F9/ For algebraic vectors the same eq. as in F8. p.9 are:

Constant vector seen from $\mathcal{F}^p$:

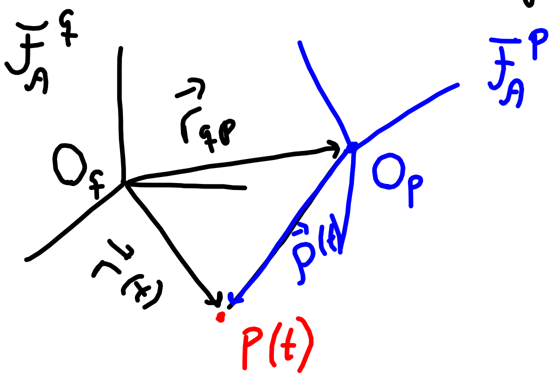
$$\boxed{\begin{aligned} \underline{r}^q &= R_p^q \underline{r}^p \\ \dot{\underline{r}}^q &= S(\underline{\omega}_p^q) R_p^q \underline{r}^p = S(\underline{\omega}_p^q) \underline{r}^q \end{aligned}}$$

$$\left( \dot{\underline{r}}^p = \underline{0} \right)$$

Time varying vector seen from $\mathcal{F}^p$:

$$\boxed{\dot{\underline{r}}^q = S(\underline{\omega}_p^q) R_p^q \underline{r}^p + R_p^q \dot{\underline{r}}^{pp}}$$

## A.5.4 Derivation of points motion in affine spaces




We are looking at the relationship between velocity and accelleration of the point $P$ seen from two frames that moves relative to each other (translation and rotation)

We can either derive the equations for the geometrical vectors and then for the algebraic vectors or derive the eq. either for the geometrical or the algebraic and then use the eq. that gives the relationship between them.

From the figure:

$$\vec r(t) = \vec r_{qp}(t) + \vec\rho(t)$$

$$\vec r(t) = P(t) - O_q$$

$$\vec\rho(t) = P(t) - O_p$$

Find: $\dot P^q(t)$ and $\ddot P^q(t)$ $\left(= \ddot P^{qq}(t)\right)$

$\iff$ $\dot{\vec r}^q(t)$ and $\ddot{\vec r}^q(t)$ $\left(= \ddot{\vec r}^{qq}(t)\right)$

Equations needed:

$$\dot{\vec c}^q = \dot{\vec c}^p + \vec\omega_p^q \times \vec c$$

$$\dot{\underline c}^{qq} = R_p^q \dot{\underline c}^{pp} + S(\underline\omega_p^{qq}) R_p^q \underline c^p$$

## Notation

$\vec v^q = \dot{\vec r}^q$ : P's velocity seen from $\mathcal{F}_A^q$

$\vec v^p = \dot{\vec\rho}^p$ : P's velocity seen from $\mathcal{F}_A^p$

$\vec a^q = \dot{\vec v}^{qq} = \ddot{\vec r}^{qq}$ : Acc. seen from $\mathcal{F}_A^q$

$\vec a^p = \dot{\vec v}^{pp} = \ddot{\vec\rho}^{pp}$ : Acc. seen from $\mathcal{F}_A^p$

Later we want to use Newton's 2. law: $\vec f = m\vec a^i$, $\mathcal{F}^i$ - inertial space

In inertial navigation (INS) we measure $\underline a^{iib}$ (b: fixed to the vehicle)

Deriving the geometrical equations:

$$\boxed{\begin{aligned}\vec{r} &= \vec{r}_{qp} + \vec{\rho} \\ \vec{v}^{q} &= \dot{\vec{r}}^{q} = \dot{\vec{r}}_{qp}^{q} + \underbrace{\vec{v}^{p} + \vec{\omega}_{p}^{q} \times \vec{\rho}}_{\dot{\vec{\rho}}^{q}}\end{aligned}}$$

$$\vec{a}^{q} = \ddot{\vec{r}}^{q} = \ddot{\vec{r}}_{qp}^{q} + \underbrace{\dot{\vec{v}}^{pp} + \omega_{p}^{q} \times \vec{v}^{p}}_{\dot{\vec{v}}^{pq}} + \dot{\vec{\omega}}_{p}^{qq} \times \vec{\rho} + \vec{\omega}_{p}^{q} \times \left(\vec{v}^{p} + \vec{\omega}_{p}^{q} \times \vec{\rho}\right)$$

$$\boxed{\vec{a}^{q} = \ddot{\vec{r}}_{pq}^{q} + \vec{a}^{p} + \dot{\vec{\omega}}_{p}^{q} \times \vec{\rho} + \vec{\omega}_{p}^{q} \times \left(\vec{\omega}_{p}^{q} \times \vec{\rho}\right) + 2\vec{\omega}_{p}^{q} \times \vec{v}^{p}}$$

$$\left.\begin{aligned}\underline{r}^{q} &= \underline{r}_{qp}^{q} + R_{p}^{q}\underline{\rho}^{p} \\ \underline{v}^{q} &= \dot{\underline{r}}_{qp}^{q} + R_{p}^{q}\left(\underline{v}^{p} + \underline{\omega}_{p}^{qp} \times \underline{\rho}^{p}\right) \\ &= \dot{\underline{r}}_{qp}^{q} + R_{p}^{q}\underline{v}^{p} + \underline{\omega}_{p}^{q} \times R_{p}^{q}\underline{\rho}^{p} \\ \underline{a}^{q} &= \ddot{\underline{r}}_{qp}^{q} + R_{p}^{q}\left(\underline{a}^{p} + \dot{\underline{\omega}}_{p}^{qpp} \times \underline{\rho}^{p} + \underline{\omega}_{p}^{qp} \times \left(\underline{\omega}_{p}^{qp} \times \underline{\rho}^{p}\right) + 2\underline{\omega}_{p}^{qp} \times \underline{v}^{p}\right) \\ &= \ddot{\underline{r}}_{qp}^{q} + R_{p}^{q}\dot{\underline{v}}^{p} + \dot{\underline{\omega}}_{p}^{q} \times R_{p}^{q}\underline{\rho}^{p} + \underline{\omega}_{p}^{q} \times \left(\underline{\omega}_{p}^{q} \times R_{p}^{q}\underline{\rho}^{p}\right) + 2\underline{\omega}_{p}^{q} \times R_{p}^{q}\underline{v}^{p}\end{aligned}\right\} \quad \text{(A- 118)}$$

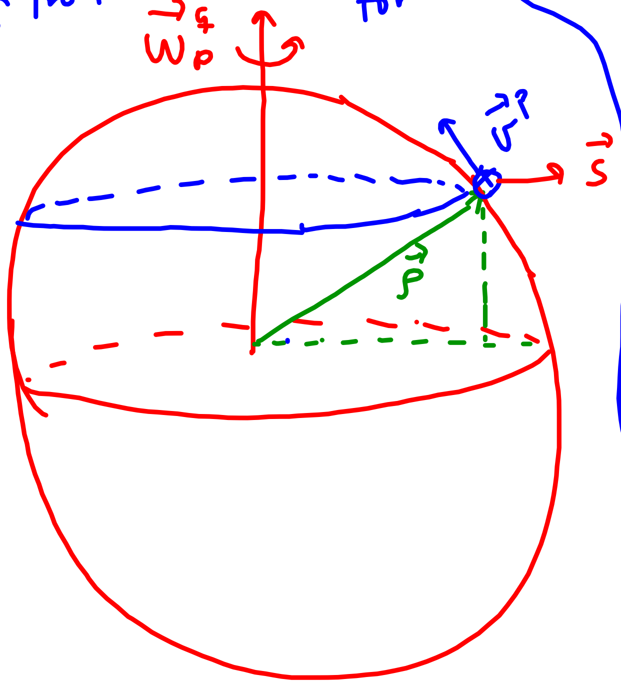
$$\vec{f} = m\,\vec{a}^q \tag{A- 119}$$

Ved å uttrykke akselerasjonen $\vec{a}^q$ vha ledda på høgre sida, får vi de kreftene som må innføres i et ikke-inertial system. Likninga ovenfor blir nå :

$$m\vec{a}^q = m\left(\ddot{\vec{r}}^{\,q}_{qp} + \vec{a}^p + \dot{\vec{\omega}}^q_p \times \vec{\rho} + \vec{\omega}^q_p \times \left(\vec{\omega}^q_p \times \vec{\rho}\right) + 2\vec{\omega}^q_p \times \vec{v}^p\right) \tag{A- 120}$$

Løser likninga mhp $m\vec{a}^p$ :

$$\underbrace{m\vec{a}^p}_{\text{Apparent force seen from } F^p} = \underbrace{m\vec{a}^q}_{\text{Outer for-}} \underbrace{- m\ddot{\vec{r}}^{\,q}_{qp}}_{\text{Force due to accelleration of } O_p \text{ seen from } F^q} \underbrace{- m\dot{\vec{\omega}}^q_p \times \vec{\rho}}_{\substack{\text{Angle accelleration term} \\ \text{(tangential force)}}} \underbrace{- m\vec{\omega}^q_p \times \left(\vec{\omega}^q_p \times \vec{\rho}\right)}_{\text{Centrifugal force } (\vec{s})} \underbrace{- m2\vec{\omega}^q_p \times \vec{v}^p}_{\text{Coriolis force} = \vec{c}} \tag{A- 121}$$



# A.6 Matrix calculation in cybernetics

Standard equation: $\dot{\underline{x}} = A\underline{x}$ , $\underline{x}(0) = \underline{x}_0$

Eigenvalues: $|\lambda I - A| = 0 \iff$ n'th order equation in $\lambda$

Matlab:

$[M, \Lambda] = \text{eig}(A)$

$$\lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0 = 0 \Rightarrow \lambda_i \ , i=1,2,\ldots,n$$

Eigenvectors: $\underbrace{(\lambda_i I - A)}_{\det = 0}\underline{m}_i = \underline{0}$ $\Bigg\}$ I.e. we have not a unique solution. We can choose for example $\|\underline{m}_i\| = 1$

If $\lambda_i \neq \lambda_j$ , for all $i \neq j$ one can prove that $\{\underline{m}_i\}$ is linearly independent and may form a basis system.

$M = [\underline{m}_1, \underline{m}_2, \ldots, \underline{m}_n]$ Eigenvector matrix

We had d.e. $\dot{\underline{x}} = A\underline{x}$, $\underline{x}(0) = \underline{x}_0$, since we want to use M as a DCM we need to introduce a clear notation for the two frames we transform between $\{m\}$ and $\{q\}$.

I.e. $\dot{\underline{x}}^q = A^q \underline{x}^q$, $\underline{x}^q(0) = \underline{x}_0^q$

$M = M_m^q = [\underline{m}_1^q, \underline{m}_2^q, \ldots, \underline{m}_n^q]$

$$\left( \dot{\underline{x}}^q = \underbrace{\dot{M}_m^q}_{=0} \underline{x}^m + M_m^q \cdot \dot{\underline{x}}^m \right)$$

$\underline{x}^q = M_m^q \underline{x}^m$

We had: $\dot{\underline{x}}^q = A^q \underline{x}^q = M_m^q \dot{\underline{x}}^m = A^q M_m^q \underline{x}^m$

$$\Leftrightarrow \dot{\underline{x}}^m = \underbrace{(M_m^q)^{-1} A^q M_m^q}_{A^m = \underline{\Lambda}^m = [\lambda_i \delta_{ij}] = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)} \underline{x}^m$$

Generally: $(M^q_m)^{-1} \neq (M^q_m)^T$

We have now: $\dot{\underline{x}}^m = \Lambda^m \underline{x}^m$, $\underline{x}^m(0) = (M^q_m)^{-1} \underline{x}^q(0)$

$$\Updownarrow$$

$\dot{x}^m_i = \lambda^m_i x^m_i$, $x^m_i(0)$ given

since $t_0 = 0$

$x^m_i(t) = e^{\lambda_i (t - t_0)} x^m_i(t_0) = e^{\lambda_i t} x^m_i(0)$

$$\Updownarrow$$

$$\boxed{\underline{x}^m(t) = \underbrace{e^{\Lambda^m t}}_{= \left[ e^{\lambda_i t} \partial_{ij} \right]} \underline{x}^m(0)}$$

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

Matlab: expm(X)

exp(X) - exp. element by element