UiO : Department of Technology Systems
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11. Mobile robots

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Lecture overview

- Mobile robots introduction (1.2.2)
- Non-holonomic constraints (11.1)
- Kinematic model (11.2)
- Chained model (11.3)
- Dynamic model (11.4)


Velodyne

Usage of mobile robots

- Vacuum cleaner (iRobot Roomba ROS compatible)
- Lawn mowers (Husqvarna Automower ROS compatible)

Usage of mobile robots

- Mars rover
- CBRN robots
- EOD robots
- Search and rescue
- Logistics
- Self-driving cars
- Agriculture
- Many more…

Telemax, produced by Telerob

Mobile robots

- Have a mobile base which allows the robot to move
- The are equipped with a locomotion system
- Many subfields
  - Ground robotics (Unmanned Ground Vehicles)
  - Marine robotics
    - Underwater robotics (Autonomous Underwater Vehicles)
    - Unmanned Surface Vehicles
  - Aerial robotics (Unmanned Aerial Vehicles)

Ground robots

- Locomotion systems
  - Wheeled
  - Legged
  - Tracked
  - Undulatory locomotion (snake motion)

BigDog by Boston Dynamics

Wheels

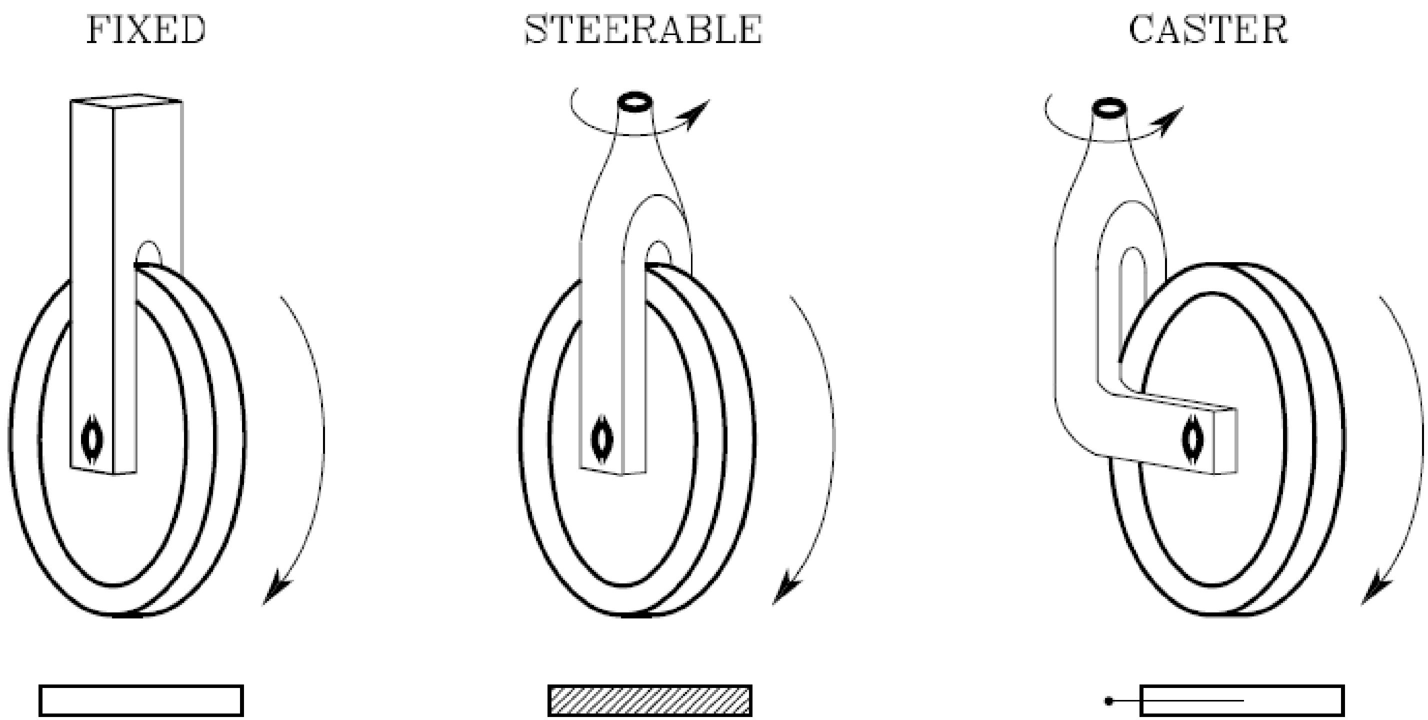


Fig. 1.12. The three types of conventional wheels with their respective icons

Common wheeled robot configurations

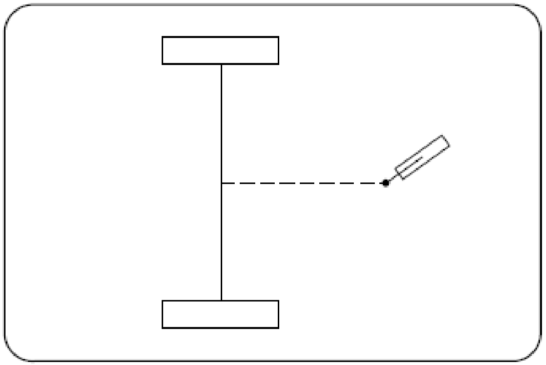


Fig. 1.13. A differential-drive mobile robot

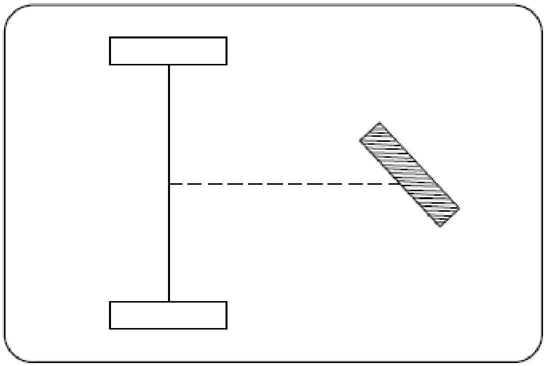


Fig. 1.15. A tricycle mobile robot

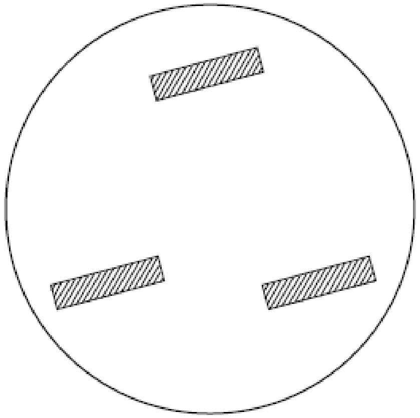


Fig. 1.14. A synchro-drive mobile robot

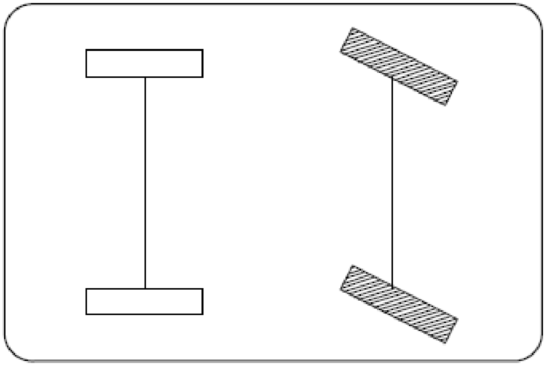


Fig. 1.16. A car-like mobile robot

Common wheeled robot configurations

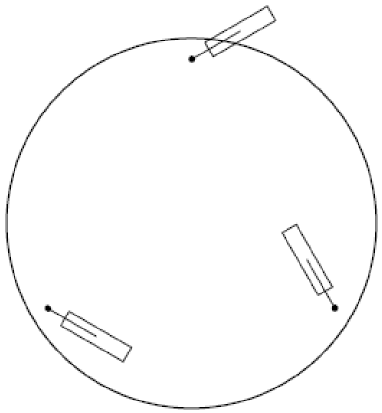


Fig. 1.17. An omnidirectional mobile robot with three independently driven caster

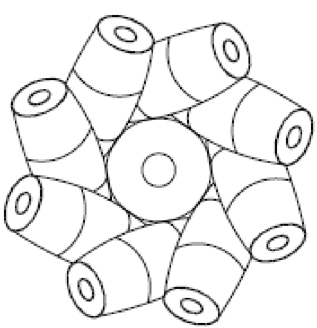
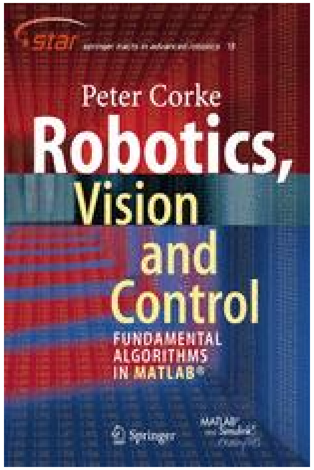


Fig. 1.18. A Mecanum (or Swedish) wheel

11.1 Non-holonomic constraints

- The textbook explains the concept very mathematically
- *Robotics, Vision and Control* by Peter Corke has a easier and simpler explanation (chap. 4.1) (link on course page)
- Not a part of the curriculum, but considered as supplementary material



Non-holonomic constraints

- Configuration space – generalized coordinates of the system
  - Train along a railway – q
  - Car – x, y, θ
- Task space – all possible poses of the vehicle
  - Train along a railway – R, R^2 or SE(3)
  - Car – SE(2) or SE(3)
- The car is under actuated, and cannot move to directly to any point in its configuration space

Non-holonomic constraints

- A holonomic constraint is an equation that can be written in the terms of the configuration variables
  - Car – x, y, θ
- A non-holonomic constraint can only be written in the terms of the derivatives of the configuration variables and can not be integrated to a constant in terms of configuration variables
- A key characteristic is that non-holonomic can not move directly from one configuration to another
- They must perform a maneuver or sequence of actions

Non-holonomic constraints

- Holonomic constraints / integrable constraints ($q \sim R^n$)

$$h_i(\boldsymbol{q}) = 0 \qquad i = 1, \ldots, k < n \tag{11.1}$$

- Non-holonomic constraints / kinematic constraints

$$a_i(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0 \qquad i = 1, \ldots, k < n$$

- Generally expressed in the Pfaffian form

$$\boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}. \tag{11.3}$$

- Where the rows of A are assumed to be smooth and linearly independent

$$A^T(q)G(q) = 0$$

11.2 Kinematic model

- The admissible trajectories for the mechanical system can be characterized as the solution to

$$\dot{q} = \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{q})u_j = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{u} \qquad m = n - k, \tag{11.10}$$

- Where G is a basis of the null space of $\boldsymbol{A}^T(\boldsymbol{q})$
- The choice of input vector fields $\boldsymbol{g}_1(\boldsymbol{q}), \ldots, \boldsymbol{g}_m(\boldsymbol{q})$ is not unique
- It is possible to choose the basis such that the u_j inputs have a physical interpretation
- The vector $\boldsymbol{u}$ may not directly be related to the actual control inputs

11.2.1 Unicycle

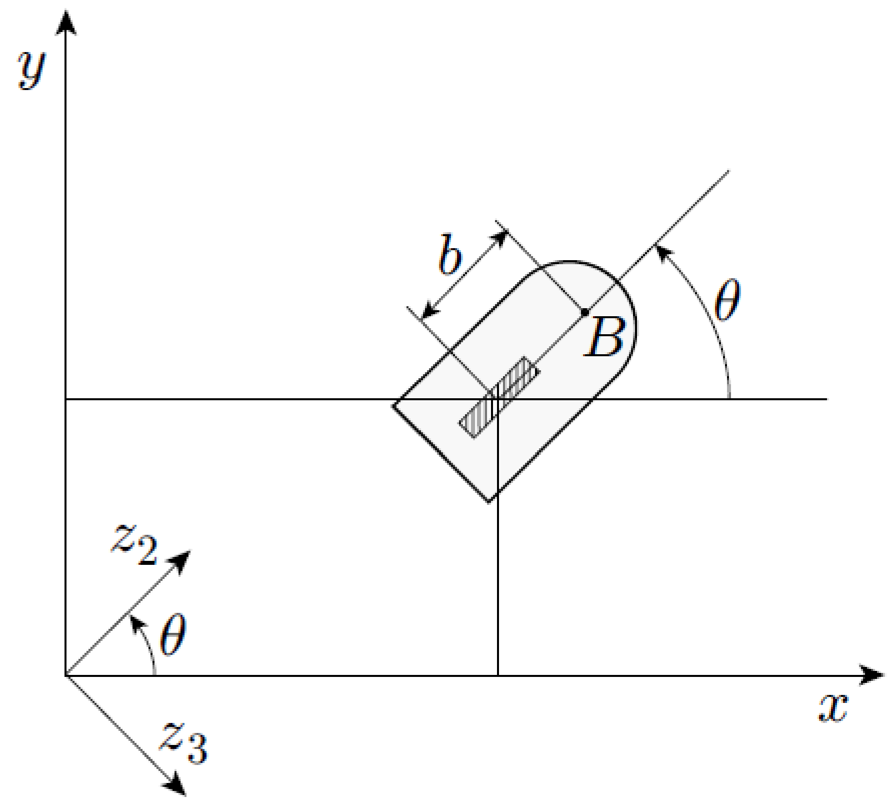


Fig. 11.3. Generalized coordinates for a unicycle

Unicycle

- A unicycle is a vehicle with a single orientable wheel
- The configuration is $\boldsymbol{q} = \begin{bmatrix} x & y & \theta \end{bmatrix}^T$
  - (x, y) is the Cartesian coordinates of the contact point
  - θ is the orientation of the wheel
- The pure rolling constraint for the wheel is expressed as

$$\dot{x}\sin\theta - \dot{y}\cos\theta = \begin{bmatrix} \sin\theta & -\cos\theta & 0 \end{bmatrix} \dot{\boldsymbol{q}} = 0, \tag{11.12}$$

- This means that the velocity is zero orthogonal to the rolling direction of the wheel

Unicycle

- Consider

$$\boldsymbol{G}(\boldsymbol{q}) = [\,\boldsymbol{g}_1(\boldsymbol{q}) \quad \boldsymbol{g}_2(\boldsymbol{q})\,] = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix},$$

- This yields the kinematic model

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \omega, \qquad (11.13)$$

- Where v is the driving velocity and omega is the steering velocity

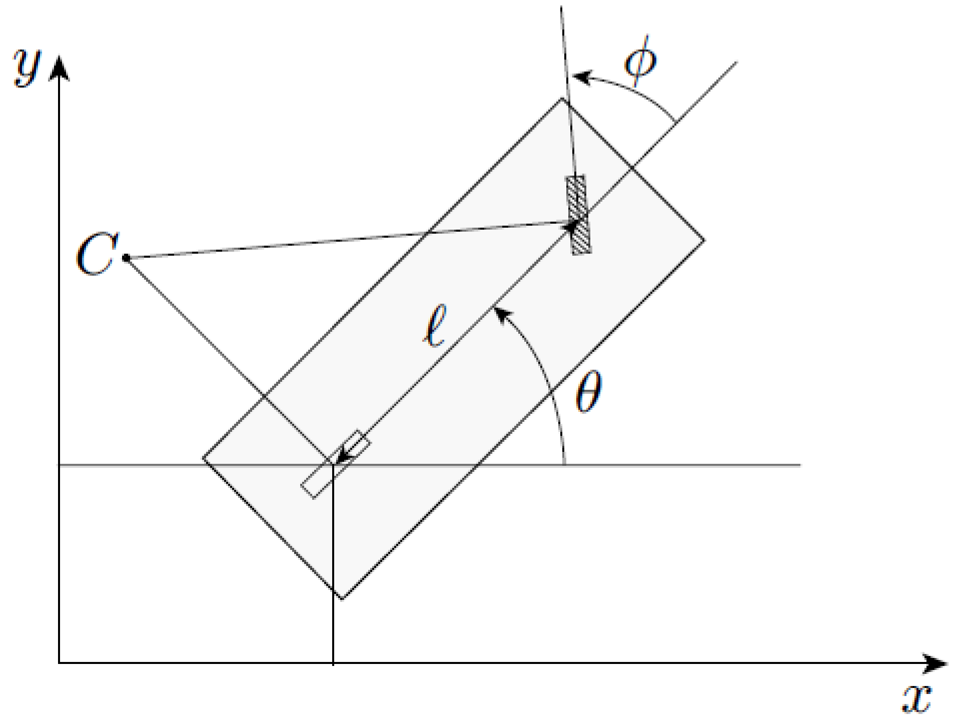
Unicycle

- The unicycle has a problem with balance
- Differential drive and synchro drive robots are kinematically equivalent
- For a differential drive robot with angular speeds ω_R and ω_L of the right and left wheel

$$v = \frac{r\left(\omega_R + \omega_L\right)}{2} \qquad \omega = \frac{r\left(\omega_R - \omega_L\right)}{d}, \tag{11.14}$$

- Where r is the radius of the wheels and d is the distance between the centres of the wheels

11.2.2 Bicycle



Bicycle

- Using generalized coordinates $\boldsymbol{q} = [x \quad y \quad \theta \quad \phi]^T$ where ϕ is the steering angle
- The motion has two constraints, one for each wheel

$$\dot{x}_f \sin(\theta + \phi) - \dot{y}_f \cos(\theta + \phi) = 0 \tag{11.15}$$

$$\dot{x} \sin\theta - \dot{y} \cos\theta = 0, \tag{11.16}$$

- Where (x_f, y_f) is the position of the center of the front wheel
- The point C is called the instantaneous center of rotation

Bicycle

- Using the rigid body constraint

$$x_f = x + \ell \cos\theta$$
$$y_f = y + \ell \sin\theta,$$

- Where ℓ is the distance between the wheels
- Constraint (10.15) can be rewritten as

$$\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - \ell\,\dot{\theta}\cos\phi = 0. \qquad (11.17)$$

- The matrix associated with the Pfaffian constraints is then

$$\boldsymbol{A}^T(\boldsymbol{q}) = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -\ell\cos\phi & 0 \end{bmatrix},$$

Bicycle

- A has a constant dimension of 2
- The dimension of the null space is then 2
- The kinematic model then has two inputs
- The kinematic model can be written as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi/\ell \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2.$$

Bicycle

- Front wheel drive

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi \\ \sin\theta\cos\phi \\ \sin\phi/\ell \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega. \quad (11.18)$$

- Rear wheel drive

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/\ell \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega, \quad (11.19)$$

$$u_1 = v/\cos\phi$$

Has singularity when ±π/2

Bicycle

- Also unstable in static conditions
- The model is kinematically equivalent to the tricycle and the car-like robot

11.3 Chained form

- It is possible to transform the kinematic model of a mobile robot into a canonical form
- A (2,n) chained form is a two-input driftless system

$$\dot{z} = \gamma_1(z)v_1 + \gamma_2(z)v_2,$$

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1 \\ &\vdots \\ \dot{z}_n &= z_{n-1} v_1. \end{aligned} \tag{11.20}$$

Chained form - unicycle

- Consider the coordinate change

$$\begin{aligned} z_1 &= \theta \\ z_2 &= x\cos\theta + y\sin\theta \\ z_3 &= x\sin\theta - y\cos\theta \end{aligned} \tag{11.23}$$

- And the input transformation

$$\begin{aligned} v &= v_2 + z_3 v_1 \\ \omega &= v_1, \end{aligned} \tag{11.24}$$

- This yields the chained form

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1. \end{aligned} \tag{11.25}$$

Chained form - bicycle

- Based on the rear-wheel drive model (11.19), using the change in coordinates

$$
\begin{aligned}
z_1 &= x \\
z_2 &= \frac{1}{\ell}\sec^3\theta\tan\phi \\
z_3 &= \tan\theta \\
z_4 &= y
\end{aligned}
$$

- And input transform

$$
\begin{aligned}
v &= \frac{v_1}{\cos\theta} \\
\omega &= -\frac{3}{\ell}v_1\sec\theta\sin^2\phi + \frac{1}{\ell}v_2\cos^3\theta\cos^2\phi,
\end{aligned}
$$

Chained form - bicycle

- Yields the chained form

$$\begin{aligned}
\dot{z}_1 &= v_1 \\
\dot{z}_2 &= v_2 \\
\dot{z}_3 &= z_2 v_1 \\
\dot{z}_4 &= z_3 v_1.
\end{aligned}$$

11.4 Dynamic model

- Derivation is similar to manipulator case
- The main difference is the presence of non-holonomic constraints
- Exact linearization is no longer possible
- As usual we define the Lagrangian

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} - \mathcal{U}(\boldsymbol{q}), \qquad (11.26)$$

Dynamic model

- The Lagrange equations are in this case

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}\right)^T = \boldsymbol{S}(\boldsymbol{q})\boldsymbol{\tau} + \boldsymbol{A}(\boldsymbol{q})\boldsymbol{\lambda}, \qquad (11.27)$$

- S(q) is an n x m matrix and is mapping the external inputs $\boldsymbol{\tau}$ to the generalized forces performing work on q
  - n – dimension of q
  - k – dimension of constraints
  - m – dimension of control inputs, m = n – k
- A(q) is the transpose of the matrix found in the Pfaffian form

$$\boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}. \qquad (11.3)$$

- $\boldsymbol{\lambda} \in \mathbb{R}^k$ are the Lagrange multipliers

Dynamic model

- From this we get the dynamic model of the constrained mechanical system

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{S}(\boldsymbol{q})\boldsymbol{\tau} + \boldsymbol{A}(\boldsymbol{q})\boldsymbol{\lambda} \tag{11.28}$$

$$\boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \boldsymbol{0}, \tag{11.29}$$

- where

$$\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\left(\frac{\partial}{\partial \boldsymbol{q}}\left(\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)\right)^T + \left(\frac{\partial \mathcal{U}(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T.$$

Dynamic model

- The matrix $\boldsymbol{G}(\boldsymbol{q})$ column vectors are a basis of the null space of $\boldsymbol{A}^T(\boldsymbol{q})$, which yields $\boldsymbol{A}^T(\boldsymbol{q})\boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{0}$
- One can replace the constraints with the kinematic model

$$\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{v} = \sum_{i=1}^{m} \boldsymbol{g}_i(\boldsymbol{q})\, v_i, \qquad (11.30)$$

- Where $\boldsymbol{v} \in \mathbb{R}^m$ is psaudo-velocities
- For example diving velocity and steering velocity for the unicycle

Dynamic model

- By pre-multiplying the Lagrange multipliers in (11.28) with $\boldsymbol{G}^T(\boldsymbol{q})$ we get the reduced dynamic model

$$\boldsymbol{G}^T(\boldsymbol{q})\left(\boldsymbol{B}(\boldsymbol{q})\ddot{q} + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})\right) = \boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{S}(\boldsymbol{q})\boldsymbol{\tau}, \qquad (11.31)$$

- The Lagrange multipliers has been eliminated
- It is now a system of m differential equations instead of n

Dynamic model

- Differentiating (10.30) yields

$$\ddot{q} = \dot{G}(q)v + G(q)\dot{v}.$$

- Pre-multiplying with $G^T(q)B(q)$ and using the reduced dynamic model yields

$$M(q)\dot{v} + m(q, v) = G^T(q)S(q)\tau, \tag{11.32}$$

- where

$$M(q) = G^T(q)B(q)G(q)$$
$$m(q, v) = G^T(q)B(q)\dot{G}(q)v + G^T(q)n(q, G(q)v),$$

Dynamic model

- This leads to the state-space reduced model

$$\dot{q} = G(q)v \tag{11.33}$$

$$\dot{v} = -M^{-1}(q)m(q,v) + M^{-1}(q)G^T(q)S(q)\boldsymbol{\tau}, \tag{11.34}$$

- Which represents a compact form of the kinematic and dynamic models of the constrained system as a set of n+m differential equations

Dynamic model

- Suppose now that

$$\det\left(\boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{S}(\boldsymbol{q})\right) \neq 0,$$

- Which is satisfied in many cases of interest
- It is possible to perform a partial linearization via feedback by letting

$$\boldsymbol{\tau} = \left(\boldsymbol{G}^T(\boldsymbol{q})\boldsymbol{S}(\boldsymbol{q})\right)^{-1}\left(\boldsymbol{M}(\boldsymbol{q})\boldsymbol{a} + \boldsymbol{m}(\boldsymbol{q}, \boldsymbol{v})\right), \tag{11.35}$$

- Where $\boldsymbol{a} \in \mathbb{R}^m$ is a pseudo-acceleration vector
- The resulting system is

$$\dot{\boldsymbol{q}} = \boldsymbol{G}(\boldsymbol{q})\boldsymbol{v} \tag{11.36}$$

$$\dot{\boldsymbol{v}} = \boldsymbol{a}. \tag{11.37}$$

Dynamic model

$$\dot{q} = G(q)v \tag{11.36}$$

$$\dot{v} = a. \tag{11.37}$$

- The n first equations are the kinematic model
- The m last equations are a dynamic extension (integrators)
- Unable to get the double integrator as in the manipulator case (unless G = S = I)
- Requires the measurements of v, or the computation of these by the kinematic model

$$v = G^{\dagger}(q)\dot{q} = \left(G^T(q)G(q)\right)^{-1} G^T(q)\dot{q}, \tag{11.38}$$

- Then q and q_dot must be measured

Dynamic model - summary

- In non-holonomic systems it is possible to *cancel* dynamic effects via nonlinear feedback
- This assumes that the dynamic parameters are exactly known
- Under these assumptions the control problem can be addressed at a pseudo-velocity level
- This means that v must be chosen according to
$$\dot{q} = G(q)v$$
for the system to behave as desired
- Since $a = \dot{v}$ the pseudo-velocities must be differentiable in time

Summary

- Mobile robots introduction (1.2.2)
- Non-holonomic constraints (11.1)
- Kinematic model (11.2)
- Chained model (11.3)
- Dynamic model (11.4)

Exercises

- Prove that (11.23) and (11.24) transforms (11.13) into (11.25)
- Show all the intermediate steps in example 11.5
- Derive the kinematic model of the front wheel of the bicycle robot. Derive the model both for front and rear wheel drive
- Exercises: 11.6