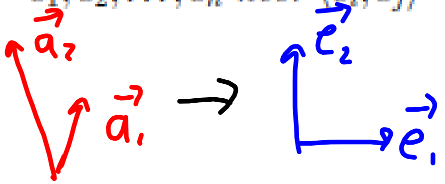
F.3

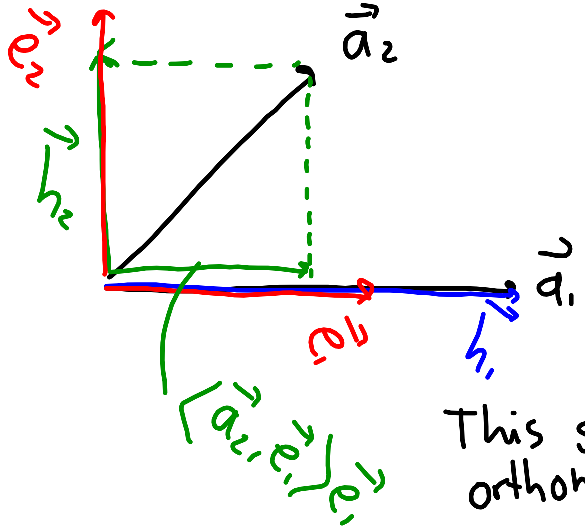
Exame date: Mon 14. Dec

Teorem A.1 *Grahm-Schmidt ortogonalisering.*
Dersom vi har et sett med basisvektorer $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ kan vi lage et ortonormalt sett av basisvektorer $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ hvor $\langle \vec{e}_i, \vec{e}_j \rangle = \delta_{ij}$ på følgende måte:



$$\left.\begin{aligned} \vec{h}_k &= \vec{a}_k - \sum_{i=1}^{k-1} \langle \vec{a}_k, \vec{e}_i \rangle \, \vec{e}_i \\ \vec{e}_k &= \vec{h}_k \left\| \vec{h}_k \right\|^{-1} \end{aligned}\right\} k = 1, 2, \dots, n \qquad \text{(A- 3)}$$

Example n=2



$$\vec{h}_1 = \vec{a}_1 - \sum_{i=1}^{0} \langle \vec{a}_1, \vec{e}_i \rangle \vec{e}_i = \vec{a}_1$$

$$\vec{e}_1 = \vec{h}_1 / \|\vec{h}_1\| \Rightarrow \|\vec{e}_1\| = 1$$

$$\vec{h}_2 = \vec{a}_2 - \sum_{i=1}^{1} \langle \vec{a}_2, \vec{e}_i \rangle \vec{e}_i = \vec{a}_2 - \langle \vec{a}_2, \vec{e}_1 \rangle \vec{e}_1$$

$$\vec{e}_2 = \vec{h}_2 / \|\vec{h}_2\| \Rightarrow \|\vec{e}_2\| = 1$$

This shows that we can always create an orthonormal (o.n.) set of basis vectors

A.2.2 Matrix representation of geometrical vectors

Problem: Given a geometrical vector $\vec{r}$ and a basis (frame) $\{\vec{p}_i\}$, what is the algebraic vector $\underline{r}^p$

Theorem A.2 Column representation (algebraic vector) av $\vec{r} \in V$:

$$\vec{r} = r_1^p \vec{p}_1 + r_2^p \vec{p}_2 + \dots + r_n^p \vec{p}_n = \sum_{i=1}^{n} r_i^p \vec{p}_i$$

where $r_i^p = \langle \vec{r}, \vec{p}_i^{\,*} \rangle$

$$\underline{r}^p = \begin{bmatrix} r_1^p \\ r_2^p \\ \vdots \\ r_n^p \end{bmatrix} = \left[r_i^p \right]$$

Proof

From linear algebra we can clearly write $\vec{r} = \sum_{i=1}^{n} r_i^p \vec{p}_i$

We first calculate the dual basis $\{\vec{p}_i^*\}$ wher $\langle \vec{p}_i, \vec{p}_j^* \rangle = \delta_{ij}$

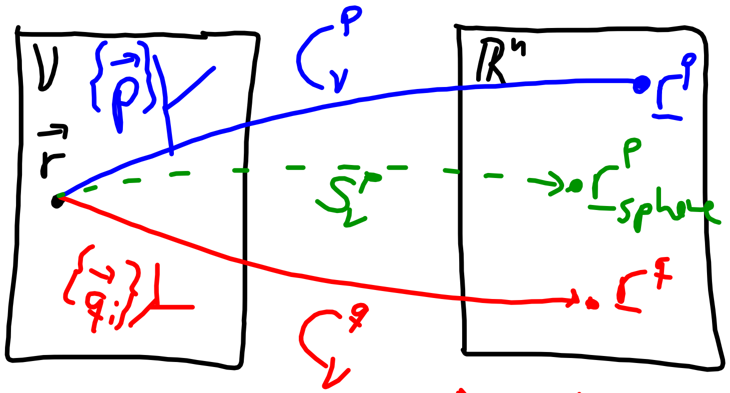
Take the inner product of $\vec{r}$ with the dual basisvector $\vec{p}_j^*$

$$\langle \vec{r}, \vec{p}_j^* \rangle = \left\langle \sum_{i=1}^{n} r_i^p \vec{p}_i, \vec{p}_j^* \right\rangle = \sum_{i=1}^{n} r_i^p \underbrace{\langle \vec{p}_i, \vec{p}_j^* \rangle}_{\delta_{ij}} = r_j^p$$

I.e. if we switch the index:

$$r_i^p = \langle \vec{r}, \vec{p}_i^* \rangle$$

I.e. given $\vec{r}$ and $\{\vec{p}_i\}$ then $\underline{r}^p = [\langle \vec{r}, \vec{p}_i^{\,*} \rangle]$ $\quad \vec{r} \overset{F_V^p}{\Longleftrightarrow} \underline{r}^p$



Independent of choise of basis-vektors, c.s.

Dependent on choise of basisvectors and type of c.s. (cartesian, spherical, polar,...)

V: n-dimentional vector space

$\mathbb{R}^n$: $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$ n-dim

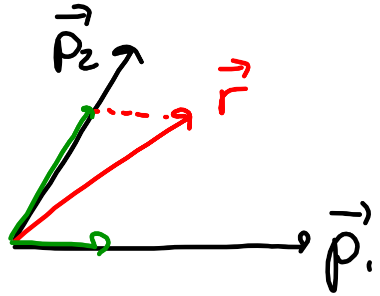
n-dimentional space of numbers $\in \mathbb{R}$

C_V^p: coordinate system (c.s)

If the basisvectors are orthonormal (o.n)

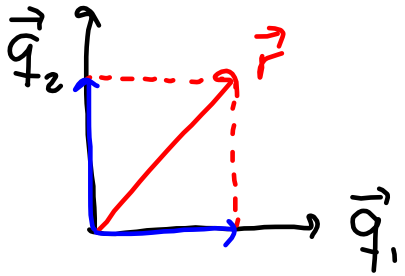
$$\langle p_i, p_j \rangle = \delta_{ij} = \langle \vec{p}_i, \vec{p}_j^{\,*} \rangle$$

$$\Rightarrow \vec{p}_j = \vec{p}_j^{\,*}$$



Low of parallelogram

Matematically we can use $\{\vec{p}_i\}$ and $\{\vec{p}_i^{\,*}\}$

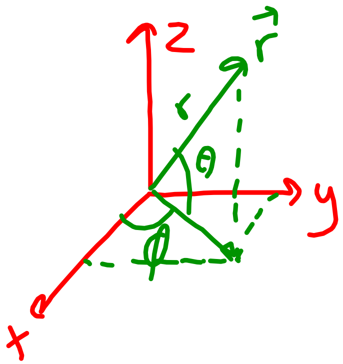


Projecting into the basis vectors

Matematically we use $\{\vec{q}_i\}$

When using polar og spherical coordinates we also need to spesify a frame (basis).

$$C_v^p(\vec{r}) = \underline{r}^p, \quad K_v^p(\vec{r}) = \begin{bmatrix} \phi \\ \theta \\ r \end{bmatrix}$$



We see that the coordinates a coordinatefunction (C_v^p, K_v^p) gives depends on the frame $\mathcal{F}_v^p$ and how the coordinates are calculated (cartesian, spherical, polar,).

A.2.3 Matrix representation of linear operators

An operator is a function that given a vector calculates a new one:

$$\mathbb{O}(\vec{r}) = \vec{a} \qquad V \to V$$

We are going to look espesially at linear operators

Def. A.10 Linear operator

The operator $\mathbb{A}$ is linear $\iff \forall \vec{x}, \vec{y} \in V$ and $\forall\, a, b \in \mathbb{R}$ we can write:

$$\boxed{\mathbb{A}(a\vec{x} + b\vec{y}) = a\mathbb{A}\vec{x} + b\mathbb{A}\vec{y}}$$

Example: $\vec{w} \times \vec{r}$, $\mathbb{A} = \vec{w}\times$

$$\mathbb{A}(a\vec{x} + b\vec{y}) = \vec{w} \times (a\vec{x} + b\vec{y}) = \vec{w} \times a\vec{x} + \vec{w} \times b\vec{y}$$

$$= a\,\underbrace{\vec{w}\times}_{\mathbb{A}}\vec{x} + b\,\underbrace{\vec{w}\times}_{\mathbb{A}}\vec{y}$$

$$= a\mathbb{A}\vec{x} + b\mathbb{A}\vec{y}$$

I.e. cross product "$\vec{w}\times$" is a linear operator

Theorem A.3 Matrix representation of a linear operator

Given $\{\vec{p}_i\} \in V$, then any linear operator $\mathbb{A}$ can be clearly represented in $\mathbb{R}^{n \times n}$ as a matrix A^p. We have that:

$$\boxed{\begin{gathered} A^p = [a^p_{ij}] = \left[\left\langle \mathbb{A}\vec{p}_j , \vec{p}_i^{\,*} \right\rangle \right] \\ \vec{y} = \mathbb{A}\,\vec{x} \underset{\mathcal{F}_V^p}{\Longleftrightarrow} \underline{y}^p = A^p \underline{x}^p \end{gathered}}$$

Proof of the theorem: Given $\mathcal{F}_V^p = \{\vec{p}_i\}$, $\vec{y} = \mathbb{A}\,\vec{x}$

We know: $\vec{y} = \sum_{i=1}^{n} y_i^p \vec{p}_i$, $\vec{x} = \sum_{j=1}^{n} x_j^p \vec{p}_j$, $\vec{y} = A\vec{x}$

$$y_i^p = \langle \vec{y}, \vec{p}_i^{\,*} \rangle = \langle A\vec{x}, \vec{p}_i^{\,*} \rangle$$
$$= \langle A\left(\sum_{j=1}^{n} x_j^p \vec{p}_j\right), \vec{p}_i^{\,*} \rangle$$
$$= \langle \sum_{j=1}^{n} x_j^p A\vec{p}_j, \vec{p}_i^{\,*} \rangle$$
$$= \sum_{j=1}^{n} x_j^p \underbrace{\langle A\vec{p}_j, \vec{p}_i^{\,*} \rangle}_{a_{ij}^p}$$

$$y_i^p = \sum_{j=1}^{n} x_j^p a_{ij}^p$$

$$\underline{y}^p = \begin{bmatrix} y_1^p \\ y_2^p \\ y_3^p \end{bmatrix} = \begin{bmatrix} a_{11}^p & a_{12}^p & a_{13}^p \\ a_{21}^p & a_{22}^p & a_{23}^p \\ a_{31}^p & a_{32}^p & a_{33}^p \end{bmatrix} \begin{bmatrix} x_1^p \\ x_2^p \\ x_3^p \end{bmatrix}$$

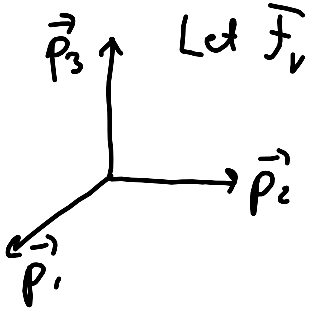
$$\underline{y}^p = A^p \quad \underline{x}^p$$

$$\underline{y}^p = A^p \underline{x}^p$$

where $A^p = [a_{ij}^p] = [\langle A\vec{p}_j, \vec{p}_i^{\,*} \rangle]$

q.e.d

Example A.4 Matrix representation of the "$\vec{w}\times$" operator



Let $\vec{\mathcal{F}}_v^p$ be orthonormal

$$\vec{w} = w_1^p \vec{p}_1 + w_2^p \vec{p}_2 + w_3^p \vec{p}_3$$

$$\underline{w} = [w_1^p;\ w_2^p;\ w_3^p]$$

$$\mathcal{A}\vec{a} = \vec{w}\times\vec{a}$$

The def. of cross product shows that "$\vec{w}\times$" is a linear operator (shown before)

i.e. $$A^p = \left[\left\langle \mathcal{A}\vec{p}_j, \vec{p}_i^{\,*}\right\rangle\right] \overset{\text{o.n}}{=} \left[\left\langle \vec{w}\times\vec{p}_j, \vec{p}_i\right\rangle\right]$$

$$= \left[\left\langle \left(w_1\vec{p}_1 + w_2\vec{p}_2 + w_3\vec{p}_3\right)\times\vec{p}_j, \vec{p}_i\right\rangle\right]$$

$$\vec{p}_i \times \vec{p}_i = 0 \qquad i = 1,2,3$$

$$\vec{p}_1 \times \vec{p}_2 = \vec{p}_3 = -\vec{p}_2 \times \vec{p}_1$$

$$\vec{p}_1 \times \vec{p}_3 = -\vec{p}_2 = -\vec{p}_3 \times \vec{p}_1$$

$$\vec{p}_2 \times \vec{p}_3 = \vec{p}_1 = -\vec{p}_3 \times \vec{p}_2$$

$$\langle \vec{p}_i, \vec{p}_i\rangle = 1, \quad i = 1,2,3$$

$$\langle \vec{p}_i, \vec{p}_j\rangle = 0, \quad i \neq j$$

$$A^p = \begin{array}{c} \\ i=1 \\ i=2 \\ i=3 \end{array} \begin{array}{c} \begin{array}{ccc} j=1 & j=2 & j=3 \end{array} \\ \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \end{array}$$

$$A^p = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} = S(\underline{w}^p)$$

$$S + S^T = 0$$

Skew symmetrical form

Note that the basisvectors is o.n.

$$\boxed{\vec{b} = \vec{w} \times \vec{a} \overset{F_p}{\Longleftrightarrow} \underline{b}^p = S(\underline{w}^p)\, \underline{a}^p}$$

We can now write geometrical equations with geometrical vectors and operators with algebraical equations of matrices.