_{\Delta t \to 0} g(t, \Delta t)$<br>$g(t, \Delta t) = \begin{cases} \frac{1}{\Delta t} & , 0 < t < \Delta t \\ 0 & , 1 < 0, t > \Delta t \end{cases}$ | $1$ |
|   | $f(t) = \begin{cases} 1 & , t \geq 0 \\ 0 & , t < 0 \end{cases}$ | $\frac{1}{s}$ |

| | | |
|---|---|---|
|  | $f(t) = \begin{cases} t & ,t \geq 0 \\ 0 & ,t < 0 \end{cases}$ | $\frac{1}{s^2}$ |
|  | $f(t) = \begin{cases} t^2 & ,t \geq 0 \\ 0 & ,t < 0 \end{cases}$ | $\frac{1}{s^3}$ |
|  | $f(t) = \begin{cases} e^{-\alpha t} & ,t \geq 0 \\ 0 & ,t < 0 \end{cases}$ | $\frac{1}{s+\alpha} = \frac{T}{1+Ts}$ |

| | | |
|---|---|---|
|  | $f(t) = \begin{cases} 1 - e^{-\alpha t} & ,t \geq 0 \\ 0 & ,t < 0 \end{cases}$ | $\frac{\alpha}{s(s+\alpha)} = \frac{1}{s(1+Ts)}$ |
|  | $f(t) = \begin{cases} \sin \beta t & ,t \geq 0 \\ 0 & ,t < 0 \end{cases}$ | $\frac{\beta}{s^2+\beta^2} = \frac{\beta}{(s+j\beta)(s-j\beta)}$ |

# 1.2 Transfer functions

- When all initial conditions of a Laplace transform are zero, the response of a linear system,Y(s), is given by its input, X(s), and its transfer function, H(s).

$$Y(s) = H(s)\,X(s)$$

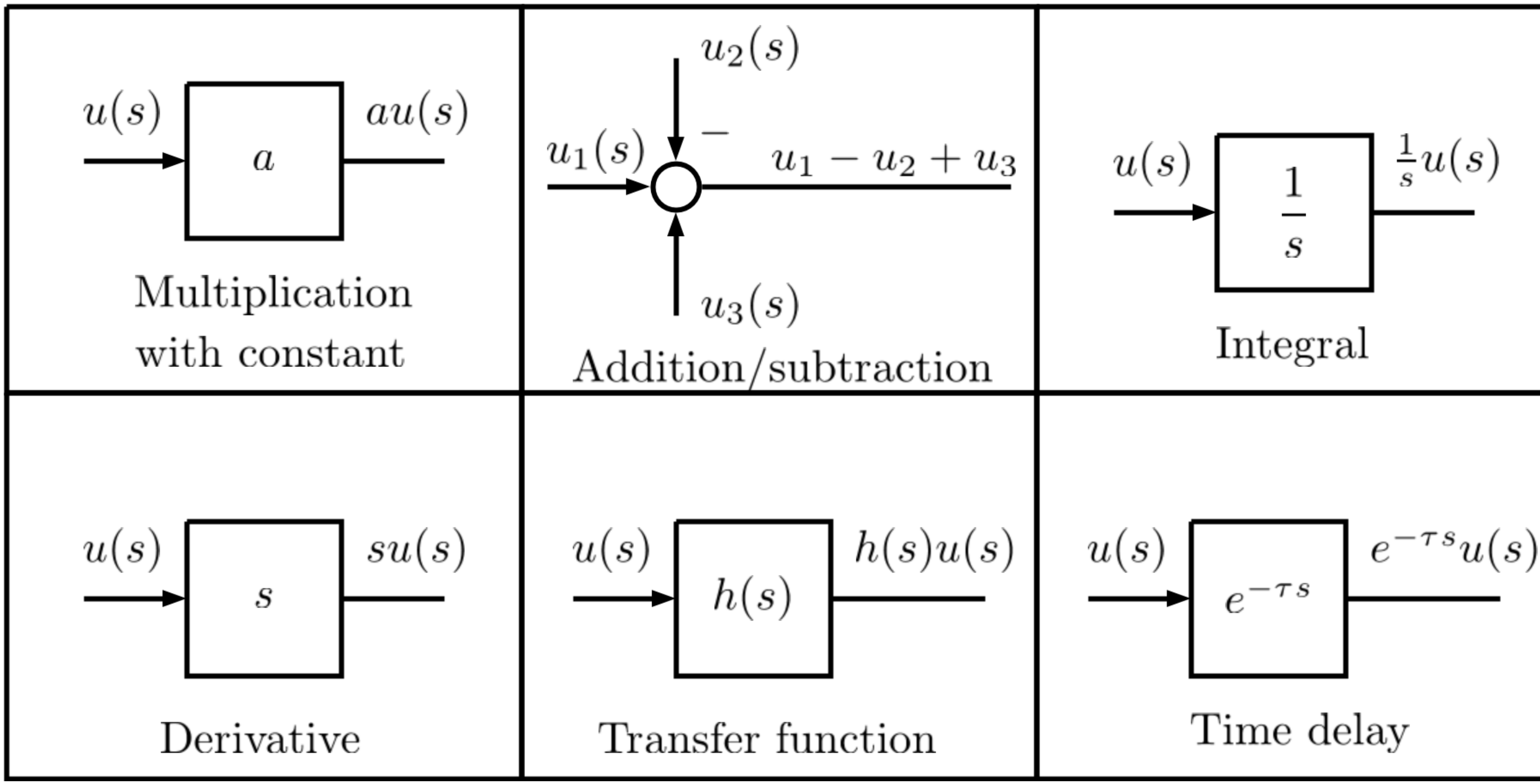
$$\frac{Y(s)}{X(s)} = H(s)$$

# Transfer functions

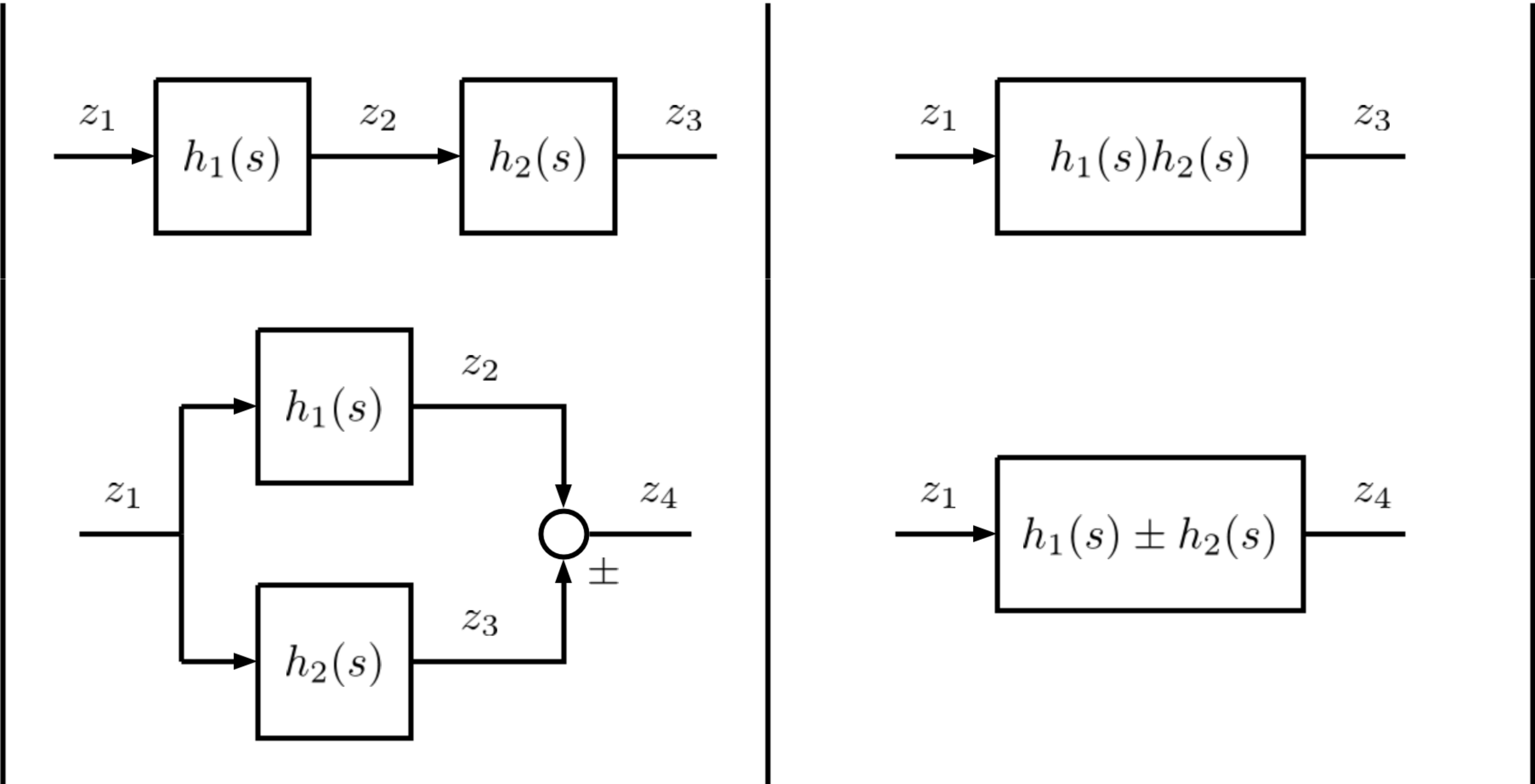
- For the mass-spring damper system the transfer function is

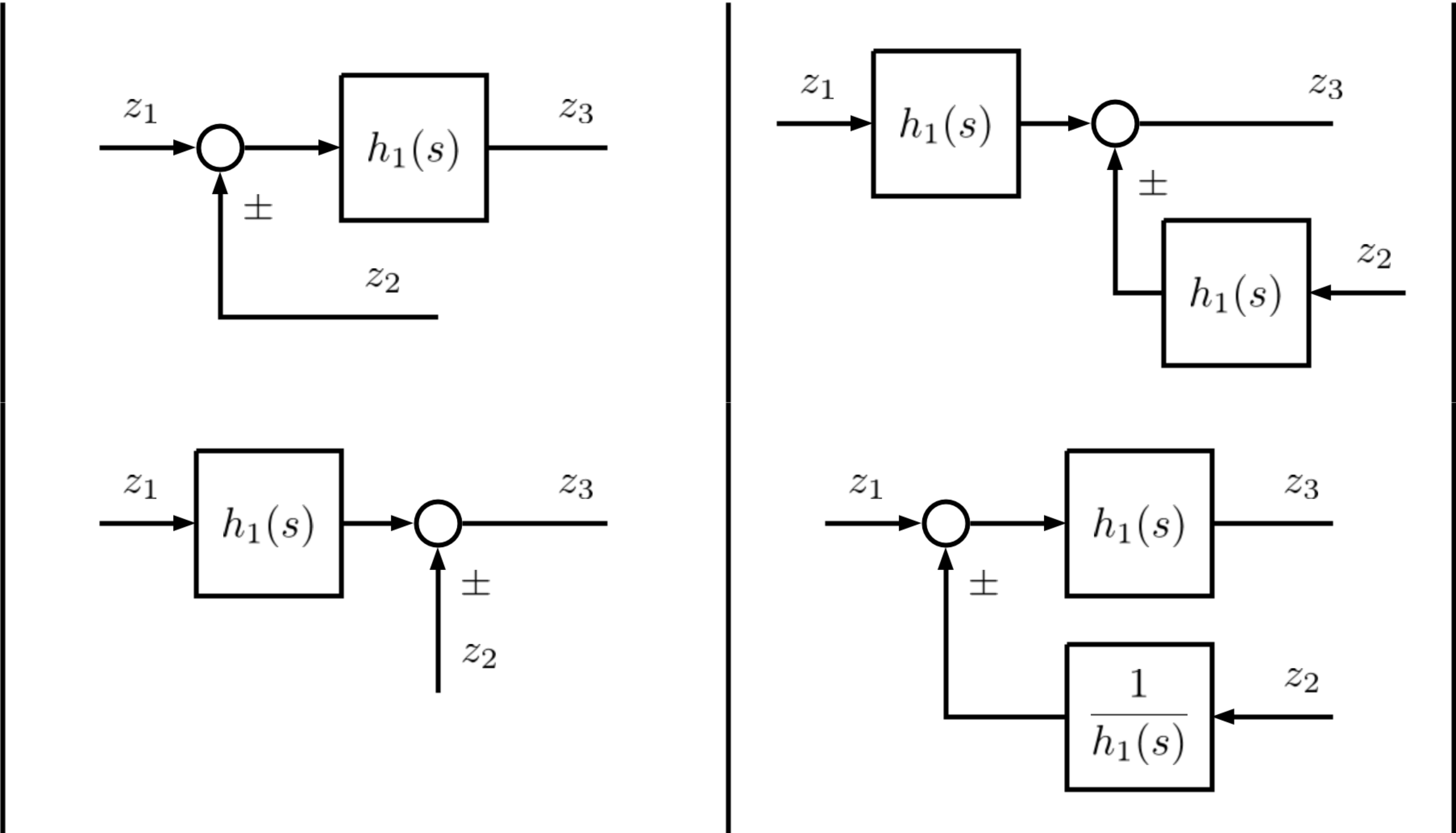
$$\frac{x}{u} = \frac{1}{ms^2 + fs + kx}$$

# Block diagrams



# Manipulation of block diagrams




$z_1$
$h_1(s)$
$z_3$
$\pm$
$z_2$
$z_1$
$h_1(s)$
$z_3$
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$h_1(s)$
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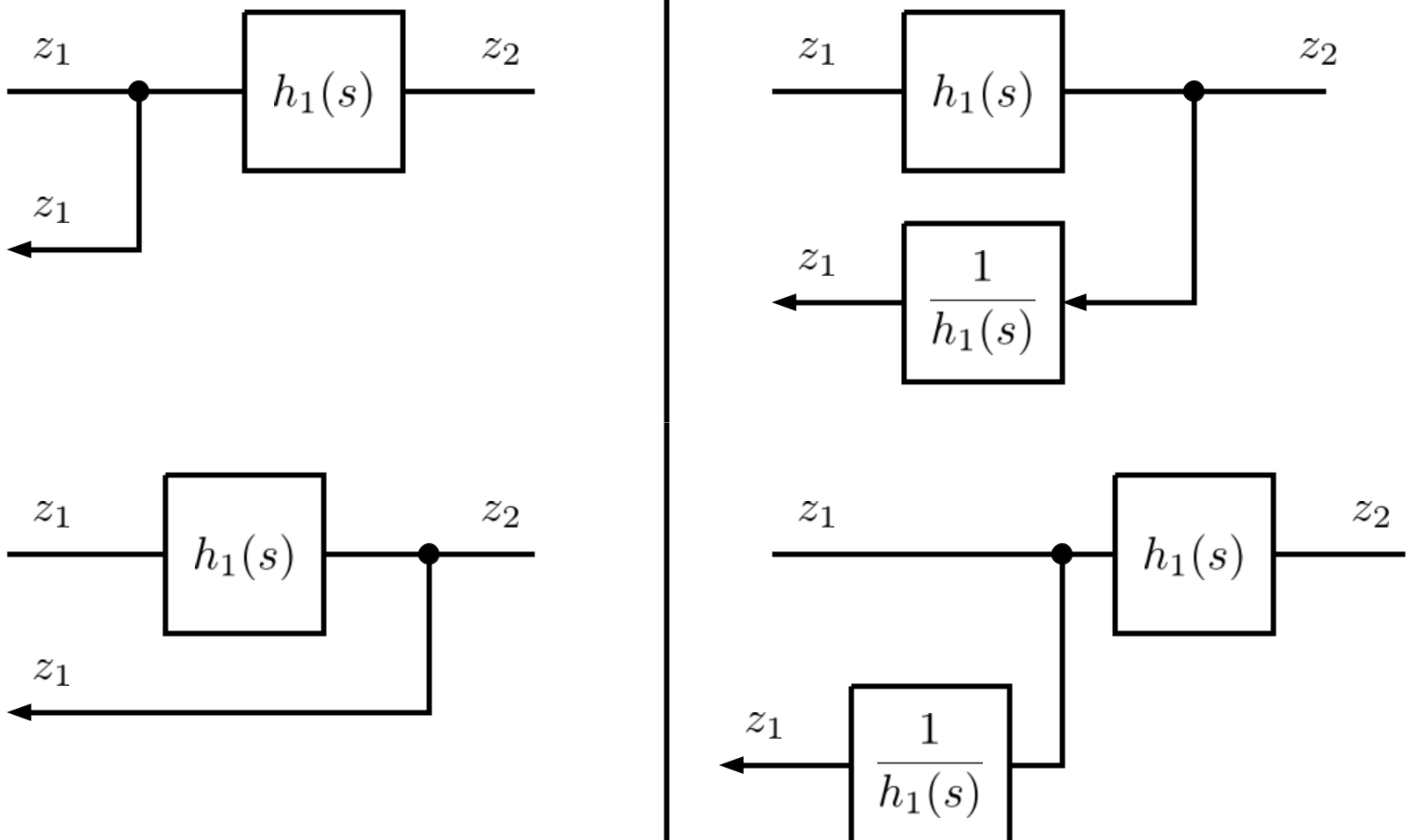
$z_2$
$z_1$
$h_1(s)$
$z_3$
$\pm$
$\frac{1}{h_1(s)}$
$z_2$


$z_1$
$h_1(s)$
$z_2$
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$\frac{1}{h_1(s)}$
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$h_1(s)$
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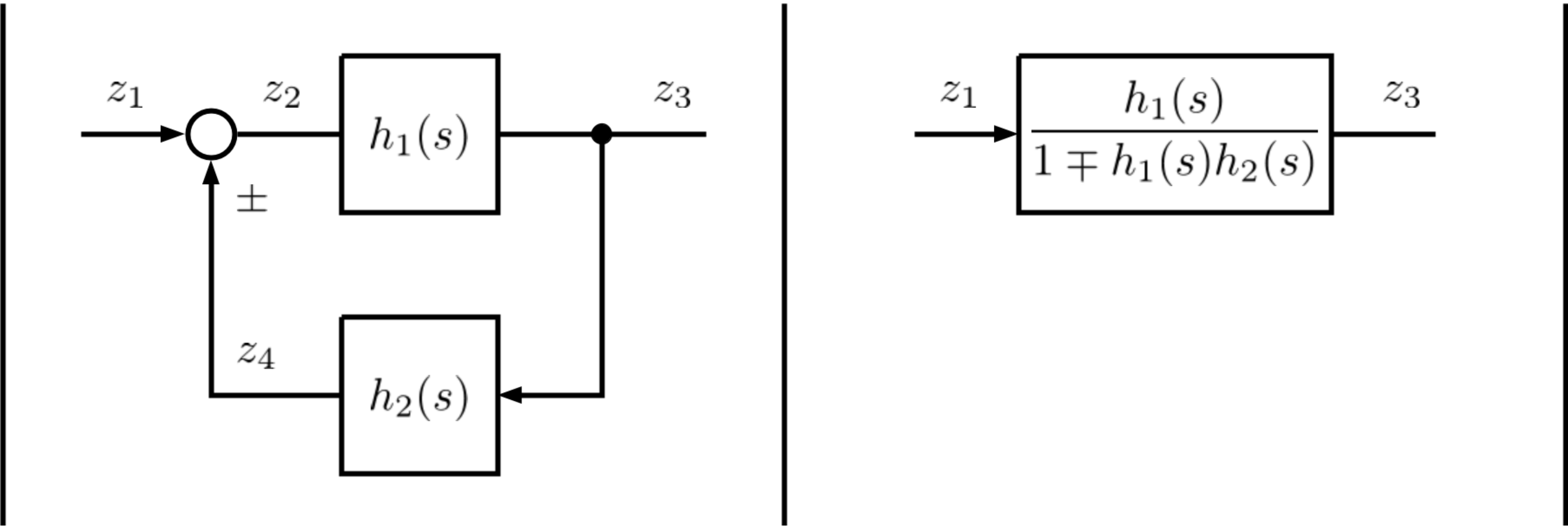
$h_1(s)$
$z_2$
$z_1$
$\frac{1}{h_1(s)}$


$z_1$
$z_2$
$h_1(s)$
$z_3$
$\pm$
$z_4$
$h_2(s)$
$z_1$
$\dfrac{h_1(s)}{1 \mp h_1(s)h_2(s)}$
$z_3$

# Drawing a block diagram of mass-spring-damper system

- $xs^2 = \frac{1}{m}(u - fxs - kx)$

# 1.2 Zeros and poles of the transfer functions

- For rational transferfunctions we denote the roots of the nominator zeros and roots of the denominator poles
- The poles gives important characteristics about the transfer function

$$h(s) = \frac{\rho_p s^p + \cdots + \rho_1 s^1 + \rho_0}{s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

$$h(s) = \frac{\rho_p (s - v_1) \cdots (s - v_n)}{(s - \lambda_1) \cdots (s - \lambda_n)}$$

# Examples

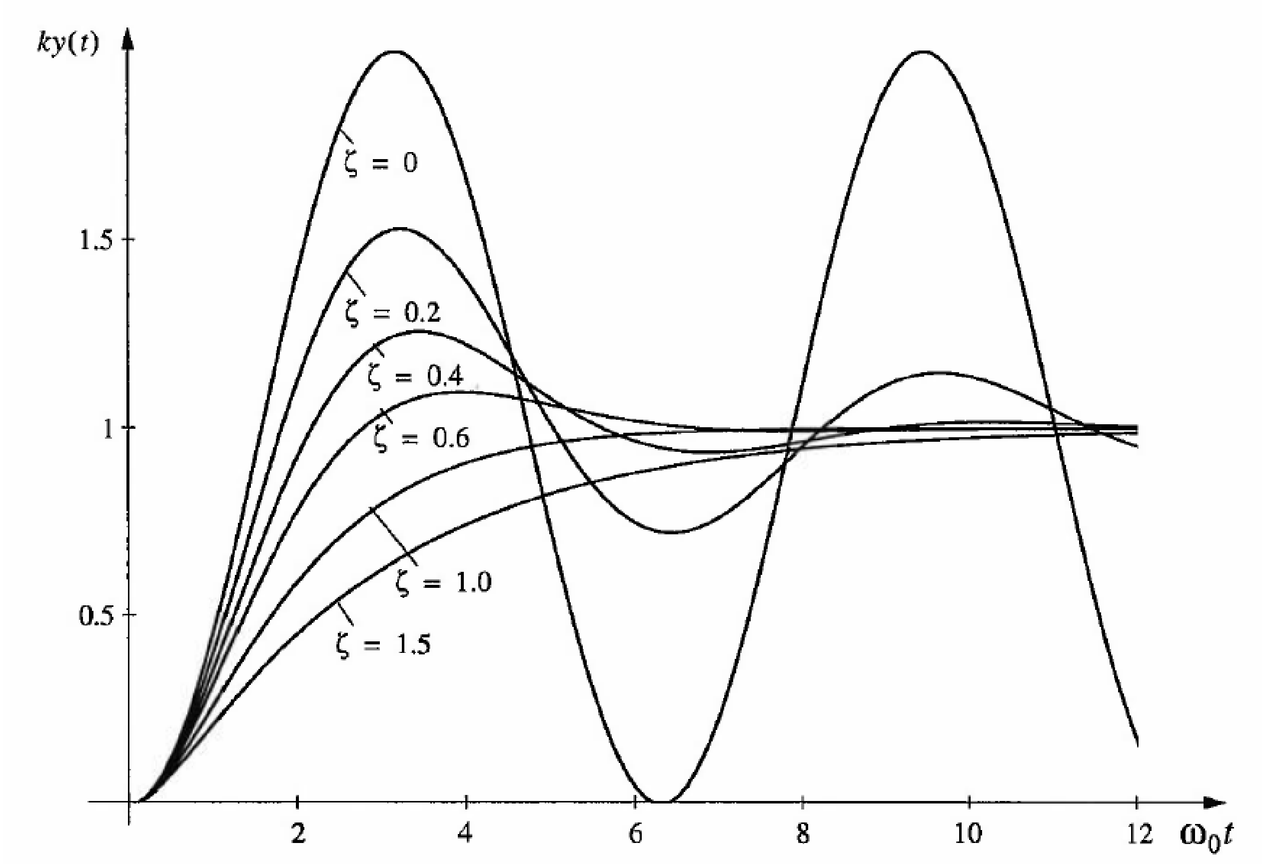
- Example 1: Given transfer function
- Example 2: Mass-spring-damper system

# Example 2

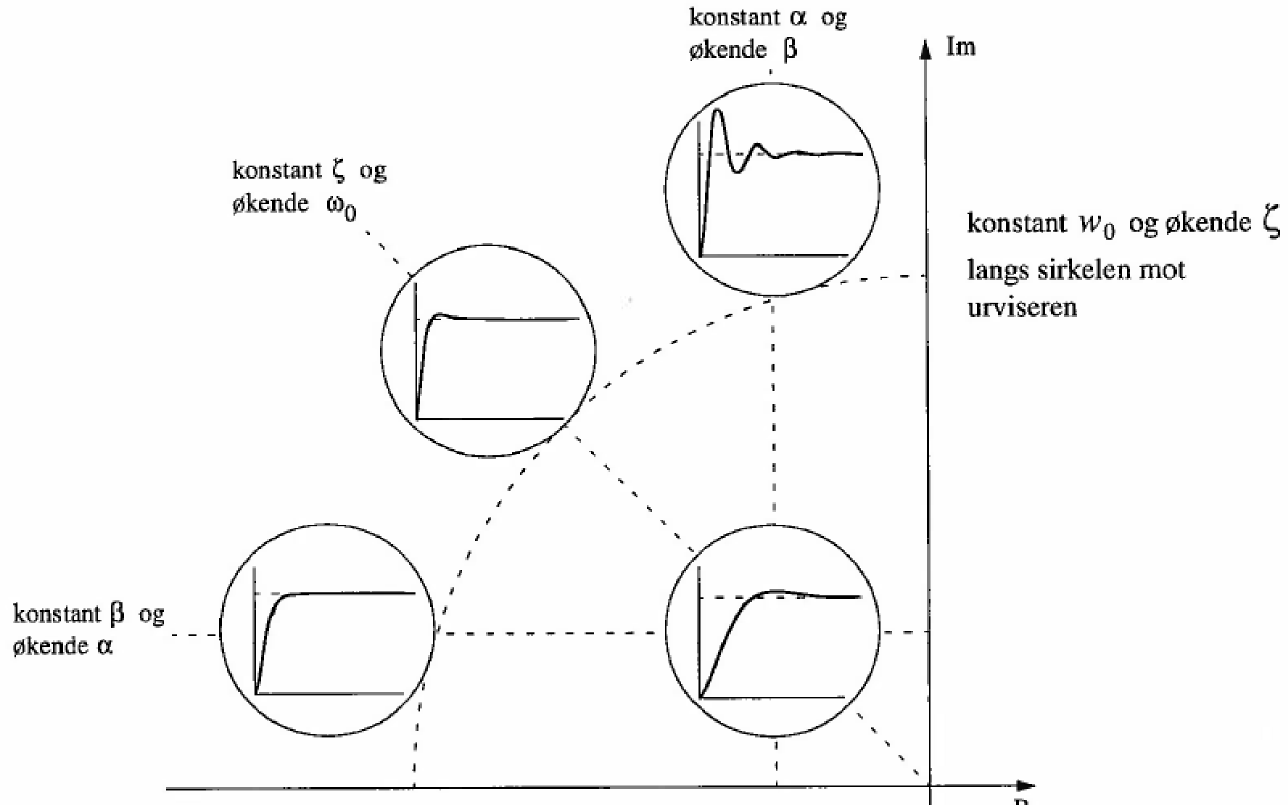
- Three cases depending on the poles
- Case I: Poles are real and distinct $\zeta > 1$
  - Over-damped system
- Case II: Poles are real and equal $\zeta = 1$
  - Critically damped system
- Case III: Poles are complex conjugates $\zeta < 1$
  - Under-damped system

$$\ddot{x} + 2\underbrace{\zeta}_{\text{Damping ratio}}\underbrace{\omega_n}_{\text{Natural frequency}}\dot{x} + \omega_n^2 x = u(t)$$

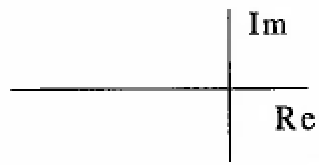
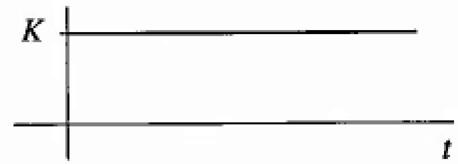
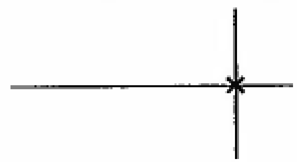
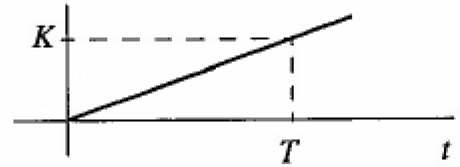
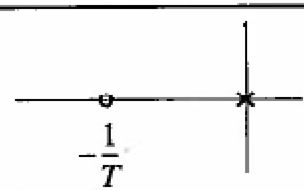
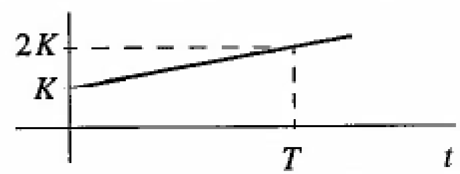
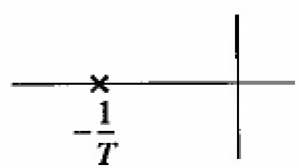
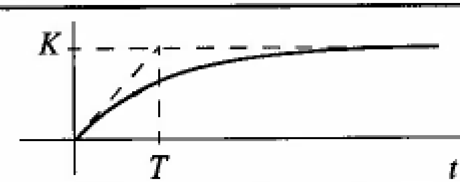
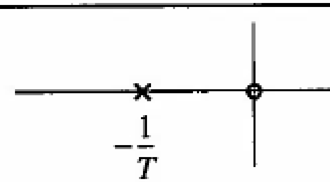
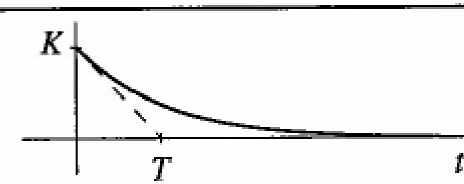
# Example 2: Time response

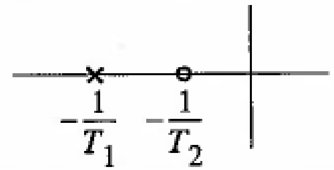
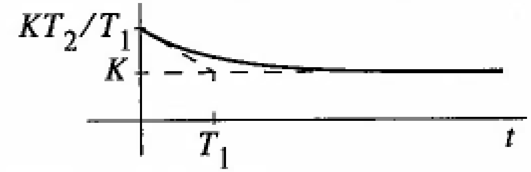
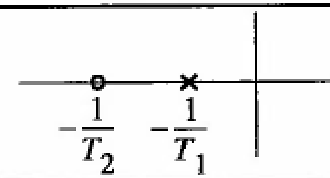
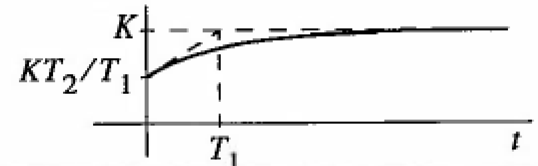
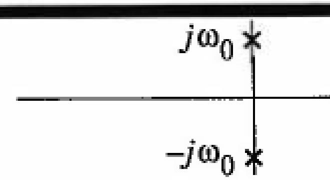
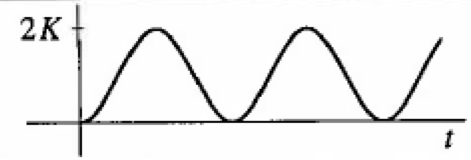
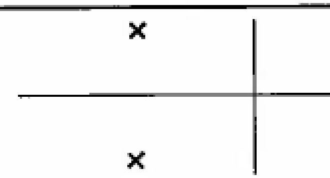
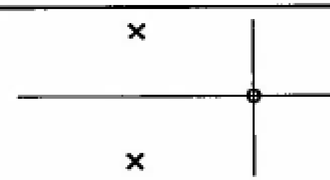
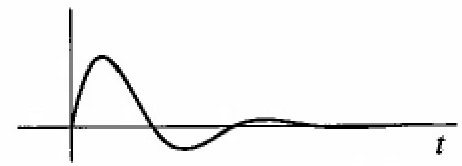
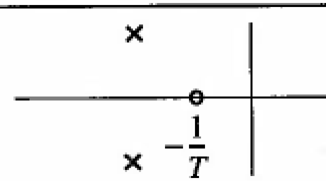
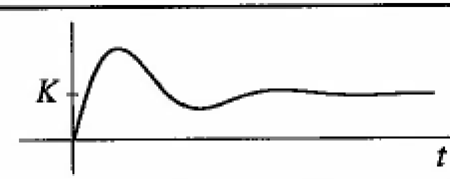


# Example 2: Effect of changes in poles



# Common transfer functions and their poles and step respones

| Transfer-funksjon $h(s)$ | Nullpunkter og poler | Sprangrepons |
|---|---|---|
| $K$ |   |   |
| $K\frac{1}{Ts}$ |  |   |
| $K\frac{1+Ts}{Ts}$ |   |   |
| $K\frac{1}{1+Ts}$ |   |   |
| $K\frac{Ts}{1+Ts}$ |   |   |

| Transfer-funksjon $h(s)$ | Nullpunkter og poler | Sprangrepons |
|---|---|---|
| $K\dfrac{1+T_2 s}{1+T_1 s}$ <br> $T_2 > T_1$ |   |   |
| $K\dfrac{1+T_2 s}{1+T_1 s}$ <br> $T_2 < T_1$ |   |   |
| $\dfrac{K}{1+\left(\dfrac{s}{\omega_0}\right)^2}$ |   |   |
| $\dfrac{K}{1+2\zeta\dfrac{s}{\omega_0}+\left(\dfrac{s}{\omega_0}\right)^2}$ |  |   |
| $\dfrac{Ks}{1+2\zeta\dfrac{s}{\omega_0}+\left(\dfrac{s}{\omega_0}\right)^2}$ |  |   |
| $\dfrac{K(1+Ts)}{1+2\zeta\dfrac{s}{\omega_0}+\left(\dfrac{s}{\omega_0}\right)^2}$ |   |   |

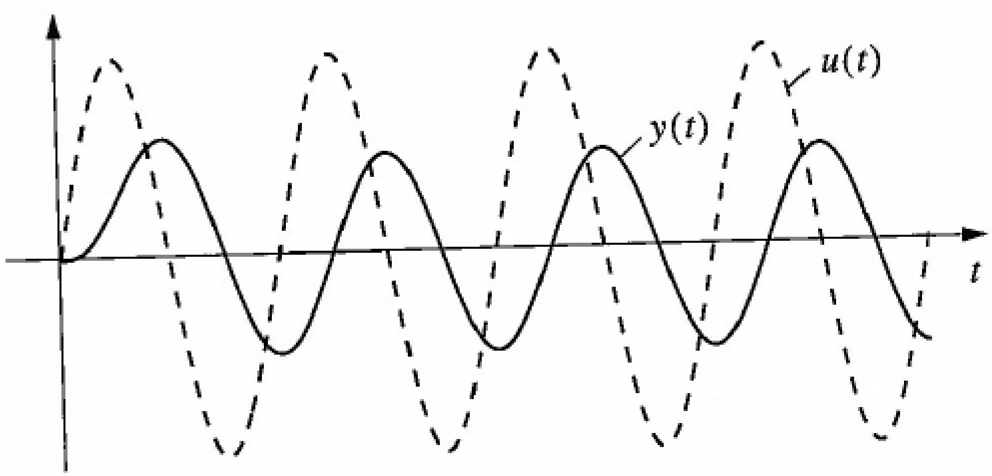
# 1.3 Root locus plots

- The paths of zeros and poles in the complex plane as a function of changed controller parameters are called root locus plots
- Example on blackboard

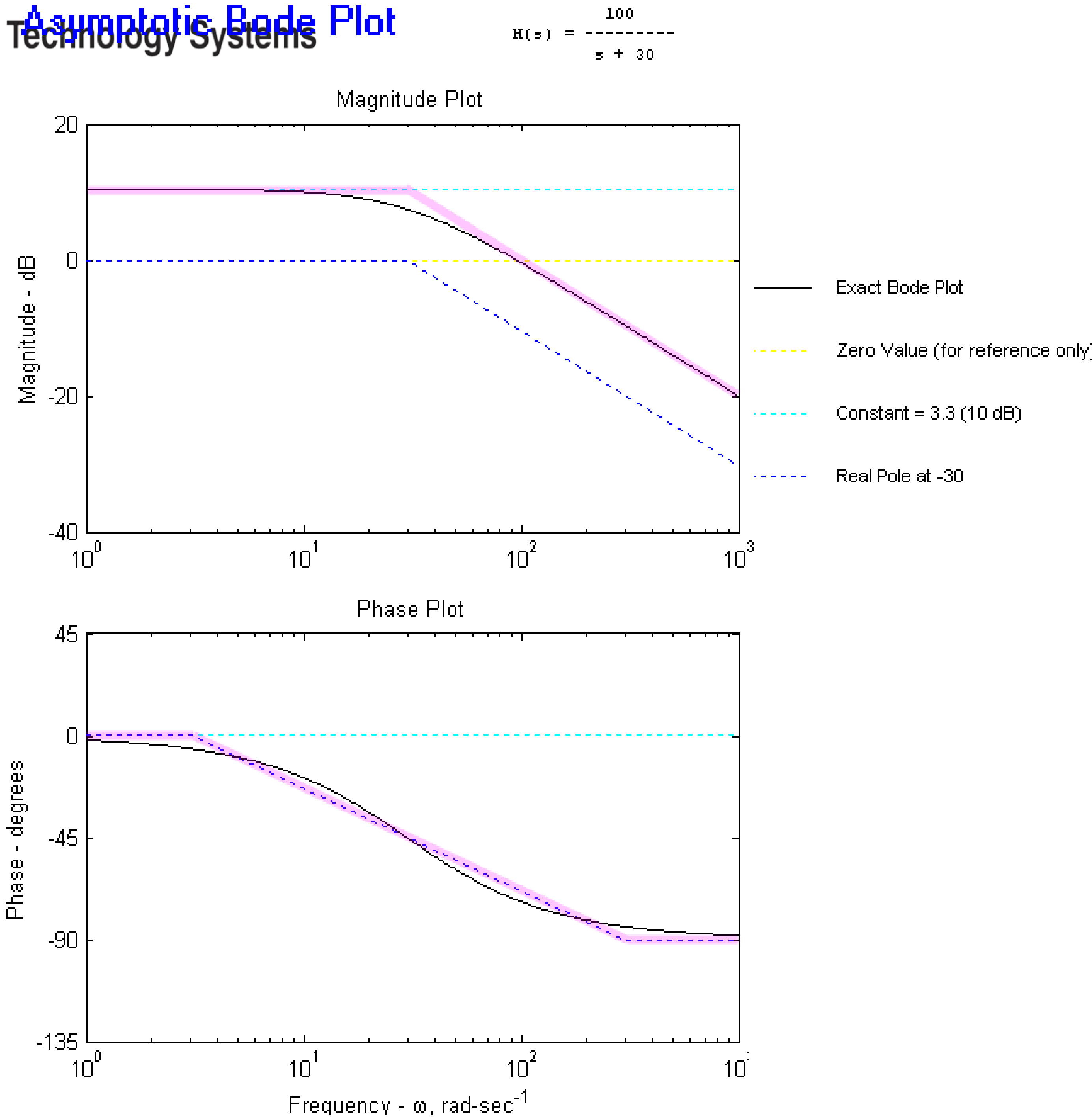
# 2. Frequency analysis

- Analysis of the frequency response of a system
- The frequency response is a mapping of the change of a sine signal from the input to the output of a system
- The frequency response if found using the transfer function
- Frequency response is only valid for linear system

# Example of amplitude and phase change



# Example Bode plot



# 3. State space systems

- Using matrix calculus to create a set of first order differential equations
- One benefit is a simple notation for complex systems

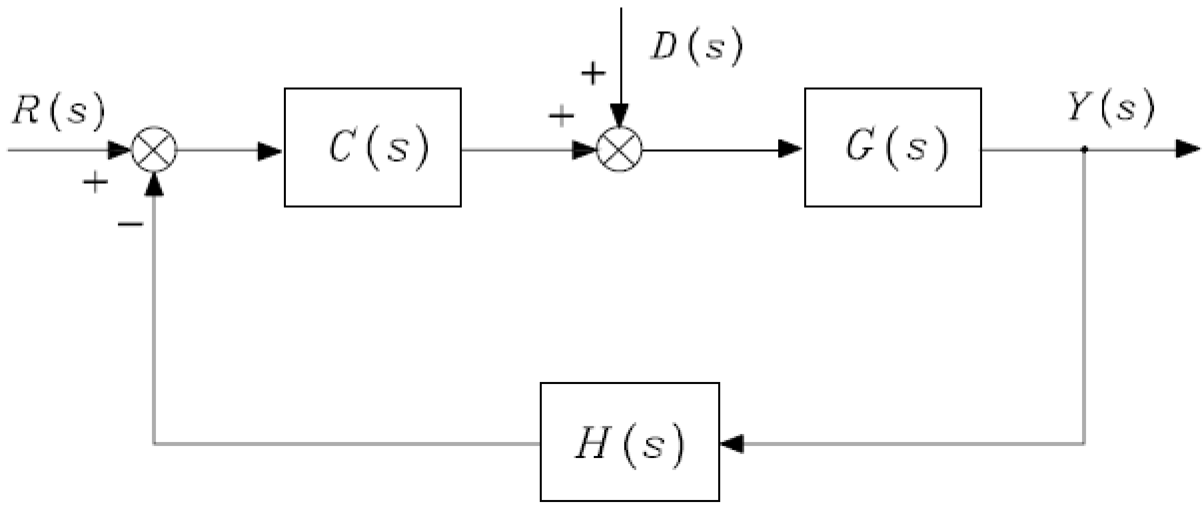
- General equation is

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

# Transforming the mass-spring-damper system to state space representation

- Blackboard

# 4. Feedback systems



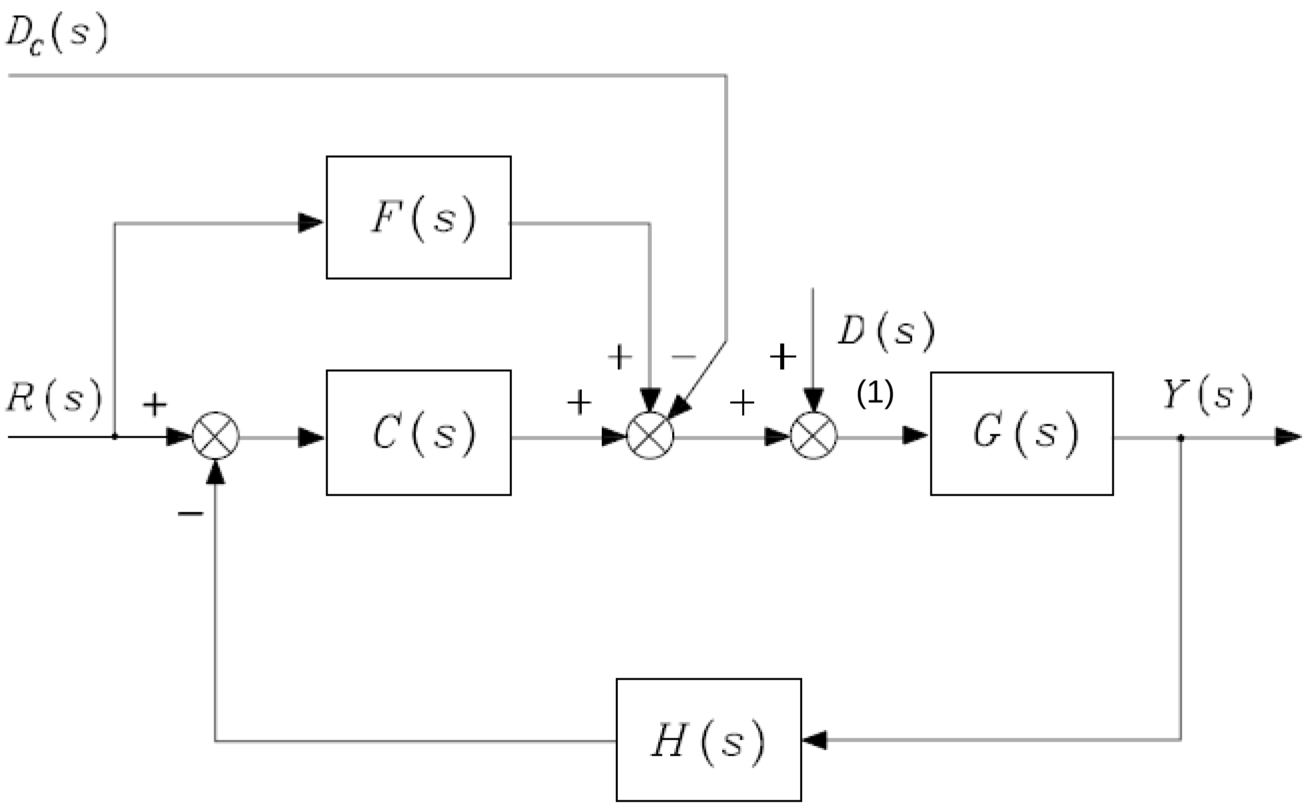
# Feedback systems

$$Y(s) = W(s)R(s) + W_D(s)D(s),$$

$$W(s) = \frac{C(s)G(s)}{1 + C(s)G(s)H(s)}$$

$$W_D(s) = \frac{G(s)}{1 + C(s)G(s)H(s)}$$

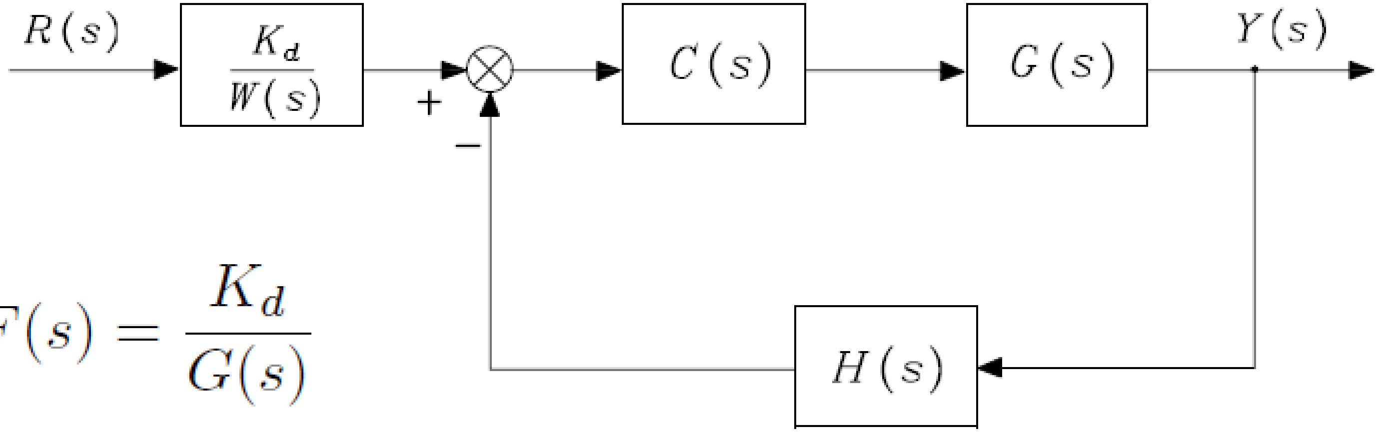
# Feedforward



# Feedforward

$$Y(s) = \left( \frac{C(s)G(s)}{1 + C(s)G(s)H(s)} + \frac{F(s)G(s)}{1 + C(s)G(s)H(s)} \right) R(s) \qquad (C.8)$$
$$+ \frac{G(s)}{1 + C(s)G(s)H(s)} \left( D(s) - D_c(s) \right).$$

# Inverse model




$$F(s) = \frac{K_d}{G(s)}$$

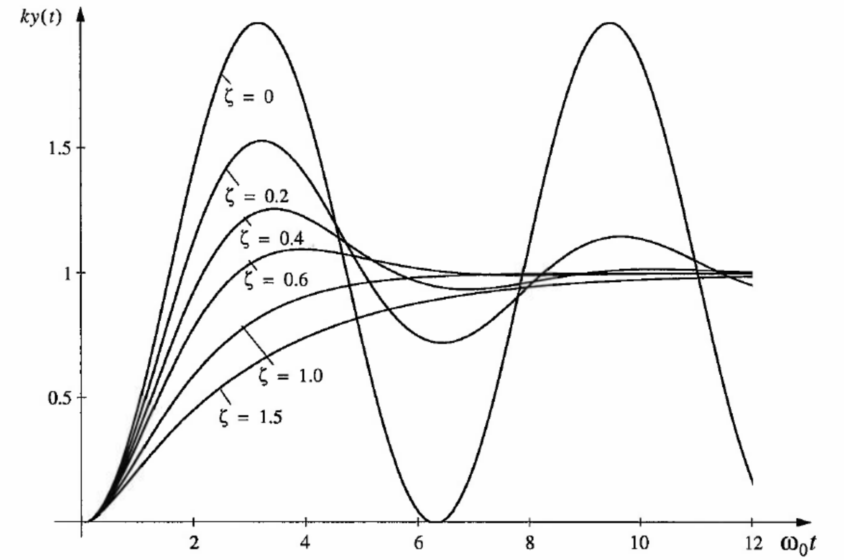
$$Y(s) = Y_d(s) + \frac{G(s)}{1 + C(s)G(s)H_0}\big(D(s) - D_c(s)\big). \tag{C.10}$$

# 5. Stability

- 5.1 Frequency domain
- 5.2 State space systems
- 5.3 Non-linear systems

# 5.1 Definition input/output stability

- Asymptotically stable if:
  - $y \to 0$ when $t \to \infty$ and u has a finite duration and amplitude
- Marginally stable if:
  - $|y| < \infty$ for all $t \geq 0$ and u has a finite duration and amplitude
- Unstable otherwise



# 5.1 Stability – Frequency domain

- Find the poles ($\lambda_i$) of the transfer function
- If $Re(\lambda_i) < 0$ for all $\lambda_i$ in H(s) the system is *asymptotically stable*
- If one or more poles has $Re(\lambda_i) = 0$, but they are not in the same point the system is *marginally stable*
- If one or more poles has $Re(\lambda_i) > 0$ the system is *unstable*

# 5.2 Definition stability of state space systems

- Asymptotically stable if:
  - $\boldsymbol{x} \rightarrow \boldsymbol{0}$ when $t \rightarrow \infty$ and u has a finite duration and amplitude
- Marginally stable if:
  - $|\boldsymbol{x}| < \infty$ for all $t \geq 0$ and u has a finite duration and amplitude
- Unstable otherwise

# 5.2 Stability state space systems

- Solve the systems characteristic equation $|\boldsymbol{A} - \lambda \boldsymbol{I}|$ the get the eigenvalues $\lambda_i$
- If $Re(\lambda_i) < 0$ for all $\lambda_i$ the system is *asymptotically stable*
- If one or more eigenvalues has $Re(\lambda_i) = 0$, but they are not in the same point the system is *marginally stable*
- If one or more poles has $Re(\lambda_i) > 0$ the system is *unstable*

# 5.3 Lyapunov direct method

- Used to test stability of non-linear systems
- Use an energy description of the system states
- If we can show that the energy of the system decreases along any system trajectory until the equilibrium is reached, the system is stable

$$\dot{x} = f(x,u)$$

$$\begin{aligned} V(\mathbf{e}) &> 0 && \forall \mathbf{e} \neq \mathbf{0} \\ V(\mathbf{e}) &= 0 && \mathbf{e} = \mathbf{0} \\ \dot{V}(\mathbf{e}) &< 0 && \forall \mathbf{e} \neq \mathbf{0} \\ V(\mathbf{e}) &\to \infty && \|\mathbf{e}\| \to \infty. \end{aligned}$$

# 5.3 Lyapunov example

- Dynamic equation

$$m\ddot{x} + b\dot{x}^3 + kx = u$$

- b is the drag coefficient