F.13/ Instead of the d.e in DCM we can use the d.e. for Euler angles (A.5)

$$\dot{\underline{\theta}} = D_b^\theta(\underline{\theta})\,\underline{\omega}_b^{ib}$$

$$\dot{\underline{\omega}}_b^{ibb} = f(\underline{\omega}_b^{ib}, \underline{n}_c^b)$$

$\underline{\theta}(t_0)$ and $\underline{\omega}_b^{ib}(t_0)$ is given

$$\underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$ (3-2-1 Euler Ang.)

$D_p^\theta(\underline{\theta})$ is given in A-105

$$D_p^\theta(\underline{\theta}) = \begin{bmatrix} 1 & \sin\theta_1 \tan\theta_2 & \cos\theta_1 \tan\theta_2 \\ 0 & \cos\theta_1 & -\sin\theta_2 \\ 0 & \sin\theta_1/\cos\theta_2 & \cos\theta_1/\cos\theta_2 \end{bmatrix}$$

Here $\underline{x} = \begin{bmatrix} \underline{\theta} \\ \underline{\omega}_b^{ib} \end{bmatrix}$

Note! D.e. for $\underline{\omega}_b^{ib}$ can be solved without solving d.e. for $\underline{\theta}$, but not the other way

B.3 Torque free motion of a rigid body

Assume $J_{xx} > J_{yy} > J_{zz}$

We want to calculate $\underline{w}_b^{ib}$ and $\underline{w}_b^{ii}$ (we find trajetories, not the time function)

Ellipsoid of inertia

Inertia matrix $J_c^b = \left[J_c^b\right]^T$

$\Rightarrow$ pos. definite matrix and by using the expression:

$J = \frac{1}{2}\underline{x}^T J_c^b \underline{x}$: ellipsoide when J is constant.

If we chose $\vec{F}^b$ in the main axis.

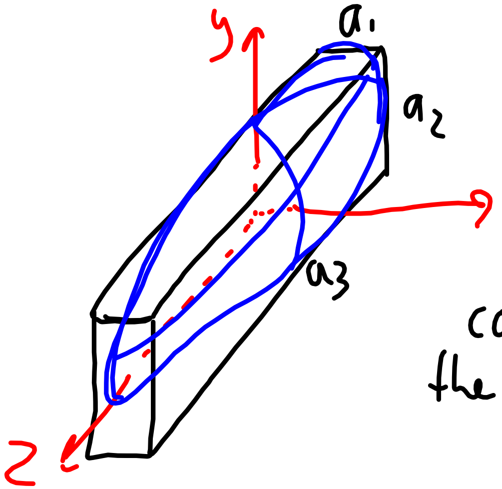
$$J = \frac{1}{2}\underline{x}^T \begin{bmatrix} J_{xx} & & \\ & J_{yy} & \\ & & J_{zz} \end{bmatrix} \underline{x} = \frac{1}{2}\sum_{i=1}^{3} J_{ii} x_i^2$$

This eq on std. form:

$$\boxed{\frac{x_1^2}{2J/J_{11}} + \frac{x_2^2}{2J/J_{22}} + \frac{x_3^2}{2J/J_{33}} = 1}$$

Half axis: $a_i = \sqrt{2J/J_{ii}}$ for $i = 1,2,3$

$a_1 < a_2 < a_3$



axis:
$1 = x$
$2 = y$
$3 = z$

The form of the ellipsoid coinsides with the form of the body

Ellipsoide of inertia represents the form of the body (w.r.t. rotation)

Kinetic rotation energy ellipsoide

Multiply the Euler equations (B-152) (assume $\underline{n}_c^b = 0$, i.e. torque free motion) with ω_x, ω_y and ω_z for line 1, 2 and 3 respectively, and add them up.

$$0 = \sum_{i=1}^{3} J_{ii}\, \dot{w}_i\, w_i \qquad \begin{array}{l} 1=x \\ 2=y \\ 3=z \end{array}$$

Integrate w.r.t time:

$$\int_{t_0}^{t} \sum_{i=1}^{3} J_{ii} \frac{dw_i}{\cancel{dt}} w_i \,\cancel{dt} = \int_{t_0}^{t} 0\, dt = 0$$

$$\Big|_{t_0}^{t} \frac{1}{2} \sum_{i=1}^{3} J_{ii}\, w_i^2(t) = 0$$

$$\frac{1}{2} \sum_{i=1}^{3} J_{ii}\, w_i^2(t) = \frac{1}{2} \sum_{i=1}^{3} J_{ii}\, w_i^2(t_0) = K_0$$

I.e.

$$\boxed{\frac{w_1^2(t)}{2K_0/J_{11}} + \frac{w_2^2(t)}{2K_0/J_{22}} + \frac{w_3^2(t)}{2K_0/J_{33}} = 1}$$

Half axis: $\sqrt{\dfrac{2K_0}{J_{ii}}}$

We see that the kinetic rotationenergy ellipsoid has the same form as the ellipsoide of inertia (can choose $J = K_0$)

$\boxed{\text{$\underline{w}_b^{ib}(t)$ has to stay on the kinetic rot. energy. ellipsoide}}$

Angular momentum ellipsoide (spinnellipsoide)

When the outer force is zero ($\vec{n}_c = \vec{0}$) $\Rightarrow$ $\dot{\vec{h}}_c^{ii} = \vec{0}$

and $\|\vec{h}_c^{i}\| = h_0$ (constant length) and the same direction in $\mathcal{F}^i$.

Angular velocity seen from $\mathcal{F}^b$

$$\underline{h}_c^{ib} = J_c^b \, \underline{w}_b^{ib} = \begin{bmatrix} J_{xx} w_x \\ J_{yy} w_y \\ J_{zz} w_z \end{bmatrix}$$

choose $\{b\}$ in the main axis

the length of $\underline{h}_c^{ib}$ is constant:

$$\left(\underline{h}_c^{ib}\right)^T \underline{h}_c^{ib} = h_0^2$$

$$\sum_{i=1}^{3} J_{ii}^2 w_i^2 = h_0^2$$

This can be witten on std. form of an ellipsoide.

$$\boxed{\frac{w_1^2(t)}{h_0^2/J_{11}^2} + \frac{w_2^2(t)}{h_0^2/J_{22}^2} + \frac{w_3^2(t)}{h_0^2/J_{33}^2} = 1}$$

Half axis: h_0/J_{ii}

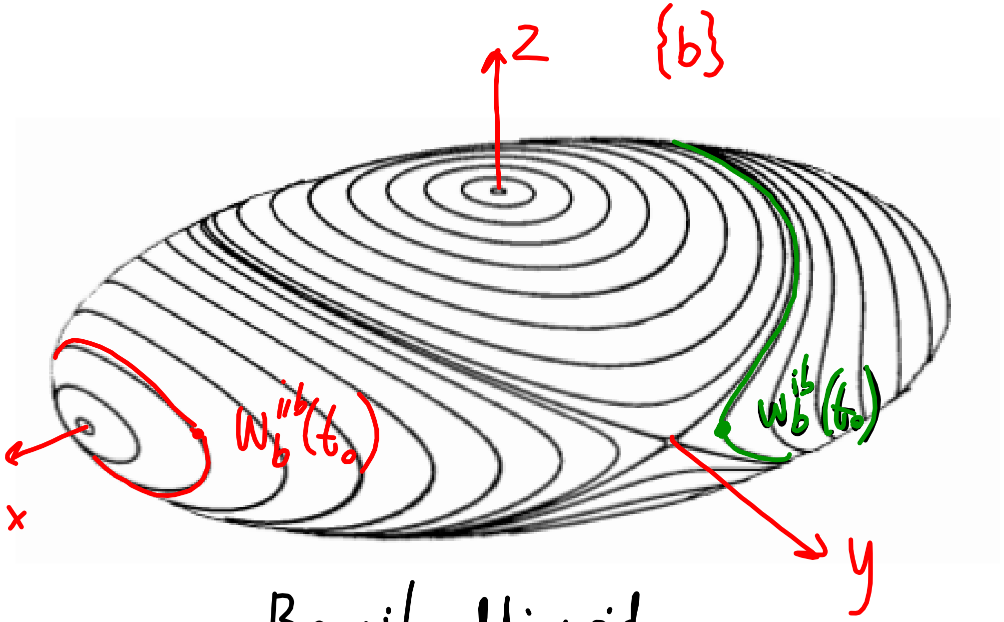
We see that the kin. rot. energy ellipsoide and the angular momentum ellipsoide do not have the same half axis

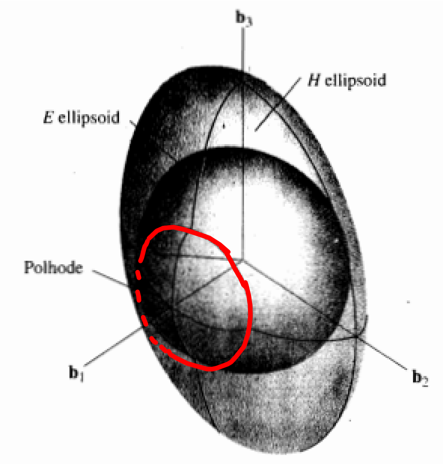
$\Rightarrow$ different form of the ellipsoids and they need to intercect.

$\underline{w}_b^{ib}(t)$ is on the intersection (pole hood) of the two ellipsoids

B.3.1 Beskrivelse av bevegelsen sett fra b-systemet

Teorem B.13 ***Bevegelsen av et stivt legeme sett fra det roterende b-systemet***
Anta b-systemet faller sammen med hovedaksene for det stive legemet. For et stivt legeme som ikke er utsatt for ytre moment beveger vinkelhastighetsvektoren ($\underline{\omega}_b^{ib}$) seg da, sett fra b-systemet, på skjeringa (polhode) mellom spinnellipsoida og den kinetiske rotasjonsenergiellipsoida. Bevegelsen til det stive legemet er i hvert øyeblikk en ren rotasjon om vinkelhastighetsvektoren.

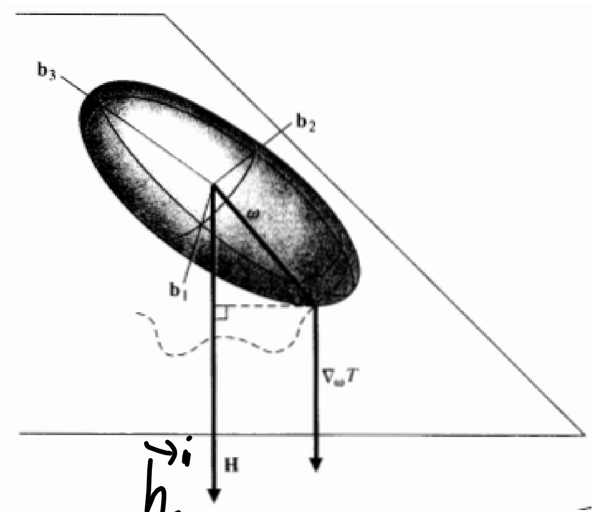




Benoit ellipsoide
(Poinsot?)

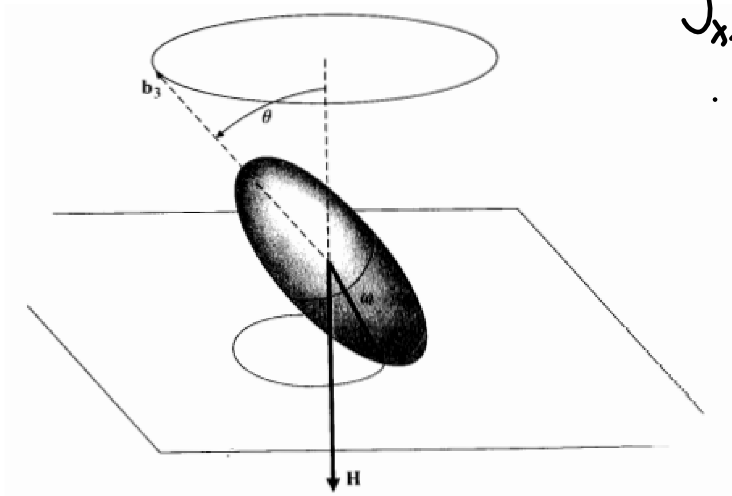
B.3.2 Beskrivelse av bevegelsen sett fra i-systemet

Teorem B.14 ***Bevegelsen av et stivt legeme sett fra treghhetssystemet* i**
Bevegelsen av et stivt legeme som ikke er utsatt for et ytre moment er beskrevet, sett fra treghetsrommet, av at den kinetiske energiellipsoida ruller på det invariable plan (plan $\perp$ spinnvektoren $\vec{h}^{\mathbf{i}}$) uten å gli. Rullinga følger polhodet på den kinetiske energiellipsoida (kontaktpunktet mellom det invariable plan og ellipsoida er dermed enden på $\vec{\omega}_b^{\mathbf{i}}$-vektoren).



$\vec{h}_c^{i}$

constant



$J_{xx} = J_{yy} > J_{zz}$

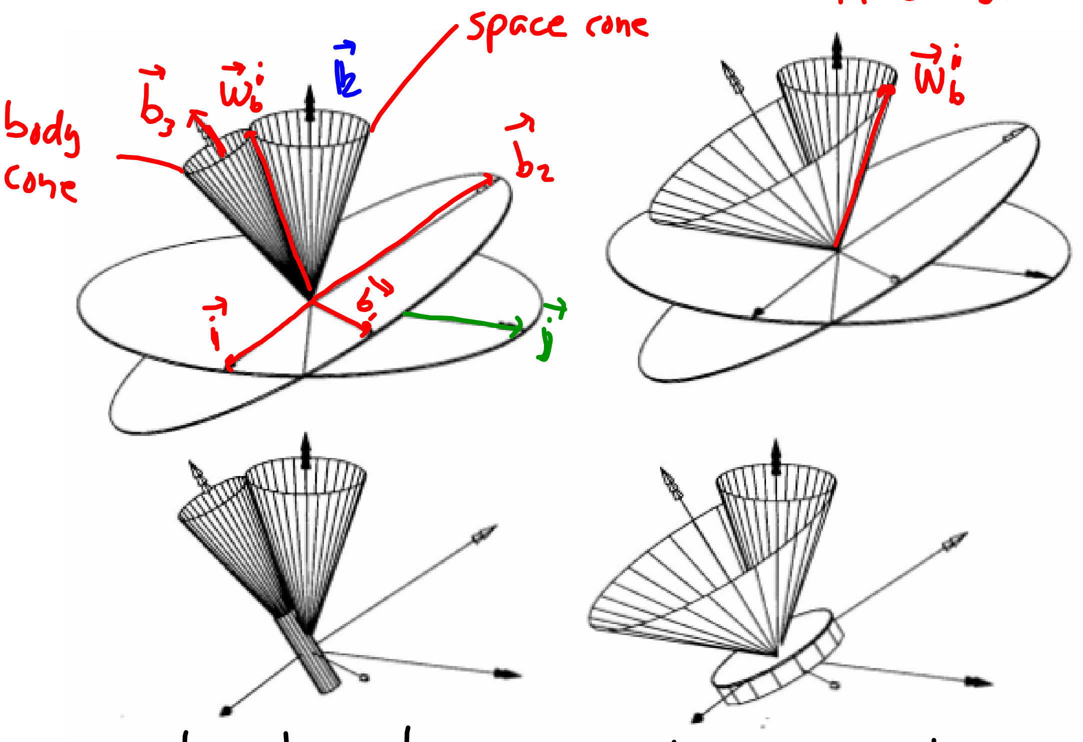
· Axis. symmetrical

B.4 Torque free motion of a axis symmetrical body.

Assume $J_{xx} = J_{yy} \neq J_{zz}$

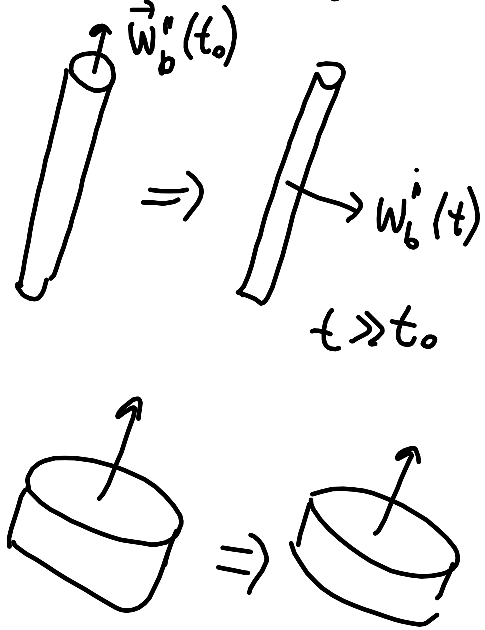
$\mathcal{F}^i = \{\vec{i}, \vec{j}, \vec{k}\}$

$\mathcal{F}^b = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$



$J_{xx} = J_{yy} > J_{zz}$ $J_{xx} = J_{yy} < J_{zz}$

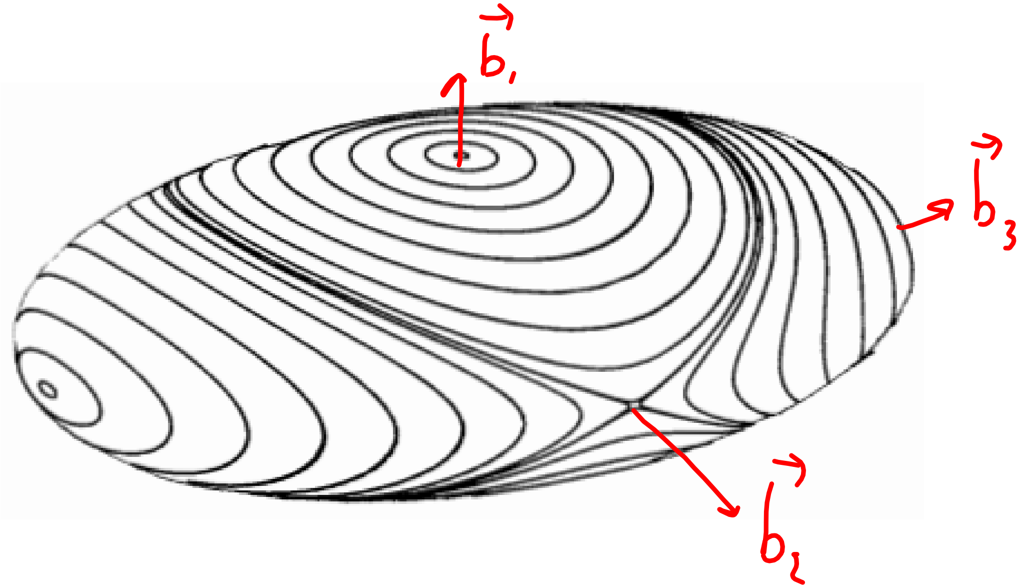
What is the stationary rotation of a cylinder?



B.3.3 Stabilitet om hovedaksene

Vi vil her undersøke stabiliteten til bevegelsen for små perturbasjoner om hovedaksene for et stivt legeme. Ovenfor så vi på bevegelsene i stort.

Teorem B.15 ***Stabiliteten om hovedaksene for et stivt legeme***
Anta b-systemet faller sammen med hovedaksene for det stive legemet og $J^b_{xx} > J^b_{yy} > J^b_{zz}$. Da gir linearisering om $\vec{b}_1$-aksen (J^b_{xx}) eller $\vec{b}_3$-aksen (J^b_{zz}) et lineært system med kompleks konjugerte egenverdier. Linearisering om $\vec{b}_2$-aksen (J^b_{yy}) gir et lineært system med to egenverdier, den ene ligger i venstre halvplan den andre i høgre.



B.3.3 Stability of main axis

1) Rotation around b_1-axis (x-axis)

$\omega_{x0} > 0$
$\alpha_1 > 0$
$\alpha_2 > 0$

Assume $|\omega_x| \gg |\omega_y| \approx |\omega_z|$, set $\omega_y \omega_z \approx 0$

Euler equations under these assumptions:

$$\dot{\omega}_x = (J_{yy} - J_{zz})\, \omega_y \omega_z / J_{xx} = 0 \Rightarrow \omega_x(t) = \omega_{x0}$$

$$\dot{\omega}_y = (J_{zz} - J_{xx})\, \omega_x \omega_z / J_{yy} = \frac{J_{zz} - J_{xx}}{J_{yy}} \omega_z \omega_{x0} = -\alpha_1 \omega_z \Rightarrow \dot{\omega}_y = -\alpha_1 \omega_z$$

$$\dot{\omega}_z = (J_{xx} - J_{yy})\, \omega_x \omega_y / J_{zz} = \frac{J_{xx} - J_{yy}}{J_{zz}} \omega_y \omega_{x0} = \alpha_2 \omega_y \Rightarrow \dot{\omega}_z = \alpha_2 \omega_y$$

$$\begin{bmatrix} \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} 0 & -\alpha_1 \\ \alpha_2 & 0 \end{bmatrix} \begin{bmatrix} \omega_y \\ \omega_z \end{bmatrix}$$

$$\dot{\underline{x}} = A \cdot \underline{x}$$

Eigenvalues:

$$|\lambda I - A| = \begin{vmatrix} \lambda & \alpha_1 \\ -\alpha_2 & \lambda \end{vmatrix} = \lambda^2 + \alpha_1 \alpha_2 = 0$$

$$\lambda_{1,2} = \pm\sqrt{-\alpha_1 \alpha_2} = \pm\sqrt{\alpha_1 \alpha_2}\; i$$