F4

$$\vec{x} \overset{\mathcal{F}_v^a}{\longleftrightarrow} \underline{x}^a$$

$$\mathbb{A} \overset{\mathcal{F}_v^b}{\longleftrightarrow} A^b$$

$$\vec{w}\times \overset{\mathcal{F}_v^a}{\longleftrightarrow} S(\underline{w}^a)$$

New exam date: Tue 15th Dec.

o.n. basis vectors

Note: $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$

$\vec{a}\to\underline{a}^p,\ \vec{b}\to\underline{b}^p,\ \vec{c}\to\underline{c}^p$

$$(\underline{a}^p\times\underline{b}^p)\times\underline{c}^p = S\big(S(\underline{a}^p)\,\underline{b}^p\big)\,\underline{c}^p$$

$$\underline{a}^p\times(\underline{b}^p\times\underline{c}^p) = S(\underline{a}^p)\big(S(\underline{b}^p)\underline{c}^p\big) = S(\underline{a}^p)\,S(\underline{b}^p)\,\underline{c}^p$$

A.2.4 Matriserepresentasjon ved bytte av basisvektorer

Problem A.3 Bestem sammenhengen mellom matriserepresentasjonene av vektoren $\vec{r}$ og operatoren $\mathbf{A}$ i hhv q- og p-systemet. Dvs sammenhengen mellom $\underline{r}^q$ og $\underline{r}^p$, A^q og A^p

Teorem A.4 ***Matriserepresentasjon ved bytte av basisvektorer.***
Gitt to basissystemer $\{\vec{q}_i\}$ og $\{\vec{p}_i\}$ i vektorrommet $\mathcal{V}$. La $\vec{r}$ og $\mathbf{A}$ være hhv en vektor og en lineær operator i $\mathcal{V}$. Da har vi følgende sammenhenger mellom matriserepresentasjonene i de to basissystemene :

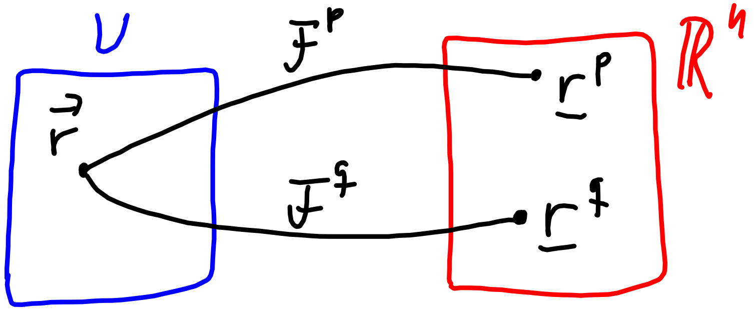
$$\underline{r}^q = C^q_p \underline{r}^p \quad hvor \quad C^q_p = [\langle \vec{p}_j, \vec{q}_i^{\,*} \rangle] \tag{A- 10}$$

$$\underline{r}^p = C^p_q \underline{r}^q \quad hvor \quad C^p_q = [\langle \vec{q}_j, \vec{p}_i^{\,*} \rangle] \tag{A- 11}$$

$$C^q_p = \left[C^p_q\right]^{-1}$$

$$A^q = C^q_p A^p C^p_q \;\; og \;\; A^p = C^p_q A^q C^q_p \tag{A- 12}$$

C^q_p og C^p_q kalles retningskosinmatriser (RKM). (Vi skal senere se at den kan brukes i mange sammenhenger og har navn deretter. Ovenfor brukes den som en koordinattransformasjonsmatrise, KTM.)



Proof: $\underline{r}^q = C_p^q \, \underline{r}^p$

$$\vec{r} = \sum_{j=1}^{n} r_j^q \vec{q}_j = \sum_{j=1}^{n} r_j^p \vec{p}_j$$

$$r_i^q = \langle \vec{r}, \vec{q}_i^{\,*} \rangle = \left\langle \sum_{j=1}^{n} r_j^q \vec{q}_j , \vec{q}_i^{\,*} \right\rangle = \left\langle \sum_{j=1}^{n} r_j^p \vec{p}_j , \vec{q}_i^{\,*} \right\rangle$$

$$= \sum_{j=1}^{n} r_j^q \underbrace{\langle \vec{q}_j , \vec{q}_i^{\,*} \rangle}_{\delta_{ij}} = \sum_{j=1}^{n} r_j^p \langle \vec{p}_j , \vec{q}_i^{\,*} \rangle$$

$$r_i^q = \sum_{j=1}^{n} \langle \vec{p}_j , \vec{q}_i^{\,*} \rangle \, r_j^p$$

$$C_p^q = \begin{array}{c|c} & j \rightarrow \\ \hline i \downarrow & \begin{bmatrix} \langle \vec{p}_1, \vec{q}_1^{\,*} \rangle & \langle \vec{p}_2, \vec{q}_1^{\,*} \rangle & \cdots \\ \langle \vec{p}_1, \vec{q}_2^{\,*} \rangle & & \\ \vdots & & \\ \langle \vec{p}_1, \vec{q}_n^{\,*} \rangle & & \end{bmatrix} \end{array}$$

$$\underbrace{\begin{bmatrix} \vdots \\ r_i^q \\ \vdots \end{bmatrix}}_{\underline{r}^q} = \underbrace{\begin{bmatrix} & & \langle \vec{p}_n, \vec{q}_i^{\,*} \rangle \\ \langle \vec{p}_1, \vec{q}_i^{\,*} \rangle & \langle \vec{p}_2, \vec{q}_i^{\,*} \rangle & \cdots \langle \cdot , \cdot \rangle \\ & & \end{bmatrix}}_{C_p^q} \underbrace{\begin{bmatrix} r_1^p \\ \vdots \\ r_i^p \\ \vdots \\ r_n^p \end{bmatrix}}_{\underline{r}^p}$$

$$\Rightarrow \boxed{\underline{r}^q = C_p^q \, \underline{r}^p, \quad C_p^q = \left[\langle \vec{p}_j , \vec{q}_i^{\,*} \rangle \right]}$$

Proof A-12

$$A^q = C_p^q A^p C_q^p$$

$\mathcal{F}_v^q$: ① $\underline{y}^q = A^q \underline{x}^q$ ② $\underline{y}^q = C_p^q \underline{y}^p$

$\mathcal{F}_v^p$: ③ $\underline{y}^p = A^p \underline{x}^p$ ④ $\underline{x}^p = C_q^p \underline{x}^q$

$$\underline{y}^q = A^q \underline{x}^q = \underbrace{C_p^q \underbrace{A^p \underbrace{C_q^p \underline{x}^q}_{\underline{x}^p}}_{\underline{y}^p}}_{\underline{y}^q}$$

$$\underline{y}^q \overset{①}{=} A^q \underline{x}^q \overset{②}{=} C_p^q \underline{y}^p \overset{③}{=} C_p^q A^p \underline{x}^p \overset{④}{=} \underbrace{C_p^q A^p C_q^p}_{A^q} \underline{x}^q$$

$$\boxed{A^q = C_p^q A^p C_q^p}$$

Similarity transformation

Eksempel A.5 Teorem A.5 ***RKM for to ortonormale basissystem***

Dersom vi har to ortonormale basisvektorsettet $\{\vec{q}_i\}$ *og* $\{\vec{p}_i\}$*, dvs*

$$\begin{aligned} \langle \vec{q}_i, \vec{q}_j \rangle &= \delta_{ij} \\ \langle \vec{p}_i, \vec{p}_j \rangle &= \delta_{ij} \end{aligned} \tag{A- 13}$$

så vil de duale basissystema være lik basissystema

$$\begin{aligned} \vec{q}_i &= \vec{q}_i^{\,*}, \quad i = 1, 2, \ldots, n \\ \vec{p}_i &= \vec{p}_i^{\,*}, \quad i = 1, 2, \ldots, n \end{aligned} \tag{A- 14}$$

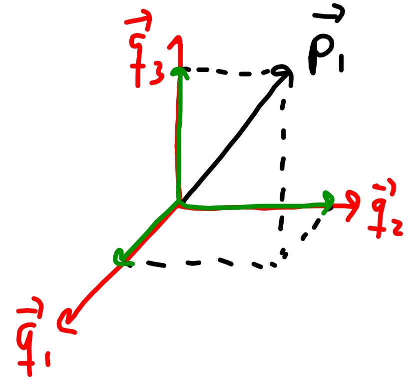
Dette gir

$$C_p^q = [\langle \vec{p}_j, \vec{q}_i \rangle] = [\cos(\angle \vec{p}_j \vec{q}_i)] = R_p^q \qquad n=3 \tag{A- 15}$$

Dette viser hvorfor C_p^q *kalles en* ***retningskosinmatrise****. Vi vil innføre en spesiell notasjon i dette tilfellet med ortonormale basissystemer og betegner en ortonormal RKM med* R_p^q*.*

$$R_p^q = \begin{bmatrix} \cos\angle \vec{p}_1 \vec{q}_1 & \cos\angle \vec{p}_2 \vec{q}_1 & \cos\angle \vec{p}_3 \vec{q}_1 \\ \cos\angle \vec{p}_1 \vec{q}_2 & \cos\angle \vec{p}_2 \vec{q}_2 & \cos\angle \vec{p}_3 \vec{q}_2 \\ \cos\angle \vec{p}_1 \vec{q}_3 & \cos\angle \vec{p}_2 \vec{q}_3 & \cos\angle \vec{p}_3 \vec{q}_3 \end{bmatrix} = \begin{bmatrix} \underline{p}_1^q & \underline{p}_2^q & \underline{p}_3^q \end{bmatrix}$$

See that ith column in R_p^q represents the unit vector $\vec{p}_i$ in the q-frame



Example (Oppg. A.1) Inner product in $\mathbb{R}^n$ o.n. basis vectors

$$\langle \vec{a}, \vec{b} \rangle = \left\langle \sum_{i=1}^{n} a_i^p \vec{p}_i , \sum_{j=1}^{n} b_j^p \vec{p}_j \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^p b_j^p \underbrace{\langle \vec{p}_i , \vec{p}_j \rangle}_{\partial_{ij}} = \sum_{i=1}^{n} a_i^p b_i^p = (\underline{a}^p)^T \underline{b}^p$$

$$[a_1 \; a_2 \; a_3]\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$n=3 \quad \langle \vec{a}, \vec{b} \rangle = (\underline{a}^p)^T \underline{b}^p = \|\vec{a}\| \, \|\vec{b}\| \cos \angle \vec{a}\vec{b}$$

Teorem A.6 ***RKM R_p^q er en ortogonal matrise***
Retningskosinmatrisa mellom to rammer som begge har ortonormale basisvektorer, R_p^q, er en ortonormal matrise. Dvs

$$(R_p^q)^{-1} = (R_p^q)^T \tag{A- 16}$$

Proof:

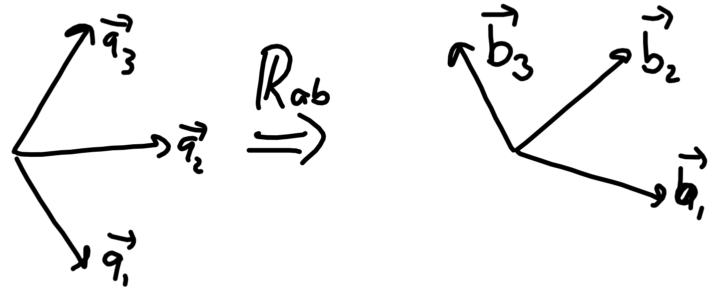
$$R_p^q = [\underline{p}_1^q \; \underline{p}_2^q \; \underline{p}_3^q], \quad (R_p^q)^T = \begin{bmatrix} (\underline{p}_1^q)^T \\ (\underline{p}_2^q)^T \\ (\underline{p}_3^q)^T \end{bmatrix} \Rightarrow (R_p^q)^T R_p^q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{I} = (R_p^q)^{-1} R_p^q$$

$$(\underline{p}_i^q)^T \underline{p}_j^q = \langle \vec{p}_i, \vec{p}_j \rangle = \partial_{ij}$$

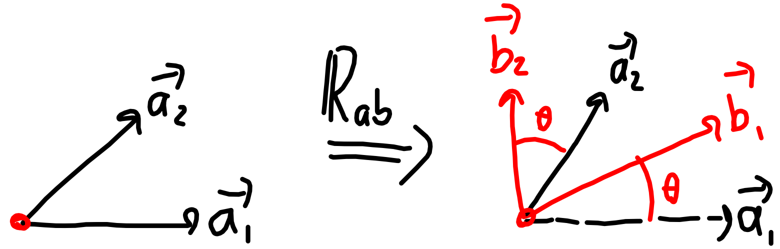
$$\Rightarrow (R_p^q)^{-1} = (R_p^q)^T$$

A2.5 Matrix representation of the rotation operator

Def. A.11 A rotation operator is a linear operator $\mathbb{R}_{ab} : V \to V$ defined by: $\vec{b}_i = \mathbb{R}_{ab}\,\vec{a}_i$, $i = 1, 2, \ldots, n$

3D) 

All basis vectors are rotated the same angle around the same rotational axes

2D) 

Question: What is the representation of $\mathbb{R}_{ab}$ in $\mathcal{F}_{\nu}^{a}$ and $\mathcal{F}_{\nu}^{b}$.

I.s. what is $[\mathbb{R}_{ab}]^{a}$ and $[\mathbb{R}_{ab}]^{b}$

Theorem A.7 Matrix rep. of $\mathbb{R}_{ab}$ in the frames $\mathcal{F}_{\nu}^{a}$ and $\mathcal{F}_{\nu}^{b}$ is:

$$\boxed{[\mathbb{R}_{ab}]^{a} = [\mathbb{R}_{ab}]^{b} = C_{b}^{a}}$$

NB! $\mathbb{R}_{ab} \Leftarrow C_{b}^{a}$

Define: $R_{ab}^{a} = [\mathbb{R}_{ab}]^{a} = C_{b}^{a}$

$R_{ab}^{b} = [\mathbb{R}_{ab}]^{b} = C_{b}^{a}$

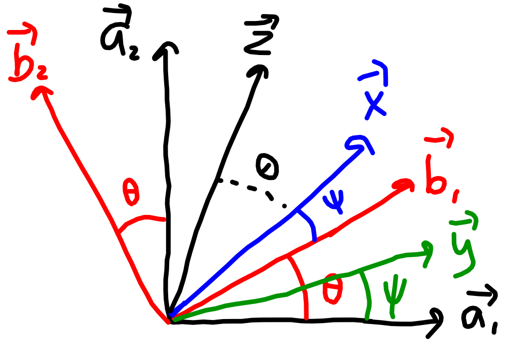
Differentiate between physically rotate a vector $(\mathbb{R}_{ab} \to C_{b}^{a})$ and to transform a vector between different frames C_{b}^{a}

Proof:

$$[\mathbb{R}_{ab}]^a = R^a_{ab} = \left[\langle \mathbb{R}_{ab}\vec{a}_j, \vec{a}_i^{\,*}\rangle\right] = \left[\langle \vec{b}_j, \vec{a}_i^{\,*}\rangle\right] = C^a_b$$

$$[\mathbb{R}_{ab}]^b = R^b_{ab} = C^b_a R^a_{ab} C^a_b = \underbrace{C^b_a C^a_b}_{I} C^a_b = C^a_b$$

Illustration of the rotation operator



R_{ab}: rotate $a \rightarrow b$ an angle θ around axis 3 ($\vec{a}_3$)

Assume $\{\vec{a}_i\}$ is o.n. $\Rightarrow$ $\{\vec{b}_i\}$ is o.n and $\|\vec{x}\| = 1$

$$\underline{x}^b = \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} \qquad \underline{x}^a = \begin{bmatrix} \cos(\theta+\psi) \\ \sin(\theta+\psi) \end{bmatrix}$$

$$\underline{x}^a = C^a_b \underline{x}^b \overset{\text{o.n.}}{=} R^a_b \underline{x}^b$$

R^a_b works as a rot. opr. when used in the same frame. The rotation is the same as rotating from $\{a\}$ to $\{b\}$. This is an active operation.

$$R^a_{ab} = R^b_{ab} = R^a_b$$

Def: $\underline{y}^a = \underline{x}^b$

$$\underline{x}^a = R^a_b \underline{x}^b = R^a_b \underline{y}^a$$

$$\underline{x}^a = R^a_{ba} \underline{y}^a$$

$$\vec{x} = R_{ab} \vec{y}$$

Def: $\underline{z}^b = \underline{x}^a$

$$\underline{z}^b = R^a_b \underline{x}^b$$

$$\underline{z}^b = R^b_{ab} \underline{x}^b$$

$$\vec{z} = R_{ab} \vec{x}$$