TEK4040 – MATEMATISK MODELLERING AV DYNAMISKE SYSTEMER

Referanser:
- O. Hallingstad: Matematisk modellering av dyn. sys.
- John J. Craig: Robotics (kap 1-6)
- Peter H. Zipfel: Modelling and Simulation of Aerospace Vehicle Dynamics

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Vi ønsker å simulere:

- Satellitten stilling i rommet
- Fly (ikke aerodynamikk)
- Roboter - mekanismer
- Treghetsnavigasjon (TNS) INS
    - * Banegenerator (gir posisjon og stilling) $\xrightarrow{\text{derivere}}$ hastighet, aks. vinkelhast.
    - * Navigasjonsligninger
    - * Ser bare på deterministiske ligninger
    - * Støy tas med i Stokastiske systemer

Vi ønsker å beskrive bevegelsen av fysiske objekter vha.:

Nok for punkt.
- Posisjon (vanligvis massesenteret)
- Hastighet
- Akselerasjon

Utstrakt legeme
- Stilling
- Vinkel hastighet
- Vinkel akselerasjon

DYNAMIKK

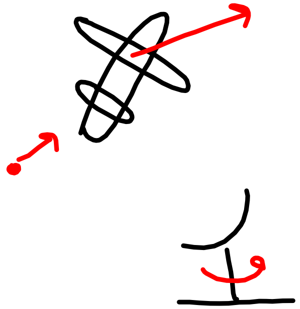
1) Kinematikk : Beskrive bevegelsen matematisk } DEL A

2) Kinetikk : Beskrive sammenhengen mellom bevegelsen og de krefter og momenter som forårsaker bevegelsen. } DEL B

Et stivt legemes bevegelse kan settes sammen av translasjon + rotasjon.

Framgangsmåte for modellering

1. Beskrive det fysiske system, velg objekter



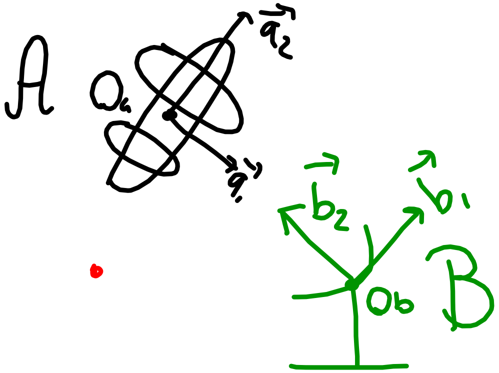
Vi må anta at objektene med tilstrekkelig nøyaktighet kan beskrives som partikler (punkter med masse) og stive legemer (molekylene har en fast stilling i forhold til hverandre)

2. Definer referanserom og treghetsrom (pga kinetikk)

Vi får referanserommet til et legeme ved å ekspandere legemet med punkter som fyller rommet og som ligger fast i forhold til molekylene i legemet.

NB! Referanserom kan defineres ulikt i ulike lærebøker

3. Definere Affine rom (modell av referanserom).



$\mathcal{A}$: affint rom

Objekter: vektorer + punkter

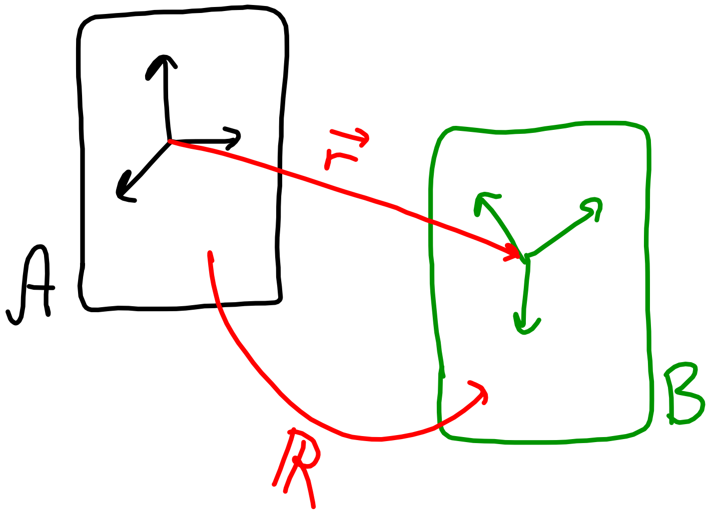
Operasjoner: $P = Q + \vec{v}$, $\vec{v} = P - Q$

4. Innføre (referanse) rammer i det affine rom

$$\mathcal{A}: \quad \mathcal{F}_{\mathcal{A}}^{a} = \{O_a, \vec{a}_1, \vec{a}_2, \vec{a}_3\} = \{a\}$$

$$\mathcal{B}: \quad \mathcal{F}_{\mathcal{B}}^{b} = \{O_b, \vec{b}_1, \vec{b}_2, \vec{b}_3\} = \{b\}$$

5. Beskrive sammenhengen mellom objekter i ulike affine rom (referanserom)



De opprinnelige stive legemer er nå beskrevet av punkter og rammer i affine rom.

Dersom vi har endringer som funksjon av tiden får vi

$\vec{r}(t)$, $\mathbb{R}(t)$

6. Definere avbildning fra affine rom til $\mathbb{R}^n$ (n-abser reelle tall)

Avbildningen gjøres ved å dekomponere vektorer ($\vec{r}$) og operatorer ($\mathcal{S}$) vha basisvektor settene (rammer)

$$\underbrace{\vec{y} = \mathcal{S}\,\vec{x}}_{\text{Affine rom}} \quad \overset{}{\underset{\vec{F}_A^a}{\Longleftrightarrow}} \quad \underbrace{\underline{y}^a = S^a \underline{x}^a}_{\mathbb{R}^n}$$

$\underline{x}^a$, $\underline{y}^a$ er kolonnematriser (kolonnevektorer)

S^a er matrise

7. Innføre tidsavhengige vektorer og operatorer. Definerer derivasjon og integrasjon.

$$\vec{r}(t),\ \mathcal{S}(t) \Longleftrightarrow \underline{r}^a(t),\ S^a(t)$$

$$\dot{\vec{r}}^b(t),\ \dot{\mathcal{S}}^b(t) \Longleftrightarrow \dot{\underline{r}}^{ba}(t),\ \dot{S}^{ba}(t)$$

Nå har vi all matematikken for å beskrive kinematikken

8. Kinetikk: finn sammenhengen mellom krefter og bevegelse

Krefter: modelleres av vektorer

$$\vec{f} \in A(V) \iff \underline{f}^a \in \mathbb{R}^n$$

Momenter: modelleres av vektorer

$$\vec{n} \in A(V) \iff \underline{n}^a \in \mathbb{R}^n$$

Massen til en partikkel eller stivt legeme: m

Newtons 2. lov: $\vec{f} = m \, \vec{a}^{ii} \iff \underline{f}^i = m \cdot \underline{a}^{iii}$

i: treghetsrom

Treghetsmatrisa for stive legemer (tilsvarer masse for partikler for roterende legemer)

$$\vec{n} = \dot{\vec{h}}^i \iff \underline{n}^i = T \, \underline{\dot{w}}^{iii}$$

$\vec{h}$: spinn T: treghetsmatrise

Kommentarer

Våre ligninger blir:

- algebraiske
- ordinære vektordifferensial ligninger (tilstandsrom ligninger)

DEL A: MATEMATISK GRUNNLAG

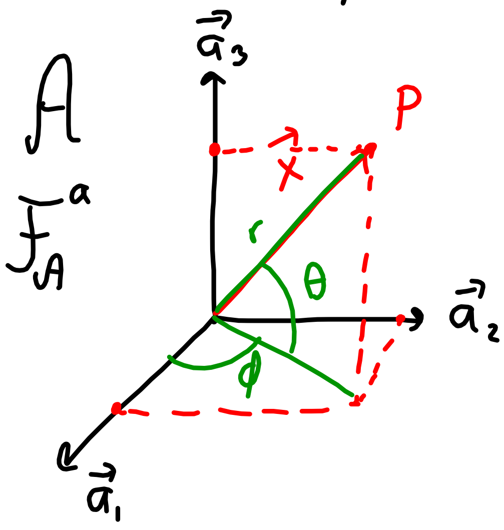
I forelesningene går vi gjennom notatet. Figurer og utledninger tas på tavla. Bygger på: lineær algebra, matriseteori og ord. diff. ligninger

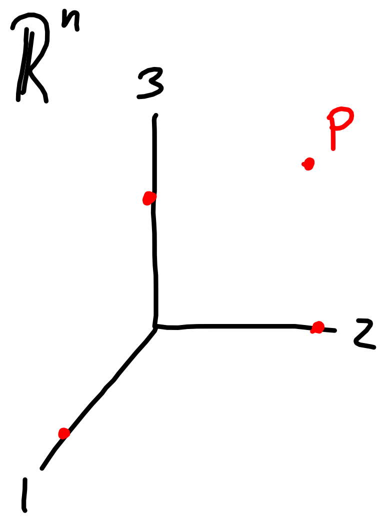
Rom	Rammer	Kommentar
Referanserom		Fysisk rom bestående av punkter som er i ro i forhold til hverandre. Eng: (observational) frame of reference
Treghetsrom		Et referanserom hvor Newtons 2. lov har sin enkleste form, $\vec{f} = m\vec{a}^i$
Vektorrom $\mathcal{V}$	$\mathcal{F}^a_{\mathcal{V}} = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \{\vec{a}_i\} = \{a\}$ ramme a i vektorrom $\mathcal{V}$ med basisvektorer $\vec{a}_i$	Matematisk definert rom med vektorer som objekter.
Affint rom $\mathcal{A}$	$\mathcal{F}^a_{\mathcal{A}} = \{O_a; \vec{a}_1, \vec{a}_2, \vec{a}_3\} = \{a\}$ ramme a i det affine rom $\mathcal{A}$ med origo O_a og basisvektorer $\vec{a}_i$	Matematisk definert rom med punkter og vektorer som objekter. Brukes som modell for referanse- og treghetsrom.

Definisjon A.1 *Et koordinatsystem $\mathcal{C}^a_{\mathcal{A}}$ for et affint rom $\mathcal{A}$ avbilder et punkt P inn i $\mathbb{R}^n$:*

$$\mathcal{C}^a_{\mathcal{A}} : P \to \underline{x}^a_P \text{ hvor } P \in \mathcal{A} \text{ og } \underline{x}^a_P \in \mathbb{R}^n$$
$$\mathcal{C}^a_{\mathcal{A}}(P) = \underline{x}^a_P$$

Koordinatsystemer





$$\underline{x}_p^a = \begin{bmatrix} x_1^a \\ x_2^a \\ x_3^a \end{bmatrix}$$ Kartesiske koordinater

$$\underline{y}_p^a = \begin{bmatrix} r \\ \phi \\ \theta \end{bmatrix}$$ Kule-koordinater

Et koordinatsystem er en funksjon (avbildning) fra et affint rom inn i $\mathbb{R}^n$. Koordinatene er avhengig av både ramma og om vi velger kule-, kartesiske- eller andre koordinater.

A.2 Vektorrom

$\mathcal{A}$: affint rom

$\mathcal{V}$: vektorrom

Jeg vil bruke følgende notasjon og forkortelser i tidsinvariante vektorrom:

$\vec{x}$ $\in \mathcal{A}$ eller $\mathcal{V}$	Geometrisk vektor
$\underline{x}$ $\in \mathbb{R}^n$	Algebraisk vektor (kolonnematrise)
$\mathbf{A}$ $\quad \vec{y} = \mathbf{A}\,\vec{x}$	Operator
$A^q = [\mathbf{A}]^q$ $\quad \underline{y}^q = A^q \underline{x}^q$	Operatoren $\mathbf{A}$ representert i q-systemet
$\{\vec{q}_i\}$	Basissystemet q
$\{\vec{q}_i^*\}$	Det duale basissystem for q
$S(\underline{\omega}^q) \equiv [\vec{\omega}\times]^q$	Matriserepresentasjon av "$\vec{\omega}\times$"-operatoren i q-systemet.
$< \vec{a}, \vec{b} >$	Indreproduktet av $\vec{a}$ og $\vec{b}$.
$\underline{x}^T = [x_1; x_2; \ldots; x_n]$	Transponert vektor (Matlab skrivemåte)
$\underline{x} = [x_i]\,, \quad x_i = [\underline{x}]_i$	Kolonnematrise med generelt element x_i
$D = [d_{ij}]\,, \quad d_{ij} = [D]_{ij}$	Matrise med generelt element d_{ij}
c_φ	$\cos(\varphi)$
s_φ	$\sin(\varphi)$
C_a^b	Retningskosinmatrise
R_a^b	Ortogonal retningskosinmatrise
k.s.	Koordinatsystem
$C_\mathcal{V}^a$	k.s. a i vektorrommet $\mathcal{V}$
$C_\mathcal{A}^a$	k.s. a i det affine rom $\mathcal{A}$
RKM	Retningskosinmatrise
$\mathbb{K}, \mathbb{R}, \mathbb{C}$	Skalarkropp, mengden av reelle tall, mengden av komplekse tall

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$$

$$\underline{x}^b = C_a^b \underline{x}^a$$

A.2.1 Definisjoner

$\mathbb{K}$: Skalarkropp: Definisjon av regneregler for bl.a. reelle og komplekse tall.

$\mathbb{N}$: naturlige tall: $1, 2, 3, \ldots$ danner ikke skalarkropp

$\mathbb{Z}$: heltall $\cdots -2, -1, 0, 1, 2, \cdots$

$\forall$: for alle

$\exists$: det finnes (eksisteres)

$\in$: element i mengden

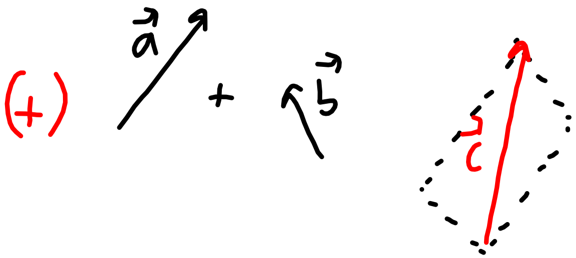
Lineært vektorrom

Et vektorrom defineres over en skalarkropp (bruker tall fra skalarkroppen)

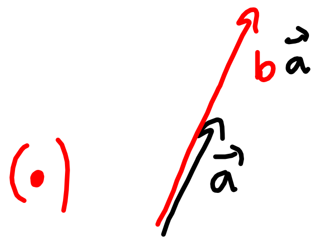
- Vektoraddisjon (+)
- Skalarmultiplikasjon (·)

Eksempler på vektorer

Piler i planet eller rommet:



$\vec{c} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$



$\|\vec{a}\| = a$

$\|\vec{b}\| = b$

Kolonnematriser med dimensjon n (n-tupler av tall)

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$(+)$ $\underline{x} + \underline{y} = [x_i] + [y_i] = [x_i + y_i]$

$(\bullet)$ $a\,\underline{x} = a[x_i] = [a x_i]$

n'te ordens polynomer

$$\vec{x} = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$
$$\vec{y} = b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x + b_0$$

$(+)$ $\vec{x} + \vec{y} = (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \dots + (a_1 + b_1) x + (a_0 + b_0)$

$(\bullet)$ $c\vec{x} = c a_n x^n + c a_{n-1} x^{n-1} + \dots + c a_1 x + c a_0$

Basis

Lineært uavhengige vektorer $\{\vec{q}_i\}$

hvis og bare hvis

$$a_1\vec{q}_1 + a_2\vec{q}_2 + \dots + a_n\vec{q}_n = 0 \quad \forall\, \vec{q}_i \neq 0 \iff \text{alle } a_i = 0$$

Gitt basisen $\{\vec{q}_i\} \in V$ da kan enhver vektor $\vec{v} \in V$ skrives entydig som: $\vec{v} = v_1\vec{q}_1 + v_2\vec{q}_2 + \dots + v_n\vec{q}_n$

Indreprodukt

Brukes bl. a. til å beregne lengden av vektorer, vinkel mellom vektorer, ortogonalitet og for å projisere en vektor ned på en annen vektor.

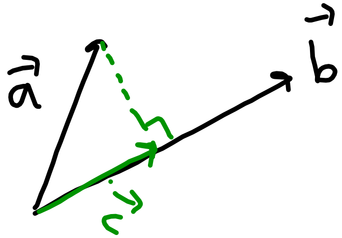
Eksempler

Algebraiske vektorer

$\underline{x}, \underline{y} \in \mathbb{R}^n \quad \langle \underline{x}, \underline{y} \rangle = \underline{x}^T \underline{y} = [x_1\ x_2 \dots x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_n + \dots + x_n y_n$

Geometriske vektorer

$\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \|\vec{b}\| \cos \angle \vec{a}\vec{b}$



$\|\vec{c}\| = \|\vec{a}\| \cos \angle \vec{a}\vec{b}$

Dersom $\|\vec{b}\| = b = 1$ gir $\vec{b} \langle \vec{a}, \vec{b} \rangle = \vec{c}$ Projisere $\vec{a}$ ned på $\vec{b}$

$\|\vec{a}\| = \langle \vec{a} . \vec{a} \rangle^{1/2} = a$ normen/lengden av $\vec{a}$

$|a|$: tallverdien til a

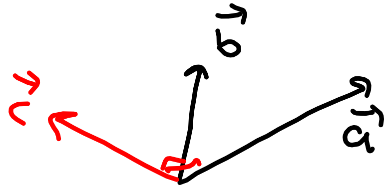
$|a| = |-a|$

Kryssprodukt

NB! $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$, $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

Eksempler

- Geometriske vektorer



$\vec{c} = \vec{a} \times \vec{b}$ $\quad \|\vec{c}\| = \left| \|\vec{a}\| \, \|\vec{b}\| \sin \angle \vec{a}\vec{b} \right|$

- Algebraiske vektorer

$$\underline{c}^q = \underline{a}^q \times \underline{b}^q = S(\underline{a}^q)\, \underline{b}^q$$

hvor $\{\vec{q}_i\}$ er ortogonal med enhets lengde

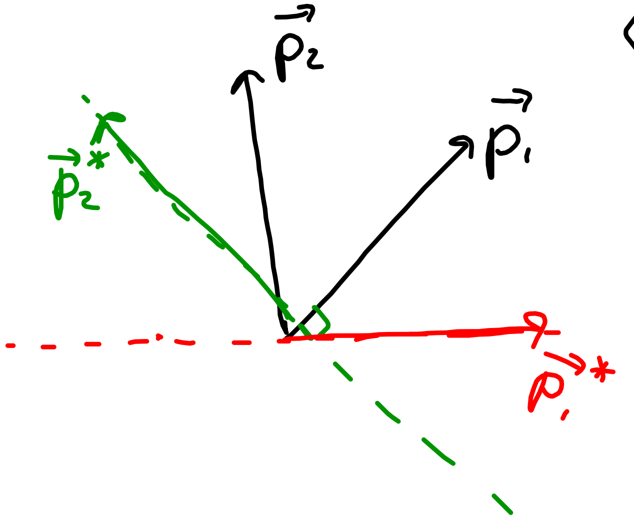
$$S(\underline{d}) = \begin{bmatrix} 0 & -d_3 & d_2 \\ d_3 & 0 & -d_1 \\ -d_2 & d_1 & 0 \end{bmatrix} : \text{Skjevsymmetrisk form av } \underline{d}$$

Def. A.9 Dual basis

Brukes til å dekomponere vektorer vha en gitt basis (ramme)

NB! Dual basis forutsetter ikke ortogonale basisvektorer

Definisjon: Den duale basisen $\{\vec{p}_i^{\,*}\}$ til basis $\{\vec{p}_i\}$ er def. ved:



$$\langle \vec{p}_i, \vec{p}_j^{\,*} \rangle = \delta_{ij} \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}, \quad \delta_{ij} \text{ kalles kronecker deltaet}$$

$$\langle \vec{p}_1, \vec{p}_1^{\,*} \rangle = 1, \quad \langle \vec{p}_2, \vec{p}_2^{\,*} \rangle = 1$$

$$\langle \vec{p}_1, \vec{p}_2^{\,*} \rangle = 0, \quad \langle \vec{p}_2, \vec{p}_1^{\,*} \rangle = 0 \quad \text{dvs } \vec{p}_1^{\,*} \perp \vec{p}_2$$

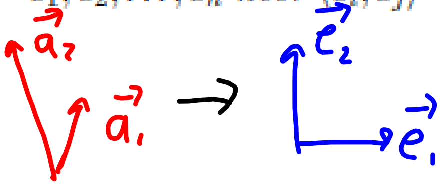
Vi skal senere vise at $\vec{v} = \sum_{i=1}^{n} v_i^p \vec{p}_i$ hvor $v_i^p = \langle \vec{v}, \vec{p}_i^{\,*} \rangle$

$$\Rightarrow \underline{v}^p = \begin{bmatrix} v_1^p \\ v_2^p \\ v_3^p \end{bmatrix}$$

F.3

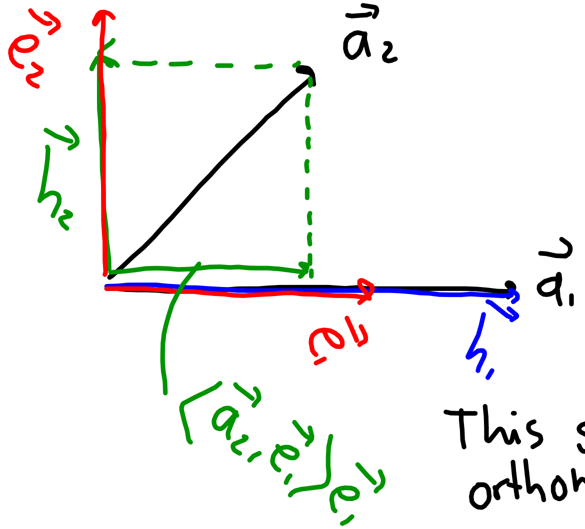
Exame date: Mon 14. Dec

Teorem A.1 ***Grahm-Schmidt ortogonalisering.***
Dersom vi har et sett med basisvektorer $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$ kan vi lage et ortonormalt sett av basisvektorer $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ hvor $\langle \vec{e}_i, \vec{e}_j \rangle = \delta_{ij}$ på følgende måte:



$$\left.\begin{aligned} \vec{h}_k &= \vec{a}_k - \sum_{i=1}^{k-1} \langle \vec{a}_k, \vec{e}_i \rangle \, \vec{e}_i \\ \vec{e}_k &= \vec{h}_k \left\| \vec{h}_k \right\|^{-1} \end{aligned}\right\} k = 1, 2, \ldots, n \qquad \text{(A- 3)}$$

Example n=2



$$\vec{h}_1 = \vec{a}_1 - \sum_{i=1}^{0} \langle \vec{a}_1, \vec{e}_i \rangle \vec{e}_i = \vec{a}_1$$

$$\vec{e}_1 = \vec{h}_1 / \|\vec{h}_1\| \implies \|\vec{e}_1\| = 1$$

$$\vec{h}_2 = \vec{a}_2 - \sum_{i=1}^{1} \langle \vec{a}_2, \vec{e}_i \rangle \vec{e}_i = \vec{a}_2 - \langle \vec{a}_2, \vec{e}_1 \rangle \vec{e}_1$$

$$\vec{e}_2 = \vec{h}_2 / \|\vec{h}_2\| \implies \|\vec{e}_2\| = 1$$

This shows that we can always create an orthonormal (o.n.) set of basis vectors

A.2.2 Matrix representation of geometrical vectors

Problem: Given a geometrical vector $\vec{r}$ and a basis (frame) $\{\vec{p}_i\}$, what is the algebraic vector $\underline{r}^p$

Theorem A.2 Column representation (algebraic vector) av $\vec{r} \in V$:

$$\vec{r} = r_1^p \vec{p}_1 + r_2^p \vec{p}_2 + \dots + r_n^p \vec{p}_n = \sum_{i=1}^{n} r_i^p \vec{p}_i$$

where $r_i^p = \langle \vec{r}, \vec{p}_i^{\,*} \rangle$

$$\underline{r}^p = \begin{bmatrix} r_1^p \\ r_2^p \\ \vdots \\ r_n^p \end{bmatrix} = \left[r_i^p \right]$$

Proof

From linear algebra we can clearly write $\vec{r} = \sum_{i=1}^{n} r_i^p \vec{p}_i$

We first calculate the dual basis $\{\vec{p}_i^{\,*}\}$ wher $\langle \vec{p}_i, \vec{p}_j^{\,*} \rangle = \partial_{ij}$

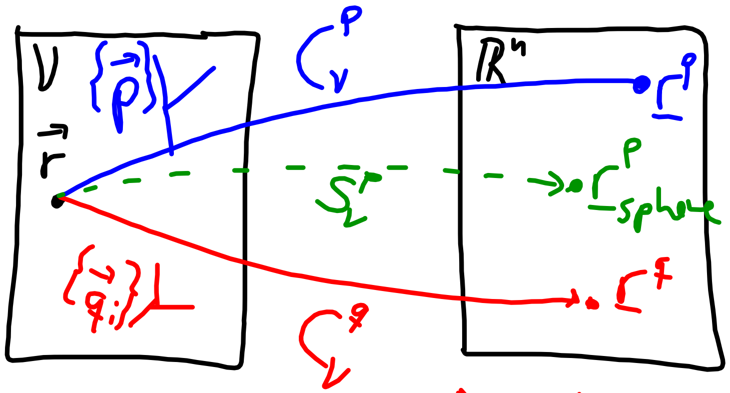
Take the inner product of $\vec{r}$ with the dual basisvector $\vec{p}_j^{\,*}$

$$\langle \vec{r}, \vec{p}_j^{\,*} \rangle = \left\langle \sum_{i=1}^{n} r_i^p \vec{p}_i, \vec{p}_j^{\,*} \right\rangle = \sum_{i=1}^{n} r_i^p \underbrace{\langle \vec{p}_i, \vec{p}_j^{\,*} \rangle}_{\partial_{ij}} = r_j^p$$

I.e. if we switch the index:

$$r_i^p = \langle \vec{r}, \vec{p}_i^{\,*} \rangle$$

i.e. given $\vec{r}$ and $\{\vec{p}_i\}$ then $\underline{r}^p = [\langle \vec{r}, \vec{p}_i^{\,*} \rangle]$ $\qquad \vec{r} \overset{F_V^p}{\Longleftrightarrow} \underline{r}^p$



Independent of choise of basis-vektors, c.s.

Dependent on choise of basisvectors and type of c.s. (cartesian, spherical, polar, ...)

V: n-dimentional vector space

$\mathbb{R}^n$: $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$ n-dim

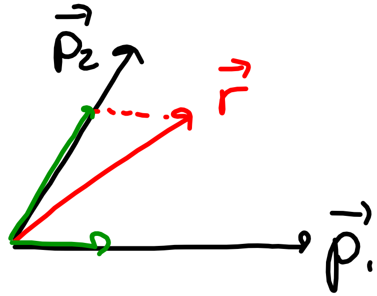
n-dimentional space of numbers $\in \mathbb{R}$

C_V^p: coordinate system (c.s)

If the basisvectors are orthonormal (o.n)

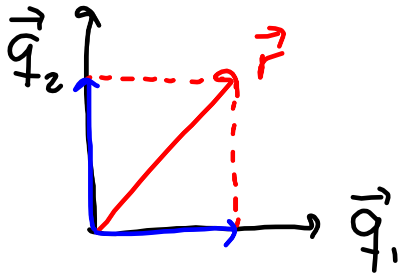
$$\langle p_i, p_j \rangle = \delta_{ij} = \langle \vec{p}_i, \vec{p}_j^{\,*} \rangle$$

$$\Rightarrow \vec{p}_j = \vec{p}_j^{\,*}$$



Low of parallelogram

Matematically we can use $\{\vec{p}_i\}$ and $\{\vec{p}_i^{\,*}\}$

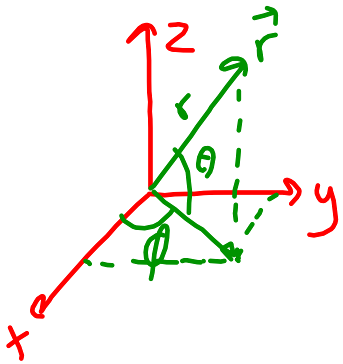


Projecting into the basis vectors

Matematically we use $\{\vec{q}_i\}$

When using polar og spherical coordinates we also need to spesify a frame (basis).

$$\mathcal{C}_v^p(\vec{r}) = \underline{r}^p, \quad \mathcal{K}_v^p(\vec{r}) = \begin{bmatrix}\phi\\ \theta\\ r\end{bmatrix}$$



We see that the coordinates a coordinate function (C_v^p, K_v^p) gives depends on the frame $\vec{\mathcal{F}}_v^p$ and how the coordinates are calculated (cartesian, spherical, polar,).

A.2.3 Matrix representation of linear operators

An operator is a function that given a vector calculates a new one:

$$\mathbb{O}(\vec{r}) = \vec{a} \qquad V \to V$$

We are going to look espesially at linear operators

Def. A.10 Linear operator

The operator $\mathbb{A}$ is linear $\iff \forall \vec{x}, \vec{y} \in V$ and $\forall a, b \in \mathbb{R}$ we can write:

$$\boxed{\mathbb{A}(a\vec{x} + b\vec{y}) = a\mathbb{A}\vec{x} + b\mathbb{A}\vec{y}}$$

Example: $\vec{w} \times \vec{r}$, $\mathbb{A} = \vec{w}\times$

$$\mathbb{A}(a\vec{x} + b\vec{y}) = \vec{w} \times (a\vec{x} + b\vec{y}) = \vec{w} \times a\vec{x} + \vec{w} \times b\vec{y}$$

$$= a\underbrace{\vec{w}\times}_{\mathbb{A}}\vec{x} + b\underbrace{\vec{w}\times}_{\mathbb{A}}\vec{y}$$

$$= a\mathbb{A}\vec{x} + b\mathbb{A}\vec{y}$$

I.e. cross product "$\vec{w}\times$" is a linear operator

Theorem A.3 Matrix representation of a linear operator

Given $\{\vec{p}_i\} \in V$, then any linear operator $\mathcal{A}$ can be clearly represented in $\mathbb{R}^{n\times n}$ as a matrix A^p. We have that:

$$\boxed{\begin{gathered} A^p = [a^p_{ij}] = \left[\langle \mathcal{A}\vec{p}_j, \vec{p}_i^{\,*} \rangle \right] \\ \vec{y} = \mathcal{A}\,\vec{x} \underset{\mathcal{F}_V^p}{\Longleftrightarrow} \underline{y}^p = A^p \underline{x}^p \end{gathered}}$$

Proof of the theorem: Given $\mathcal{F}_V^p = \{\vec{p}_i\}$, $\vec{y} = \mathcal{A}\,\vec{x}$

We know: $\vec{y} = \sum_{i=1}^{n} y_i^p \vec{p}_i$, $\vec{x} = \sum_{j=1}^{n} x_j^p \vec{p}_j$, $\vec{y} = A\vec{x}$

$$y_i^p = \langle \vec{y}, \vec{p}_i^{\,*} \rangle = \langle A\vec{x}, \vec{p}_i^{\,*} \rangle$$

$$= \langle A \left(\sum_{j=1}^{n} x_j^p \vec{p}_j\right), \vec{p}_i^{\,*} \rangle$$

$$= \langle \sum_{j=1}^{n} x_j^p A \vec{p}_j , \vec{p}_i^{\,*} \rangle$$

$$= \sum_{j=1}^{n} x_j^p \underbrace{\langle A\vec{p}_j, \vec{p}_i^{\,*} \rangle}_{a_{ij}^p}$$

$$y_i^p = \sum_{j=1}^{n} x_j^p a_{ij}^p$$

$$\underline{y}^p = \begin{bmatrix} y_1^p \\ y_2^p \\ y_3^p \end{bmatrix} = \begin{bmatrix} a_{11}^p & a_{12}^p & a_{13}^p \\ a_{21}^p & a_{22}^p & a_{23}^p \\ a_{31}^p & a_{32}^p & a_{33}^p \end{bmatrix} \begin{bmatrix} x_1^p \\ x_2^p \\ x_3^p \end{bmatrix}$$

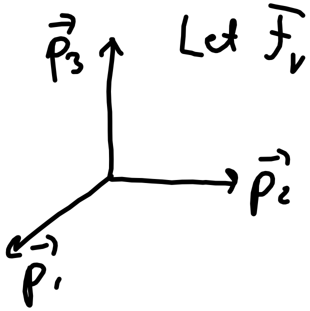
$$\underline{y}^p = A^p \quad \underline{x}^p$$

$$\underline{y}^p = A^p \underline{x}^p$$

where $A^p = [a_{ij}^p] = [\langle A\vec{p}_j, \vec{p}_i^{\,*} \rangle]$

q.e.d

Example A.4 Matrix representation of the "$\vec{w}\times$" operator



Let $\mathcal{F}_v^p$ be orthonormal

$$\vec{w} = w_1^p \vec{p}_1 + w_2^p \vec{p}_2 + w_3^p \vec{p}_3$$

$$\underline{w} = [w_1^p;\ w_2^p;\ w_3^p]$$

$$\mathcal{A}\vec{a} = \vec{w} \times \vec{a}$$

The def. of cross product shows that "$\vec{w}\times$" is a linear operator (shown before)

i.e. $$A^p = \left[\langle \mathcal{A}\vec{p}_j, \vec{p}_i^{\,*} \rangle\right] \underset{\text{o.n}}{=} \left[\langle \vec{w}\times\vec{p}_j, \vec{p}_i \rangle\right]$$

$$= \left[\langle (w_1\vec{p}_1 + w_2\vec{p}_2 + w_3\vec{p}_3)\times\vec{p}_j, \vec{p}_i \rangle\right]$$

$$\vec{p}_i \times \vec{p}_i = 0 \quad i = 1,2,3$$

$$\vec{p}_1 \times \vec{p}_2 = \vec{p}_3 = -\vec{p}_2 \times \vec{p}_1$$

$$\vec{p}_1 \times \vec{p}_3 = -\vec{p}_2 = -\vec{p}_3 \times \vec{p}_1$$

$$\vec{p}_2 \times \vec{p}_3 = \vec{p}_1 = -\vec{p}_3 \times \vec{p}_2$$

$$\langle \vec{p}_i, \vec{p}_i \rangle = 1, \quad i = 1,2,3$$

$$\langle \vec{p}_i, \vec{p}_j \rangle = 0, \quad i \neq j$$

$$A^p = \begin{array}{c|ccc} & j=1 & j=2 & j=3 \\ \hline i=1 & 0 & -w_3 & w_2 \\ i=2 & w_3 & 0 & -w_1 \\ i=3 & -w_2 & w_1 & 0 \end{array}$$

$$A^p = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} = S(\underline{w}^p)$$

$$S + S^T = 0$$

Skew symmetrical form

Note that the basisvectors is o.n.

$$\boxed{\vec{b} = \vec{w} \times \vec{a} \xLeftrightarrow{\mathcal{F}_p} \underline{b}^p = S(\underline{w}^p)\,\underline{a}^p}$$

We can now write geometrical equations with geometrical vectors and operators with algebraical equations of matrices.

F4

$$\vec{x} \overset{\mathcal{F}_v^a}{\longleftrightarrow} \underline{x}^a$$

$$\mathbb{A} \overset{\mathcal{F}_v^b}{\longleftrightarrow} A^b$$

$$\vec{w}\times \overset{\mathcal{F}_v^a}{\longleftrightarrow} S(\underline{w}^a)$$

o.n. basis vectors

New exam date: Tue 15th Dec.

Note: $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

$\vec{a}\to\underline{a}^p, \quad \vec{b}\to\underline{b}^p, \quad \vec{c}\to\underline{c}^p$

$$(\underline{a}^p\times\underline{b}^p)\times\underline{c}^p = S\big(S(\underline{a}^p)\,\underline{b}^p\big)\,\underline{c}^p$$

$$\underline{a}^p\times(\underline{b}^p\times\underline{c}^p) = S(\underline{a}^p)\big(S(\underline{b}^p)\,\underline{c}^p\big) = S(\underline{a}^p)\,S(\underline{b}^p)\,\underline{c}^p$$

A.2.4 Matriserepresentasjon ved bytte av basisvektorer

Problem A.3 Bestem sammenhengen mellom matriserepresentasjonene av vektoren $\vec{r}$ og operatoren $\mathbf{A}$ i hhv q- og p-systemet. Dvs sammenhengen mellom $\underline{r}^q$ og $\underline{r}^p$, A^q og A^p

Teorem A.4 ***Matriserepresentasjon ved bytte av basisvektorer.***
Gitt to basissystemer $\{\vec{q}_i\}$ og $\{\vec{p}_i\}$ i vektorrommet $\mathcal{V}$. La $\vec{r}$ og $\mathbf{A}$ være hhv en vektor og en lineær operator i $\mathcal{V}$. Da har vi følgende sammenhenger mellom matriserepresentasjonene i de to basissystemene :

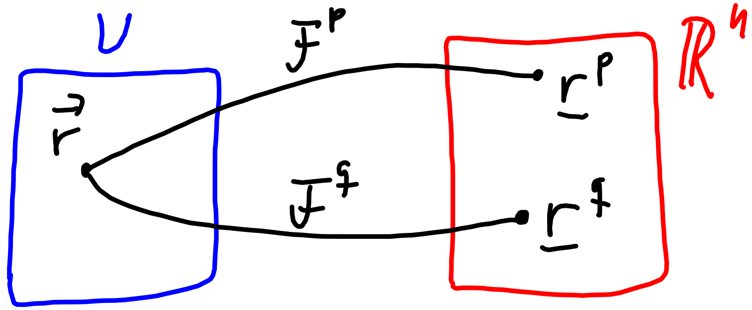
$$\underline{r}^q = C^q_p \underline{r}^p \quad hvor \quad C^q_p = [(\vec{p}_j, \vec{q}_i)] \tag{A- 10}$$

$$C^q_p = \left[C^p_q\right]^{-1}$$

$$\underline{r}^p = C^p_q \underline{r}^q \quad hvor \quad C^p_q = [(\vec{q}_j, \vec{p}_i)] \tag{A- 11}$$

$$A^q = C^q_p A^p C^p_q \text{ og } A^p = C^p_q A^q C^q_p \tag{A- 12}$$

C^q_p og C^p_q kalles retningskosinmatriser (RKM). (Vi skal senere se at den kan brukes i mange sammenhenger og har navn deretter. Ovenfor brukes den som en koordinattransformasjonsmatrise, KTM.)



Proof: $\underline{r}^q = C_p^q \, \underline{r}^p$

$$\vec{r} = \sum_{j=1}^{n} r_j^q \vec{q}_j = \sum_{j=1}^{n} r_j^p \vec{p}_j$$

$$r_i^q = \langle \vec{r}, \vec{q}_i^{\,*} \rangle = \left\langle \sum_{j=1}^{n} r_j^q \vec{q}_j , \vec{q}_i^{\,*} \right\rangle = \left\langle \sum_{j=1}^{n} r_j^p \vec{p}_j , \vec{q}_i^{\,*} \right\rangle$$

$$= \sum_{j=1}^{n} r_j^q \underbrace{\langle \vec{q}_j , \vec{q}_i^{\,*} \rangle}_{\delta_{ij}} = \sum_{j=1}^{n} r_j^p \langle \vec{p}_j , \vec{q}_i^{\,*} \rangle$$

$$r_i^q = \sum_{j=1}^{n} \langle \vec{p}_j , \vec{q}_i^{\,*} \rangle \, r_j^p$$

$$\underbrace{\begin{bmatrix} \vdots \\ r_i^q \\ \vdots \end{bmatrix}}_{\underline{r}^q} = \underbrace{\begin{bmatrix} & & & \langle \vec{p}_n, \vec{q}_i^{\,*} \rangle \\ \langle \vec{p}_1, \vec{q}_i^{\,*} \rangle & \langle \vec{p}_2, \vec{q}_i^{\,*} \rangle & \cdots & \langle \cdot, \cdot \rangle \\ & & & \end{bmatrix}}_{C_p^q} \underbrace{\begin{bmatrix} r_1^p \\ \vdots \\ r_i^p \\ \vdots \\ r_n^p \end{bmatrix}}_{\underline{r}^p}$$

$$C_p^q = \begin{array}{c} \xrightarrow{\quad j \quad} \\ \begin{bmatrix} \langle \vec{p}_1, \vec{q}_1^{\,*} \rangle & \langle \vec{p}_2, \vec{q}_1^{\,*} \rangle & \cdots \\ \langle \vec{p}_1, \vec{q}_2^{\,*} \rangle & & \\ \vdots & & \\ \langle \vec{p}_1, \vec{q}_n^{\,*} \rangle & & \end{bmatrix} \end{array} \; i \downarrow$$

$$\Rightarrow \boxed{\underline{r}^q = C_p^q \, \underline{r}^p, \quad C_p^q = \left[\langle \vec{p}_j , \vec{q}_i^{\,*} \rangle \right]}$$

Proof A-12

$$A^q = C_p^q A^p C_q^p$$

$\mathcal{F}_v^q$: (1) $\underline{y}^q = A^q \underline{x}^q$ | (2) $\underline{y}^q = C_p^q \underline{y}^p$

$\mathcal{F}_v^p$: (3) $\underline{y}^p = A^p \underline{x}^p$ | (4) $\underline{x}^p = C_q^p \underline{x}^q$

$$\underline{y}^q = A^q \underline{x}^q = C_p^q A^p \underbrace{C_q^p \underline{x}^q}_{}$$

$$\underbrace{C_p^q A^p \underline{x}^p}_{}$$

$$\underbrace{C_p^q \underline{y}^p}_{\underline{y}^q}$$

$$\underline{y}^q \overset{(1)}{=} A^q \underline{x}^q \overset{(2)}{=} C_p^q \underline{y}^p \overset{(3)}{=} C_p^q A^p \underline{x}^p \overset{(4)}{=} \underbrace{C_p^q A^p C_q^p}_{A^q} \underline{x}^q$$

$$\boxed{A^q = C_p^q A^p C_q^p}$$

Similarity transformation

Eksempel A.5 Teorem A.5 ***RKM for to ortonormale basissystem***

Dersom vi har to ortonormale basisvektorsettet $\{\vec{q}_i\}$ og $\{\vec{p}_i\}$, dvs

$$\begin{aligned} \langle \vec{q}_i, \vec{q}_j \rangle &= \delta_{ij} \\ \langle \vec{p}_i, \vec{p}_j \rangle &= \delta_{ij} \end{aligned} \tag{A- 13}$$

så vil de duale basissystema være lik basissystema

$$\begin{aligned} \vec{q}_i &= \vec{q}_i^{\,*}, \quad i = 1, 2, \ldots, n \\ \vec{p}_i &= \vec{p}_i^{\,*}, \quad i = 1, 2, \ldots, n \end{aligned} \tag{A- 14}$$

Dette gir

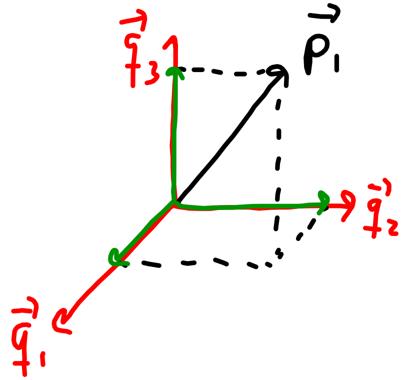
$$C_p^q = [\langle \vec{p}_j, \vec{q}_i \rangle] = [\cos(\angle \vec{p}_j \vec{q}_i)] = R_p^q \tag{A- 15}$$

$n=3$

Dette viser hvorfor C_p^q kalles en ***retningskosinmatrise****. Vi vil innføre en spesiell notasjon i dette tilfellet med ortonormale basissystemer og betegner en ortonormal RKM med R_p^q.*

$$R_p^q = \begin{bmatrix} \cos\angle\vec{p}_1\vec{q}_1 & \cos\angle\vec{p}_2\vec{q}_1 & \cos\angle\vec{p}_3\vec{q}_1 \\ \cos\angle\vec{p}_1\vec{q}_2 & \cos\angle\vec{p}_2\vec{q}_2 & \cos\angle\vec{p}_3\vec{q}_2 \\ \cos\angle\vec{p}_1\vec{q}_3 & \cos\angle\vec{p}_2\vec{q}_3 & \cos\angle\vec{p}_3\vec{q}_3 \end{bmatrix} = \begin{bmatrix} \underline{p}_1^q & \underline{p}_2^q & \underline{p}_3^q \end{bmatrix}$$

See that ith column in R_p^q represents the unit vector $\vec{p}_i$ in the q-frame



Example (Oppg A.1) Inner product in $\mathbb{R}^n$ o.n. basis vectors

$$\langle \vec{a}, \vec{b} \rangle = \left\langle \sum_{i=1}^{n} a_i^p \vec{p}_i , \sum_{j=1}^{n} b_j^p \vec{p}_j \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^p b_j^p \underbrace{\langle \vec{p}_i , \vec{p}_j \rangle}_{\partial_{ij}} = \sum_{i=1}^{n} a_i^p b_i^p = (\underline{a}^p)^T \underline{b}^p$$

$$[a_1 \; a_2 \; a_3]\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$n=3 \quad \langle \vec{a}, \vec{b} \rangle = (\underline{a}^p)^T \underline{b}^p = \|\vec{a}\| \, \|\vec{b}\| \cos \angle \vec{a}\vec{b}$$

Teorem A.6 ***RKM R_p^q er en ortogonal matrise***
Retningskosinmatrisa mellom to rammer som begge har ortonormale basisvektorer, R_p^q, er en ortonormal matrise. Dvs

$$(R_p^q)^{-1} = (R_p^q)^T \tag{A- 16}$$

Proof:

$$R_p^q = [\underline{p}_1^q \; \underline{p}_2^q \; \underline{p}_3^q], \quad (R_p^q)^T = \begin{bmatrix} (\underline{p}_1^q)^T \\ (\underline{p}_2^q)^T \\ (\underline{p}_3^q)^T \end{bmatrix} \Rightarrow (R_p^q)^T R_p^q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{I} = (R_p^q)^{-1} R_p^q$$

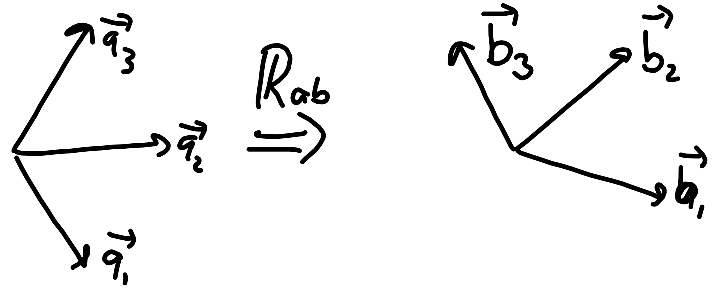
$$(\underline{p}_i^q)^T \underline{p}_j^q = \langle \vec{p}_i, \vec{p}_j \rangle = \partial_{ij}$$

$$\Rightarrow (R_p^q)^{-1} = (R_p^q)^T$$

A2.5 Matrix representation of the rotation operator

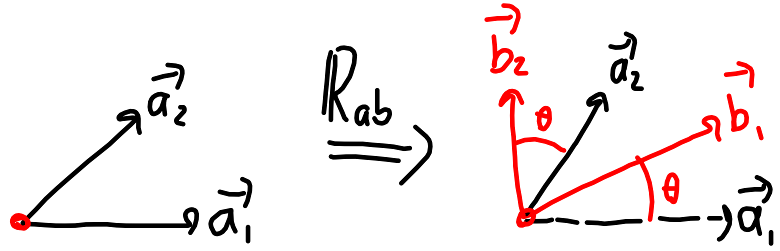
Def. A.11 A rotation operator is a linear operator $\mathbb{R}_{ab} : V \to V$ defined by: $\vec{b}_i = \mathbb{R}_{ab}\,\vec{a}_i \;,\; i = 1, 2, \dots, n$

3D)



All basis vectors are rotated the same angle around the same rotational axes

2D)



Question: What is the representation of $\mathbb{R}_{ab}$ in $\mathcal{F}_v^a$ and $\mathcal{F}_v^b$.
I.s. what is $[\mathbb{R}_{ab}]^a$ and $[\mathbb{R}_{ab}]^b$

Theorem A.7 Matrix rep. of $\mathbb{R}_{ab}$ in the frames $\mathcal{F}_v^a$ and $\mathcal{F}_v^b$ is:

$$\boxed{[\mathbb{R}_{ab}]^a = [\mathbb{R}_{ab}]^b = C_b^a}$$

NB! $\mathbb{R}_{ab} \Leftrightarrow C_b^a$

Define: $R_{ab}^a = [\mathbb{R}_{ab}]^a = C_b^a$
$R_{ab}^b = [\mathbb{R}_{ab}]^b = C_b^a$

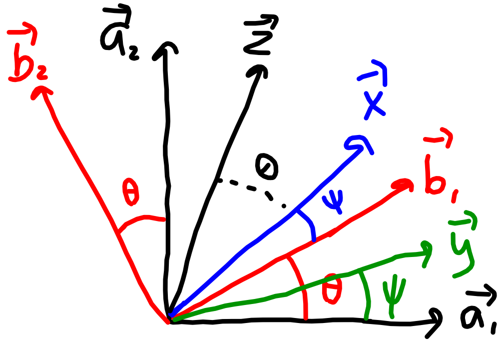
Differentiate between physically rotate a vector ($\mathbb{R}_{ab} \to C_b^a$) and to transform a vector between different frames C_b^a

Proof:

$$[\mathbb{R}_{ab}]^a = R^a_{ab} = \left[\langle \mathbb{R}_{ab}\vec{a}_j, \vec{a}_i^{\,*}\rangle\right] = \left[\langle \vec{b}_j, \vec{a}_i^{\,*}\rangle\right] = C^a_b$$

$$[\mathbb{R}_{ab}]^b = R^b_{ab} = C^b_a R^a_{ab} C^a_b = \underbrace{C^b_a C^a_b}_{I} C^a_b = C^a_b$$

Illustration of the rotation operator



R_{ab} : rotate $a \to b$ an angle θ around axis 3 ($\vec{a}_3$)

Assume $\{\vec{a}_i\}$ is o.n. $\Rightarrow$ $\{\vec{b}_i\}$ is o.n and $\|\vec{x}\| = 1$

$$\underline{x}^b = \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} \qquad \underline{x}^a = \begin{bmatrix} \cos(\theta+\psi) \\ \sin(\theta+\psi) \end{bmatrix} \qquad \underline{x}^a = C^a_b \underline{x}^b \overset{\text{o.n.}}{=} R^a_b \underline{x}^b$$

Def: $\underline{y}^a = \underline{x}^b$

$\underline{x}^a = R^a_b \underline{x}^b = R^a_b \underline{y}^a$

$\underline{x}^a = R^a_{ba} \underline{y}^a$

$\vec{x} = R_{ab} \vec{y}$

Def: $\underline{z}^b = \underline{x}^a$

$\underline{z}^b = R^a_b \underline{x}^b$

$\underline{z}^b = R^b_{ab} \underline{x}^b$

$\vec{z} = R_{ab} \vec{x}$

R^a_b works as a rot. opr. when used in the same frame. The rotation is the same as rotating from $\{a\}$ to $\{b\}$. This is an active operation.

$R^a_{ab} = R^b_{ab} = R^a_b$

F5

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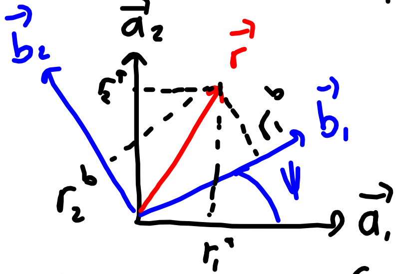
Exam 15. Dec

A.2.6 Interpretation of the direction cosine matrix (DCM) (RKM)

1. C_b^a is a coordinate transformation matrix (CTM)

$$\underline{r}^a = C_b^a \underline{r}^b$$



$$C_b^a \overset{\text{an}}{=} R_b^a = \begin{bmatrix} \underline{b}_1^a & \underline{b}_2^a \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$$

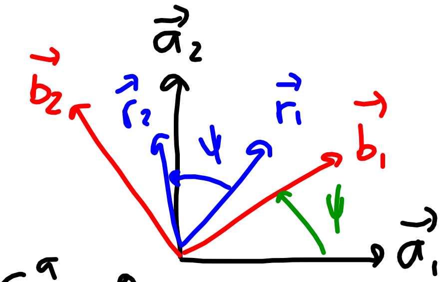
This is a passive operation

2. C_b^a is an attitude matrix (stillingsmatrise)

$$C_b^a = \left[\langle \vec{b}_j, \vec{a}_i \rangle\right] = \begin{bmatrix} \langle \vec{b}_1, \vec{a}_1 \rangle & \langle \vec{b}_2, \vec{a}_1 \rangle & \langle \vec{b}_3, \vec{a}_1 \rangle \\ \langle \vec{b}_1, \vec{a}_2 \rangle & \langle \vec{b}_2, \vec{a}_2 \rangle & \langle \vec{b}_3, \vec{a}_2 \rangle \\ \langle \vec{b}_1, \vec{a}_3 \rangle & \langle \vec{b}_2, \vec{a}_3 \rangle & \langle \vec{b}_3, \vec{a}_3 \rangle \end{bmatrix} = \begin{bmatrix} \underline{b}_1^a & \underline{b}_2^a & \underline{b}_3^a \end{bmatrix}$$

3. C_b^a is a rotation matrix (RM)

$$C_b^a = [\mathbb{R}_{ab}]^a = [\mathbb{R}_{ab}]^b$$



$$\vec{r}_2 = \mathbb{R}_{ab}\vec{r}_1 \iff \underline{r}_2^a = [\mathbb{R}_{ab}]^a \underline{r}_1^a = R_{ab}^a \underline{r}_1^a = C_b^a \underline{r}_1^a$$

$$\underline{r}_2^b = [\mathbb{R}_{ab}]^b \underline{r}_1^b = R_{ab}^b \underline{r}_1^b = C_b^a \underline{r}_1^b$$

This is an active operation. The vector is rotated.

The coordinate transformation matrix (CTM) C_b^a that transforms a vector in the b-fram to the a-frame acts as an rotationmatrix in one and the same frame, and rotates the vector in the same way as the a-frame needs to rotate to coinside with the b-frame.

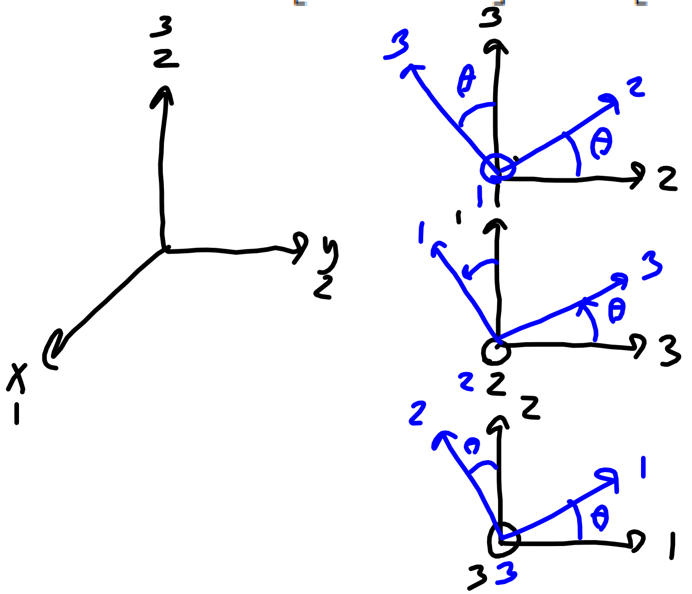
A.2.7 Representasjon av ortogonale RKM

Vi skal i dette avsnittet se på ulike måter å representere ortogonale RKM.

Eulervinkelrepresentasjon av RKM

Elementære RKM. Gitt rammene q og p. Dersom en tenker seg at rammene opprinnelig var sammenfallende fås den endelige p-ramma ved å dreie den en vinkel θ om q_i-aksen. Vi har følgende elementære RKMer:

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix}, \; R_2 = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}, \; R_3 = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(A- 30)}$$



$$R_1(\theta) = R^{\text{black}}_{\text{blue}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculation rules for the elementary DCM

$$R_i(\theta_1+\theta_2) = R_i(\theta_1)R_i(\theta_2) = R_i(\theta_2)R_i(\theta_1)$$

$$\left(R_i(\theta)\right)^{-1} = \left(R_i(\theta)\right)^T = R_i(-\theta)$$

$$\Downarrow$$

$$\left(R_b^a\right)^T = R_a^b$$

Generally

$$R_a^c = R_b^c R_a^b \neq R_a^b R_b^c$$

DCM are non-comutative

Rotation sequences

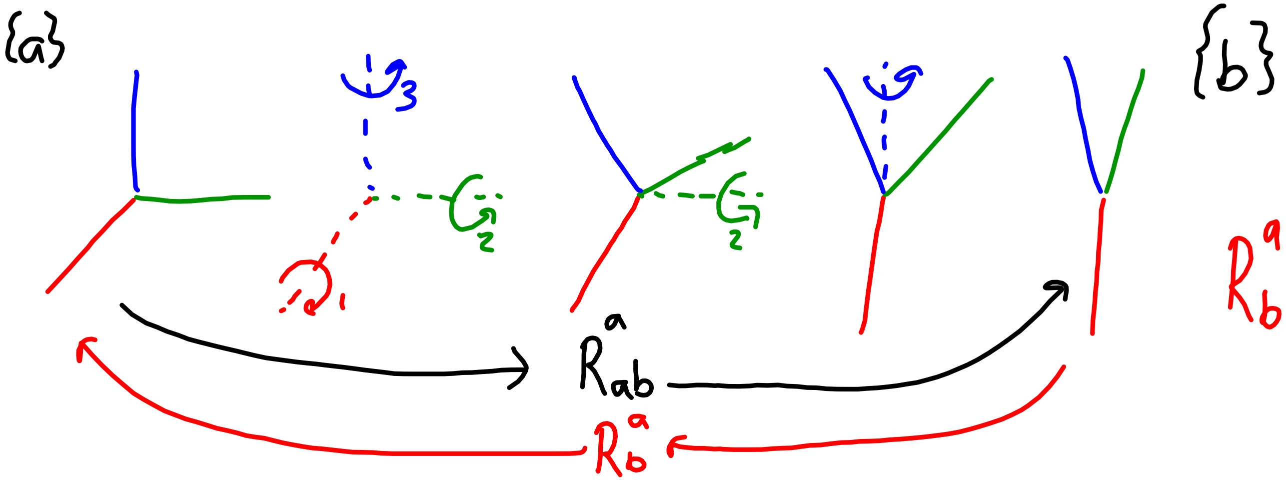
1. Rotation around new axis (Euler angles)

3-2-1 Euler angles

Rot. 1	1	1	2	2	3	3	1	1	2	2	3	3
Rot. 2	2	3	1	3	1	2	2	3	1	3	1	2
Rot. 3	3	2	3	1	2	1	1	1	2	2	3	3

} 12 sequences

2. Rotation around fixed axis (12 sequences)



R_{ab}^{a}: rot. a → b

R_{b}^{a}: transforms b → a

$$R_{b}^{a} = R_{ab}^{a} = R_{ab}^{b}$$

Example

What is R_b^a when rotating ψ arond axis 3, θ around axis 2 and ϕ arond axis 1?

Fixed axis: $R_b^a = R_1(\phi)\,R_2(\theta)\,R_3(\psi)$

New axis : $R_b^a = R_3(\psi)\,R_2(\theta)\,R_1(\phi)$

Teorem A.8 ***Sammenheng mellom rotasjon om nye og faste akser***
Tre rotasjoner om nye akser (eulervinkler) gir den samme endelige stilling som de samme rotasjoner tatt i omvendt rekkefølge om faste akser, dvs :

$$ {}^{E}R_{p}^{q}(\alpha_i, \beta_j, \gamma_k) = {}^{F}R_{p}^{q}(\gamma_k, \beta_j, \alpha_i) \tag{A- 35}$$

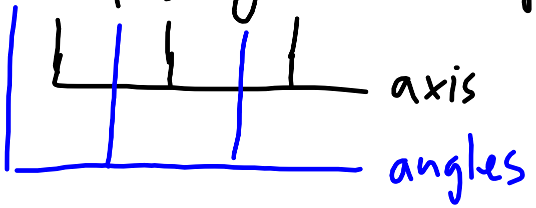
$$ {}^{E}R_{p}^{q}(\alpha_i, \beta_j, \gamma_k) = R_i(\alpha)\, R_j(\beta)\, R_k(\gamma) $$

$$ {}^{F}R_{p}^{q}(\gamma_k, \beta_j, \alpha_i) = R_i(\alpha)\, R_j(\beta)\, R_k(\gamma) $$

Direct problem

Given the rotation sequence and angles (fixed or new axis) find the DCM

$$\alpha_i \; \beta_j \; \gamma_k \rightarrow R_p^q$$



Gives always a clear R_p^q-matrix

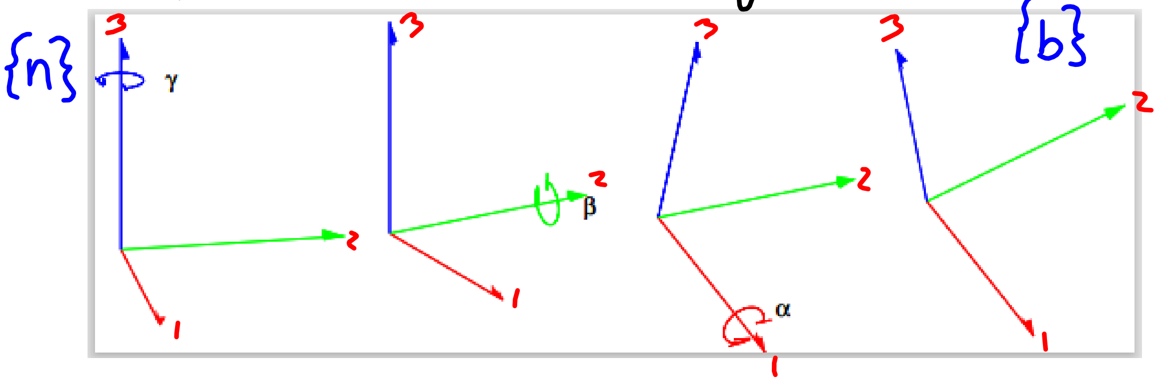
Inverse problem

Given R_p^q-matrix find α, β, γ for a given rotation sequence

Cannot always find a unique solution $\Rightarrow$ we have singularities (for 3-2-1 euler angles if we rotate 90° around axis 2.)

Example A.6 3-2-1 Euler angles.

$R_b^n = R_3(\gamma) R_2(\beta) R_1(\alpha)$



b: body frame

n: navigation frame

Eksempel A.6 3-2-1 Eulervinkler.

Ved simulering av fly og båter bruker en ofte følgende stillingsmatrise:

$$R_b^n = {}^E R_b^n(\theta_3, \theta_2, \theta_1) = R_3(\theta_3) R_2(\theta_2) R_1(\theta_1) \tag{A- 36}$$

Multipliseres de elementære RKM sammen får vi :

$${}^E R_b^n(\theta_3, \theta_2, \theta_1) = \begin{bmatrix} c_{\theta_3} c_{\theta_2} & c_{\theta_3} s_{\theta_2} s_{\theta_1} - s_{\theta_3} c_{\theta_1} & c_{\theta_3} s_{\theta_2} c_{\theta_1} + s_{\theta_3} s_{\theta_1} \\ s_{\theta_3} c_{\theta_2} & s_{\theta_3} s_{\theta_2} s_{\theta_1} + c_{\theta_3} c_{\theta_1} & s_{\theta_3} s_{\theta_2} c_{\theta_1} - c_{\theta_3} s_{\theta_1} \\ -s_{\theta_2} & c_{\theta_2} s_{\theta_1} & c_{\theta_2} c_{\theta_1} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{A- 37}$$

$r_{31} = -\sin(\theta_2)$

Eksempel A.8 Det inverse problem for 3-2-1 Eulervinkler,:
Gitt R_b^n finn vinklene, har løsningen (Craig 1989, s. 47)

$$\begin{aligned}\theta_1 &= \operatorname{atan2}\left(r_{32}/c_{\theta_2}, r_{33}/c_{\theta_2}\right)\\ \theta_2 &= \operatorname{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right) \leftarrow \text{Solve first} \\ \theta_3 &= \operatorname{atan2}\left(r_{21}/c_{\theta_2}, r_{11}/c_{\theta_2}\right)\end{aligned} \tag{A- 39}$$

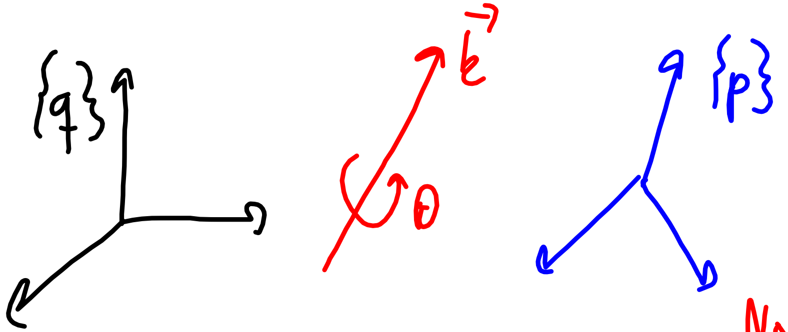
Vi løser først for θ_2. For $\theta_2 = \pm 90^\circ$ har vi singularitet og bare summen av θ_1 og θ_3 kan beregnes.

Angle-axis representation of the direction cosine matrix (DCM)

Teorem A.9 ***Eulers rotasjonsteorem***
En vilkårlig retningskosinmatrise R_p^q kan fås ved å rotere p-systemet en vinkel θ om aksen $\underline{k}^q = [k_1^q, k_2^q, k_3^q]$ ($\|\underline{k}^q\| = 1$). Dvs vi har :

$$R_p^q = R_{\underline{k}^q}(\theta) \tag{A- 40}$$

Denne representasjonen kalles for vinkel-akse representasjon.



Choose 1: $\|\vec{k}\| = \|\underline{k}^q\| = 1$, rot. θ

Choose 2: $\|\vec{k}\| = \|\underline{k}^q\| = \theta$

Note: R_p^q is represented by 3 parameters

Teorem A.10 ***Det direkte problem for vinkel-akse***

Når vinkel-akserepresentasjonen er gitt kan retningskosinmatrisa beregenes på følgende måte:

$$R_p^q = R_{\underline{k}}(\theta) = I + S(\underline{k}^q)\sin\theta + S^2(\underline{k}^q)(1-\cos\theta) \qquad \text{(A- 41)}$$

$$= I\cos\theta + S(\underline{k}^q)\sin\theta + \underline{k}^q\,(\underline{k}^q)^T\,(1-\cos\theta) \qquad \text{(A- 42)}$$

No proof

Fortegnet til θ bestemmes ut fra høgrehåndsregelen.

Teorem A.11 ***Det inverse problem for vinkel-akse***

Gitt retningskosinmatrise

$$R_p^q = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{221} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad \text{(A- 43)}$$

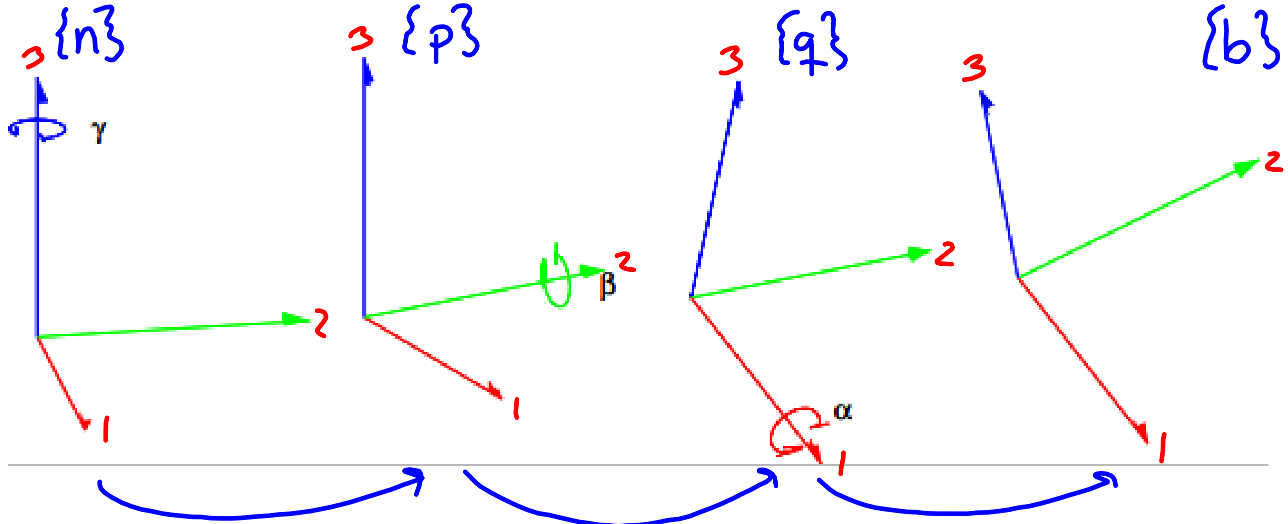
da er vinkel-akserepresentasjonen gitt ved

$$\theta - \arccos(\frac{r_{11}+r_{22}+r_{33}-1}{2}); \quad \underline{k}^q - \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix} \qquad \text{(A- 44)}$$

No proof

Vi får her en $\theta \in [0^\circ, 180^\circ\rangle$, det finns en annen løsning $(-\underline{k}^q, -\theta)$ som gir samme retningskosinmatrise. NB: for små vinkler θ kan de numeriske feilene ved bestemmelse av $\underline{k}^q$ bli store.

F.6/



$R_3(\gamma)$ $\quad$ $R_2(\beta)$ $\quad$ $R_1(\alpha)$

$R_{np} = R_p^n$ $\quad$ $R_{pq} = R_q^p$ $\quad$ $R_{qb} = R_b^q$

$$R_{nb} = R_b^n = R_p^n R_q^p R_b^q = R_3(\gamma)\, R_2(\beta)\, R_1(\alpha)$$

Eulers symmetrical parameter representation of the DCM

Definisjon A.12 *Eulers symmetriske parametre*
Vha vinkel-akse representasjonen kan vi definere en 4-parameter representasjon på følgende måte :

$$\underline{k}^q = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}, \theta$$

$$\left. \begin{matrix} \varepsilon_1 = k_1 \sin(\theta/2) & \varepsilon_3 = k_3 \sin(\theta/2) \\ \varepsilon_2 = k_2 \sin(\theta/2) & \varepsilon_0 = \cos(\theta/2) \end{matrix} \right\} \Rightarrow \varepsilon_0^2 + \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 = 1 \qquad \text{(A- 45)}$$

$$\sin^2\left(\frac{\theta}{2}\right) \underbrace{\left(k_1^2 + k_2^2 + k_3^2\right)}_{1} + \cos^2\left(\frac{\theta}{2}\right) = 1$$

Quaternion representation of the DCM

A quaternion (those with unit length) can represent the DCM R_p^q if it is defined using Eulers symmetrical parametres in the following way :

$$\boxed{\mathcal{E}_p^q = \varepsilon_0 + \varepsilon_1 \mathbb{i} + \varepsilon_2 \mathbb{j} + \varepsilon_3 \mathbb{k}}$$

$$\|\mathcal{E}_p^q\| = \left(\sum_{i=0}^{3} \varepsilon_i^2\right)^{1/2} = 1$$

$$\mathbb{i}\mathbb{i} = \mathbb{j}\mathbb{j} = \mathbb{k}\mathbb{k} = -1$$

E.1. viewed as a complex number

$$\mathbb{i}\mathbb{j} = -\mathbb{j}\mathbb{i} = \mathbb{k}$$

$$\mathbb{j}\mathbb{k} = -\mathbb{k}\mathbb{j} = \mathbb{i}$$

$$\mathbb{k}\mathbb{i} = -\mathbb{i}\mathbb{k} = \mathbb{j}$$

View $\mathbb{i}, \mathbb{j}$ and $\mathbb{k}$ as vectors in 3D (o.n.) and the product of two vectors as the "$\times$"-product.

Determinant of R_p^q: $|R_p^q| = 1 \iff \|\mathcal{E}_p^q\| = 1$

Renormalization of $\mathcal{E}_p^q$ is much simpler than renormalization of R_p^q $\left(|R_p^q| = 1,\ \underline{p}_1^q \perp \underline{p}_2^q \perp \underline{p}_3^q \right)$

Calculation rules for quaternions.

$$\boxed{\begin{aligned} R^a_c = R^a_b R^b_c \quad &\Longleftrightarrow \quad \mathcal{E}^a_c = \mathcal{E}^b_c \mathcal{E}^a_b \\ R^a_b \underline{r}^b \quad &\Longleftrightarrow \quad (\mathcal{E}^a_b)^* \, \mathbb{r}^b \, \mathcal{E}^a_b \end{aligned}}$$

where:

$$(\mathcal{E}^a_b)^* = \varepsilon_0 - \varepsilon_1 \mathbb{i} - \varepsilon_2 \mathbb{j} - \varepsilon_3 \mathbb{k}$$

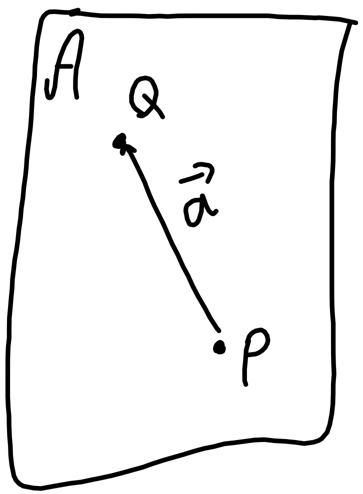
$$\mathbb{r}^b = r^b_1 \mathbb{i} + r^b_2 \mathbb{j} + r^b_3 \mathbb{k}, \qquad \underline{r}^b = \begin{bmatrix} r^b_1 \\ r^b_2 \\ r^b_3 \end{bmatrix}$$

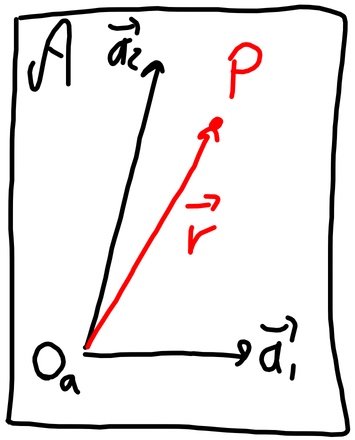
Parametrisering	Notasjon	Fordel	Ulempe	Vanilige anvendelser
RKM	C_p^q (R_p^q) o.n.	Ingen singulariteter, ingen trigonometriske funksjoner, enkel produktregel for suksesive rotasjoner	Seks redundante parametre	I analysen, for å transformere vektorer fra et k.s. til et annet
Eulervinkler	φ, θ, ψ	Ingen redundante parametre, klar fysisk tolkning	Trigonometriske funksjoner, singulariteter for visse vinkler, ingen enkel produktregel for suksessive rotasjoner	Analytiske studier, 3-akset stillingskontrol av legemer
Vinkel-akse	$\underline{k}, \theta$	Klar fysisk tolkning	En redundant parameter, aksen er udefinert når $\sin\theta = 0$, trigonometriske funksjoner	Reorienteringsmanøvre (slew)
Kvaternioner	ε	Ingen singulariteter, ingen trigonometriske funksjoner, enkel produktregel for suksessive rotasjoner	En redundant parameter, ingen klar fysisk tolkning	Treghetsnavigasjonsberegninger

$\|\vec{k}\| = \theta$

A.3 Affine spce

Definisjon A.14 ***Affint rom***
La $\mathcal{A}$ være en ikke-tom mengde av punkter, og la $\mathcal{V}$ være et vektorrom over skalarkroppen $\mathbb{K}$. Anta at for vilkårlige punkt $P \in \mathcal{A}$ og $\vec{a} \in \mathcal{V}$ er det definert en addisjon $P + \vec{a} \in \mathcal{A}$ som tilfredstiller følgende betingelser :
1. $P + \vec{0} = P$ ($\vec{0}$ er nullvektoren i $\mathcal{V}$)
2. $(P + \vec{a}) + \vec{b} = P + (\vec{a} + \vec{b})$ for $\forall \vec{a}, \vec{b} \in \mathcal{V}$
3. For enhver $Q \in \mathcal{A}$ eksisterer en entydig vektor $\vec{a} \in \mathcal{V}$ slik at $Q = P + \vec{a}$
Da er $\mathcal{A}$ et affint rom.





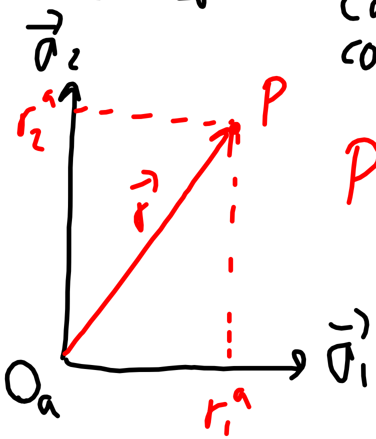
Given the frame $\mathcal{F}_A^a = \{O_a, \vec{a}_1, \vec{a}_2, \vec{a}_3\}$

There is a clear relation between P, $\vec{r}$, $\underline{r}^a$

A3.2 Coordinate systems and frames

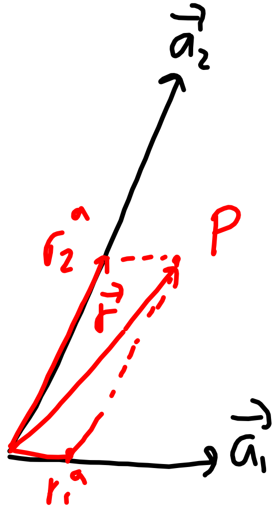
Definisjon A.15 ***Affine koordinater***
La $\mathcal{A}$ være et $n-$dimensjonalt affint rom og la $\mathcal{F}^e = (O_e; \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n)$ være en **ramme**, *hvor O_e, kallt origo, er et punkt i $\mathcal{A}$, og vektorene $\{\vec{e}_i\}$ er et sett basisvektorer for for det tilhørende vektorrom $\mathcal{V}$. Da er de inhomogene* **koordinatene** *til et vilkårlig punkt $P \in \mathcal{A}$ med hensyn til ramma $\mathcal{F}^e$ gitt av n-tuppelet $\{p_1^e, p_2^e, \ldots, p_n^e\}$ (vi vil sette disse sammen til en algebraisk vektor, $\underline{p}^e$), hvor $P = O_e + \sum_{i=1}^{n} p_i^e \vec{e}_i$. Dersom $\mathcal{A}$ også har strukturen til et Euclidsk rom (se nedefor) og vi lar ramma $\mathcal{F}^e = (O_e; \vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n)$ bestå av ortogonale basisvektorer sier vi at vi har et* **rektangulært koordinatsystem** *(eller ortogonalt k.s.), dersom basisvektorer har lengde 1, $\langle \vec{e}_i, \vec{e}_j \rangle = \delta_{ij}$,kalles k.s. for* **kartesisk**. *Generelt vil $\langle \vec{e}_i, \vec{e}_j \rangle = c_{ij}$ og vi sier vi har et* **oblikt koordinatsystem** *(eller ikke-ortogonalt k.s.). Dersom vi har en oblik ramme hvor lengden på basisvektorene er $\langle \vec{e}_i, \vec{e}_i \rangle = 1$ er vinkelen mellom basisvektoren gitt av $c_{ij} = \cos \angle \vec{e}_i \vec{e}_j$*

Example



Cartesian coordinate system

$P, \vec{r} \to \underline{r}^a$



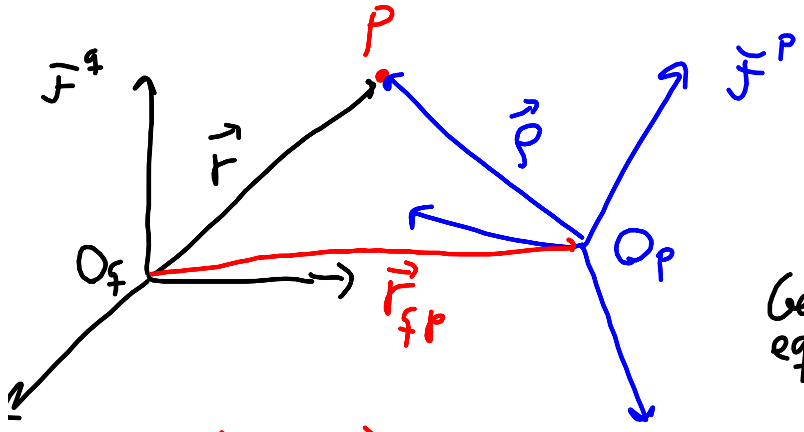
Obligt c.s.

$P, \vec{r} \to \underline{r}^a$

A.3.3 Matrix representation of points and vectors

Inhomogenuos representation.

Def. A.17 Position vector



$$P = O_q + \vec{r} = O_p + \vec{\rho}$$

$$\vec{r} - \vec{\rho} = O_p - O_q = \vec{r}_{qp}$$

Def. $\vec{r}_{qp} = \vec{r}_{O_q O_p}$

Geometrical equation: $\vec{r} = \vec{r}_{qp} + \vec{\rho}$

Algebraic equation: $\underline{r}^q = \underline{r}_{qp}^q + \underline{\rho}^q = \underline{r}_{qp}^q + R_p^q \underline{\rho}^p$

We see that when representing points in two different frames the coordinates transform as:

$$\boxed{\underline{r}^q = \underline{r}^q_{qp} + R^q_p \underline{\rho}^p}$$ Inhomogenous form

But vectors in vector space transform as:

$$\boxed{\underline{v}^q = R^q_p \underline{v}^p}$$ Homogenous form

Homogenous representation of coordinates to a point.

$$(1a) \quad \boxed{\underline{r}^q_p = \underline{r}^q_{qp} + R^q_p \underline{\rho}^p_p}$$

$$\underset{\sim}{\underline{r}}^q_p = \begin{bmatrix} r_1^q \\ r_2^q \\ r_3^q \\ 1 \end{bmatrix}$$

Define:

$$\underset{\sim}{\underline{r}}^q_p = [\underline{r}^q_p \,;\, 1]$$

$$\underset{\sim}{\underline{\rho}}^p_p = [\underline{\rho}^p_p \,;\, 1]$$

; 4 dim. matrix rep. of the point p in $\mathcal{F}^p$ and $\mathcal{F}^q$

We get:

$$(1b) \quad \boxed{\underset{\sim}{\underline{r}}^q_p = T^q_p \underset{\sim}{\underline{\rho}}^p_p ,}$$

$$T^q_p = \left[\begin{array}{c|c} R^q_p & \underline{r}^q_{qp} \\ \hline [0,0,0] & 1 \end{array}\right]$$

T^q_p : transformation matrix

For normal vectors (not matrix rep. of points) we have:

$$\boxed{\underline{v}^q = R^q_p \underline{v}^p} \quad (2a)$$

Define:

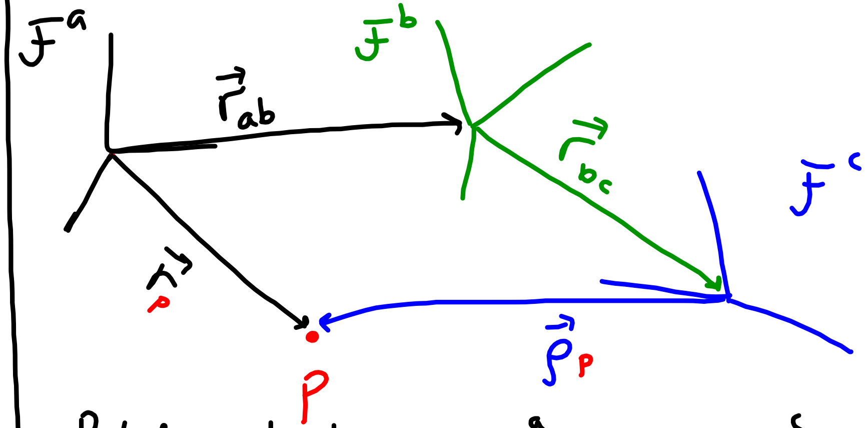
$$\underline{\tilde{v}}^q = [\underline{v}^q ; 0]$$

$$\underline{\tilde{v}}^p = [\underline{v}^p ; 0]$$

These vectors transform as:

$$\boxed{\underline{\tilde{v}}^q = T^q_p \underline{\tilde{v}}^p} \quad (2b)$$

Example: 3 frames.



Relation between $\underline{r}^a_p$ and $\underline{\rho}^c_p$

$$\vec{r}_p = \vec{r}_{ab} + \vec{r}_{bc} + \vec{\rho}_p$$

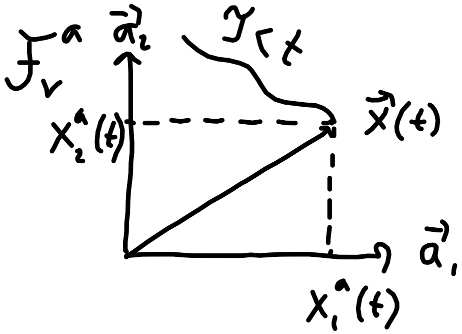
$$\boxed{\underline{r}^a_p = \underline{r}^a_{ab} + R^a_b \underline{r}^b_{bc} + R^a_b R^b_c \underline{\rho}^c_p} \quad (3a)$$

$$\boxed{\underline{\tilde{r}}^a_p = T^a_b T^b_c \underline{\tilde{\rho}}^c_p} \quad (3b)$$

F.7 / **A.4 Time in vector space and affine space.**

Up to now we have looked at points and vectors as static. Given the vector space V and the frame $\mathcal{F}_V^a$ we can describe a time varying vector as:

$$\boxed{\vec{x}(t) = \sum_{i=1}^{n} x_i^a(t)\,\vec{a}_i}$$



Clear relation:

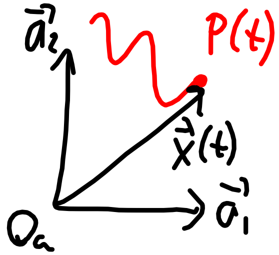
$\vec{x}(t)$ and $\underline{x}^a(t)$

Given the affine space $\mathcal{A}$ with frame $\mathcal{F}_{\mathcal{A}}^a = \{O_a, \vec{a}_1, \vec{a}_2, \vec{a}_3\}$ a time varying point is defined by:

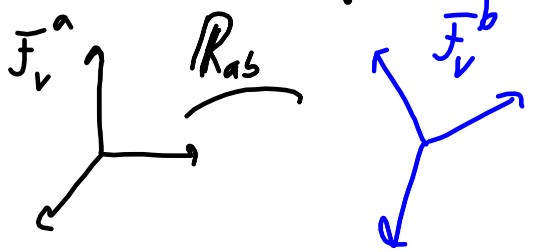
$$\boxed{P(t) = O_a + \sum_{i=1}^{n} x_i^a(t)\,\vec{a}_i}$$

Clear relation:

$P(t)$ and $\underline{x}^a(t)$



Relative attitude between two frames can be described using the rotation operator $\mathcal{R}_{ab}$



From before:

$$\underbrace{[\mathcal{R}_{ab}]^a}_{R_{ab}^a} = \underbrace{[\mathcal{R}_{ab}]^b}_{R_{ab}^b} = R_b^a$$

Representation of the rotation operator kan be made time variant:

$$\boxed{[\mathcal{R}_{ab}(t)]^a = R_b^a(t)}$$

For example using Euler angles (3-2-1)

$$R_b^a(t) = R_3(\psi(t))R_2(\theta(t))R_1(\phi(t))$$

To find the relation between the representation of a point $P(t)$ in the frames $\vec{\mathcal{F}}_A^a$ and $\vec{\mathcal{F}}_A^b$ we can use time varying transformation matrices:

$$\boxed{\begin{aligned} T_b^a(t) &= \begin{bmatrix} R_b^a(t) & \underline{r}_{ab}^a(t) \\ \underline{0}^T & 1 \end{bmatrix} \\ \tilde{\underline{r}}_p^a(t) &= T_b^a(t)\,\tilde{\underline{r}}_p^b(t) \end{aligned}}$$

If we have 3 frames: $\vec{\mathcal{F}}_A^a, \vec{\mathcal{F}}_A^b$ and $\vec{\mathcal{F}}_A^c$

$$\tilde{\underline{r}}^a = T_b^a T_c^b \tilde{\underline{r}}^c \quad \text{(all function of } t)$$

NB! In classical mechanics we use Galileo transformations. We can add relative velocities.

NB! When calculating the distance between points position at two different times we need to choose <u>one</u> affine space:

$$\overset{A}{P}(t_2) - \overset{A}{P}(t_1) = \vec{r}_{\overset{A}{P}(t_2)\overset{A}{P}(t_1)}$$

A.5. Derivation in vector- and affine space.

Notation need to take into account two aspects:

1. In which frame do we represent the vector: $\underline{x}^{a}(t)$
2. In which frame do we see the derivation from: $\{b\}$: $\underline{\dot{x}}^{ba}(t)$

In math

Given $f(x,y)$ the partial derivation is:

$$\frac{\partial f(x,y)}{\partial x} = f_x(x,y) \qquad \frac{\partial f(x,y)}{\partial y} = f_y(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = f_{xy}(x,y) \qquad \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right) = f_{yx}(x,y)$$

Here we had two free variables and derivation w.r.t. these. We will only have one free variable (time - t), but we can se the time variations from different frames that are moving relative to each other.

Introduce notation:

$$\frac{d^a}{dt}\vec{x}(t) = \dot{\vec{x}}^a(t)$$ Derivation seen from $\mathcal{F}^a$

$$\frac{d^b}{dt}\vec{x}(t) = \dot{\vec{x}}^b(t)$$ ——''—— $\mathcal{F}^b$

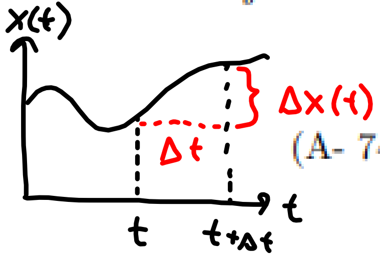
$$\frac{d^b}{dt}\left(\frac{d^a}{dt}\vec{x}(t)\right) = \ddot{\vec{x}}^{ab}(t)$$

Represent in $\mathcal{F}^c$

$$\left[\dot{\vec{x}}^a(t)\right]^c = \underline{\dot{x}}^{ac}(t), \quad \left[\ddot{\vec{x}}^{ab}(t)\right]^c = \underline{\ddot{x}}^{abc}(t)$$

A.5.1 Definisjon av deriverte i vektorrom og affine rom.

Når vi skal definere de deriverte av vektorer og punkter må vi starte med det vi kjenner fra matematikken nemlig derivasjon i $\mathbb{R}$ og så generalisere til $\mathbb{R}^n$, $\mathcal{V}$ og $\mathcal{A}$:



1. Derivasjon i $\mathbb{R}$:

$$\dot{x}(t) = \lim_{\Delta t \to 0} \left(\frac{1}{\Delta t} \overbrace{(x(t + \Delta t) - x(t))}^{\Delta x} \right) \tag{A- 74}$$

2. Derivasjon i $\mathbb{R}^n$:

$$\dot{\underline{x}}(t) = [\dot{x}_i(t)] \tag{A- 75}$$

3. Derivasjon i vektorrommet $\mathcal{V}$ sett fra en fast ramme $\mathcal{F}_{\mathcal{V}}^a$:

$$\dot{\vec{x}}^a = \sum_{i=1}^{n} \dot{x}_i^a(t)\, \vec{a}_i \qquad \vec{x}(t) = \sum_{i=1}^{n} x_i^a(t)\, \vec{a}_i \tag{A- 76}$$

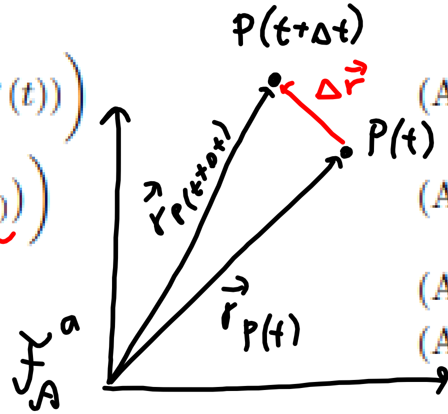
4. Derivasjon i det affine rom $\mathcal{A}$ sett fra en fast ramme $\mathcal{F}_{\mathcal{A}}^a$:

$$\dot{P}^a(t) = \lim_{\Delta t \to 0} \left(\frac{1}{\Delta t} (P(t + \Delta t) - P(t)) \right) \tag{A- 77}$$

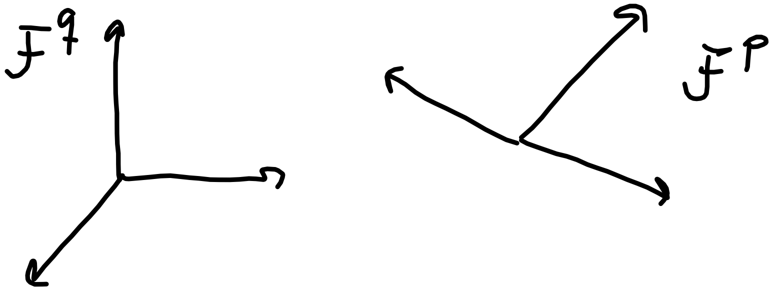
$$= \lim_{\Delta t \to 0} \left(\frac{1}{\Delta t} \underbrace{(\vec{r}_{P(t+\Delta t)} - \vec{r}_{P(t)})}_{\Delta \vec{r}} \right) \tag{A- 78}$$

$$= \dot{\vec{r}}_P^a \tag{A- 79}$$

$$= \vec{v}_P^a \tag{A- 80}$$



A5.2 Derivation of DCM



$R_p^q(t)$ gives the relative attitude

Assume R_p^q is a o.n. matrix $\Rightarrow (R_p^q)^T = (R_p^q)^{-1}$

Let $R = R_p^q \Rightarrow R \cdot R^T = I$

Derive on both sides:

$$\frac{d}{dt}\left(R(t)\, R^T(t)\right) = \frac{d}{dt} I$$

$$\dot{R}(t)\, R^T(t) + R(t)\frac{d}{dt}\left(R^T(t)\right) = 0$$

① $$\dot{R}(t)\, R^T(t) + R(t)\left(\dot{R^T}(t)\right) = 0$$

Multiply from right side with $R(t)$

$$\dot{R}\underbrace{R^T R}_{I} + R(\dot{R^T})R = 0$$

② $$\dot{R} + R(\dot{R^T})R = 0$$

Is $(\dot{R})^T = (\dot{R^T})$?

$$\frac{d}{dt}\left(R^T\right) = \frac{d}{dt}\left(\left[\underline{r}_1^q, \underline{r}_2^q, \underline{r}_3^q\right]^T\right)$$

$$= \frac{d}{dt}\begin{pmatrix} \underline{r}_1^{q\,T} \\ \underline{r}_2^{q\,T} \\ \underline{r}_3^{q\,T} \end{pmatrix} = \begin{bmatrix} \dot{\underline{r}}_1^{q\,T} \\ \dot{\underline{r}}_2^{q\,T} \\ \dot{\underline{r}}_3^{q\,T} \end{bmatrix} = \left(\frac{d}{dt}R\right)^T$$

I.e.

$$\frac{d}{dt}\left(R^T\right) = \frac{d}{dt}\left(R^{-1}\right) = \left(\frac{d}{dt}R\right)^T$$

Generally: $\frac{d}{dt}A^{-1} \neq \left(\frac{d}{dt}A\right)^T$

Equation 1 can be written as:

$$\dot{R}R^T + \left(\dot{R}R^T\right)^T = 0$$

$$S := \dot{R}R^T \Rightarrow S + S^T = 0$$

I.e. S: Scew symmetrical matrix

$$S = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} = S(\underline{w})$$

$$\vec{w} \times \vec{a} \rightarrow S(\underline{w}^q)\,\underline{a}^q$$

Equation ② :

$$\dot{R} + \underbrace{R\dot{R}^T}_{S^T} R = 0$$

$$\dot{R} = -S^T R = SR$$

$$\boxed{\dot{R}_p^q = S(\underline{w}) R_p^q}, \quad \underset{\text{given}}{R_p^q(t_0)}$$

R_p^q is an attitude matrix

$$R_p^q = [\underline{p}_1^q, \underline{p}_2^q, \underline{p}_3^q]$$

$$\dot{R}_p^q = [\dot{\underline{p}}_1^q, \dot{\underline{p}}_2^q, \dot{\underline{p}}_3^q]$$

$$= S(\underline{w}) [\underline{p}_1^q, \underline{p}_2^q, \underline{p}_3^q]$$

$$= [S(\underline{w})\underline{p}_1^q, S(\underline{w})\underline{p}_2^q, S(\underline{w})\underline{p}_3^q]$$

$$\dot{\underline{p}}_i^q = S(\underline{w}^q)\,\underline{p}_i^q$$

$$= \underline{w}^q \times \underline{p}_i^q$$

We see that $\underline{w}$ is part of the calculation of the derivative of the rotating basis vectors ($\underline{p}_i^q$) seen from the q-system. We interpret therefore $\underline{w}$ as the angular velocity of the p-system seen from the q-system, and use the notation $\underline{w}_p^q = \underline{w}_p^{qq}$ because we derive seen from the q-system and represent in the q-system:

We therefore write : $\boxed{\dot{R}_p^q = S(\underline{w}_p^q)\,R_p^q}$

$S(\underline{\omega}_p^q) = S(\underline{\omega}_p^{qq})$ is the representation of the operator " $\vec{\omega}_p^q \times$ " in the q-frame. But, linear operators can also be represented in other frames using the similarity transformation:

$$S(\underline{\omega}_p^{qq}) = R_p^q \, S(\underline{\omega}_p^{qp}) \, R_q^p$$

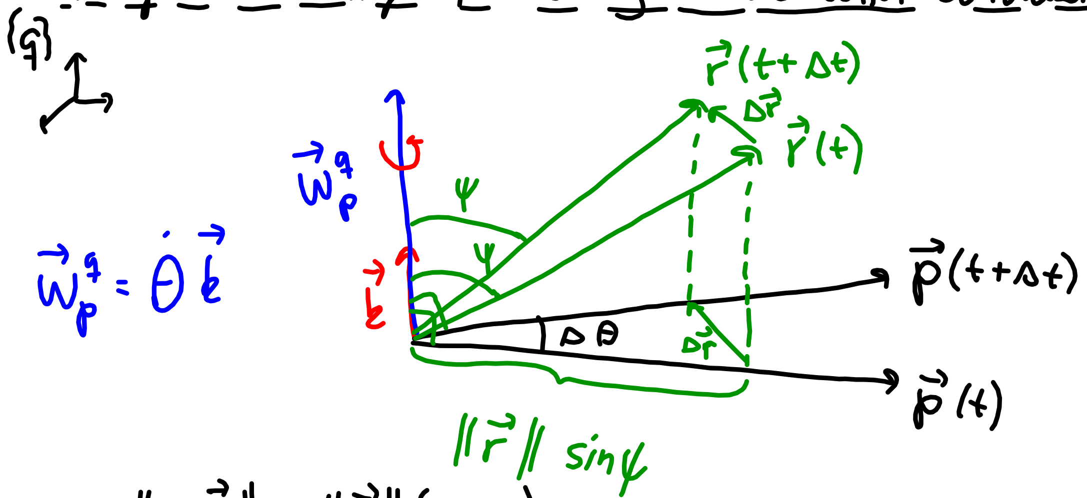
$$\dot{R}_p^q = S(\underline{\omega}_p^{qq}) R_p^q = R_p^q \, S(\underline{\omega}_p^{qp}) \underbrace{R_q^p \, R_p^q}_{I}$$

$$\boxed{\dot{R}_p^q = S(\underline{\omega}_p^q) R_p^q = R_p^q \, S(\underline{\omega}_p^{qp})}$$

Theorem A.15

$$\underline{\omega}_p^{qq} = R_p^q \, \underline{\omega}_p^{qp}$$

Angular velocity of rotating vectors with constant length



* $\vec{r}$ is fixed to $\{p\}$
* $\|\vec{r}\|$ is constant
* $\{p\}$ rotates relative to $\{q\}$ with $\vec{\omega}_p^q$
* $\|\vec{k}\| = 1$

$$\|\Delta\vec{r}\| = \|\vec{r}\|(\sin\psi)\cdot\Delta\theta$$

The direction og $\Delta\vec{r}$ shall be $\perp$ to both $\vec{r}$ and $\vec{k}$, and have the direction given by the r.h.r. for rotation around $\vec{k}$. With unit length this becoms:

$$\frac{\vec{k}\times\vec{r}}{\|\vec{k}\times\vec{r}\|} \quad \text{where } \|\vec{k}\times\vec{r}\| = \|\vec{k}\|\,\|\vec{r}\|\sin\psi \;\Rightarrow\; \frac{\vec{k}\times\vec{r}}{\|\vec{r}\|\sin\psi}$$

$$\Rightarrow \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta\theta}{\Delta t}\sin\psi\,\|\vec{r}\|\frac{\vec{k}\times\vec{r}}{\|\vec{r}\|\sin\psi} = \frac{\Delta\theta}{\Delta t}\,\vec{k}\times\vec{r} \;\Rightarrow\; \boxed{\dot{\vec{r}}^{\,q} = \dot{\theta}\,\vec{k}\times\vec{r} = \vec{\omega}_p^q\times\vec{r}}$$

$$\underline{\dot{r}}^{qq} = \underline{\omega}_p^{qq}\times\underline{r}^q$$

F.8/ Proof of Theorem A.17 Diff. eq. of DCM.

$$C_p^q = [p_1^q, p_2^q, p_3^q]$$

$$\dot{C}_p^q = [\dot{p}_1^{qq}, \dot{p}_2^{qq}, \dot{p}_3^{qq}] = [\underline{w}_p^{qq} \times p_1^q, \ \underline{w}_p^q \times p_2^q, \ \underline{w}_p^q \times p_3^q]$$

$$= [S(\underline{w}_p^q) p_1^q, \ S(\underline{w}_p^q) p_2^q, \ S(\underline{w}_p^q) p_3^q]$$

$$= S(\underline{w}_p^q)[p_1^q, p_2^q, p_3^q] = S(\underline{w}_p^q) C_p^q$$

$$\dot{C}_p^q = S(\underline{w}_p^{qq}) C_p^q = C_p^q S(\underline{w}_p^{qp}) \underbrace{C_q^p C_p^q}_{I} = C_p^q S(\underline{w}_p^{qp})$$

$$\boxed{\dot{C}_p^q = S(\underline{w}_p^q) C_p^q = C_p^q S(\underline{w}_p^{qp}), \qquad \underline{w}_p^{qq} = C_p^q \underline{w}_p^{qp}}$$

<u>The kinematic problem for 3-2-1 Euler angles</u> : Given $\vec{\omega}_p^{\,q}$, what is the d.e. for the attitude matrix or special representation of the attitude matrix.

1. If $R_p^q = [\underline{e}_1^q, \underline{e}_2^q, \underline{e}_3^q] \Rightarrow \boxed{\dot{R}_p^q = S(\underline{\omega}_p^q) R_p^q}$

2. If $R_p^q = R_3(\theta_3) R_2(\theta_2) R_1(\theta_1)$: 3-2-1 Euler angles

$$\boxed{\begin{aligned} \dot{\underline{\theta}} &= D_p^{\theta}(\underline{\theta})\, \underline{\omega}_p^{qp} \\ &= D_q^{\theta}(\underline{\theta})\, \underline{\omega}_p^{q} \end{aligned} \quad , \quad \underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}}$$

We solve the d.e. (differential equations) using numerical methods.

1. order Euler method:

a) $\dot{x}(t) = f(x(t))$, $x(t_0)$ given

scalar

$$\frac{\Delta x}{\Delta t} = \frac{x(t_{k+1}) - x(t_k)}{\Delta t} = f(x(t_k)) \quad , \quad \Delta t = t_{k+1} - t_k$$

$$x(t_{k+1}) = x(t_k) + \Delta t \cdot f(x(t_k)) \quad , \quad t_k = \Delta t \cdot k \ , \ x(t_k) = x_k$$

$$\boxed{x_{k+1} = x_k + \Delta t \, f(x_k) \quad , \quad x_0 \text{ given}}$$

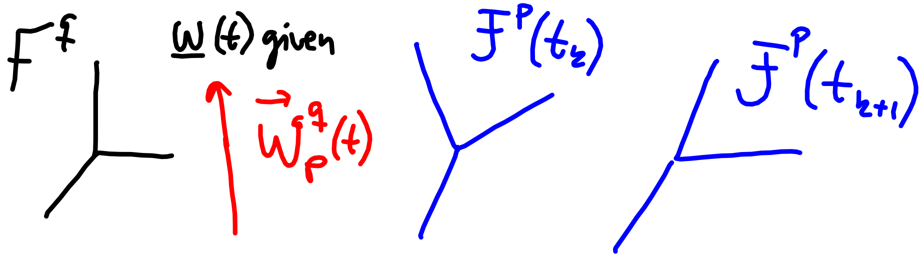
vector

b) $\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t))$, $\underline{x}(t_0)$ given

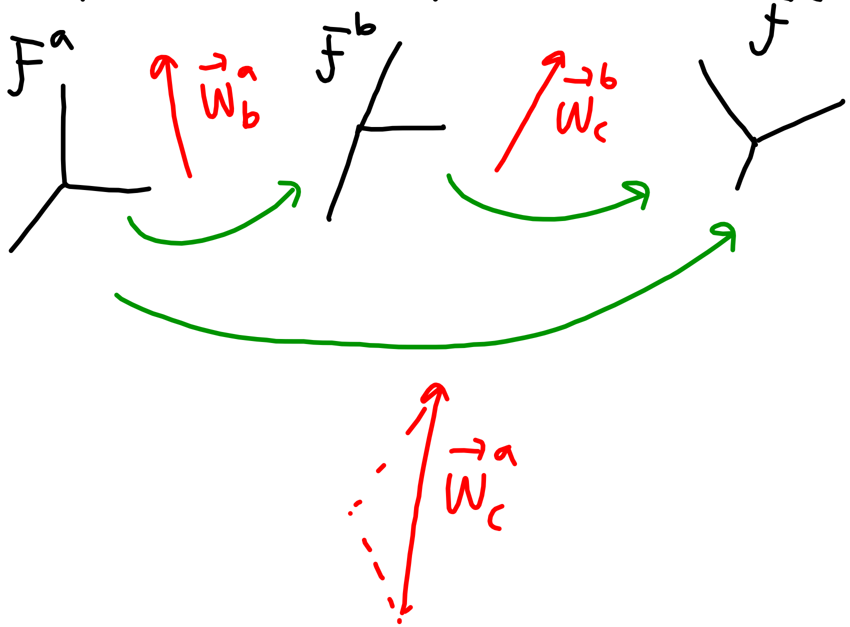
$$\boxed{\underline{x}_{k+1} = \underline{x}_k + \Delta t \cdot \underline{f}(\underline{x}_k) \ , \ \underline{x}_0 \text{ given}}$$

c) Matrix $\dot{R}(t) = S(\underline{w}(t))R(t)$, $R(t_0)$ given

$$\boxed{R_{k+1} = R_k + \Delta t \, S(\underline{w}_k) R_k \ , \ R_0 \text{ given}}$$



Angular velocity and ang. accelleration in different basis systems



$$\vec{\omega}_c^a = \vec{\omega}_b^a + \vec{\omega}_c^b$$

Derivation seen from F^a

$$\dot{\vec{\omega}}_c^{aa} = \dot{\vec{\omega}}_b^{aa} + \dot{\vec{\omega}}_c^{ba}$$

What is $\dot{\vec{\omega}}_c^{ba}$ expressed with $\dot{\vec{\omega}}_c^{bb}$

$$\vec{\omega}_c^b = \sum_{i=1}^{3} \omega_{ci}^{bb} \vec{b}_i$$

$$\frac{d^a}{dt}\vec{\omega}_c^b = \frac{d^a}{dt}\left(\sum \omega_{ci}^{bb} \vec{b}_i\right)$$

$$= \sum \dot{\omega}_{ci}^{bbb} \vec{b}_i + \underbrace{\sum \omega_{ci}^{bb} \overbrace{\vec{\omega}_b^a \times \vec{b}_i}^{\dot{\vec{b}}_i^a}}_{\vec{\omega}_b^a \times \sum \omega_{ci}^{bb} \vec{b}_i}$$

$$\dot{\vec{\omega}}_c^{ba} = \dot{\vec{\omega}}_c^{bb} + \vec{\omega}_b^a \times \vec{\omega}_c^b$$

$$\boxed{\dot{\vec{\omega}}_c^{aa} = \dot{\vec{\omega}}_b^{aa} + \dot{\vec{\omega}}_c^{bb} + \vec{\omega}_b^{a} \times \vec{\omega}_c^{b}}$$

Natural to represent also the derivation (actually 2. derivation) in the same frame as the 1. derivation

Angular velocities and their derivations in case of algebraic vectors can either be find by representing the equations for the geometrical vectors in a desired frame or by derivating:

$$\underline{\omega}_c^{aa} = \underline{\omega}_b^{aa} + \underline{\omega}_c^{ba}$$

$$= \underline{\omega}_b^{aa} + R_b^a\, \underline{\omega}_c^{bb}$$

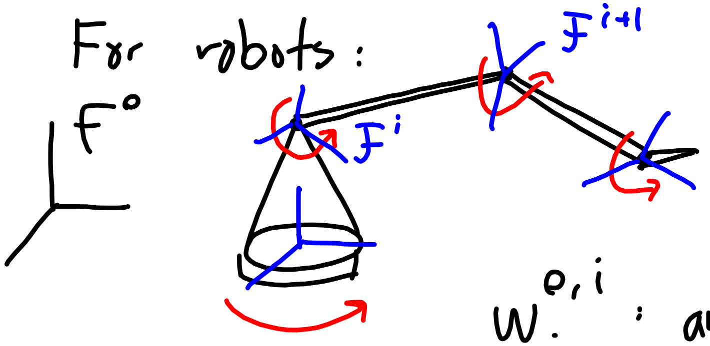
: Eaisier to derivate and represent in the same frame

We had:
$$\underline{w}_c^{aa} = \underline{w}_b^{aa} + R_b^a \, \underline{w}_c^{bb}$$

Derivate:
$$\boxed{\dot{\underline{w}}_c^{aaa} = \dot{\underline{w}}_b^{aaa} + S(\underline{w}_b^{aa}) R_b^a \, \underline{w}_c^{bb} + R_b^a \, \dot{\underline{w}}_c^{bbb}}$$

$$\dot{\underline{w}}_c^{aaa} = \dot{\underline{w}}_b^{aaa} + R_b^a \, S(\underbrace{\underline{w}_b^{ab}}_{R_a^b \underline{w}_b^{aa}}) \underline{w}_c^{bb} + R_b^a \, \dot{\underline{w}}_c^{bbb}$$

For robots :



$\underline{\omega}^{0,i}_{i}$: angular velocity for link i (frame i) seen from the link 0 (frame 0) represented in link i (frame i)

$$\underline{\omega}^{0,i+1}_{i+1} = \underline{\omega}^{0,i+1}_{i} + \underline{\omega}^{i,i+1}_{i+1}$$

$$\boxed{\underline{\omega}^{0,i+1}_{i+1} = R^{i+1}_{i}\,\underline{\omega}^{0,i}_{i} + \underline{\omega}^{i,i+1}_{i+1}}$$

$$\boxed{\dot{\underline{\omega}}^{0,i+1,i+1}_{i+1} = S\left(\underline{\omega}^{i+1,i+1}_{i}\right) R^{i+1}_{i}\,\underline{\omega}^{0,i}_{i} + R^{i+1}_{i}\,\dot{\underline{\omega}}^{0,i,i}_{i} + \dot{\underline{\omega}}^{i,i+1,i+1}_{i+1}}$$

Theorem A.18 Derivation of angular velocities

$$\boxed{\dot{\underline{w}}_b^{abb} = R_a^b \, \dot{\underline{w}}_b^{aaa}}$$

Proof: $\underline{w}_b^{ab} = R_a^b \, \underline{w}_b^{aa}$

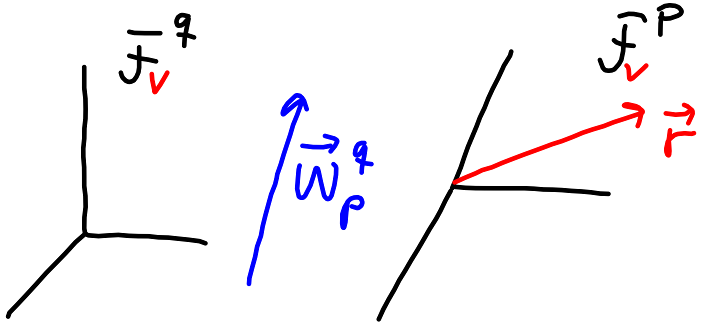
$$\dot{\underline{w}}_b^{abb} = \underbrace{\dot{R}_a^b \, \underline{w}_b^{aa}}_{0\,?} + R_a^b \, \dot{\underline{w}}_b^{aaa}$$

$\vec{w}_a^{\,b} = -\vec{w}_b^{\,a}$

$$\dot{R}_a^b \, \underline{w}_b^{aa} = S(\underline{w}_a^{bb}) R_a^b \, \underline{w}_b^{aa} = S(\underline{w}_a^{bb}) \, \underline{w}_b^{ab} = \underline{w}_a^{bb} \times \underline{w}_b^{ab}$$

$$= -\underline{w}_a^{bb} \times \underline{w}_a^{bb} = 0$$

A 5.3 Derivation of vectors



Assume $\|\vec r\|$ = constant and fixed to the p-frame.

Proved earlier:

$$\boxed{\dot{\vec r}^{\,q} = \vec{\omega}_p^q \times \vec r}$$

Assume $\vec r$ varies seen from $\vec{\mathcal{F}}_v^p$

$$\vec r = \sum r_i^p \vec p_i$$

$$\dot{\vec r}^{\,q} = \sum \dot r_i^{pp} \vec p_i + \sum r_i^p \dot{\vec p}_i^{\,q}$$

$$= \sum \dot r_i^p \vec p_i + \sum r_i^p \vec{\omega}_p^q \times \vec p_i$$

$$\boxed{\dot{\vec r}^{\,q} = \dot{\vec r}^{\,p} + \vec{\omega}_p^q \times \vec r}$$

$\vec p_i$ const. length.

F9/ For algebraic vectors the same eq. as in F8. p.9 are:

Constant vector seen from $\mathcal{F}^p$:

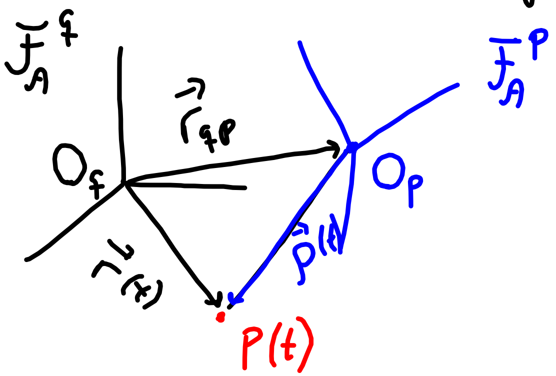
$$\boxed{\begin{aligned} \underline{r}^q &= R_p^q \underline{r}^p \\ \dot{\underline{r}}^q &= S(\underline{\omega}_p^q) R_p^q \underline{r}^p = S(\underline{\omega}_p^q) \underline{r}^q \end{aligned}}$$

$(\dot{\underline{r}}^p = \underline{0})$

Time varying vector seen from $\mathcal{F}^p$:

$$\boxed{\dot{\underline{r}}^q = S(\underline{\omega}_p^q) R_p^q \underline{r}^p + R_p^q \dot{\underline{r}}^{pp}}$$

A.5.4 Derivation of points motion in affine spaces



We are looking at the relationship between velocity and accelleration of the point P seen from two frames that moves relative to each other (translation and rotation)

We can either derive the equations for the geometrical vectors and then for the algebraic vectors or derive the eq. either for the geometrical or the algebraic and then use the eq. that gives the relationship between them.

From the figure:

$$\vec{r}(t) = \vec{r}_{qp}(t) + \vec{\rho}(t)$$

$$\vec{r}(t) = P(t) - O_q$$

$$\vec{\rho}(t) = P(t) - O_p$$

Find: $\dot{P}^q(t)$ and $\ddot{P}^q(t)$ $\left(= \ddot{P}^{qq}(t)\right)$

$\Leftrightarrow$ $\dot{\vec{r}}^q(t)$ and $\ddot{\vec{r}}^q(t)$ $\left(= \ddot{\vec{r}}^{qq}(t)\right)$

Equations needed:

$$\dot{\vec{c}}^q = \dot{\vec{c}}^p + \vec{\omega}_p^q \times \vec{c}$$

$$\dot{\underline{c}}^{qq} = R_p^q \dot{\underline{c}}^{pp} + S(\underline{\omega}_p^{qq}) R_p^q \underline{c}^p$$

Notation

$\vec{v}^q = \dot{\vec{r}}^q$: P's velocity seen from $\mathcal{F}_A^q$

$\vec{v}^p = \dot{\vec{\rho}}^p$: P's velocity seen from $\mathcal{F}_A^p$

$\vec{a}^q = \dot{\vec{v}}^{qq} = \ddot{\vec{r}}^{qq}$: Acc. seen from $\mathcal{F}_A^q$

$\vec{a}^p = \dot{\vec{v}}^{pp} = \ddot{\vec{\rho}}^{pp}$: Acc. seen from $\mathcal{F}_A^p$

Later we want to use Newton's 2. law: $\vec{f} = m\vec{a}^i$, $\mathcal{F}^i$ - inertial space

In inertial navigation (INS) we measure $\underline{a}^{iib}$ (b: fixed to the vehicle)

Deriving the geometrical equations:

$$\boxed{\begin{aligned}\vec{r} &= \vec{r}_{qp} + \vec{\rho}\\ \vec{v}^{q} &= \dot{\vec{r}}^{q} = \dot{\vec{r}}_{qp}^{\,q} + \overbrace{\vec{v}^{p} + \vec{\omega}_{p}^{q}\times\vec{\rho}}^{\dot{\vec{\rho}}^{q}}\end{aligned}}$$

$$\vec{a}^{q} = \ddot{\vec{r}}^{q} = \ddot{\vec{r}}_{qp}^{\,q} + \underbrace{\dot{\vec{v}}^{pp} + \omega_{p}^{q}\times\vec{v}^{p}}_{\dot{\vec{v}}^{pq}} + \dot{\vec{\omega}}_{p}^{qq}\times\vec{\rho} + \vec{\omega}_{p}^{q}\times\left(\vec{v}^{p} + \vec{\omega}_{p}^{q}\times\vec{\rho}\right)$$

$$\boxed{\vec{a}^{q} = \ddot{\vec{r}}_{pq}^{\,q} + \vec{a}^{p} + \dot{\vec{\omega}}_{p}^{q}\times\vec{\rho} + \vec{\omega}_{p}^{q}\times\left(\vec{\omega}_{p}^{q}\times\vec{\rho}\right) + 2\vec{\omega}_{p}^{q}\times\vec{v}^{p}}$$

$$\left.\begin{aligned}
\underline{r}^{q} &= \underline{r}_{qp}^{q} + R_{p}^{q}\underline{\rho}^{p}\\
\underline{v}^{q} &= \dot{\underline{r}}_{qp}^{q} + R_{p}^{q}\left(\underline{v}^{p} + \underline{\omega}_{p}^{qp}\times\underline{\rho}^{p}\right)\\
&= \dot{\underline{r}}_{qp}^{q} + R_{p}^{q}\underline{v}^{p} + \underline{\omega}_{p}^{q}\times R_{p}^{q}\underline{\rho}^{p}\\
\underline{a}^{q} &= \ddot{\underline{r}}_{qp}^{q} + R_{p}^{q}\left(\underline{a}^{p} + \dot{\underline{\omega}}_{p}^{qpp}\times\underline{\rho}^{p} + \underline{\omega}_{p}^{qp}\times\left(\underline{\omega}_{p}^{qp}\times\underline{\rho}^{p}\right) + 2\underline{\omega}_{p}^{qp}\times\underline{v}^{p}\right)\\
&= \ddot{\underline{r}}_{qp}^{q} + R_{p}^{q}\dot{\underline{v}}^{p} + \dot{\underline{\omega}}_{p}^{q}\times R_{p}^{q}\underline{\rho}^{p} + \underline{\omega}_{p}^{q}\times\left(\underline{\omega}_{p}^{q}\times R_{p}^{q}\underline{\rho}^{p}\right) + 2\underline{\omega}_{p}^{q}\times R_{p}^{q}\underline{v}^{p}
\end{aligned}\right\} \quad \text{(A- 118)}$$

$$\vec{f} = m\,\vec{a}^q \tag{A- 119}$$

Ved å uttrykke akselerasjonen $\vec{a}^q$ vha ledda på høgre sida, får vi de kreftene som må innføres i et ikke-inertial system. Likninga ovenfor blir nå :

$$m\vec{a}^q = m\left(\ddot{\vec{r}}^q_{qp} + \vec{a}^p + \dot{\vec{\omega}}^q_p \times \vec{\rho} + \vec{\omega}^q_p \times \left(\vec{\omega}^q_p \times \vec{\rho}\right) + 2\vec{\omega}^q_p \times \vec{v}^p\right) \tag{A- 120}$$

Løser likninga mhp $m\vec{a}^p$:

$$m\vec{a}^p = m\vec{a}^q - m\ddot{\vec{r}}^q_{qp} - m\dot{\vec{\omega}}^q_p \times \vec{\rho} - m\vec{\omega}^q_p \times \left(\vec{\omega}^q_p \times \vec{\rho}\right) - m2\vec{\omega}^q_p \times \vec{v}^p \tag{A- 121}$$

Apparent force seen from F^p

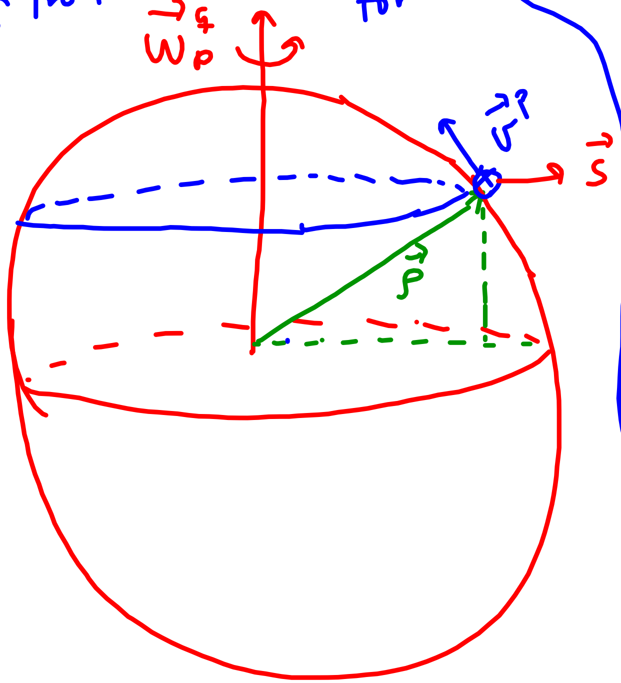
Outer for-

Coriolis force = $\vec{c}$

Centrifugal force ($\vec{s}$)

Angle accelleration term (tangential force)

Force due to accelleration of O_p seen from F^q



A.6 Matrix calculation in cybernetics

Standard equation: $\dot{\underline{x}} = A\underline{x}$, $\underline{x}(0) = \underline{x}_0$

Eigen values: $|\lambda I - A| = 0 \iff$ n'th order equation in λ

$$\lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0 = 0 \Rightarrow \lambda_i \ , i = 1, 2, \dots, n$$

Matlab:
$[M, \Lambda] = \mathrm{eig}(A)$

Eigenvectors: $\underbrace{(\lambda_i I - A)}_{\det = 0}\underline{m}_i = \underline{0}$ } I.e. we have not a uniqe solution. We can choose for example $\|\underline{m}_i\| = 1$

If $\lambda_i \neq \lambda_j$, for all $i \neq j$ one can prove that $\{\underline{m}_i\}$ is linearly independent and may form a basis system.

$M = [\underline{m}_1, \underline{m}_2, \ldots, \underline{m}_n]$ Eigenvector matrix

We had d.e. $\dot{\underline{x}} = A\underline{x}$, $\underline{x}(0) = \underline{x}_0$, since we want to use M as a DCM we need to introduce a clear notation for the two frames we transform between $\{m\}$ and $\{q\}$.

I.e. $\dot{\underline{x}}^q = A^q \underline{x}^q$, $\underline{x}^q(0) = \underline{x}_0^q$

$M = M_m^q = [\underline{m}_1^q, \underline{m}_2^q, \ldots, \underline{m}_n^q]$

$$\left(\dot{\underline{x}}^q = \underbrace{\dot{M}_m^q}_{=0} \underline{x}^m + M_m^q \cdot \dot{\underline{x}}^m \right)$$

$\underline{x}^q = M_m^q \underline{x}^m$

We had: $\dot{\underline{x}}^q = A^q \underline{x}^q = M_m^q \dot{\underline{x}}^m = A^q M_m^q \underline{x}^m$

$\Leftrightarrow \dot{\underline{x}}^m = \underbrace{(M_m^q)^{-1} A^q M_m^q}_{} \underline{x}^m$

$$A^m = \underline{\Lambda}^m = [\lambda_i \delta_{ij}] = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

Generally: $(M^q_m)^{-1} \neq (M^q_m)^T$

We have now: $\underline{\dot{x}}^m = \Lambda^m \underline{x}^m$, $\underline{x}^m(0) = (M^q_m)^{-1} \underline{x}^q(0)$

$\Updownarrow$

$\dot{x}_i^m = \lambda_i^m x_i^m$, $x_i^m(0)$ given

$x_i^m(t) = e^{\lambda_i(t-t_0)} x_i^m(t_0) = e^{\lambda_i t} x_i^m(0)$ since $t_0 = 0$

$\Updownarrow$

$$\boxed{\underline{x}^m(t) = \underbrace{e^{\Lambda^m t}}_{= \left[e^{\lambda_i t} \delta_{ij} \right]} \underline{x}^m(0)}$$

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

Matlab: expm(X)

exp(X) - exp. element by element

F10/ From F.9 : $\dot{\underline{x}}^q = A^q \underline{x}^q$, $\underline{x}^q(0)$ given

This eq. has the solution: $\underline{x}^q(t) = M^q_m \, \underline{x}^m(t)$

$$\underline{x}^q(t) = M^q_m \, e^{\Lambda^m t} \left(M^q_m\right)^{-1} \underline{x}^q(0)$$

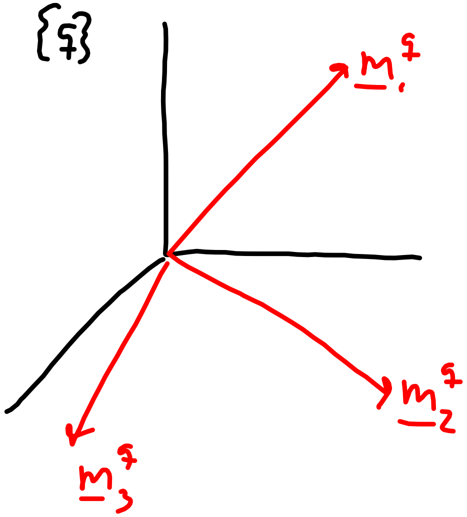
$$\underline{x}^q(t) = e^{M^q_m \Lambda^m (M^q_m)^{-1} t} \, \underline{x}^q(0)$$

$$\underline{x}^q(t) = e^{A^q t} \, \underline{x}^q(0)$$

If M is invertible:

$$e^{MXM^{-1}} = M e^{X} M^{-1}$$

$$\text{where } e^{A^q t} = \underbrace{M^q_m \, e^{\Lambda^m t} \left(M^q_m\right)^{-1}}_{\phi(t,0)} = \underbrace{\left(I + \frac{1}{1!} A^q t + \frac{1}{2!} A^q A^q t^2 + \dots \right)}_{\text{Used in numerical calculations}}$$



$$\underline{x}^q(t) = M^q_m \underline{x}^m(t) = \underline{m}^q_1 x^m_1(t) + \ldots + \underline{m}^q_n x^m_n(t)$$
$$= \underline{m}^q_1 e^{\lambda_1 t} x^m_1(0) + \ldots + \underline{m}^q_n e^{\lambda_n t} x^m_n(0)$$

NB! We have assumed that we have distinctive eigenvalues $\Rightarrow$ linearly independent eigenvectors. But we can also have complex conjugated eigenvalues (which gives us complex eigenvectors). In the figure above we have assumed that the eigenvalues also are real.

Part B DYNAMICS

Dynamics includes:

1) Kinematics
   - Describing the motion by mathematics (Part A)

2) Kinetic
   - Relation between the motion and the forces that makes the motion (math + physics), f. ehs Newton's laws.

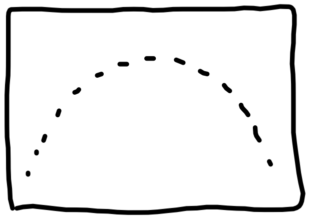
Terms

- Reference space = inertial frame
- Conected to a physical system
- Coordinate systems (frame, units, function)
- Particles (modeled by points and mass)
- Pos., vel., acc. (modeled by vectors)
- Rigid bodies (modeled by a frame and mass)
- Attitude (modeled by frames)

- Affine space
- Mathematical model of a reference space
- $\underset{\rightarrow}{\mathcal{F}}_A^a : \{O_a, \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$
- P: points
- $\vec{v}$: vectors

You can take a look at: Grunnleggende prinsipper i klassisk mekanikk.

Aristotles.



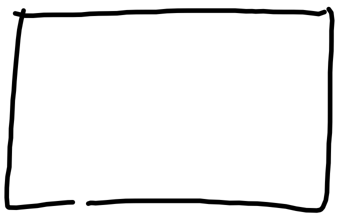
Absolute points

Newton

$$\vec{f} = \frac{d}{dt}(m \cdot \vec{v})$$

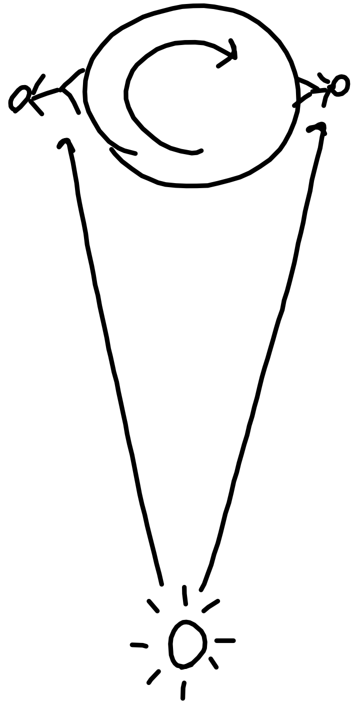
$$\vec{v}_3 = \vec{v}_1 + \vec{v}_2$$

Einstein



$$\|v\| \leq c \text{ (speed of light)}$$

$$\vec{v}_3 \neq \vec{v}_1 + \vec{v}_2$$

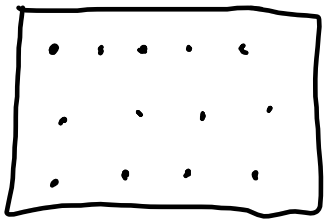


Part B Dynamics

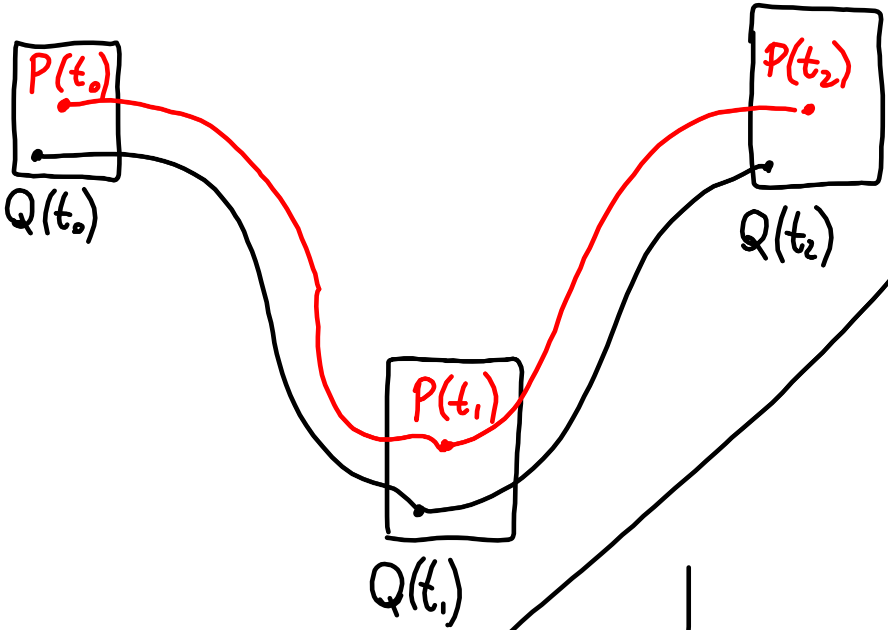
B.1 Kinematics

B.1.1 Kinematic description of particles

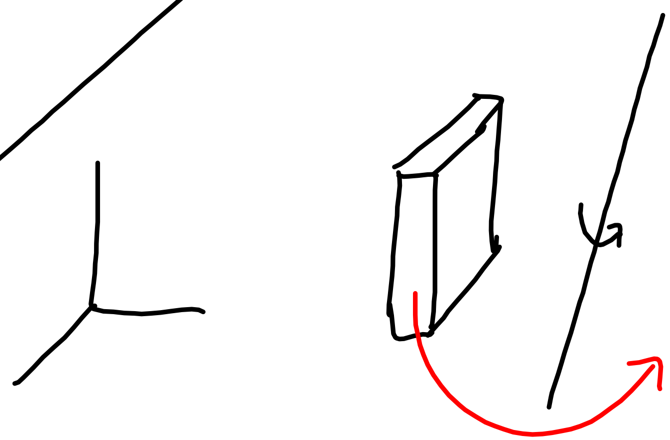
Def. B.1 Rigid body



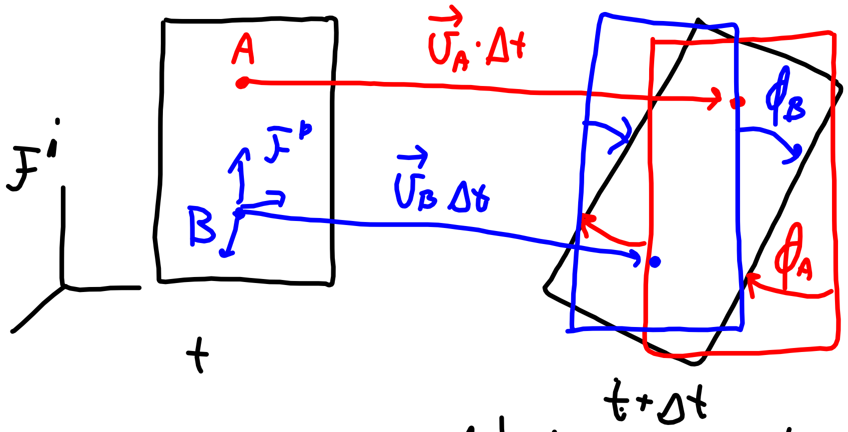
Def. B.2 Pure translation of rigid bodies



Def B.3 Rotation of a rigid body



Teorem B.1 (Chasley's teorem) ***Dekomponering i translasjon og rotasjon***
Bevegelsen av et stivt legeme relativt et k.s. kan settes sammen av translasjon og rotasjon. Dette kan gjøres på følgende måte :
1) Velg et punkt A (B) i legemet. Anta at alle punktene i legemet har samme hastighet, $\vec{v}_A\,(\vec{v}_B)$, hvor $\vec{v}_A\,(\vec{v}_B)$ er hastigheten relativt vårt k.s.
2) Superponer en ren rotasjon om punktet A med vinkelhastighet $\vec{\omega}$ relativt vårt k.s. (NB : $\vec{\omega} = \vec{\omega}_A = \vec{\omega}_B$, mens generelt er $\vec{v}_A \neq \vec{v}_B$ ($\vec{v}_A = \lim_{\Delta t \to 0}(\Delta \vec{r}_A/\Delta t)$).)



$W_B = \lim \frac{\phi_B}{\Delta t}$

$W_A = \lim \frac{\phi_A}{\Delta t}$

$\vec{v}_A \neq \vec{v}_B \;,\; W_A = W_B$

The motion of a rigid body is put together by $\vec{v}^{\,i}_{Ob}(t)$ and $\vec{W}^{\,i}_b(t)$, i is inertial frame

I.e. we can model the motion of a rigid body as the motion of the frame F^b relative F^i

B.2 Kinetic

Newton's laws for a particle

Teorem B.2 (Newtons 1.lov) *Dersom en partikkel er langt borte fra innflytelsen fra alle andre partiler i universet, vil den bevege seg med konstant hastighet mht et treghetssystem,* **i** *(kan egentlig utledes fra Newtons 2.lov).* (N.1 is a special case of N.2)

Teorem B.3 (Newtons 2.lov) *Dersom det lineære moment,* $\vec{p}^{\mathbf{i}}$*, for en partikkel i et treghetssystem* **i** *endres med tiden, sies partikkelen å være påvirket av en kraft,* $\vec{f}$*, gitt ved :*

$$\vec{f} = \dot{\vec{p}}^{\mathbf{i}} \quad hvor \quad \vec{p}^{\mathbf{i}} = m\vec{v}^{\mathbf{i}} \tag{B- 139}$$

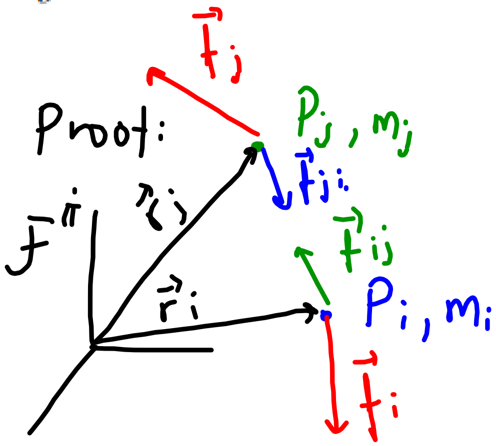
Teorem B.4 (Newtons 3.lov) *Dersom to isolerte partikler interakterer med hverandre vil den krafta partikkel nr 1 utsetter partikkel nr 2 for være lik i størrelse, men motsatt rettet den krafta partikkel nr 2 utsetter partikkel 1 for. Dvs : aksjon = reaksjon eller kraft = motkraft.*

Teorem B.5 ***Newtons 2. lov for et system av partikler***

Vi antar at Newtons 3. lov gjelder for krafta mellom partiklene, dvs $\vec{f}_{ij} = -\vec{f}_{ji}$. *Da vil den totale ytre kraft,* $\vec{F}$, *være lik total masse,* M, *ganger med massesenterets akselerasjon,* $\vec{a}_c^{\,\mathrm{i}}$, *sett fra treghetsramma :*

$$\boxed{\vec{F} = M\frac{{}^{\mathrm{i}}d\,{}^{\mathrm{i}}d}{dt^2}\vec{r}_c = M\,\vec{a}_c^{\,\mathrm{i}}} \qquad M = \sum_{i=1}^{n} m_i\,, \quad \vec{F} = \sum_{i=1}^{n} \vec{f}_i \tag{B- 140}$$

Proof:



$$\vec{r}_c = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i \Rightarrow M\vec{r}_c = \sum_{i=1}^{n} m_i\vec{r}_i$$

$\vec{r}_c$: center of mass

$\vec{f}$: outer force

$\vec{f}_{ij}$: inner force (according to N.3 law)

m_i : mass of particle i

$\vec{r}_i$: position vector of particle i

Newtons 2. law works for all particles.

$$P_i : \vec{f}_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} f_{ij} = m_i a_i^i = m_i \ddot{\vec{r}}_i^{\,ii}$$

$$P_1 : \vec{f}_1 + 0 + \vec{f}_{12} + \vec{f}_{13} + \ldots\ldots + \vec{f}_{1n} = m_1 \frac{d^i}{dt}\left(\frac{d^i}{dt}\vec{r}_1\right)$$

$$P_2 : \vec{f}_2 + \vec{f}_{21} + 0 + \vec{f}_{23} + \ldots\ldots + \vec{f}_{2n} = m_2 \frac{d^i}{dt}\left(\frac{d^i}{dt}\vec{r}_2\right)$$

$$P_3 : \vec{f}_3 + \vec{f}_{31} + \vec{f}_{32} + 0 + \ldots\ldots + \vec{f}_{3n} = m_3 \frac{d^i}{dt}\left(\frac{d^i}{dt}\vec{r}_3\right)$$

$$\vdots$$

$$P_n : \vec{f}_n + \vec{f}_{n1} + \vec{f}_{n2} + \vec{f}_{n3} + \ldots\ldots + 0 = m_n \frac{d^i}{dt}\left(\frac{d^i}{dt}\vec{r}_n\right)$$

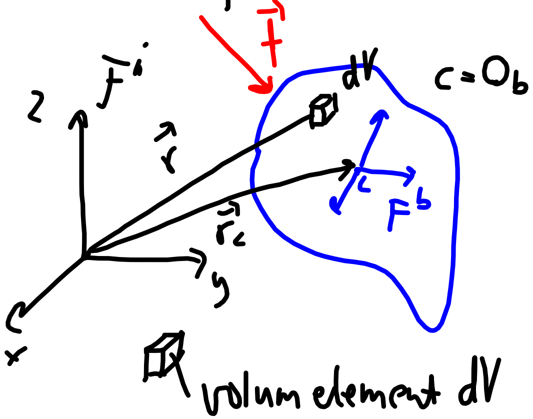
$$\sum P_i : \underbrace{\sum_{i=1}^{n}\vec{f}_i}_{\vec{F}} + \vec{0} = \underbrace{\frac{d^i}{dt}\left(\frac{d^i}{dt}\sum m_i \vec{r}_i\right)}_{M\ddot{\vec{r}}_c^{\,ii}}$$

$$\boxed{\vec{F} = M\vec{a}_c^{\,i}}$$

Since a rigid body can be viewed as a sum of particles (molecyl) we have:

$$\boxed{\vec{f} = m\,\vec{a}_c^{\,i}}$$

where $\vec{f}$ is the total outer-force acting on the body, m is the mass and c is the center of mass.



$$m = \iiint_V k(\vec{r})\,dV$$ where $k(\vec{r})$ is the mass density.

$$\vec{r}_c = \frac{1}{m}\iiint_V \vec{r}\,k(\vec{r})\,dV = \frac{1}{m}\int_z\int_y\int_x \vec{r}\,k(\vec{r})\,dx\,dy\,dz$$

F.11 / B.2.2 Law of angular momentum ("Spinn satsen")

We want to define the law of angular momentum of a rigid body that gives us the relation between outer torque, the inertia matrix and angular accelleration. We do this in three steps.

1. Law of angular momentum for one particle.
2. —— " —— n particles.
3. —— " —— rigid body

Teorem B.6 ***Spinnsatsen for en partikkel***
Gitt en partikkel, P, som er utsatt for en kraft, F . La A være et vilkårlig punkt i treghetssystemet **i** *($\vec{r} = \vec{r}_A + \vec{\rho}_A$). Da er sammenhengen mellom momentet og spinnet om punktet A:*

(≙ Definition)

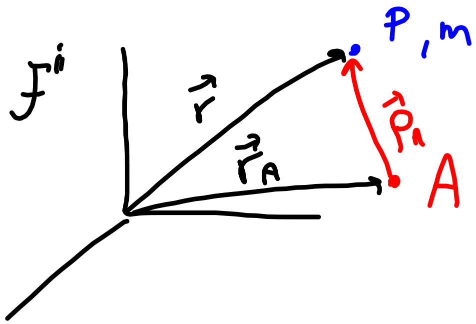
$$\vec{n}_A = \dot{\vec{h}}^{\mathbf{i}}_A + \vec{\rho}_A \times (m\ddot{\vec{r}}^{\mathbf{i}}_A) \quad hvor$$

$$\vec{n}_A \triangleq \vec{\rho}_A \times \vec{F} \tag{B- 141}$$

$$\vec{h}^{\mathbf{i}}_A \triangleq \vec{\rho}_A \times (m\dot{\vec{\rho}}^{\mathbf{i}}_A) \tag{B- 142}$$

Dersom $\vec{\rho}_A \times (m\ddot{\vec{r}}^{\mathbf{i}}_A) = \vec{0}$, dvs bla når A oppfyller 1 eller 2 :
1). $\ddot{\vec{r}}^{\mathbf{i}}_A = \vec{0}$: A har konstant hastighet i treghetssystemet (ligger f.eks i ro).
2). $\vec{\rho}_A \parallel \ddot{\vec{r}}^{\mathbf{i}}_A$: A akselererer mot/fra partikkel P
så kan spinnsatsen skrives :

$$\boxed{\vec{n}_A = \dot{\vec{h}}^{\mathbf{i}}_A} \tag{B- 143}$$



$$\vec{r} = \vec{r}_A + \vec{\rho}_A$$

Def:

Torque (moment) around A: $\vec{n}_A = \vec{\rho} \times \vec{f}$

Angular momentum (spin) –ıı– : $\vec{h}_A^{\,i} = \vec{\rho} \times (m \dot{\vec{\rho}}^{\,i})$

Note: Torque and angular momentum can be defined differently in different text books.

What is the relationship between torque $\vec{n}_A$ and the angular momentum $\vec{h}_A^{\,i}$?

Answer: $\vec{n}_A = \dot{\vec{h}}_A^{\,i} + \underbrace{\vec{\rho}_A \times (m \ddot{\vec{r}}_A^{\,i})}_{\text{When } = 0?\ \text{How to choose } A.}$

Proof:

$$\vec{n}_A \underset{\text{Def.}}{\triangleq} \vec{\rho} \times \vec{f} \underset{\text{N.2}}{=} \vec{\rho} \times \left(m \ddot{\vec{r}}^{\,i}\right) = \vec{\rho} \times \left(m \frac{d^{i\,2}}{dt^2}\left(\vec{r}_A + \vec{\rho}\right)\right)$$

$$= \vec{\rho} \times m \ddot{\vec{r}}_A^{\,i} + \underbrace{\vec{\rho} \times m \ddot{\vec{\rho}}^{\,i}}_{= \dot{\vec{h}}_A^{\,i}\ ?}$$

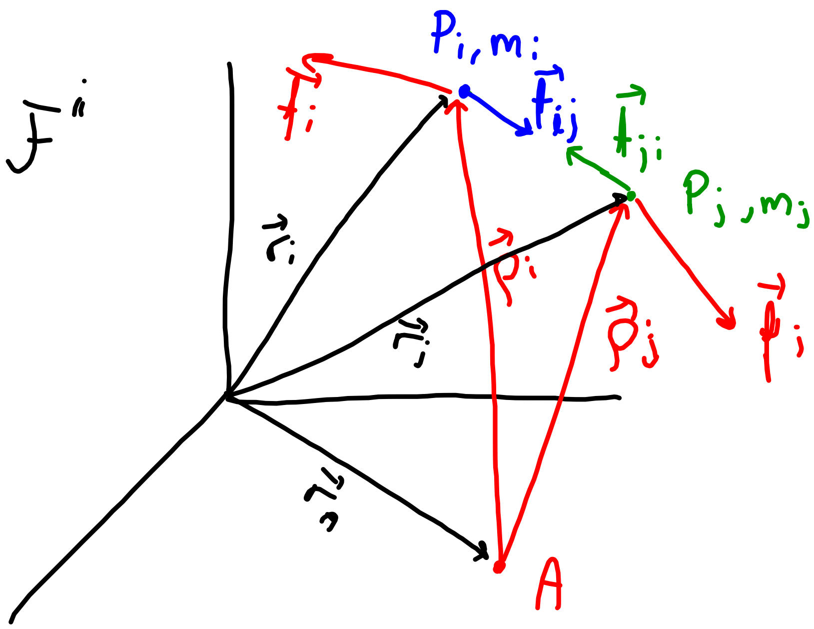
Yes, they are.

$$\vec{h}_A^{\,i} \triangleq \vec{\rho} \times \left(m \dot{\vec{\rho}}^{\,i}\right)$$

$$\dot{\vec{h}}_A^{\,i} = \underbrace{\dot{\vec{\rho}}^{\,i} \times \left(m \dot{\vec{\rho}}^{\,i}\right)}_{=0} + \vec{\rho} \times \left(m \ddot{\vec{\rho}}^{\,i}\right)$$

$$\Rightarrow \boxed{\vec{n}_A = \dot{\vec{h}}_A^{\,i} + \vec{\rho} \times m \ddot{\vec{r}}_A^{\,i}} \quad \text{q.e.d}$$

Law of ang. mom. for a n-particle system



A: arbitrary point

$\mathcal{F}^i$: inertial frame

P_i : particle nr. i

m_i : mass of P_i

$\vec{f}_i$: outer force on P_i

$\vec{f}_{ij}$: force P_j acts on P_i

Assume that $\vec{f}_{ij} = -\vec{f}_{ji}$ and

$\vec{f}_{ij} \parallel \vec{r}_i - \vec{r}_j = \vec{r}_{ij}$, i.e.

central forces

For P_i we have:

$$\vec{n}_{Ai} = \vec{\rho}_i \times \left(\vec{f}_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} \vec{f}_{ij}\right)$$

$$\dot{\vec{h}}_{Ai} = \vec{\rho}_i \times (m\,\ddot{\vec{\rho}}_i)$$

For n-particles:

$$\vec{n}_A = \sum_{i=1}^{n} \vec{n}_{Ai} \quad : \text{Total torque around } A$$

$$\dot{\vec{h}}_A = \sum_{i=1}^{n} \dot{\vec{h}}_{Ai} \quad : \text{Total angular momentum (spin) around } A$$

What is the relationship between $\vec{n}_A$ and $\dot{\vec{h}}_A$?

Teorem B.7 ***Spinnsatsen for et n-partikkel system***

Gitt et system av n partikler hvor partikkel i er utsatt for den ytre kraft $\vec{F}_i$ og krafta $\vec{f}_{ij}$ fra partikkel j ($j = 1, \ldots, n$) antas å oppfylle Newtons 3.lov ($\vec{f}_{ij} = -\vec{f}_{ji}$) og i tillegg være en sentralkraft (ligger langs $\vec{r}_i - \vec{r}_j$). Da vil for et vilkårlig punkt A i treghetssystemet **i***:*

$$\vec{n}_A = \dot{\vec{h}}^{\mathbf{i}}_A + \sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{r}}^{\mathbf{i}}_A \quad \textit{hvor} \tag{B- 144}$$

$$\vec{n}_A = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \vec{F}_i \quad \textit{totalt ytre moment om A} \tag{B- 145}$$

$$\vec{h}_A = \sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \dot{\vec{\rho}}^{\mathbf{i}}_{Ai} \quad \textit{totalt spinn om A} \tag{B- 146}$$

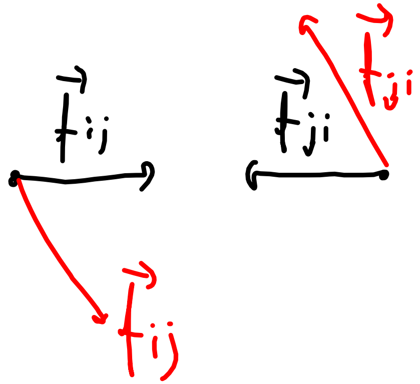
Dersom $\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{r}}^{\mathbf{i}}_A = 0$, dvs bl.a. når A oppfyller 1, 2 eller 3 :
1). $\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} = \vec{0}$: A er i massesenteret. normal thing to do
2). $\ddot{\vec{r}}^{\mathbf{i}}_A = \vec{0}$: A har konstant hastighet i treghetsrommet (f.eks. i ro).
3). $\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \parallel \ddot{\vec{r}}^{\mathbf{i}}_A$: A akselererer mot massesenteret.
så kan spinnsatset for et n-partikkel system skrives :

$$\vec{n}_A = \dot{\vec{h}}^{\mathbf{i}}_A \tag{B- 147}$$

Proof:

Sum $\vec{h}_{Ai}^{i}$ over all i and do the same with $\vec{n}_{Ai}$.

Use Newtons. 2nd law and see that all cross terms dissapears because $\vec{f}_{ij} + \vec{f}_{ji} = 0$ and they are central forces.



$\vec{f}_{ij} = -\vec{f}_{ji}$ are central forces.

$\vec{f}_{ij} = -\vec{f}_{ji}$ are not central forces

Proof of B.7

$$\vec{n}_A = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \Big(\vec{f}_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \vec{f}_{ij} \Big)$$

$$= \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \vec{f}_i + \underbrace{\vec{\rho}_{A1} \times (0 + \vec{f}_{12} + \dots + \vec{f}_{1n}) + \vec{\rho}_{A2} \times (\vec{f}_{21} + 0 + \dots + \vec{f}_{2n}) + \dots + \vec{\rho}_{An} \times (\vec{f}_{n1} + \vec{f}_{n2} + \dots + 0)}_{= \vec{0}}$$

$$\boxed{\vec{n}_A = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \vec{f}_i}$$

$= \vec{0}$ because:

$$\vec{\rho}_{Ai} \times \vec{f}_{ij} + \vec{\rho}_{Aj} \times \vec{f}_{ji}$$

$$= \vec{\rho}_{Ai} \times \vec{f}_{ij} - \vec{\rho}_{Aj} \times \vec{f}_{ij}$$

$$= (\vec{\rho}_{Ai} - \vec{\rho}_{Aj}) \times \vec{f}_{ij}$$

$$= \vec{0}$$

because $\vec{\rho}_{Ai} - \vec{\rho}_{Aj} \parallel \vec{f}_{ij}$

(central forces)

$$\vec{n}_A = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \vec{f}_i = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \left(m_i \ddot{\vec{r}}_i^{\,i}\right) = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \left(m_i \frac{d^{2\,i}}{dt^2}\left(\vec{r}_i\right)\right)$$

$$= \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \left(m_i \frac{d^{2\,i}}{dt^2}\left(\vec{r}_A + \vec{\rho}_{Ai}\right)\right)$$

$$= \sum_{i=1}^{n} \vec{\rho}_{Ai} \times m_i \ddot{\vec{r}}_A^{\,i} + \underbrace{\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{\rho}}_{Ai}^{\,i}}_{= \dot{\vec{h}}_A^{\,i} \,?}$$

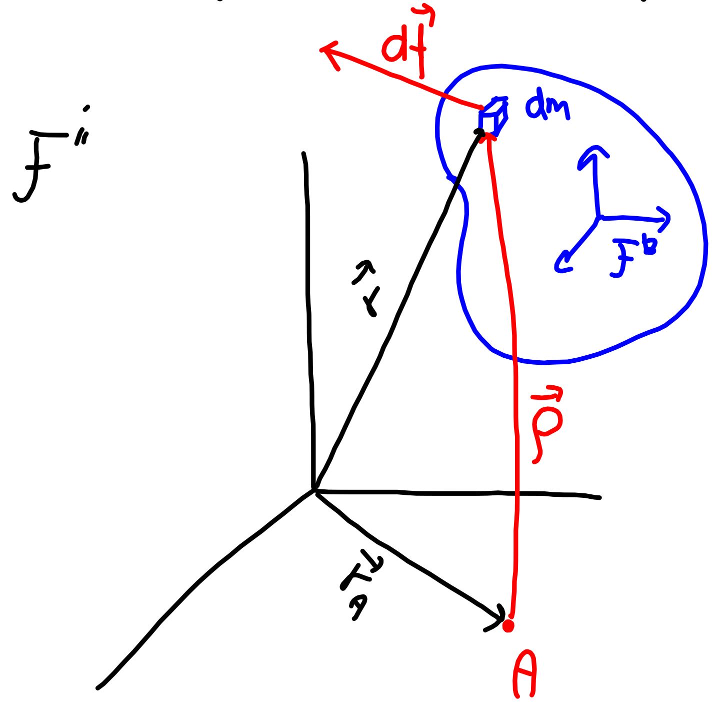
$$\dot{\vec{h}}_A^{\,i} = \frac{d^i}{dt}\left(\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \dot{\vec{\rho}}_{Ai}^{\,i}\right)$$

Yes, they are!

$$= \underbrace{\sum_{i=1}^{n} m_i \dot{\vec{\rho}}_{Ai}^{\,i} \times \dot{\vec{\rho}}_{Ai}^{\,i}}_{= \vec{0}} + \sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{\rho}}_{Ai}^{\,i}$$

$$\Rightarrow \boxed{\vec{n}_A = \dot{\vec{h}}_A^{\,i} + \sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{r}}_A^{\,i}} \quad \text{q.e.d.}$$

Law of angular momentum (spinnsatzen) for a rigid body



$\mathcal{F}^i$: inertial frame

$\mathcal{F}^b$: frame fixed to the body

$d\vec{f}$: outer force acting on the mass element dm

$dm = k(\vec{r})\, dV$ where

$k(\vec{r})$ is the mass density.

b- body

The body is rigid, i.e. the distance between the molecyls (particles) is const.
We have already found the law of angular momentum for a n-particle system.
We want to use integrals instead of sums (Σ)

We let $n \to \infty$: $\Sigma \to \iiint_M dm$ or $\iiint_V k(\vec{r})\, dV$

M: total mass of body
V: volume of body

Particles $\to$ mass differentials

$$\vec{h}_{Ai}^{\,i} = \vec{\rho}_i \times \left(m_i \dot{\vec{\rho}}_i^{\,i}\right) \Rightarrow d\vec{h}_A^{\,i} = \vec{\rho} \times \left(\dot{\vec{\rho}}^{\,i}\, dm\right)$$

$$\vec{h}_A^{\,i} = \iiint d\vec{h}_A^{\,i} = \iiint_M \vec{\rho} \times \dot{\vec{\rho}}^{\,i}\, dm$$

$$= \iiint_M \vec{\rho} \times \left(\dot{\vec{\rho}}^{\,b} + \vec{\omega}_b^{\,i} \times \vec{\rho}\right) dm$$

If A is fixed to the body:

$$\vec{h}_A^{\,i} = \iiint_M \vec{\rho} \times (\vec{\omega}_b^{\,i} \times \vec{\rho})\, dm = \underbrace{-\iiint_M dm\, \vec{\rho} \times (\vec{\rho} \times}_{\mathbb{J}_A} \vec{\omega}_b^{\,i}) = \mathbb{J}_A\, \vec{\omega}_b^{\,i}$$

Total outer torque:

one-particle: $\vec{n}_{Ai} = \vec{\rho}_i \times \vec{f}_i$

n-particles: $\vec{n}_A = \sum \vec{\rho}_i \times \vec{f}_i \implies \vec{n}_A = \iiint_M \vec{\rho} \times d\vec{f}$

What is the relationship between $\vec{n}_A$ and $\vec{h}_A^{\,i}$?

Teorem B.8 ***Spinnsatsen for stive legemer***

Gitt treghetssystemet **i** *og et k.s. b som ligger fast i legemet og har sitt origo i A. Dersom A tilfredstiller 1 eller 2 :*

1). A ligger i massesenteret.

2). A ligger i ro i treghetsrommet.

er spinnsatsen på en koordinatuavhengig form :

Theorem A.19

$$\vec{n}_A = \dot{\vec{h}}^{\mathbf{i}}_A = \dot{\vec{h}}^{\mathbf{ib}}_A + \vec{\omega}^{\mathbf{i}}_b \times \vec{h}^{\mathbf{i}}_A \qquad \text{(B- 148)}$$

eller representert i b-systemet :

Theorem A.18

$$\underline{n}^b_A = \dot{\underline{h}}^{\mathbf{i}bb}_A + \underline{\omega}^{\mathbf{i}b}_b \times \underline{h}^{\mathbf{i}b}_A = J^b \dot{\underline{\omega}}^{\mathbf{i}bb}_b + \underline{\omega}^{\mathbf{i}b}_b \times (J^b \underline{\omega}^{\mathbf{i}b}_b) \qquad \text{(B- 149)}$$

$$\dot{\underline{\omega}}^{\mathbf{i}bb}_b = R^b_{\mathbf{i}} \dot{\underline{\omega}}^{\mathbf{i}}_b$$

hvor spinnet er definert ved :

$$\underline{h}^{\mathbf{i}b}_A = J^b \underline{\omega}^{\mathbf{i}b}_b$$

$$\vec{n}_A = \iiint_M \vec{\rho} \times d\vec{f} \quad \text{(only outer forces)}$$

$$\vec{h}^{i}_A = -\iiint_M \vec{\rho} \times (\vec{\rho} \times \vec{\omega}^{i}_b)\, dm = \mathbb{J}_A \vec{\omega}^{i}_b$$

$$J^b = [\mathbb{J}_A]^b$$

Treghetsmatrisa J^b *beregnes via treghetsmomenta,* J^b_{ii}, *og treghetsproduktа,* J^b_{ij} :

$$J^b = \begin{bmatrix} J^b_{xx} & -J^b_{xy} & -J^b_{xz} \\ -J^b_{yx} & J^b_{yy} & -J^b_{yz} \\ -J^b_{zx} & -J^b_{zy} & J^b_{zz} \end{bmatrix} \qquad \text{(B- 150)}$$

$$\begin{bmatrix} J^b_{xx} & -J^b_{xy} & -J^b_{xz} \\ -J^b_{yx} & J^b_{yy} & -J^b_{yz} \\ -J^b_{zx} & -J^b_{zy} & J^b_{zz} \end{bmatrix} = \begin{bmatrix} \int_M (y^2+z^2)\,dm & -\int_M xydm & -\int_M xzdm \\ -\int_M xydm & \int_M (x^2+z^2)\,dm & -\int_M yzdm \\ -\int_M xzdm & -\int_M yzdm & \int_M (x^2+y^2)\,dm \end{bmatrix} \qquad \text{(B- 151)}$$

Dvs treghetsmatrisa er symmetrisk.

NB! Here is $\rho^b = \begin{bmatrix} \rho_1^b \\ \rho_2^b \\ \rho_3^b \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

F.12/ How to calculate the angular momentum $\vec{h}_c^i = \mathbb{J}_c \vec{\omega}_b^i$

If we represent $\vec{\rho}$ in $\mathcal{F}^i$ the matrix representation of $\mathbb{J}_c$ becomes tim variant, but represented in the body frame $\mathcal{F}^b$ it wil be time invariant.

We therefore choose to calculate $\vec{h}_c^i$ in $\mathcal{F}^b$.

Ang. mom:

$$\vec{h}_c^i = -\iiint_M \vec{\rho} \times (\vec{\rho} \times \vec{\omega}_b^i)\, dm = \mathbb{J}_c \vec{\omega}_b^i, \quad \text{where } O_b = A = C$$

$$\underline{h}_c^{ib} = -\iiint_M \rho^b \times (\rho^b \times \underline{\omega}_b^{ib})\, dm = -\iiint S(\rho^b) S(\rho^b) \underline{\omega}_b^{ib}\, dm$$

$$= \underbrace{\left(-\iiint_M S(\rho^b) S(\rho^b)\, dm\right)}_{J_c^b = [\mathbb{J}_c]^b} \underline{\omega}_b^{ib} = J_c^b \underline{\omega}_b^{ib}$$

Summary Kinetic equations for the center of mass ($C = A = O_b$)

$$\vec{p}_c^{\,i} = m \cdot \vec{v}_c^{\,i} \quad , \quad \vec{h}_c^{\,i} = -\iiint_M \vec{\rho} \times (\vec{\rho} \times \vec{\omega}_b^{\,i})\,dm = \mathbb{J}_c \vec{\omega}_b^{\,i}$$

N.2 $\quad \vec{f} = \dot{\vec{p}}_c^{\,i} = \frac{d}{dt}\left(m\,\vec{v}_c^{\,i}\right) = \dot{\vec{p}}_c^{\,ib} + \vec{\omega}_b^{\,i} \times \vec{p}_c^{\,i} \quad \Big| \quad \underline{f}^i = m\,\dot{\underline{v}}_c^i \; , \; \underline{f}^b = \dot{\underline{p}}_c^{ibb} + \overbrace{\underline{\omega}_b^{ib}}^{S(\underline{\omega}_b^{ib})} \times \underline{p}^{ib}$

Ang. mom. $\quad \vec{n}_c = \dot{\vec{h}}_c^{\,i} = \frac{d}{dt}\left(\mathbb{J}_c \vec{\omega}_b^{\,i}\right) = \dot{\vec{h}}_c^{\,ib} + \vec{\omega}_b^{\,i} \times \vec{h}_c^{\,i} \quad \Big| \quad \underline{n}_c^i = \dot{\underline{h}}_c^i \; , \; \underline{n}_c^b = \dot{\underline{h}}_c^{ibb} + \underbrace{\underline{\omega}_b^{ib}}_{S(\underline{\omega}_b^{ib})} \times \underline{h}_c^{ib}$

$$\boxed{\begin{aligned} \underline{h}_c^{ib} &= J_c^b \, \underline{w}_b^{ib} \\ \underline{n}_c^b &= J_c^b \, \underline{\dot{w}}_b^{ibb} + S(\underline{w}_b^{ib}) \, J_c^b \, \underline{w}_b^{ib} \end{aligned}}$$ — Gives B-149

J_c^b : Inertia matrix

$$J_c^b = -\iiint_M S(\underline{\rho}^b) S(\underline{\rho}^b) \, dm \quad , \quad \underline{\rho}^b = [\rho_1 ; \rho_2 ; \rho_3]$$

$$= -\iiint_M \begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix} dm$$

$$= + \iiint_M \begin{bmatrix} \rho_2^2 + \rho_3^2 & -\rho_1 \rho_2 & -\rho_1 \rho_3 \\ -\rho_2 \rho_1 & \rho_1^2 + \rho_3^2 & -\rho_2 \rho_3 \\ -\rho_3 \rho_1 & -\rho_3 \rho_2 & \rho_1^2 + \rho_2^2 \end{bmatrix} dm$$

$$J_c^b = [J_c^b]^T$$

Moment of inertia

$$[J_c^b]_{ii} = \iiint_M (\rho_j^2 + \rho_k^2) \, dm, \quad j \neq k, \; j \neq i, \; k \neq i$$

$$[J_c^b]_{ij} = -\iiint_M \rho_i \rho_j \, dm, \quad i \neq j$$

Product of inertia

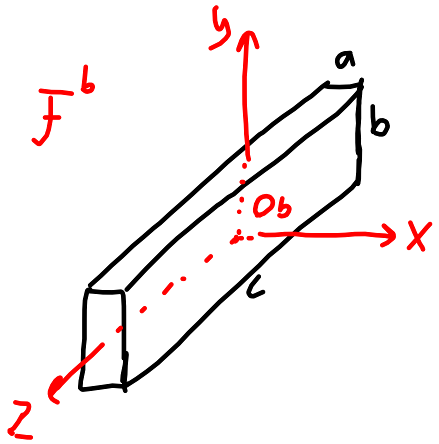
J_c^b is a real, symmetric, positive definite matrix ($Det(J_c^b) > 0$)

$\Rightarrow$ matrix has real eigenvalues and orthogonal eigenvectors

Eigenvectors are called the main axis of the body. I.e. if $\mathcal{F}^b$ has the basis vectors along the main axis:

$$J_c^b = diag(J_{xx}^b, J_{yy}^b, J_{zz}^b)$$

Example: Inertia matrix of a brickwall "murstein"



Assume: $a < b < c$

$\mathcal{F}^b$ has origo in the center of mass O_b

$$\underline{r}^b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Crossterms:

$$\left[J_c^s\right]_{xy} = -\iiint_M xy\,dm = -\iiint_M xy\,k\,dx\,dy\,dz \quad , \; k: \text{mass density (const.)}$$

$$= -k \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} y \left(\int_{-a/2}^{a/2} x\,dx \right) dy\,dz = 0 \quad \text{because of symmetry}$$

$$\left[J_c^s\right]_{xx} = k \iiint (y^2+z^2)\,dx\,dy\,dz = k \iint (y^2+z^2) \left(\int_{-a/2}^{a/2} 1\,dx \right) dy\,dz$$

$$= k\,a \iint (y^2+z^2)\,dy\,dz = ka \iint y^2\,dy\,dz + ka \iint z^2\,dy\,dz$$

$$= k\,a\,c \int_{-b/2}^{b/2} y^2\,dy + k\,a\,b \int_{-c/2}^{c/2} z^2\,dz$$

$$= akc \left[\frac{1}{3} y^3\right]_{-b/2}^{b/2} + akb \left[\frac{1}{3} z^3\right]_{-c/2}^{c/2} = akc\,\frac{2}{3}\,\frac{b^3}{8} + akb\,\frac{2}{3}\,\frac{c^3}{8}$$

$$= \underbrace{k\,abc}_{M} \frac{b^2}{12} + \underbrace{k\,abc}_{M} \frac{c^2}{12} = \frac{M}{12}(b^2+c^2)$$

$$J_c^b = \mathrm{diag}\left(\frac{M}{12}(b^2+c^2) \ , \ \frac{M}{12}(a^2+c^2) \ , \ \frac{M}{12}(a^2+b^2) \right)$$

We had $a < b < c \Rightarrow J_{xx}^b > J_{yy}^b > J_{zz}^b$

Euler equation

When we put A in the centre of mass C, $\mathcal{F}^b$ is fixed to the body and $O_b = C = A$, the law of angular momentum becomes:

① $$\underline{n}_c^b = J_c^b \dot{\underline{w}}_b^{ibb} + S(\underline{w}_b^{ib}) J_c^b \underline{w}_b^{ib}$$

Assume $\mathcal{F}^b$ coinsides with the main axis, i.e. J_c^b is diagonal, and $\underline{n}_c^b = [n_x ; n_y ; n_z]$, $\underline{w}_b^{ib} = [w_x ; w_y ; w_z]$, $J_c^b = \mathrm{diag}(J_{xx}^b , J_{yy}^b , J_{zz}^b)$.

In this case we get the Euler equations.

Teorem B.9 ***Eulerlikningene***
Dersom k.s. b velges fast i legemet med origo i A, med akser langs hovedaksene for legemet og A i tillegg tilfredstiller 1 eller 2 :
1). A ligger i massesenteret.
2). A ligger i ro i treghetsrommet. $A = C = O_b$
kan spinnsatsen skrives på følgende enkle form :

$$\left.\begin{aligned} n_x &= J^b_{xx}\dot{\omega}_x + \omega_y\omega_z(J^b_{zz} - J^b_{yy}) \\ n_y &= J^b_{yy}\dot{\omega}_y + \omega_z\omega_x(J^b_{xx} - J^b_{zz}) \\ n_z &= J^b_{zz}\dot{\omega}_z + \omega_x\omega_y(J^b_{yy} - J^b_{xx}) \end{aligned}\right\}, \quad \underline{n}^b_A = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}, \quad \underline{\omega}^{ib}_b = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{B- 152}$$

Proof: Insert into eq. 1

$$\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} J^b_{xx}\dot{w}_x \\ J^b_{yy}\dot{w}_y \\ J^b_{zz}\dot{w}_z \end{bmatrix} + \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix} \begin{bmatrix} J^b_{xx} w_x \\ J^b_{yy} w_y \\ J^b_{zz} w_z \end{bmatrix} = \begin{bmatrix} J^b_{xx}\dot{w}_x + w_y w_z (J^b_{zz} - J^b_{yy}) \\ J^b_{yy}\dot{w}_y + w_x w_z (J^b_{xx} - J^b_{zz}) \\ J^b_{zz}\dot{w}_z + w_x w_y (J^b_{yy} - J^b_{xx}) \end{bmatrix}$$

Euler equations can be used in 2 ways:

1) Given the forces ($\underline{n}_c^b$) find the motion ($\underline{w}_b^{ib}$). Differential eq.

2) Given the motion ($\underline{w}_b^{ib}$) find the forces ($\underline{n}_c^b$). Algebraic eq.

Solution of 1)

(2)
$$\boxed{\begin{aligned} \dot{w}_x &= \frac{1}{J_{xx}}\left[(J_{yy}-J_{zz})w_y w_z + n_x\right] \\ \dot{w}_y &= \frac{1}{J_{yy}}\left[(J_{zz}-J_{xx})w_x w_z + n_y\right] \\ \dot{w}_z &= \frac{1}{J_{zz}}\left[(J_{xx}-J_{yy})w_x w_y + n_z\right] \end{aligned}}$$

Eq. (2) is on standard form:

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}), \quad \underline{x}(t_0) \text{ given}$$

For d.e. (2): $\underline{x} = \underline{w}_b^{ib}$, $\underline{u} = \underline{n}_c^b$

To find the orientation/attitude:

$$\boxed{\begin{aligned} &\dot{R}_b^i = R_b^i S(\underline{w}_b^{ib}), \quad R_b^i(t_0) \text{ given} \\ &\dot{\underline{w}}_b^{ibb} = \underline{f}(\underline{w}_b^{ib}, \underline{n}_c^b), \quad \underline{w}_b^{ib}(t_0) \text{ given} \end{aligned}}$$

F.13/ Instead of the d.e in DCM we can use the d.e. for Euler angles (A.5)

$$\dot{\underline{\theta}} = D_b^{\theta}(\underline{\theta})\,\underline{w}_b^{ib}$$

$$\dot{\underline{w}}_b^{ibb} = \underline{f}(\underline{w}_b^{ib}, \underline{n}_c^{b})$$

$\underline{\theta}(t_0)$ and $\underline{w}_b^{ib}(t_0)$ is given

$$\underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} \quad \text{(3-2-1 Euler Ang.)}$$

$D_p^{\theta}(\underline{\theta})$ is given in A-105

$$D_p^{\theta}(\underline{\theta}) = \begin{bmatrix} 1 & \sin\theta_1 \tan\theta_2 & \cos\theta_1 \tan\theta_2 \\ 0 & \cos\theta_1 & -\sin\theta_2 \\ 0 & \sin\theta_1/\cos\theta_2 & \cos\theta_1/\cos\theta_2 \end{bmatrix}$$

Here $\underline{X} = \begin{bmatrix} \underline{\theta} \\ \underline{w}_b^{ib} \end{bmatrix}$

Note! D.e. for $\underline{w}_b^{ib}$ can be solved without solving d.e. for $\underline{\theta}$, but not the other way

B.3 Torque free motion of a rigid body

Assume $J_{xx} > J_{yy} > J_{zz}$

We want to calculate $\underline{\omega}_b^{ib}$ and $\underline{\dot{\omega}}_b^{ib}$ (we find trajetories, not the time function)

Ellipsoid of inertia

Inertia matrix $J_c^b = \left[J_c^b \right]^T$

$\Rightarrow$ pos. definite matrix and by using the expression:

$J = \frac{1}{2} \underline{x}^T J_c^b \underline{x}$: ellipsoide when J is constant.

If we chose $\vec{F}^b$ in the main axis.

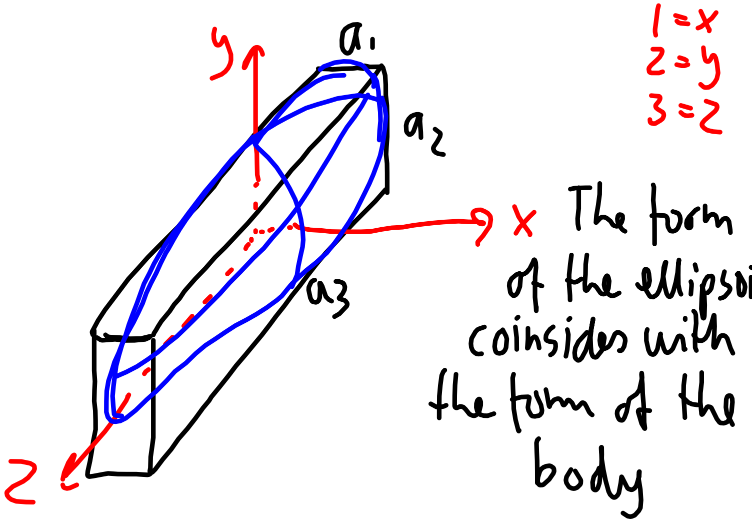
$$J = \frac{1}{2} \underline{x}^T \begin{bmatrix} J_{xx} & & \\ & J_{yy} & \\ & & J_{zz} \end{bmatrix} \underline{x} = \frac{1}{2} \sum_{i=1}^{3} J_{ii} x_i^2$$

This eq on std. form :

$$\boxed{\frac{x_1^2}{2J/J_{11}} + \frac{x_2^2}{2J/J_{22}} + \frac{x_3^2}{2J/J_{33}} = 1}$$

Half axis: $a_i = \sqrt{2J/J_{ii}}$ for $i=1,2,3$

$a_1 < a_2 < a_3$



The form of the ellipsoid coinsides with the form of the body

Ellipsoide of inertia represents the form of the body (w.r.t. rotation)

Kinetic rotation energy ellipsoide

Multiply the Euler equations (B-152) (assume $\underline{n}_c^b = 0$, i.e. torque free motion) with ω_x, ω_y and ω_z for line 1, 2 and 3 respectively, and add them up.

$$0 = \sum_{i=1}^{3} J_{ii}\, \dot{w}_i\, w_i \qquad \begin{array}{|l} 1=x \\ 2=y \\ 3=z \end{array}$$

Integrate w.r.t time:

$$\int_{t_0}^{t} \sum_{i=1}^{3} J_{ii} \frac{dw_i}{\cancel{dt}} w_i \cancel{dt} = \int_{t_0}^{t} 0\, dt = 0$$

$$\Big| _{t_0}^{t} \frac{1}{2} \sum_{i=1}^{3} J_{ii}\, w_i^2(t) = 0$$

$$\frac{1}{2} \sum_{i=1}^{3} J_{ii}\, w_i^2(t) = \frac{1}{2} \sum_{i=1}^{3} J_{ii}\, w_i^2(t_0) = K_0$$

I.e.

$$\boxed{\frac{w_1^2(t)}{2K_0/J_{11}} + \frac{w_2^2(t)}{2K_0/J_{22}} + \frac{w_3^2(t)}{2K_0/J_{33}} = 1}$$

Half axis: $\sqrt{\dfrac{2K_0}{J_{ii}}}$

We see that the kinetic rotationenergy ellipsoid has the same form as the ellipsoide of inertia (can choose $J = K_0$)

$\boxed{\underline{w}_b^{ib}(t) \text{ has to stay on the kinetic rot. energy. ellipsoide}}$

Angular momentum ellipsoide (spinnellipsoide)

When the outer force is zero ($\vec{n}_c = \vec{0}$) $\Rightarrow$ $\dot{\vec{h}}_c^{ii} = \vec{0}$

and $\| \vec{h}_c^{i} \| = h_0$ (constant length) and the same direction in $\mathcal{F}^{i}$.

Angular velocity seen from $\mathcal{F}^b$

$$\underline{h}_c^{ib} = J_c^b \, \underline{w}_b^{ib} = \begin{bmatrix} J_{xx} w_x \\ J_{yy} w_y \\ J_{zz} w_z \end{bmatrix}$$

choose {b} in the main axis

the length of $\underline{h}_c^{ib}$ is constant:

$$\left(\underline{h}_c^{ib}\right)^T \underline{h}_c^{ib} = h_0^2$$

$$\sum_{i=1}^{3} J_{ii}^2 w_i^2 = h_0^2$$

This can be writen on std. form of an ellipsoide.

$$\boxed{\frac{w_1^2(t)}{h_0^2/J_{11}^2} + \frac{w_2^2(t)}{h_0^2/J_{22}^2} + \frac{w_3^2(t)}{h_0^2/J_{33}^2} = 1}$$

Half axis: h_0/J_{ii}

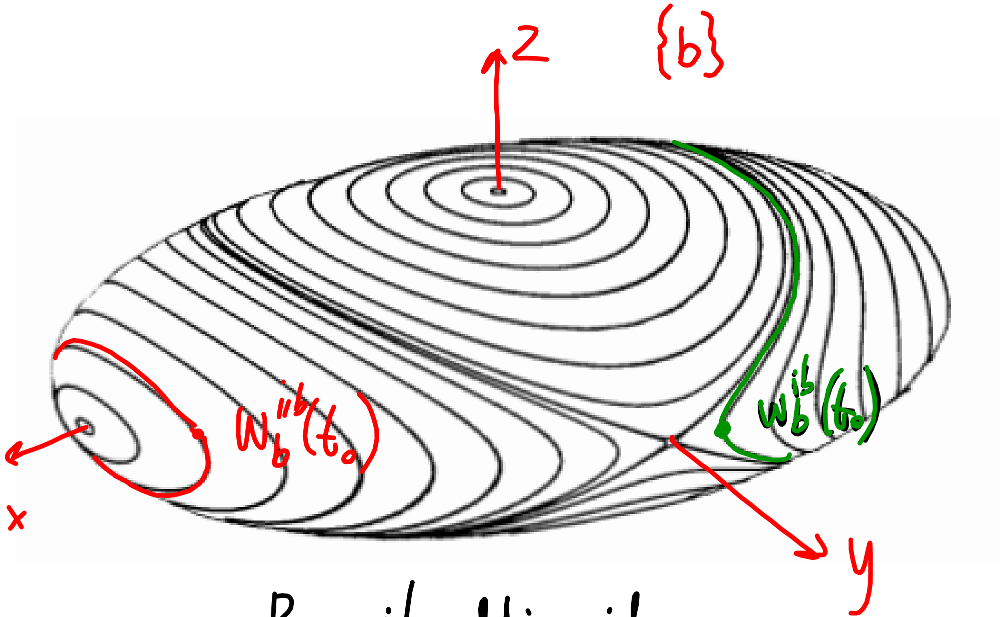
We see that the kin. rot. energy ellipsoide and the angular momentum ellipsoide do not have the same half axis

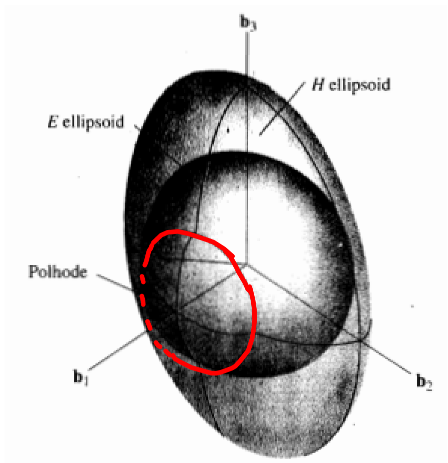
=> different form of the ellipsoids and they need to intercect.

$\underline{w}_b^{ib}(t)$ is on the intersection (pole hod) of the two ellipsoids

B.3.1 Beskrivelse av bevegelsen sett fra b-systemet

Teorem B.13 ***Bevegelsen av et stivt legeme sett fra det roterende b-systemet***
Anta b-systemet faller sammen med hovedaksene for det stive legemet. For et stivt legeme som ikke er utsatt for ytre moment beveger vinkelhastighetsvektoren ($\underline{\omega}_b^{ib}$) *seg da, sett fra b-systemet, på skjeringa (polhode) mellom spinnellipsoida og den kinetiske rotasjonsenergiellipsoida. Bevegelsen til det stive legemet er i hvert øyeblikk en ren rotasjon om vinkelhastighetsvektoren.*





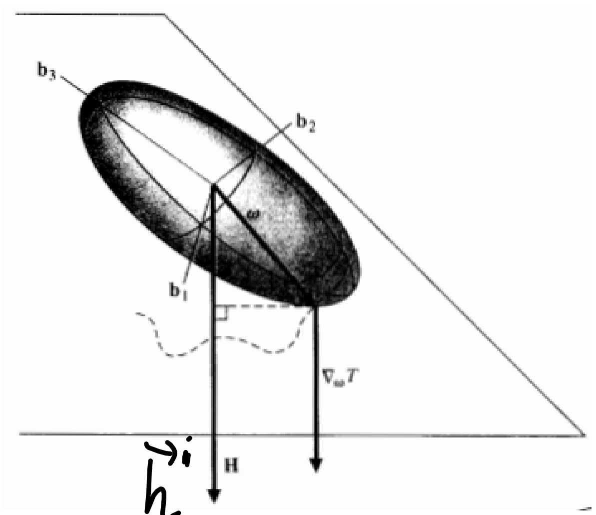
Benoit ellipsoide
(Poinsot?)

B.3.2 Beskrivelse av bevegelsen sett fra i-systemet

Teorem B.14 ***Bevegelsen av et stivt legeme sett fra treghhetssystemet* i**
Bevegelsen av et stivt legeme som ikke er utsatt for et ytre moment er beskrevet, sett fra treghet-srommet, av at den kinetiske energiellipsoida ruller på det invariable plan (plan ⊥ spinnvektoren $\vec{h}^{\mathbf{i}}$) uten å gli. Rullinga følger polhodet på den kinetiske energiellipsoida (kontaktpunktet mellom det invariable plan og ellipsoida er dermed enden på $\vec{\omega}_b^{\mathbf{i}}$-vektoren).

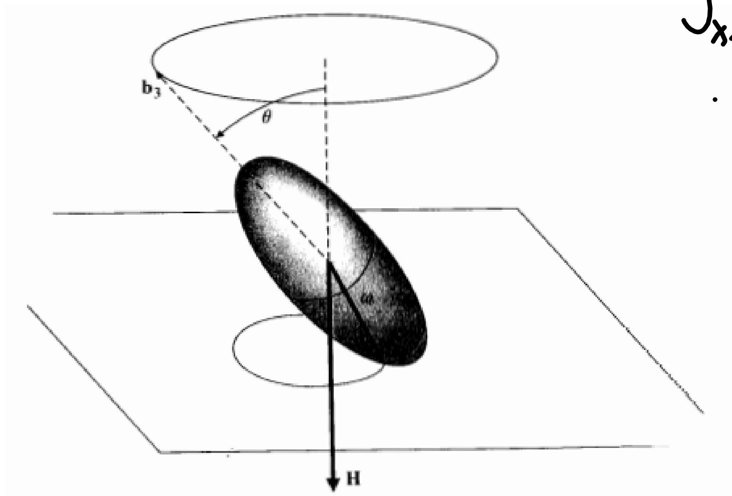
$J_{xx} = J_{yy} > J_{zz}$

· Axis. symetrical



$\vec{h}_c^{\,i}$

constant

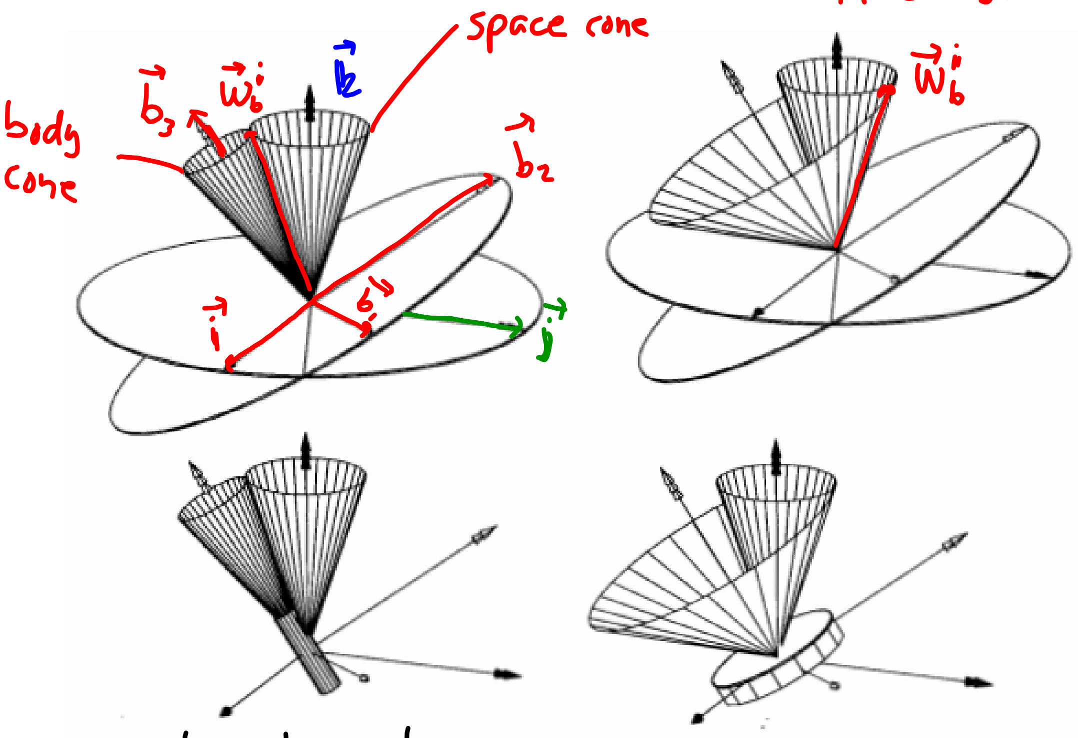


B.4 Torque free motion of a axis symmetrical body.

Assume $J_{xx} = J_{yy} \neq J_{zz}$

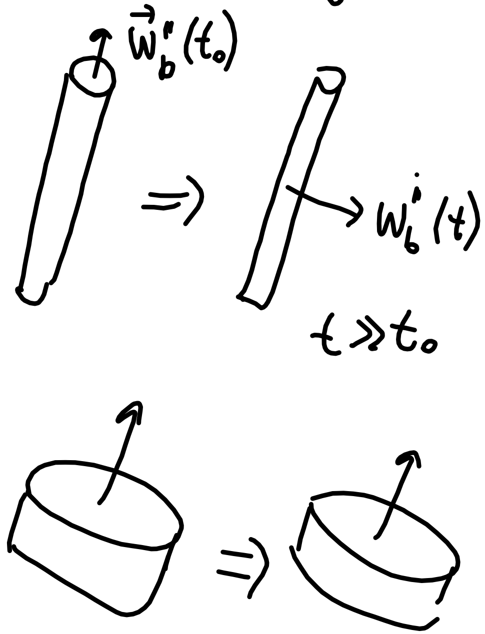
$\mathcal{F}^i = \{\vec{i}, \vec{j}, \vec{k}\}$

$\mathcal{F}^b = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$



$J_{xx} = J_{yy} > J_{zz}$ $J_{xx} = J_{yy} < J_{zz}$

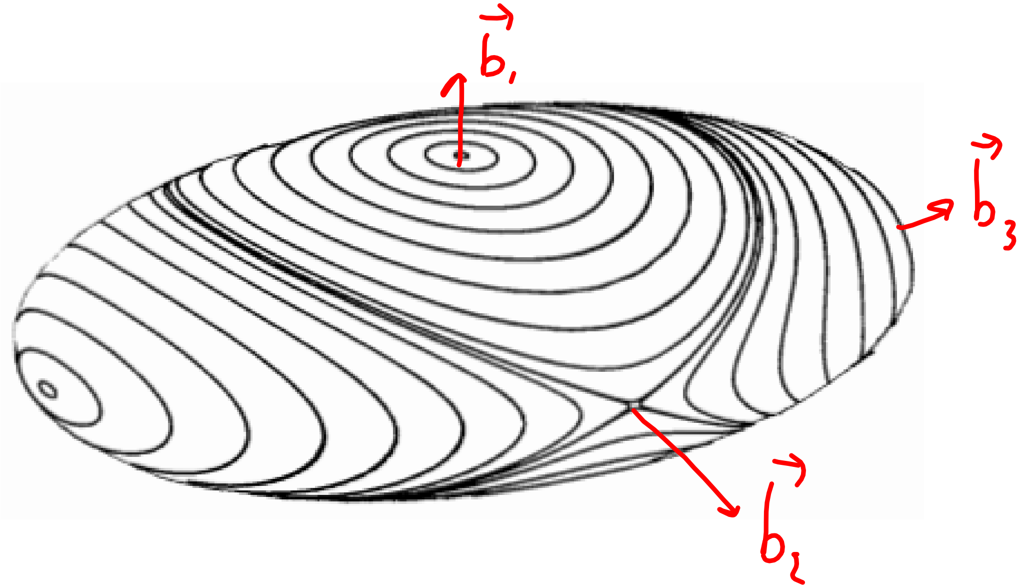
What is the stationary rotation of a cylinder?



B.3.3 Stabilitet om hovedaksene

Vi vil her undersøke stabiliteten til bevegelsen for små perturbasjoner om hovedaksene for et stivt legeme. Ovenfor så vi på bevegelsene i stort.

Teorem B.15 ***Stabiliteten om hovedaksene for et stivt legeme***
Anta b-systemet faller sammen med hovedaksene for det stive legemet og $J^b_{xx} > J^b_{yy} > J^b_{zz}$. Da gir linearisering om $\vec{b}_1$-aksen (J^b_{xx}) eller $\vec{b}_3$-aksen (J^b_{zz}) et lineært system med kompleks konjugerte egenverdier. Linearisering om $\vec{b}_2$-aksen (J^b_{yy}) gir et lineært system med to egenverdier, den ene ligger i venstre halvplan den andre i høgre.



B.3.3 Stability of main axis

1) Rotation around b_1-axis (x-axis)

$\omega_{x0} > 0$

$\alpha_1 > 0$

$\alpha_2 > 0$

Assume $|\omega_x| \gg |\omega_y| \approx |\omega_z|$, set $\omega_y \omega_z \approx 0$

Euler equations under these assumptions:

$$\dot{\omega}_x = (J_{yy} - J_{zz})\,\omega_y \omega_z / J_{xx} = 0 \Rightarrow \omega_x(t) = \omega_{x0}$$

$$\dot{\omega}_y = (J_{zz} - J_{xx})\,\omega_x \omega_z / J_{yy} = \frac{J_{zz} - J_{xx}}{J_{yy}} \omega_z \omega_{x0} = -\alpha_1 \omega_z \Rightarrow \dot{\omega}_y = -\alpha_1 \omega_z$$

$$\dot{\omega}_z = (J_{xx} - J_{yy})\,\omega_x \omega_y / J_{zz} = \frac{J_{xx} - J_{yy}}{J_{zz}} \omega_y \omega_{x0} = \alpha_2 \omega_y \Rightarrow \dot{\omega}_z = \alpha_2 \omega_y$$

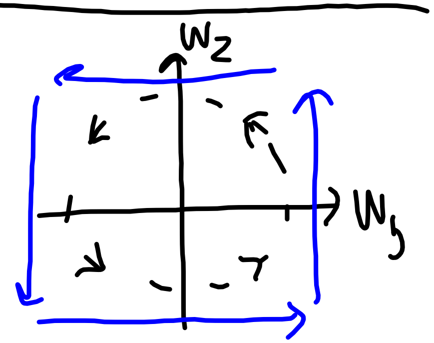
$$\begin{bmatrix} \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} 0 & -\alpha_1 \\ \alpha_2 & 0 \end{bmatrix} \begin{bmatrix} \omega_y \\ \omega_z \end{bmatrix}$$

$$\dot{\underline{x}} = A \cdot \underline{x}$$

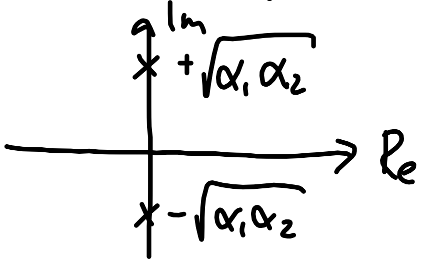
Eigenvalues:

$$|\lambda I - A| = \begin{vmatrix} \lambda & \alpha_1 \\ -\alpha_2 & \lambda \end{vmatrix} = \lambda^2 + \alpha_1 \alpha_2 = 0$$

$$\lambda_{1,2} = \pm\sqrt{-\alpha_1 \alpha_2} = \pm\sqrt{\alpha_1 \alpha_2}\; i$$



F.14 / From F.13, page 11.



We know the solution has the form:

$$\boxed{w_y(t) = A\sin(wt+\phi)}, \quad w = 2\pi f$$

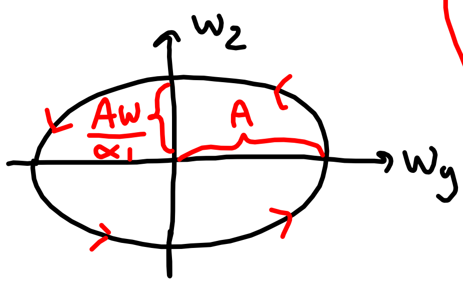
From d.e. $\dot{w}_y = -\alpha_1 w_z$

$$\boxed{w_z(t) = -\frac{1}{\alpha_1} A\, w \cos(wt+\phi)}$$

$$\dot{w}_z(t) = \frac{1}{\alpha_1} A w^2 \sin(wt+\phi) = \alpha_2 w_y$$

$$\Rightarrow w^2 = \alpha_1 \alpha_2$$

Given $w_b^{ib}(t_0)$



We can show:

$$w = w_{x0}\sqrt{\frac{(J_{xx}-J_{zz})(J_{xx}-J_{yy})}{J_{yy}J_{zz}}}$$

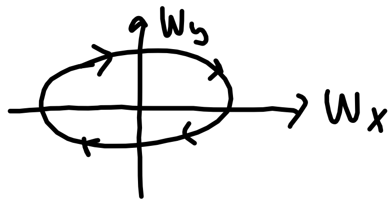
$$A = \frac{w_y(0)}{\sin\phi}$$

$$\phi = \arctan\left(\frac{-w\, w_y(0)}{\alpha_1 w_z(0)}\right)$$

2) Rotation around b_3-axis.

Assume $|w_z| \gg |w_x| \approx |w_y|$, $w_x w_y \approx 0$

$\Rightarrow$ Complex conjugated eigenvalues and $w_z(t) = w_{z0}$



3) Rotation around b_2-axis (y-axis)

Assume $|\omega_y| \gg |\omega_x| \approx |\omega_z| \Rightarrow \omega_x \omega_z \approx 0$

$$\dot{\omega}_x = \frac{J_{yy}-J_{zz}}{J_{xx}} \omega_{y0} \omega_z = \beta_1 \omega_z \,, \quad \beta_1 > 0$$

$$\dot{\omega}_y = \frac{J_{zz}-J_{xx}}{J_{yy}} \omega_x \omega_z = 0 \Rightarrow \omega_y(t) = \omega_{y0}$$

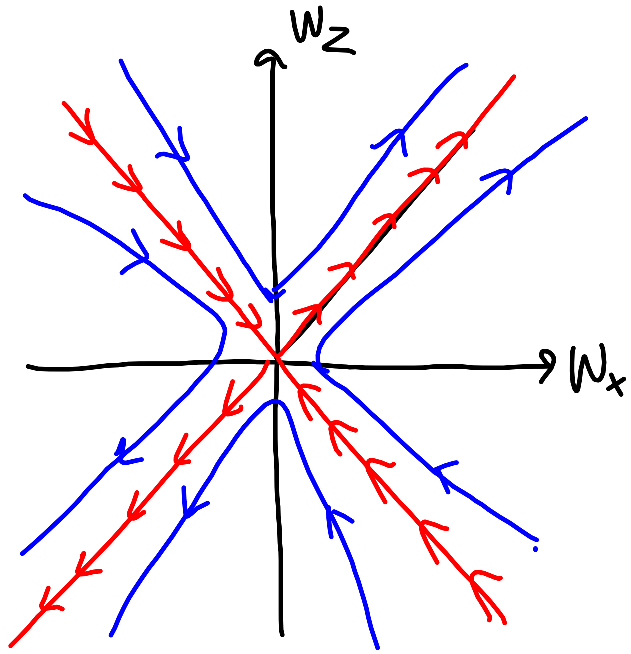
$$\dot{\omega}_z = \frac{J_{xx}-J_{yy}}{J_{zz}} \omega_{y0} \omega_x = \beta_2 \omega_x \,, \quad \beta_2 > 0$$

$$\begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} 0 & \beta_1 \\ \beta_2 & 0 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_z \end{bmatrix}$$

$$\dot{\underline{x}} = A \underline{x}$$

$$\dot{\omega}_x = \beta_1 \omega_z$$

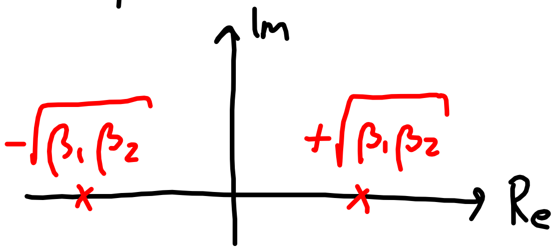
$$\dot{\omega}_z = \beta_2 \omega_x$$



Eigenvalues :

$$|\lambda I - A| = \begin{vmatrix} \lambda & -\beta_1 \\ -\beta_2 & \lambda \end{vmatrix} = \lambda^2 - \beta_1\beta_2 = 0$$

$$\lambda_{1,2} = \pm\sqrt{\beta_1\beta_2}$$

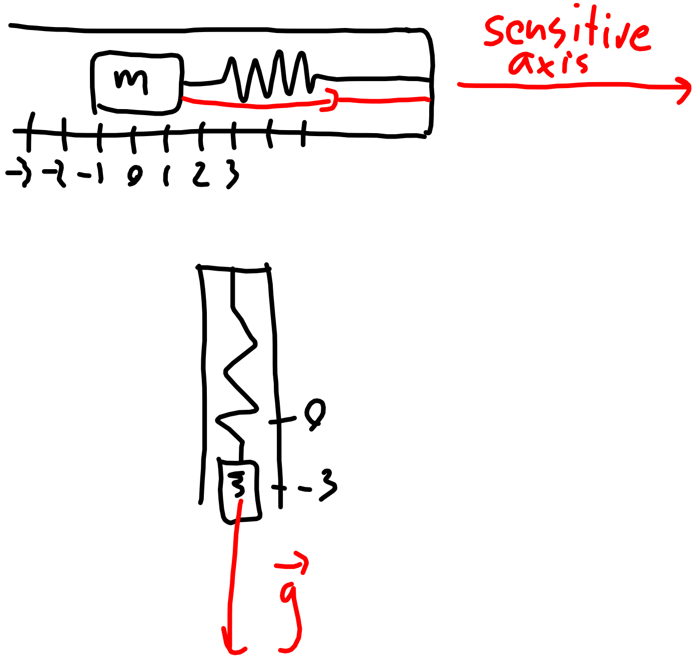


$$\begin{bmatrix} W_x(t) \\ W_y(t) \end{bmatrix} = M \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{-\lambda_1 t} \end{bmatrix} M^{-1} \begin{bmatrix} W_x(0) \\ W_y(0) \end{bmatrix}$$

Unstable !

Part E: Inertial navigation system (INS)

Accellerometer



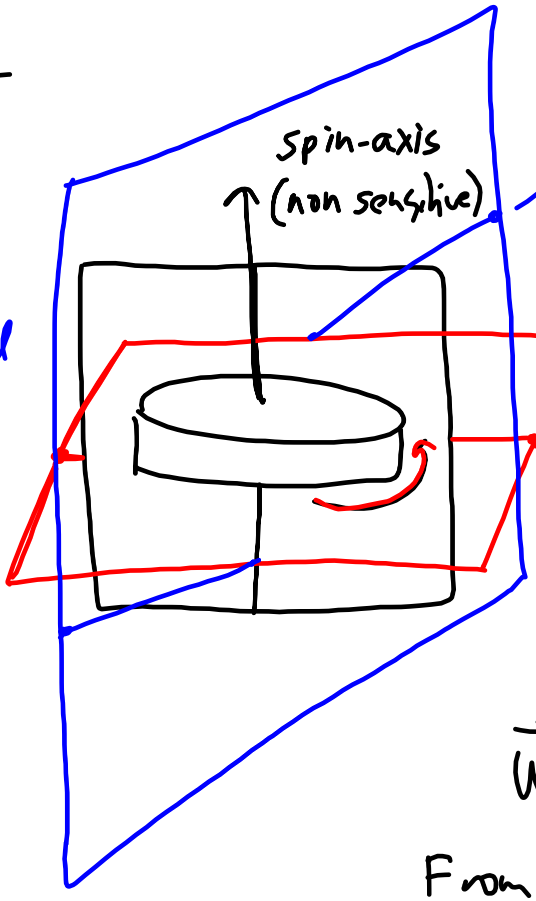
Measuring specific force:

$$\underline{f}^a = \underline{a}^{iia} - \underline{g}^a$$

$$\underline{a}^{iia} = \underline{f}^a + \underline{g}^a$$

Gyro

Mechanical gyro



sensitive axis

sensitive axis

Types of gyros:

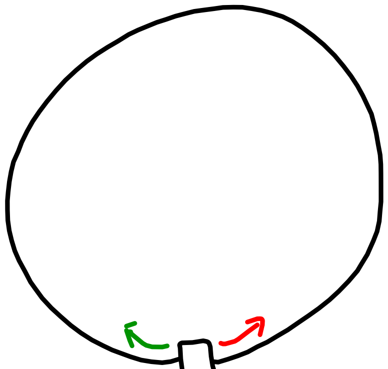
Mehanical gyros

Ring laser gyro (RLG)

Fiber optics gyro (FOG)

MEMS

USING LIGHT



light source

Measureing:

$$\vec{w}_g^{\,i} - \underline{w}_g^{ig}$$

From before:

$$\dot{R}_g^i = R_g^i \, S(\underline{w}_g^{ig})$$

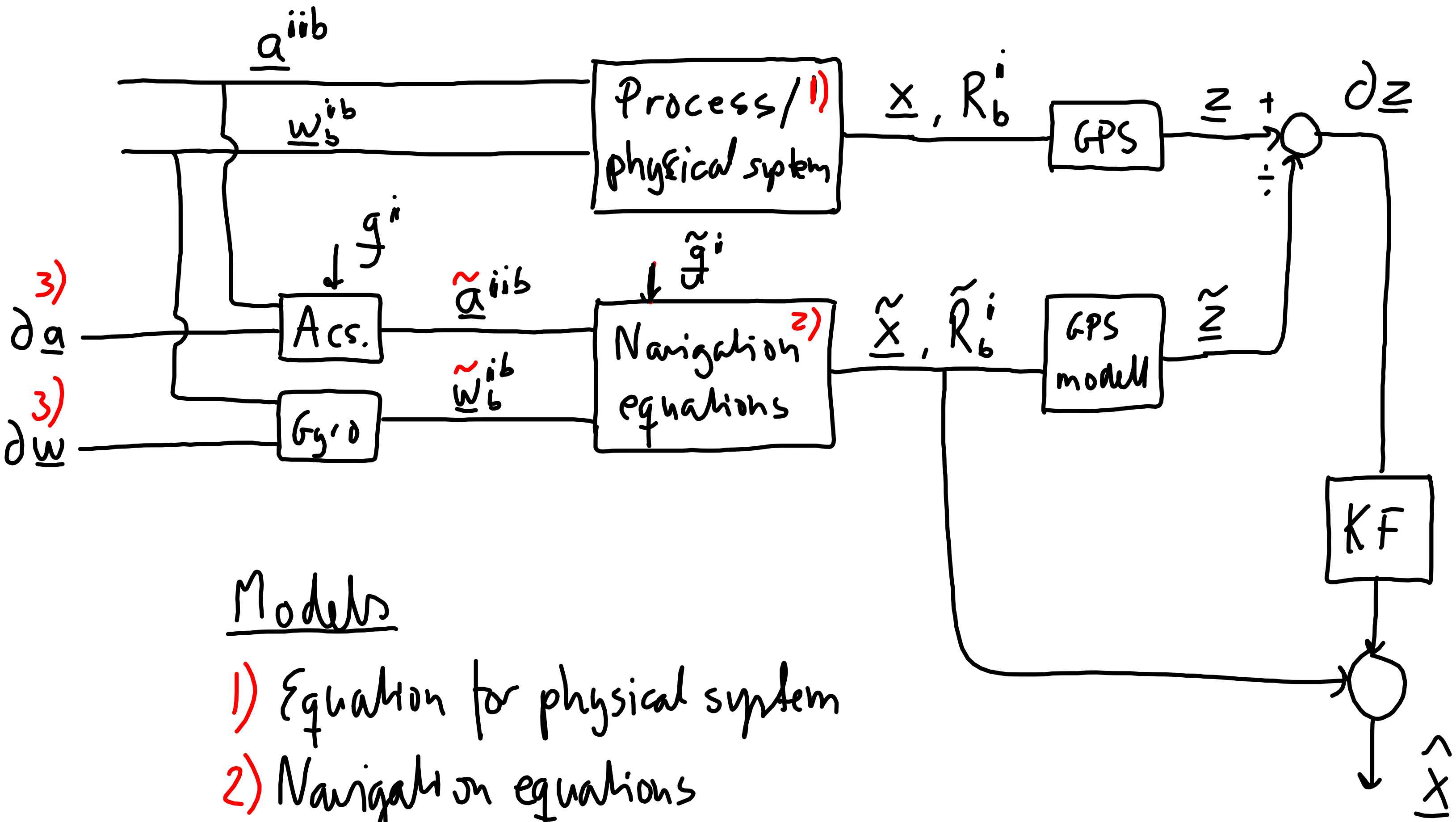
. Navigation equations

$\mathcal{F}^a = \mathcal{F}^g = \mathcal{F}^b$ - body frame

$$\boxed{\begin{aligned} \dot{\tilde{\underline{p}}}^i &= \tilde{\underline{v}}^i \\ \dot{\tilde{\underline{v}}}^i &= \tilde{R}^i_b \, \tilde{\underline{f}}^b + \tilde{\underline{g}}^i \\ \dot{\tilde{R}}^i_b &= \tilde{R}^i_b \, S(\tilde{\underline{w}}^{ib}_b) \end{aligned}}$$

$\tilde{(\cdot)}$: measured component

$\tilde{(\cdot)}$: calculated component



Models

1) Equation for physical system
2) Navigation equations
3) Error models

Part C: Mathematical modelling of airplane

This part builds on:

Chin-Fang Lin

Advanced Control Systems Design

chapter 13.4

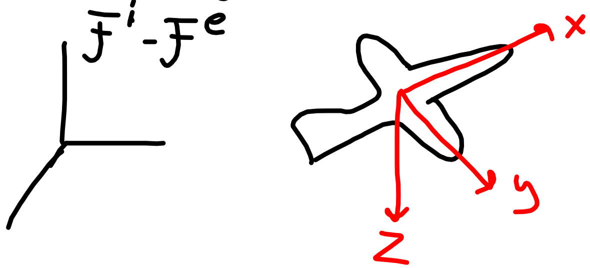
We will write Newtons 2. law and the law of angular momentum in the form (notation) that is normal to give the d.e. for airplanes.

We will also give the relationship between a state vector as in our equations (given in part B) and the one used as standard for airplanes.

We will not deal with airodynamics, i.e. the relation between the planes orientation, velocity and "pådrag" and those forces and torques this results in due to the air flow.

application

Law of angular momentum in $\mathcal{F}^b$



From part B:

$$\underline{n}_c^b = J_c^b \, \dot{\underline{w}}_b^{ibb} + S(\underline{w}_b^{ib}) \, J_c^b \, \underline{w}_b^{ib}$$

$\mathcal{F}^i$: fixed to the earth

$\mathcal{F}^b$: fixed to the airplane

Assume that $\mathcal{F}^b$ coinsides with the main axis of the plane $\Rightarrow$ Eulerequations

$$(1) \quad \begin{aligned} n_x &= J^b_{xx}\dot{\omega}_x + \omega_y\omega_z(J^b_{zz} - J^b_{yy}) \\ n_y &= J^b_{yy}\dot{\omega}_y + \omega_z\omega_x(J^b_{xx} - J^b_{zz}) \\ n_z &= J^b_{zz}\dot{\omega}_z + \omega_x\omega_y(J^b_{yy} - J^b_{xx}) \end{aligned}$$

Compare (1) with (13-72)

$$\underline{n}_c^b = [n_x, n_y, n_z]^T = [L, M, N]^T$$

$$\underline{w}_b^{ib} = [w_x, w_y, w_z]^T = [P, Q, R]$$

$$J_c^b = \mathrm{diag}(J_{xx}, J_{yy}, J_{zz}) = \mathrm{diag}(I_{xx}, I_{yy}, I_{zz})$$

F15 N.2.

Questions Friday 11. dec
13.15 - 16.00

From Part B:

(2) $$\underline{f}^{b} = m\,\dot{\underline{v}}^{ibb} + m\,S(\underline{w}_b^{i})\,\underline{v}^{ib}$$

Write out eq. (2) and compare with (13-73)

$$\begin{bmatrix} f_x^b \\ f_y^b \\ f_z^b \end{bmatrix} = m\begin{bmatrix} \dot{V}_x^{ibb} \\ \dot{V}_y^{ibb} \\ \dot{V}_z^{ibb} \end{bmatrix} + m\begin{bmatrix} 0 & -W_z & W_y \\ W_z & 0 & -W_x \\ -W_y & W_x & 0 \end{bmatrix}\begin{bmatrix} V_x^{ib} \\ V_y^{ib} \\ V_z^{ib} \end{bmatrix}$$

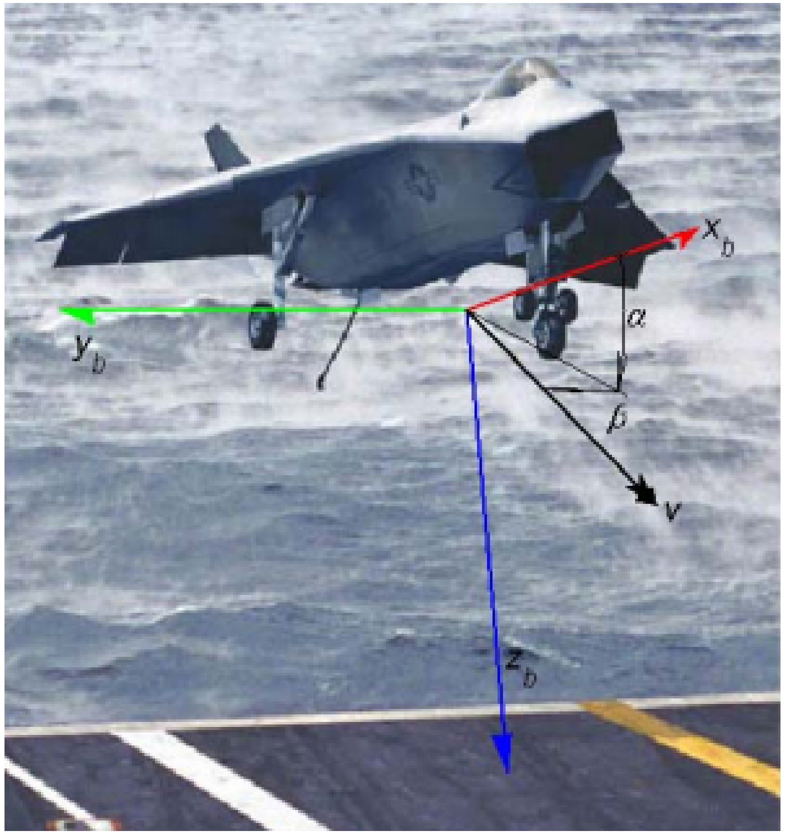
$$\begin{bmatrix} f_x^b \\ f_y^b \\ f_z^b \end{bmatrix} = m\begin{bmatrix} \dot{V}_x + W_yV_z - W_zV_y \\ \dot{V}_y + W_zV_x - W_xV_z \\ \dot{V}_z + W_xV_y - W_yV_x \end{bmatrix}$$

$$\underline{w}_b^{ib} = [P, Q, R]^T, \quad \underline{v}^{ib} = [U, V, W]^T$$

$$\underline{f}^{b} = [F_x + T_x + g_x,\ F_y + g_y,\ F_z + g_z]^T$$

F_x, F_y, F_z : aerodynamic forces
T_x : thrust from engine
g_x, g_y, g_z : gravity comp. in $\mathcal{F}^b$

$$\underline{g}^b = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = R_i^b \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} \underline{g}^i, \quad g = 9.81\,m/s^2$$



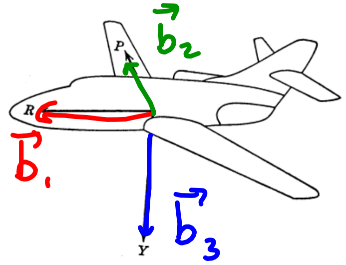


Figure 3.2 Body frame.

P : pitch
R : roll
Y : yaw

α: angle of attac
β: sideslip angle
V_m: speed

From the figure and $\underline{v}^{ib} = [U, V, W]$

$$V_m = \sqrt{U^2 + V^2 + W^2} = \| \underline{v}^{ib} \|$$

$$\alpha = \arctan\left(\frac{W}{U}\right)$$

$$\beta = \arcsin\left(\frac{V}{V_m}\right)$$

We have a uniqe relation between $\underline{v}^{ib}$ and V_m, α, β

We will write N.2 with V_m, α, β, and the law of angular momentum with orientation / attitude given by 3-2-1 Euler angles $\underline{\Theta} = [\phi, \theta, \psi]$

$$\dot{V}_m = \frac{1}{m}\{\cos\alpha\cos\beta(F_x + g_x + T_x) + \sin\beta(F_y + g_y) + \sin\alpha\cos\beta(F_z + g_z)\}$$

$$\dot{\alpha} = Q + \frac{1}{V_m\cos\beta}\{-PV_m\cos\alpha\sin\beta - RV_m\sin\alpha\sin\beta - \sin\alpha(F_x + g_x + T_x) + \cos\alpha(F_z + g_z)\} \quad (13\text{-}74)$$

$$\dot{\beta} = P\sin\alpha - R\cos\alpha + \frac{1}{mV_m}\{-\cos\alpha\sin\beta(F_x + g_x + T_x) + \cos\beta(F_y + g_y) - \sin\alpha\sin\beta(F_z + g_z)\}$$

○: aero. dyn. forces

Newtons 2. law (N.2) ②

$$\left.\begin{aligned}\dot{\Theta} &= \cos\Phi Q - \sin\Phi R\\ \dot{\Phi} &= P + \sin\Phi\tan\Theta Q + \cos\Phi\tan\Theta R\\ \dot{\Psi} &= \sin\Phi\sec\Theta Q + \cos\Phi\sec\Theta R\end{aligned}\right\}$$

Derivative of the Euler angles, $\underline{\Theta} = [\Phi, \theta, \Psi]$

$$\dot{\underline{\Theta}} = D_b^{e}(\underline{\Theta})\,\underline{w}_b^{ib} \quad (A\text{-}105)$$

$$\left.\begin{aligned}\dot{P} &= -\frac{I_{zz} - I_{yy}}{I_{xx}}QR + \frac{L}{I_{xx}}\\ \dot{Q} &= -\frac{I_{xx} - I_{zz}}{I_{yy}}PR + \frac{M}{I_{yy}}\\ \dot{R} &= -\frac{I_{yy} - I_{xx}}{I_{zz}}PQ + \frac{N}{I_{zz}}\end{aligned}\right\}$$

Law of angular momentum ①

$$\boxed{\begin{aligned}\underline{n}_c^b &= [L, M, N]\\ \underline{F}^b &= [F_x, F_y, F_z]\end{aligned}}$$

Aero. dyn. forces rep. in $\mathcal{F}^b$

T_x: thrust

The aero dynamical forces depends on speed, angle of attac, sideslip angle and fin control deflections. $M_m = \frac{V_m}{V_0}$, where V_0 is speed of sound.

$$
\begin{aligned}
F_x &= k_F \rho V_m^2 C_x \\
F_y &= k_F \rho V_m^2 C_y \\
F_z &= k_F \rho V_m^2 C_z \\
L &= k_M \rho V_m^2 C_l \\
M &= k_M \rho V_m^2 C_m \\
N &= k_M \rho V_m^2 C_n
\end{aligned}
$$

Measured in wind tunnels

$\underline{x} = [V_m, \alpha, \beta, \phi, \theta, \psi, P, Q, R]^T$

$\underline{u} : [\delta_r, \delta_e, \delta_a, T_x]$

ρ: atmospherical density

k_F, k_n: constans depending on the vehical geometry

$$
\begin{aligned}
C_x &= C_{x0}(\alpha, \beta, M_m) + C_{x\delta_e}(\alpha, \delta_e, M_m) + C_{x\delta_a}(\alpha, \delta_a, M_m) + C_{x\delta_r}(\alpha, \delta_r, M_m) \\
C_y &= C_{y0}(\alpha, \beta, M_m) + C_{y\delta_e}(\alpha, \delta_e, M_m) + C_{y\delta_a}(\alpha, \delta_a, M_m) + C_{y\delta_r}(\alpha, \delta_r, M_m) \\
C_z &= C_{z0}(\alpha, \beta, M_m) + C_{z\delta_e}(\alpha, \delta_e, M_m) + C_{z\delta_a}(\alpha, \delta_a, M_m) + C_{z\delta_r}(\alpha, \delta_r, M_m) \\
C_l &= C_{l0}(\alpha, \beta, M_m) + C_{l\delta_e}(\alpha, \delta_e, M_m) + C_{l\delta_a}(\alpha, \delta_a, M_m) + C_{l\delta_r}(\alpha, \delta_r, M_m) \\
C_m &= C_{m0}(\alpha, \beta, M_m) + C_{m\delta_e}(\alpha, \delta_e, M_m) + C_{m\delta_a}(\alpha, \delta_a, M_m) + C_{m\delta_r}(\alpha, \delta_r, M_m) \\
C_n &= C_{n0}(\alpha, \beta, M_m) + C_{n\delta_e}(\alpha, \delta_e, M_m) + C_{n\delta_a}(\alpha, \delta_a, M_m) + C_{n\delta_r}(\alpha, \delta_r, M_m)
\end{aligned}
$$

δ_r : rudder (yaw)

δ_e : elevator (pitch)

δ_a : aileron (roll)

The equations can be written as:

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$$

I.e. a non-linear and determanistic equations.

If we want to create an autopilot that force the vehicle to follow a given trajectory we expand the state vector with the vehicle position in $\mathcal{F}^i$. i.e.

$$\underline{x} := [\underline{x}, \underline{p}^i]$$

If we want to follow a trajectory with a spesific velocity.

Given $\tilde{\underline{p}}(t)$ and $\tilde{\underline{v}}(t)$

$$\partial \underline{p} = \underline{p}(t) - \tilde{\underline{p}}(t) \quad , \quad \partial \underline{v} = \underline{v}(t) - \tilde{\underline{v}}(t)$$

We desire an autopilot that forces:

$\partial \underline{p} \to 0$, $\partial \underline{v} \to 0$

Create a nominal solution:

$$\dot{\tilde{\underline{x}}} = \underline{f}(\tilde{\underline{x}}, \tilde{\underline{u}})$$

Linearize around $\tilde{\underline{x}}$ and $\tilde{\underline{u}}$:

$$\partial \underline{x} = \underline{x} - \tilde{\underline{x}} \quad , \quad \partial \underline{u} = \underline{u} - \tilde{\underline{u}}$$

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$$

$$(\dot{\tilde{\underline{x}}} + \partial \underline{x}) = \underline{f}(\tilde{\underline{x}} + \partial \underline{x}, \tilde{\underline{u}} + \partial \underline{u})$$

$$\dot{\tilde{\underline{x}}} + \partial \dot{\underline{x}} = f(\tilde{\underline{x}}, \tilde{\underline{u}}) + \overbrace{\left.\frac{\partial \underline{f}}{\partial \underline{x}^T}\right|_{\tilde{x}, \tilde{u}}}^{F(\tilde{x}, \tilde{u})} \cdot \partial \underline{x} + \overbrace{\left.\frac{\partial \underline{f}}{\partial \underline{u}^T}\right|_{\tilde{x}, \hat{u}}}^{L(\tilde{x}, \tilde{u})} \partial \underline{u} + h.o.t$$

1 order approximation:

$$\partial \dot{\underline{x}} = F(\tilde{\underline{x}}, \tilde{\underline{u}}) \partial \underline{x} + L(\tilde{\underline{x}}, \tilde{\underline{u}}) \partial \underline{u}$$

Regulation problem:

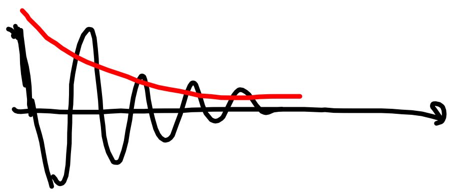
Designe a feedback that forces $\partial \underline{x} \to 0$

$$\partial \underline{u} = G(t) \partial \underline{x}$$

$$\underline{u} = \tilde{\underline{u}} + G(t) \partial \underline{x}$$

$\underline{x}(t)$: calculated by KF (INS) $\hat{\underline{x}}(t)$

$$\partial \dot{\underline{x}} = (F + LG) \partial \underline{x}$$



Part D: Mathematical modelling of robots

We go throug the document: "Matematisk modellering av roboter" written by Oddvar Hallingstad.

- Based on Craig.

- Craig only use algebraic vectors. $\underline{r}^b$ (not geometric $\vec{r}$)

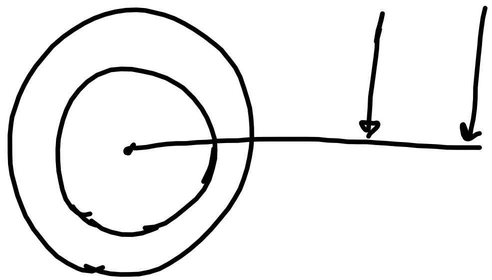
$$① \quad \underline{r}^a = R_b^a \underline{r}^b$$

$$② \quad R_b^a = \begin{bmatrix} \underline{x}_b^a & \underline{y}_b^a & \underline{z}_b^a \end{bmatrix}$$

$$③ \quad \underline{y}^b = R_b^a \underline{x}^b$$

$$\tilde{R}_p^q = \begin{bmatrix} R_p^q & \underline{r}_{pq}^q \\ 0,0,0 & 1 \end{bmatrix}, \quad \tilde{\underline{r}}^p = \begin{bmatrix} \underline{r}^p \\ 1 \end{bmatrix}, \quad \tilde{\underline{r}}^q = \begin{bmatrix} \underline{r}^q \\ 1 \end{bmatrix}$$

$$\left(R_b^a\right)^{-1} = R_a^b = \left(R_b^a\right)^T$$ <u>o.n basis</u>



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