UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in:	TEK4030
Day of exam:	4th of December
Exam hours:	9.15 - 13.15 (4 hours)

This examination paper consists of 10 page(s).

Appendices:	Formulas
Permitted materials:	None

Make sure that your copy of this examination paper is complete before answering.

Problem 1 - Actuators and sensors (8%)

a) (4 %) What is the effect of having a high gear reduction ratio?

> It reduce the impact of the load reaction torque, and the moment of intertia and friction coefficient of the load.

b) (4 %) When modeling an electric drive what are the three main components that the model is derived from?

> Mechanical balance, electrical balanse, power amplifier model.
>
> ---
>
> - 4 points for all
> - 1,5 points subtracted for each wrong answer

Problem 2 - Independent joint control (16%)

Assume that we have the following dynamic model for a singe joint of a manipulator

$$I\ddot{\theta} + F\dot{\theta} + d = u \tag{1}$$

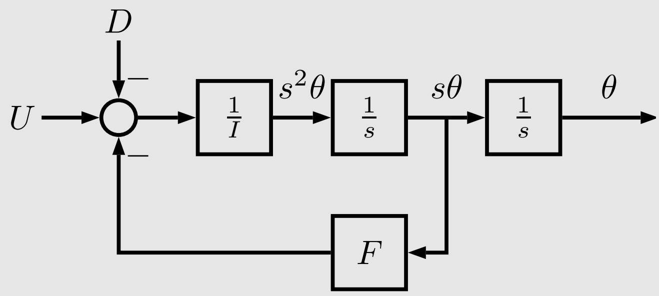
where I is the moment of intertia, F is the viscous friction coefficient, d is the disturbance, and θ is the joint position.

a) (4 %) Transform the above model into the Lapace domain and draw the block diagram of the system using only constant and intregral blocks.

$$Is^2\theta + Fs\theta + D = U$$

Rewriting to easier make block diagram

$$s^2\theta = \frac{1}{I}(U - Fs\theta - D)$$



- 2 points correct transfer function
- 2 points for correct block diagram

b) (4 %) Use a propotional controller to control the joint to a constant desired position θ_d. Find the transfer function of the system and add the controller to the block diagram of the system.

A proportional controller has the following equation

$$U = K_P(\theta_d - \theta)$$

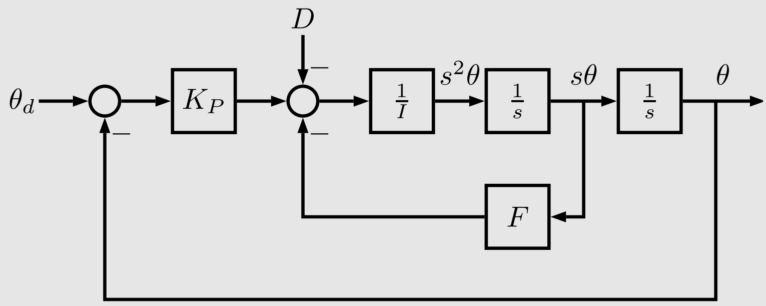
Inserting this into the system equation yields

$$\begin{aligned} Is^2\theta + Fs\theta + D &= K_P(\theta_d - \theta) \\ Is^2\theta + Fs\theta + K_P\theta &= K_P\theta_d - D \\ (Is^2 + Fs + K_P)\theta = K_P\theta_d - D\theta &= \frac{K_P}{(Is^2 + Fs + K_P)}\theta_d - \frac{1}{(Is^2 + Fs + K_P)}D \end{aligned}$$

The transfer function is

$$\frac{\theta}{\theta_d} = \frac{K_P}{(Is^2 + Fs + K_P)}$$

The block diagram should be as follows



- 2 points correct transfer function
- 2 points for correct block diagram

c) (4 %) Find the roots of the system. Is the system stable? Why/why not?

Using the solution for quadratic equations

$$s = -\frac{F}{2I} \pm \frac{\sqrt{F^2 - 4IK_P}}{2I}$$

The system is stable as long as the following condition is met.

$$4IK_P > 0$$

These conditions enures that the poles are in the negative real half plane, and thus the system is stable.

d) (4 %) Will the error of the system go to zero? Why/why not?

In general the error of the system will not go to zero because of the disturbance. Given a constant disturbance D_c the error will be $\frac{D_c}{K_P}$.

Problem 3 - Centralized control (24%)

a) (4 %) In operational space control there are two general schemes for controlling the robot. What are the two schemes called?

Jacobian inverse and Jacobian transpose

b) (4 %) Briefly explain the difference between computed torque feedforward control and inverse dynamics control.

Computed torque feedforward control use an inverse model strategy, but uses the model parameters for the desired joint trajectory and can be computed offline. While inverse model control also uses an inverse model strategy, but uses the actual joint values for calculating the model parameters.

c) (16 %) A robotic manipulator has the following dynamic model

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u} \tag{2}$$

where $\boldsymbol{q}$ is the joint position, $\dot{\boldsymbol{q}}$ is the joint velocities, $\ddot{\boldsymbol{q}}$ is the joint accelerations and $\boldsymbol{u}$ is the control input. The controller is given by

$$\boldsymbol{u} = \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_P\tilde{\boldsymbol{q}} \tag{3}$$

where $\tilde{\boldsymbol{q}} = \boldsymbol{q}_d - \boldsymbol{q}$, and $\boldsymbol{q}_d$ is the desired joint positions which is constant. The Lyapunov function candidate for the controller is

$$\boldsymbol{V}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T\boldsymbol{K}_P\tilde{\boldsymbol{q}} \tag{4}$$

Check the stability of the system using Lyapunovs direct method. What causes the system to be stable, or what lacks in the controller or dynamic model to prove the stability?

To check the stability we need to check that the four conditions of the Lyapunov direct methos is satisfied. The conditions are

$$
\begin{aligned}
&\boldsymbol{V}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}}) > 0 && \forall\, \dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}} \neq 0 \\
&\boldsymbol{V}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}}) = 0 && \dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}} = 0 \\
&\boldsymbol{V}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}}) \to \infty && ||\dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}}|| \to \infty \\
&\dot{\boldsymbol{V}}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}}) < 0 && \forall\, \dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}} \neq 0
\end{aligned}
$$

The three first conditions are satisified as $\boldsymbol{V}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}})$ is a quadratic function. Taking the time derivative of $\boldsymbol{V}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}})$ yields

$$
\begin{aligned}
\dot{\boldsymbol{V}}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}}) &= \frac{1}{2}\ddot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\tilde{\boldsymbol{q}}}^T \boldsymbol{K}_P \tilde{\boldsymbol{q}} + \frac{1}{2}\tilde{\boldsymbol{q}}^T \boldsymbol{K}_P \dot{\tilde{\boldsymbol{q}}} \\
&= \dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \dot{\tilde{\boldsymbol{q}}}^T \boldsymbol{K}_P \tilde{\boldsymbol{q}}
\end{aligned}
$$

Insering the dynamic model (2) and that $\dot{\tilde{\boldsymbol{q}}} = -\dot{\boldsymbol{q}}$

$$
\begin{aligned}
&= \dot{\boldsymbol{q}}^T \left(\boldsymbol{u} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right) + \frac{1}{2}\dot{\boldsymbol{q}}^T \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T \boldsymbol{K}_P \tilde{\boldsymbol{q}} \\
&= \frac{1}{2}\dot{\boldsymbol{q}}^T \left(\dot{\boldsymbol{B}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right)\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T \left(\boldsymbol{u} - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{K}_P \tilde{\boldsymbol{q}}\right) \\
&= \dot{\boldsymbol{q}}^T \left(\boldsymbol{u} - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{K}_P \tilde{\boldsymbol{q}}\right)
\end{aligned}
$$

Inserting the control law (3)

$$
\begin{aligned}
\dot{\boldsymbol{V}}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}}) &= \dot{\boldsymbol{q}}^T \left(\boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{K}_P \tilde{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q}) - \boldsymbol{K}_P \tilde{\boldsymbol{q}}\right) \\
\dot{\boldsymbol{V}}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}}) &= 0
\end{aligned}
$$

Stability cannot be proved using Lyapunovs direct method. Either a damping term must be added in the control low or there have to be viscous friction in the dynamic model.

- 4 points for showing the four conditions, and stating that three of them are satisfied
- 8 points for showing that $\dot{\boldsymbol{V}}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{q}}) = 0$
- 4 points for stating what lack in the control law or dynamic model

Problem 4 - Force control (16%)

a) (4 %) Briefly explain the difference between impedance and admittance control.

With admitance control one have an outer impedance control and an inner motion control law, so that the impedance parameters does not affect the motion control properties of the system.

b) (4 %) The control law for active compliance is very similar to the operational space PD controller with gravity compensation. What is the principal difference?

The error definition. In the active compliance case it is referred to the desired frame and in the PD controller with gravity compensation it it referred to the base frame.

c) (8 %) The dynamic model of a manipulator interacting with the environment is given by

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u} - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e \tag{5}$$

Assume that the following controller is used to do an exact linearization

$$\boldsymbol{u} = \boldsymbol{B}(\boldsymbol{q})\boldsymbol{y} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e \tag{6}$$

Assume that only position variables are controlled, implying that the analytic and geometric Jacobian are the same. Find the ~~error~~ system dynamics using the following equation

$$\boldsymbol{y} = \boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{M}_d^{-1}\left(-\boldsymbol{K}_D\dot{\boldsymbol{x}}_e + \boldsymbol{K}_P(\boldsymbol{x}_F - \boldsymbol{x}_e) - \boldsymbol{M}_d\dot{\boldsymbol{J}}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}\right) \tag{7}$$

where $\boldsymbol{x}_e$ is the end-effector position, and $\boldsymbol{x}_F$ a reference to be related to the force error.

Combining (5) and (6)

$$\begin{aligned}
&\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \\
&\boldsymbol{B}(\boldsymbol{q})\boldsymbol{y} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) + \cancel{\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e} - \cancel{\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e}
\end{aligned}$$

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \cancel{\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q})} = \boldsymbol{B}(\boldsymbol{q})\boldsymbol{y} + \cancel{\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q})}$$

$$\cancel{\boldsymbol{B}(\boldsymbol{q})}\ddot{\boldsymbol{q}} = \cancel{\boldsymbol{B}(\boldsymbol{q})}\boldsymbol{y}$$

Now inserting (7)

$$\ddot{\boldsymbol{q}} = \boldsymbol{y} = \boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{M}_d^{-1}\left(-\boldsymbol{K}_D\dot{\boldsymbol{x}}_e + \boldsymbol{K}_P(\boldsymbol{x}_F - \boldsymbol{x}_e) - \boldsymbol{M}_d\dot{\boldsymbol{J}}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}\right)$$

$$\boldsymbol{M}_d\boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} = -\boldsymbol{K}_D\dot{\boldsymbol{x}}_e + \boldsymbol{K}_P(\boldsymbol{x}_F - \boldsymbol{x}_e) - \boldsymbol{M}_d\dot{\boldsymbol{J}}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$$

Now using the following fact

$$\dot{\boldsymbol{x}}_e = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}} \implies \ddot{\boldsymbol{x}}_e = \boldsymbol{J}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Inserting this yields

$$\boldsymbol{M}_d \left(\ddot{\boldsymbol{x}}_e - \cancel{\dot{\boldsymbol{J}}(\boldsymbol{q})}\dot{\boldsymbol{q}} \right) = -\boldsymbol{K}_D \dot{\boldsymbol{x}}_e + \boldsymbol{K}_P(\boldsymbol{x}_F - \boldsymbol{x}_e) - \cancel{\boldsymbol{M}_d \dot{\boldsymbol{J}}(\boldsymbol{q}, \dot{\boldsymbol{q}})}\dot{\boldsymbol{q}}$$
$$\boldsymbol{M}_d \ddot{\boldsymbol{x}}_e = -\boldsymbol{K}_D \dot{\boldsymbol{x}}_e + \boldsymbol{K}_P(\boldsymbol{x}_F - \boldsymbol{x}_e)$$

Rearranging yields

$$\boldsymbol{M}_d \ddot{\boldsymbol{x}}_e + \boldsymbol{K}_D \dot{\boldsymbol{x}}_e + \boldsymbol{K}_P \boldsymbol{x}_e = \boldsymbol{K}_P \boldsymbol{x}_F$$

- 3 points for showing that $\ddot{\boldsymbol{q}} = \boldsymbol{y}$
- 4 points for correctly transforming the equation into the operational space
- 1 points for setting the equation on the appropiate form

Problem 5 - Tele-operations (8%)

a) (4 %) Explain what haptic guidance is with regards to tele-operation.

The generation of virtual forces to guide the movements of operator of the system.

b) (4 %) The relationship between the forces and velocities of a master and slave system is often represented using a hybrid matrix, as given below

$$\begin{bmatrix} f_m \\ -v_s \end{bmatrix} = \boldsymbol{H} \begin{bmatrix} v_m \\ f_s \end{bmatrix} \tag{8}$$

How does the hybrid matrix for a transparent system look like? What does this imply for the relationship between the master and slave system?

The hybrid matrix for a transparent system looks like this

$$\boldsymbol{H} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

This implies that the velocity on the master and slave side are equal, and that the forces on the master and slave side are equal.

- 2 points for correct hybrid matrix
- 2 points for correct explainition

Problem 6 - Visual servoing (12%)

a) (8 %) Assume we have a non-moving point P that with respect to the camera fram c is defined as

$$\boldsymbol{r}_c^c = \boldsymbol{R}_c^T(\boldsymbol{p} - \boldsymbol{o}_c) \tag{9}$$

where $\boldsymbol{p}$ is the point in with respect to the base frame and $\boldsymbol{o}_c$ is the origin of the camera frame with respect to the base frame. The normalized image coordinates X and Y are used as a feature vector, which yields

$$\boldsymbol{s}(\boldsymbol{r}_c^c) = \frac{1}{z_c}\begin{bmatrix} x_c \\ y_c \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \tag{10}$$

where $\boldsymbol{r}_c^c = [x_c\ y_c\ z_c]$.

Find the interaction matrix $\boldsymbol{L}_s(\boldsymbol{s}, z_c)$ for the point P. Since $\boldsymbol{p}$ is constant you may use the relation

$$\dot{\boldsymbol{r}}_c^c = [-\boldsymbol{I}\ \boldsymbol{S}(\boldsymbol{r}_c^c)]\boldsymbol{v}_c^c \tag{11}$$

In order to find the interaction matrix we need to find the time derivative of $\boldsymbol{s}(\boldsymbol{r}_c^c)$

$$\dot{\boldsymbol{s}} = \frac{\partial \boldsymbol{s}(\boldsymbol{r}_c^c)}{\partial \boldsymbol{r}_c^c}\dot{\boldsymbol{r}}_c^c$$

The first term is

$$\frac{\partial \boldsymbol{s}(\boldsymbol{r}_c^c)}{\partial \boldsymbol{r}_c^c} = \frac{1}{z_c}\begin{bmatrix} 1 & 0 & -x_c/z_c \\ 0 & 1 & -y_c/z_c \end{bmatrix} = \frac{1}{z_c}\begin{bmatrix} 1 & 0 & -X \\ 0 & 1 & -Y \end{bmatrix}$$

Using (11) with the above yields

$$\begin{aligned}
\dot{\boldsymbol{s}} &= \frac{1}{z_c}\begin{bmatrix} 1 & 0 & -X \\ 0 & 1 & -Y \end{bmatrix}[-\boldsymbol{I}\ \boldsymbol{S}(\boldsymbol{r}_c^c)]\boldsymbol{v}_c^c \\
\dot{\boldsymbol{s}} &= \frac{1}{z_c}\begin{bmatrix} 1 & 0 & -X \\ 0 & 1 & -Y \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 & -z_c & y_c \\ 0 & -1 & 0 & z_c & 0 & -x_c \\ 0 & 0 & -1 & -y_c & x_c & 0 \end{bmatrix}\boldsymbol{v}_c^c \\
\dot{\boldsymbol{s}} &= \frac{1}{z_c}\begin{bmatrix} -1 & 0 & X & Xy_c & -z_c - Xx_c & y_c \\ 0 & -1 & Y & z_c + Yy_c & -Yx_c & -x_c \end{bmatrix}\boldsymbol{v}_c^c \\
\dot{\boldsymbol{s}} &= \begin{bmatrix} -1/z_c & 0 & X & XY & -1-X^2 & Y \\ 0 & -1/z_c & Y & 1+Y^2 & -YX & -X \end{bmatrix}\boldsymbol{v}_c^c \\
\dot{\boldsymbol{s}} &= \boldsymbol{L}_s(\boldsymbol{s}, z_c)\boldsymbol{v}_c^c
\end{aligned}$$

b) (4 %) If you have a set of points, as given in a), how would you do visual servoing?

You would stack the points together in a feature vector and combine all the interaction matrices by stacking them on top of each other. In practise it would look like this

$$\begin{bmatrix} \boldsymbol{s}_1 \\ \vdots \\ \boldsymbol{s}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_{\boldsymbol{s}_1}(\boldsymbol{s}_1, z_{c,1}) \\ \vdots \\ \boldsymbol{L}_{\boldsymbol{s}_n}(\boldsymbol{s}_n, z_{c,n}) \end{bmatrix} \tag{12}$$

Problem 7 - Mobile robots (16%)

a) (4 %) Name two mobile robots that are kinematically equivalent to a unicycle.

The differential-drive and syncro-drive robots.

b) (4 %) What is the differential flatness property? Does the unicycle and bicycle have this property?

For a system with this property there exist a set of outputs, that are called flat outputs, such that the state and control input can be expressed algebraically as a function of the flat outputs. Both the unicycle and bicycle have this property.

c) (4 %) Briefly explain the input/output linearization trajectory tracking method, and state one benefit with this method.

The method linearize the output with regards to the input by tracking a point relative to the base of the robot (either in front or behind). The benefit is that the robot can handle jumps in orientation without stopping.

d) (4 %) What is odometric localization? Mention one method to do odometric localization, and briefly explain how it works.

Odemetric localization is finding the position and orientation of the robot by integrating the measured velocity of the robot. One can use either integrate using Eulers method, a Runge-Kutta method or exact integration. A one sentence explenation of the method is sufficient.

A Formulas

Solution for quadratic equations

$$ax^2 + bx + c = 0 \qquad \Rightarrow \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Properties of dynamic models of robots
Given the following dynamic system

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u}$$

The following properties hold

$$\begin{aligned} \dot{\boldsymbol{q}}^T \left(\dot{\boldsymbol{B}} - 2\boldsymbol{C}\right) \dot{\boldsymbol{q}} &= \boldsymbol{0} \\ \boldsymbol{B} &= \boldsymbol{B}^T \\ \boldsymbol{q}^T \boldsymbol{B} \boldsymbol{q} &> 0 \\ \boldsymbol{q}^T \boldsymbol{F} \boldsymbol{q} &> 0 \end{aligned}$$

Linear algebra
A matrix is symmetric if it satisfies the condition

$$\boldsymbol{A} = \boldsymbol{A}^T$$

For symmetric matrices the following holds

$$\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{y} = \boldsymbol{y}^T \boldsymbol{A} \boldsymbol{x}$$

A skew-symmetric matrix is from the vector $\boldsymbol{x} = [x\ y\ z]^T$ is defined as

$$\boldsymbol{S}(\boldsymbol{x}) = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \tag{13}$$

Trigonometry
Definitions

$$\sin\theta = \frac{opposite}{hypotenuse} \qquad \cos\theta = \frac{adjacent}{hypotenuse}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{opposite}{adjacent}$$

Addition formulas

$$\begin{aligned} \sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\ \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \end{aligned}$$

Derivatives

$$\frac{d}{dx} \sin x = \cos x \tag{14}$$

$$\frac{d}{dx} \cos x = -\sin x \tag{15}$$