UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: TEK4030
Day of exam: 17th of December
Exam hours: 9.15 - 13.15 (4 hours)
This examination paper consists of 11 page(s).
Appendices: Formulas
Permitted materials: None

Make sure that your copy of this examination paper is complete before answering.

Problem 1 - Independent joint control (20%)

a) (4 %) Briefly describe the principal difference between decentralized and centralized control schemes.

> In decentralized control each joint is controlled only using measurements from that joint. All effect from other joints are considered as disturbance. In centralized control the control law is using measurements from all the joints and uses a model based on all the joints.

b) (4 %) In independent motion control schemes, what are the main motivation for adding feed forward compensation of joint velocities and acceleration?

> This is added in order to track trajectories.

c) (12 %) Given the model of the manipulator with drives below

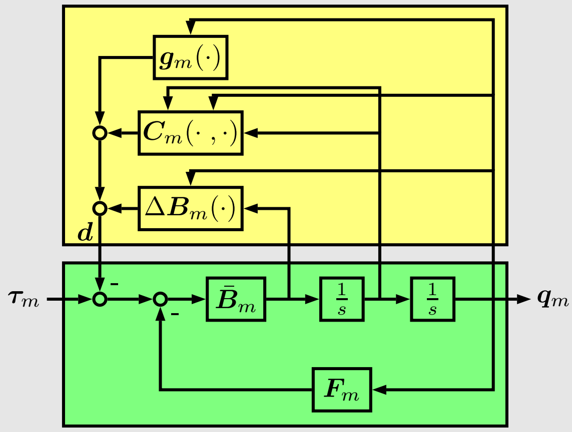
$$\boldsymbol{B}_m(\boldsymbol{q})\ddot{\boldsymbol{q}}_m + \boldsymbol{C}_m(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_m + \boldsymbol{F}_m\dot{\boldsymbol{q}}_m + \boldsymbol{g}_m(\boldsymbol{q}) = \boldsymbol{\tau}_m \tag{1}$$

where $\boldsymbol{q}_m$ is the vector of joint actuator displacement, $\boldsymbol{q}$ is the vector of the joint positions, $\boldsymbol{\tau}_m$ is the vector of actuator driving torque and

$$\boldsymbol{B}_m(\boldsymbol{q}) = \boldsymbol{K}_r^{-1}\boldsymbol{B}(\boldsymbol{q})\boldsymbol{K}_r^{-1} \qquad \boldsymbol{C}_m(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{K}_r^{-1}\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\boldsymbol{K}_r^{-1}$$
$$\boldsymbol{F}_m = \boldsymbol{K}_r^{-1}\boldsymbol{F}_v\boldsymbol{K}_r^{-1} \qquad \boldsymbol{g}_m(\boldsymbol{q}) = \boldsymbol{K}_r^{-1}\boldsymbol{g}(\boldsymbol{q})$$

where $\boldsymbol{K}_r$ is a diagonal matrix of gear reduction ratios. Show how the model (1) can be divided into one linear decoupled part and one nonlinear coupled part. Draw a block diagram of the system showing the division.

> Divide the inertia matrix into two parts, one constant and one configuration dependent
>
> $$\boldsymbol{B}_m(\boldsymbol{q}) = \bar{\boldsymbol{B}}_m + \Delta\boldsymbol{B}_m(\boldsymbol{q})$$
>
> The dynamic equation the becomes
>
> $$\bar{\boldsymbol{B}}_m\ddot{\boldsymbol{q}}_m + \Delta\boldsymbol{B}_m(\boldsymbol{q})\ddot{\boldsymbol{q}}_m + \boldsymbol{C}_m(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_m + \boldsymbol{F}_m\dot{\boldsymbol{q}}_m + \boldsymbol{g}_m(\boldsymbol{q}) = \boldsymbol{\tau}_m$$
>
> Extracting the nonlinear coupled term to a disturbance $\boldsymbol{d}$
>
> $$\boldsymbol{d} = \Delta\boldsymbol{B}_m(\boldsymbol{q})\ddot{\boldsymbol{q}}_m + \boldsymbol{C}_m(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}_m + \boldsymbol{g}_m(\boldsymbol{q})$$
>
> The model now becomes
>
> $$\bar{\boldsymbol{B}}_m\ddot{\boldsymbol{q}}_m + \boldsymbol{F}_m\dot{\boldsymbol{q}}_m + \boldsymbol{d} = \boldsymbol{\tau}_m$$
>
> Which is linear and decoupled, assuming no disturbance.



In the above sketch the yellow (upper) box indicates the non-linear coupled part, and the green (lower) box indicates the linear decoupled part.

- 2 %-points given for correct division of $\boldsymbol{B}_m$
- 2 %-points given for correct definition of $\boldsymbol{d}$
- 4 %-points given for showing the equations for the non-linear coupled part and the linear decoupled part
- 4 %-points given for correct block diagram

Problem 2 - Centralized control (20%)

a) (4 %) Briefly explain the difference between joint space control and operation space control.

In joint space control you first first the desired joint setpoint using inverse kinematics (in some cases also first and second order differential kinematics), and then have the control law in joint space. In operational space control you have the setpoint and feedback in operational space, and there is no need for inverse kinematics. The Jacobian is in the control law for transforming between the two spaces.

b) (16 %) The following controller is a PD controller with gravity compensation in operational space

$$\boldsymbol{u} = \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}_P\tilde{\boldsymbol{x}} - \boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}_D\boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{2}$$

where $\tilde{\boldsymbol{x}} = \boldsymbol{x}_d - \boldsymbol{x}_e$. Here $\boldsymbol{x}_d$ is the desired end-effector pose and $\boldsymbol{x}_e$ is the current end-effector pose. The desired end-effector pose $\boldsymbol{x}_d$ is assumed to be constant, and this yields the time derivative of the error to be given as

$$\dot{\tilde{\boldsymbol{x}}} = -\boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{3}$$

The controller will be used to control the robotic system

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u} \tag{4}$$

Use the following Lyapunov function candidate

$$V(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{x}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\tilde{\boldsymbol{x}}^T\boldsymbol{K}_P\tilde{\boldsymbol{x}} \tag{5}$$

and show that the system is asymptotically stable under the assumption that $\boldsymbol{K}_P$ and $\boldsymbol{K}_D$ are~~is~~ a symmetric positive definite constant matrix and that the Jacobian has full rank. Use Lyapunov's direct method. If $\dot{V}$ becomes negative-semi definite, check if the equilibrium posture has zero error. If the error is zero and the other conditions are met, the system is asymptotically stable.

The three conditions that are satisfied are

$$\begin{aligned}
\boldsymbol{V}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{x}}) &> 0 && \forall\ \dot{\boldsymbol{q}}, \tilde{\boldsymbol{x}} \neq 0 \\
\boldsymbol{V}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{x}}) &= 0 && \dot{\boldsymbol{q}}, \tilde{\boldsymbol{x}} = 0 \\
\boldsymbol{V}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{x}}) &\to \infty && ||\dot{\boldsymbol{q}}, \tilde{\boldsymbol{x}}|| \to \infty
\end{aligned}$$

Now checking if

$$\dot{\boldsymbol{V}}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{x}}) < 0 \qquad \forall\ \dot{\boldsymbol{q}}, \tilde{\boldsymbol{x}} \neq 0$$

Time differentiating $\boldsymbol{V}$

$$\begin{aligned}
\dot{\boldsymbol{V}}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{x}}) &= \frac{1}{2}\ddot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\tilde{\boldsymbol{x}}}^T\boldsymbol{K}_P\tilde{\boldsymbol{x}} + \frac{1}{2}\tilde{\boldsymbol{x}}^T\boldsymbol{K}_P\dot{\tilde{\boldsymbol{x}}} \\
&= \dot{\boldsymbol{q}}^T\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \dot{\tilde{\boldsymbol{x}}}^T\boldsymbol{K}_P\tilde{\boldsymbol{x}}
\end{aligned}$$

Now inserting the manipulator dynamics (i.e. inserting for $\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}}$)

$$\begin{aligned}
\dot{\boldsymbol{V}}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{x}}) &= \dot{\boldsymbol{q}}^T\left(\boldsymbol{u} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \boldsymbol{F}\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right) + \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{\boldsymbol{B}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \dot{\tilde{\boldsymbol{x}}}^T\boldsymbol{K}_P\tilde{\boldsymbol{x}} \\
\dot{\boldsymbol{V}}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{x}}) &= \cancel{\frac{1}{2}\dot{\boldsymbol{q}}^T\left(\dot{\boldsymbol{B}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q}\right)}\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T\left(\boldsymbol{u} - \boldsymbol{F}\dot{\boldsymbol{q}} - \boldsymbol{g}(\boldsymbol{q})\right) + \dot{\tilde{\boldsymbol{x}}}^T\boldsymbol{K}_P\tilde{\boldsymbol{x}}
\end{aligned}$$

Inserting for the control law $\boldsymbol{u}$

$$\begin{aligned}
\dot{\boldsymbol{V}}(\dot{\boldsymbol{q}}, \tilde{\boldsymbol{x}}) &= \dot{\boldsymbol{q}}^T\left(\cancel{\boldsymbol{g}(\boldsymbol{q})} + \boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}_P\tilde{\boldsymbol{x}} - \boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}_D\boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{F}\dot{\boldsymbol{q}} - \cancel{\boldsymbol{g}(\boldsymbol{q})}\right) + \dot{\tilde{\boldsymbol{x}}}^T\boldsymbol{K}_P\tilde{\boldsymbol{x}} \\
&= \dot{\boldsymbol{q}}^T\left(\boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}_P\tilde{\boldsymbol{x}} - \boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}_D\boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}} - \boldsymbol{F}\dot{\boldsymbol{q}}\right) + \dot{\tilde{\boldsymbol{x}}}^T\boldsymbol{K}_P\tilde{\boldsymbol{x}} \\
&= \cancel{\dot{\tilde{\boldsymbol{x}}}^T\boldsymbol{K}_P\tilde{\boldsymbol{x}}} - \dot{\boldsymbol{q}}^T\boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}_D\boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T\boldsymbol{F}\dot{\boldsymbol{q}} + \cancel{\dot{\tilde{\boldsymbol{x}}}^T\boldsymbol{K}_P\tilde{\boldsymbol{x}}} \\
&= -\dot{\boldsymbol{q}}^T\boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}_D\boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T\boldsymbol{F}\dot{\boldsymbol{q}}
\end{aligned}$$

$\dot{\boldsymbol{V}}$ is negative semi-definite. Inserting $\dot{\boldsymbol{q}} = 0$ and $\ddot{\boldsymbol{q}} = 0$ into the dynamic model to check the equilibrium posture.

$$\cancel{\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}}} + \cancel{\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}} + \cancel{\boldsymbol{F}\dot{\boldsymbol{q}}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{g}(\boldsymbol{q}) + \boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}_P\tilde{\boldsymbol{x}} - \cancel{\boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}_D\boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}}$$
$$\cancel{\boldsymbol{g}(\boldsymbol{q})} = \cancel{\boldsymbol{g}(\boldsymbol{q})} + \boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}_P\tilde{\boldsymbol{x}}$$
$$\boldsymbol{0} = \boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}_P\tilde{\boldsymbol{x}}$$

The position error is zero with the assumptions given, and this shows that the system is asymptotically stable.

- 2 %-points given for showing the three first conditions
- 2 %-points given correctly differentiating $\boldsymbol{V}$
- 4 %-points given for inserting the manipulator dynamics and reducing the expression
- 4 %-points given for inserting the control law and showing that the resulting expression is negative semi-definite
- 4 %-points given checking the equilibrium posture correctly

Problem 3 - Force control (16%)

a) (4 %) The dynamic model of a manipulator interacting with the environment is given by

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u} - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e \quad (6)$$

Do an exact linearization of the system, as done in the inverse dynamics control for motion control. Why is the resulting system not an double integrator, as in the inverse dynamics control for motion control case?

Defining $\boldsymbol{n}$ to be

$$\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q})$$

Now performing an exact linearization by using $\boldsymbol{u} = \boldsymbol{B}(\boldsymbol{q})\boldsymbol{y} + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})$

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \cancel{\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})} = \boldsymbol{B}(\boldsymbol{q})\boldsymbol{y} + \cancel{\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})} - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e$$
$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} = \boldsymbol{B}(\boldsymbol{q})\boldsymbol{y} - \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e$$

The resulting system is not a double integrator since is does not compensate for interaction forces with the environment.

- 3 %-points given for showing the correct equation
- 1 %-points given for answering the question correctly

b) (4 %) Now assume that we measure the contact forces. Use the following equation to do en exact linearization of the system.

$$\boldsymbol{u} = \boldsymbol{B}(\boldsymbol{q})\boldsymbol{y} + \boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e \tag{7}$$

Show that the system becomes an double integrator.

$$\begin{aligned}
\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \cancel{\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})} &= \boldsymbol{B}(\boldsymbol{q})\boldsymbol{y} + \cancel{\boldsymbol{n}(\boldsymbol{q}, \dot{\boldsymbol{q}})} + \cancel{\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e} - \cancel{\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{h}_e} \\
\cancel{\boldsymbol{B}(\boldsymbol{q})}\ddot{\boldsymbol{q}} &= \cancel{\boldsymbol{B}(\boldsymbol{q})}\boldsymbol{y} \\
\ddot{\boldsymbol{q}} &= \boldsymbol{y}
\end{aligned}$$

c) (8 %) Now we will use the following controller on the system

$$\boldsymbol{y} = \boldsymbol{J}_{A_d}^{-1}\boldsymbol{M}_d^{-1}\left(\boldsymbol{K}_D\dot{\tilde{\boldsymbol{x}}} + \boldsymbol{K}_P\tilde{\boldsymbol{x}} - \boldsymbol{M}_d\dot{\boldsymbol{J}}_{A_d}\dot{\boldsymbol{q}} + \boldsymbol{M}_d\dot{\boldsymbol{b}} - \boldsymbol{h}_e^d\right) \tag{8}$$

The acceleration error of system is given as

$$\ddot{\tilde{\boldsymbol{x}}} = -\boldsymbol{J}_{A_d}\ddot{\boldsymbol{q}} - \dot{\boldsymbol{J}}_{A_d}\dot{\boldsymbol{q}} + \dot{\boldsymbol{b}} \tag{9}$$

Find the error dynamics of the system. What kind of controller is this?

$$\begin{aligned}
\ddot{\boldsymbol{q}} &= \boldsymbol{J}_{A_d}^{-1}\boldsymbol{M}_d^{-1}\left(\boldsymbol{K}_D\dot{\tilde{\boldsymbol{x}}} + \boldsymbol{K}_P\tilde{\boldsymbol{x}} - \boldsymbol{M}_d\dot{\boldsymbol{J}}_{A_d}\dot{\boldsymbol{q}} + \boldsymbol{M}_d\dot{\boldsymbol{b}} - \boldsymbol{h}_e^d\right) \\
\boldsymbol{J}_{A_d}\boldsymbol{M}_d\ddot{\boldsymbol{q}} &= \boldsymbol{K}_D\dot{\tilde{\boldsymbol{x}}} + \boldsymbol{K}_P\tilde{\boldsymbol{x}} - \boldsymbol{M}_d\dot{\boldsymbol{J}}_{A_d}\dot{\boldsymbol{q}} + \boldsymbol{M}_d\dot{\boldsymbol{b}} - \boldsymbol{h}_e^d \\
\boldsymbol{M}_d\left(\boldsymbol{J}_{A_d}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_{A_d}\dot{\boldsymbol{q}} - \dot{\boldsymbol{b}}\right) &= \boldsymbol{K}_D\dot{\tilde{\boldsymbol{x}}} + \boldsymbol{K}_P\tilde{\boldsymbol{x}} - \boldsymbol{h}_e^d \\
-\boldsymbol{M}_d\ddot{\tilde{\boldsymbol{x}}} &= \boldsymbol{K}_D\dot{\tilde{\boldsymbol{x}}} + \boldsymbol{K}_P\tilde{\boldsymbol{x}} - \boldsymbol{h}_e^d \\
\boldsymbol{M}_d\ddot{\tilde{\boldsymbol{x}}} + \boldsymbol{K}_D\dot{\tilde{\boldsymbol{x}}} + \boldsymbol{K}_P\tilde{\boldsymbol{x}} &= \boldsymbol{h}_e^d
\end{aligned}$$

This is an impedance controller. One can specify desired mass, damping and spring constant.

- 6 %-points given for finding the correct error dynamics
- 2 %-points given for specifying the correct controller

Problem 4 - Tele-operations (8%)

a) (4 %) What possible advantages are there in using tele-operations in surgery? Two advantages are sufficient.

> Any two of the following give full score (2 %-points for each advantage)
>
> - Higher accuracy - Scaling of operator movements
> - Elimination of tremor
> - Improved dexterity
> - Computer controlled dexterity of instruments inside the body
> - "Converts" keyhole surgery to open technique (instrument tip control)
> - Improved Ergonomics
> - Possibility for remote control
>
> ---
>
> 2 %-points for each advantage

b) (4 %) What does it mean that a tele-operated system is transparent?

> A transparent system has both the master force equal the slave force and the master velocity equal the slave velocity. In a transparent system the user perceives the actual environment impedance.

Problem 5 - Visual servoing (8%)

a) (4 %) Briefly explain the principal difference between eye-in-hand and eye-to-hand configuration for a visual system.

> In eye-in-hand configuration the robot moves the camera, while in eye-to-hand the cameras are stationary and image the robot.

b) (4 %) Image based visual servoing is based on the following relation between change in image features and the velocities of an object and the camera

$$\dot{\boldsymbol{s}} = \boldsymbol{J}_s \boldsymbol{v}_o^c + \boldsymbol{L}_s \boldsymbol{v}_c^c \tag{10}$$

What assumption is made in the above equation in order to use this relation to image based visual servoing. Use the assumption to simplify the above equation.

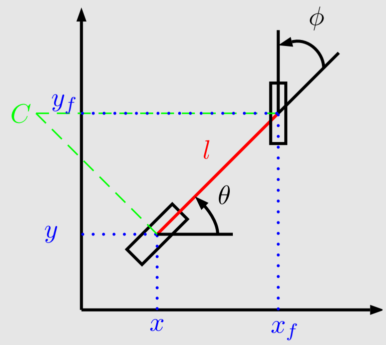
It is assumed that the object velocity is zero (i.e. $\boldsymbol{v}_o^c = \mathbf{0}$). Then the equation reduces to

$$\dot{\boldsymbol{s}} = \boldsymbol{L}_s \boldsymbol{v}_c^c$$

- 2 %-points given for specifying the correct assumption
- 2 %-points given reducing the equation correctly

Problem 6 - Mobile robots (12%)

a) (4 %) Draw a sketch showing all the parameters of the bicycle model. This includes the position of the back wheel (x, y), position of the front wheel (x_f, y_f), the orientation of the vehicle (θ), the orientation of the front wheel (ϕ), the spacing between the wheels (l) and the instantaneous center of rotation (C).



Assuming the figure is roughly correct

- 0.5 %-points subtracted for each wrong or missing parameter (6 parameters in total)

b) (8 %) Find the non-holonomic contraints for the bicycle model expressed in the Pfaffian form. The generic non-holonomic constraint for a wheel is given as

$$v_x \sin\gamma - v_y \cos\gamma = 0 \tag{11}$$

where v_x is the wheel's velocity in along the x-axis, v_y is the wheel's velocity in along the y-axis, and γ is the wheel's orientation. You may also use the following relation

$$\sin\theta \sin(\theta + \phi) + \cos\theta \cos(\theta + \phi) = \cos\phi \tag{12}$$

By using the generic non-holonomic constraint for a wheel given in the exercise we get

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$
$$\dot{x}_f \sin(\theta+\phi) - \dot{y}_f \cos(\theta+\phi) = 0$$

Then we use the rigid transform between the wheels

$$x = x_f + l\cos\theta$$
$$y = y_f + l\sin\theta$$

Time differentiating yields

$$\dot{x} = \dot{x}_f - \dot{\theta} l \sin\theta$$
$$\dot{y} = \dot{y}_f + \dot{\theta} l \cos\theta$$

Inserting this into the second non-holonomic constraint yields

$$\begin{aligned}
(\dot{x} - \dot{\theta} l \sin\theta)\sin(\theta+\phi) - (\dot{y} + \dot{\theta} l \cos\theta)\cos(\theta+\phi) = 0 \\
\dot{x}\sin\theta - \dot{y}\cos(\theta+\phi) - \dot{\theta} l(\sin\theta\sin(\theta+\phi) + \cos\theta\cos(\theta+\phi)) = 0 \\
\dot{x}\sin\theta - \dot{y}\cos(\theta+\phi) - \dot{\theta} l\cos\phi = 0
\end{aligned}$$

Now expressing the contraints in the Pfaffian form $\boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \mathbf{0}$

$$\begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -l\cos\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \mathbf{0}$$

- 2 %-points given for defining the two non-holonomic constraints (one for the back wheel and one for the front wheel)
- 2 %-points given for defining and time differentiating the rigid transformation between the wheels
- 2 %-points given for finding the redefined non-holonomic constraint (i.e. the front wheel constraint expressed in the back wheel's coordinates)
- 2 %-points given for expressing the constraints in the Pfaffian form

Problem 7 - Motion planning (16%)

a) (4 %) Given an obstacle in a mobile robot's workspace, briefly explain how we can find the obstacle's configuration space image (i.e. the $\mathcal{C}$-obstacle). What is the benefit of representing obstacles in configuration space?

> The $\mathcal{C}$-obstacle can be found by expanding the obstacle so that the new border of the obstacle corresponds to the positions of the robot where no collisions occur, but for one or more poses the robot touches the obstacle. This is done on order to treat the robot as a point in space, rather than body in space, when planning the robot's trajectory.
>
> ---
>
> - 2 %-points given for explaining how to find $\mathcal{C}$-obstacles
> - 2 %-points given for explaining the benefit

b) (12 %) In the course these four methods for motion planning was covered:

- Retraction
- Probabilistic roadmap method
- Bidirectional rapidly-exploring random tree
- Artificial potential fields

Choose one and explain the concept of the method. Also specify if the method can be considered a single- or multi-quarry method.

> The methods are explained in the textbook
>
> - Retraction - p. 532
> - Probabilistic roadmap method - p. 541
> - Bidirectional rapidly-exploring random tree - p. 543
> - Artificial potential fields - p. 546
>
> ---
>
> - 10 %-points given for describing the concept of the method. The description should contain all the steps in the method, but it is not necessary with a very detailed explanation.
> - 2 %-points given indicating correctly if the method is a single- or multi-quarry method.

A Formulas

Solution for quadratic equations

$$ax^2 + bx + c = 0 \qquad \Rightarrow \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Properties of dynamic models of robots
Given the following dynamic system

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{F}\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{u}$$

The following properties hold

$$\begin{aligned} \dot{\boldsymbol{q}}^T \left(\dot{\boldsymbol{B}} - 2\boldsymbol{C}\right) \dot{\boldsymbol{q}} &= \boldsymbol{0} \\ \boldsymbol{B} &= \boldsymbol{B}^T \\ \boldsymbol{q}^T \boldsymbol{B} \boldsymbol{q} &> 0 \\ \boldsymbol{q}^T \boldsymbol{F} \boldsymbol{q} &> 0 \end{aligned}$$

Linear algebra
A matrix is symmetric if it satisfies the condition

$$\boldsymbol{A} = \boldsymbol{A}^T$$

For symmetric matrices the following holds

$$\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{y} = \boldsymbol{y}^T \boldsymbol{A} \boldsymbol{x}$$

Trigonometry
Definitions

$$\sin\theta = \frac{opposite}{hypotenuse} \qquad \cos\theta = \frac{adjacent}{hypotenuse}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{opposite}{adjacent}$$

Addition formulas

$$\begin{aligned} \sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y \\ \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y \end{aligned}$$

Derivatives

$$\frac{d}{dx} \sin x = \cos x \tag{13}$$
$$\frac{d}{dx} \cos x = -\sin x \tag{14}$$