8.3.1 PD control with gravity compensation (4-9)

Constant desired joint varable

$$\underline{q}_d = \begin{bmatrix} q_{d_1} \\ \vdots \\ q_{d_n} \end{bmatrix}$$

Determine the control input based on the Lyapunov direct method

Using the state vector

$$\left[\underline{\tilde{q}}, \underline{\dot{q}}\right]^T$$

$$\underline{\tilde{q}} = \underline{q}_d - \underline{q} \quad \text{(position error)}, \qquad (8.46)$$

Choose the Lyapunov candidate function

$$V(\tilde{q},\dot{q}) = \underbrace{\frac{1}{2}\dot{q}^T B(q)\dot{q}}_{\text{Kinetic energy}} + \underbrace{\frac{1}{2}\tilde{q}^T K_P \tilde{q}}_{\text{Potential energy (of a spring)}} \quad (8.47)$$

$V > 0 \quad \forall\ \tilde{q}, \dot{q} \neq 0$

$V = 0 \quad \tilde{q}, \dot{q} = 0$

$V \to \infty \quad \| [\tilde{q}, \dot{q}]^T \| \to \infty$

$$V = \frac{1}{2}\dot{q}^T B \dot{q} + \frac{1}{2}\tilde{q}^T K_p \tilde{q}$$

$$\boxed{(\ddot{q}^T B \dot{q})^T = \dot{q}^T B^T (\ddot{q}^T)^T}$$

$$\dot{V} = \frac{1}{2}\ddot{q}^T B \dot{q} + \frac{1}{2}\dot{q}^T \dot{B} \dot{q} + \frac{1}{2}\dot{q}^T B \ddot{q}$$

$$+ \frac{1}{2}\dot{\tilde{q}}^T K_p \tilde{q} + \frac{1}{2}\tilde{q}^T K_p \dot{\tilde{q}}$$

$$= \dot{q}^T B \ddot{q} + \frac{1}{2}\dot{q}^T \dot{B} \dot{q} + \dot{\tilde{q}}^T K_p \tilde{q}$$

$$\tilde{q} = q_d - q \implies \dot{\tilde{q}} = \dot{q}_d - \dot{q} = -\dot{q}$$

$$\dot{V} = \dot{q}^T B \ddot{q} + \frac{1}{2}\dot{q}^T \dot{B} \dot{q} - \dot{q}^T K_p \tilde{q} \qquad (8.48)$$

Recall

$$B(q)\ddot{q} + C(q,\dot{q})\dot{q} + F\dot{q} + g(q) = u \quad (8.7)$$

Solve for $B(q)\ddot{q}$ and insert into (8.48)

$$\dot{V} = \dot{q}^T\left(-\underline{C\dot{q}} - F\dot{q} - \underline{g} + \underline{u}\right) + \underline{\tfrac{1}{2}\dot{q}^T\dot{B}\dot{q}} - \dot{q}^T \underline{K_p\tilde{q}}$$

$$= \tfrac{1}{2}\dot{q}^T(\dot{B} - 2C)\dot{q} - \dot{q}^T F\dot{q}$$

$$+ \dot{q}^T(u - g - K_p\tilde{q})$$

From dynamics of manipulators

$$\dot{q}^T(\dot{B}-2C)\dot{q}=0 \qquad (7.47) \text{ and } (7.49)$$

$$\dot{V} = \underbrace{-\dot{q}^T F \dot{q}}_{\text{always } \leq 0} + \dot{q}^T(u - g - K_p\tilde{q})$$

$$u = g + K_p\tilde{q}$$

$$\dot{V} = -\dot{q}^T F \dot{q}$$

negativ semi-definit $\dot{V}$

This controller leads to a negative semi-definit $\dot{V}$, since

$$\dot{V} = 0 \qquad \dot{q} = 0 \quad \forall \tilde{q}$$

Finding equilibrium posture

$$\dot{V} = 0 \quad \text{only if} \quad \dot{q} = 0 \quad (\text{and } \ddot{q} = 0)$$

System dynamics are

$$\cancel{B\ddot{q}} + \cancel{C\dot{q}} + \cancel{F\dot{q}} + \cancel{g} = \cancel{g} + K_p\tilde{q}$$

$$0 = K_p\tilde{q}$$

We expand the controller with a derivative term

$$u = g + K_P \tilde{q} - K_D \dot{q} \qquad (8.51)$$

This yields

$$\dot{V} = -\dot{q}^T (F + K_D) \dot{q}$$

8.5.2 Inverse dynamics control (10-17)

We now want to design a controller that can track a joint space trajectory

$$(q_d, \dot{q}_d, \ddot{q}_d)$$ (all functions of tim)

Begin by rewriting the dynamics (8.7)

$$B(q)\ddot{q} + n(q, \dot{q}) = u \qquad (8.55)$$

$$n(q, \dot{q}) = C(q, \dot{q})\dot{q} + F\dot{q} + g(q) \qquad (8.56)$$

We will do an exact linearization based on nonlinear state feedback

$$u = B(q)y + n(q,\dot{q})$$

Insert into (8.55)

$$B(q)\ddot{q} + \cancel{n(q,\dot{q})} = B(q)y + \cancel{n(q,\dot{q})} \quad \Big| \cdot B^{-1}$$

$$\ddot{q} = y$$

We will now find the control for y

Choose $$y = -K_P q - K_D \dot{q} + r \tag{8.58}$$

This yields

$$\ddot{q} = -K_P q - K_D \dot{q} + r$$

$$\ddot{q} + K_D \dot{q} + K_P q = r \tag{8.59}$$

The system is asymtotically stable if

$$K_P > 0 \quad K_D > 0$$

Choose

$$K_P = \mathrm{diag}(\omega_{n1}^2, \cdots \omega_{nn}^2)$$

$$K_D = \mathrm{diag}(2\zeta_1\omega_{n1}, \cdots 2\zeta_n\omega_{nn})$$

Given any trajectory $q_d(t)$ tracking is ensured by choosing

$$r = \ddot{q}_d + K_D \dot{q}_d + K_P q_d \qquad (8.60)$$

Insert into (8.59),

$$\ddot{q} + K_D \dot{q} + K_P q = \ddot{q}_d + K_D \dot{q}_d + K_P q_d$$

$$\ddot{\tilde{q}} + K_D \dot{\tilde{q}} + K_P \tilde{q} = 0$$