F.8/ Proof of Theorem A.17 Diff. eq. of DCM.

$$C_p^q = [\underline{p}_1^q, \underline{p}_2^q, \underline{p}_3^q]$$

$$\dot{C}_p^q = [\dot{\underline{p}}_1^{qq}, \dot{\underline{p}}_2^{qq}, \dot{\underline{p}}_3^{qq}] = [\underline{w}_p^{qq} \times \underline{p}_1^q, \ \underline{w}_p^q \times \underline{p}_2^q, \ \underline{w}_p^q \times \underline{p}_3^q]$$

$$= [S(\underline{w}_p^q)\underline{p}_1^q, \ S(\underline{w}_p^q)\underline{p}_2^q, \ S(\underline{w}_p^q)\underline{p}_3^q]$$

$$= S(\underline{w}_p^q)[\underline{p}_1^q, \underline{p}_2^q, \underline{p}_3^q] = S(\underline{w}_p^q) C_p^q$$

$$\dot{C}_p^q = S(\underline{w}_p^{qq}) C_p^q = C_p^q S(\underline{w}_p^{qp}) \underbrace{C_q^p C_p^q}_{I} = C_p^q S(\underline{w}_p^{qp})$$

$$\boxed{\dot{C}_p^q = S(\underline{w}_p^q) C_p^q = C_p^q S(\underline{w}_p^{qp}), \qquad \underline{w}_p^{qq} = C_p^q \underline{w}_p^{qp}}$$

The kinematic problem for 3-2-1 Euler angles: Given $\vec{\omega}_p^q$, what is the d.e. for the attitude matrix or special representation of the attitude matrix.

1. If $R_p^q = [\underline{p}_1^q, \underline{p}_2^q, \underline{p}_3^q] \Rightarrow$ $\boxed{\dot{R}_p^q = S(\underline{\omega}_p^q) R_p^q}$

2. If $R_p^q = R_3(\theta_3) R_2(\theta_2) R_1(\theta_1)$: 3-2-1 Euler angles

$$\boxed{\begin{aligned} \dot{\underline{\theta}} &= D_p^{\theta}(\underline{\theta})\, \underline{\omega}_p^{qp} \\ &= D_q^{\theta}(\underline{\theta})\, \underline{\omega}_p^{q} \end{aligned} \quad , \quad \underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}}$$

We solve the d.e. (differential equations) using numerical methods.

1. order Euler method:

a) $\dot{x}(t) = f(x(t))$, $x(t_0)$ given

scalar

$$\frac{\Delta x}{\Delta t} = \frac{x(t_{k+1}) - x(t_k)}{\Delta t} = f(x(t_k)), \quad \Delta t = t_{k+1} - t_k$$

$$x(t_{k+1}) = x(t_k) + \Delta t \cdot f(x(t_k)), \quad t_k = \Delta t \cdot k, \quad x(t_k) = x_k$$

$$\boxed{x_{k+1} = x_k + \Delta t\, f(x_k), \quad x_0 \text{ given}}$$

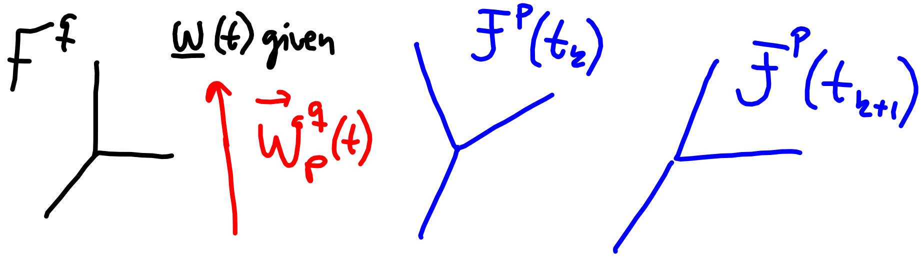
vector

b) $\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t))$, $\underline{x}(t_0)$ given

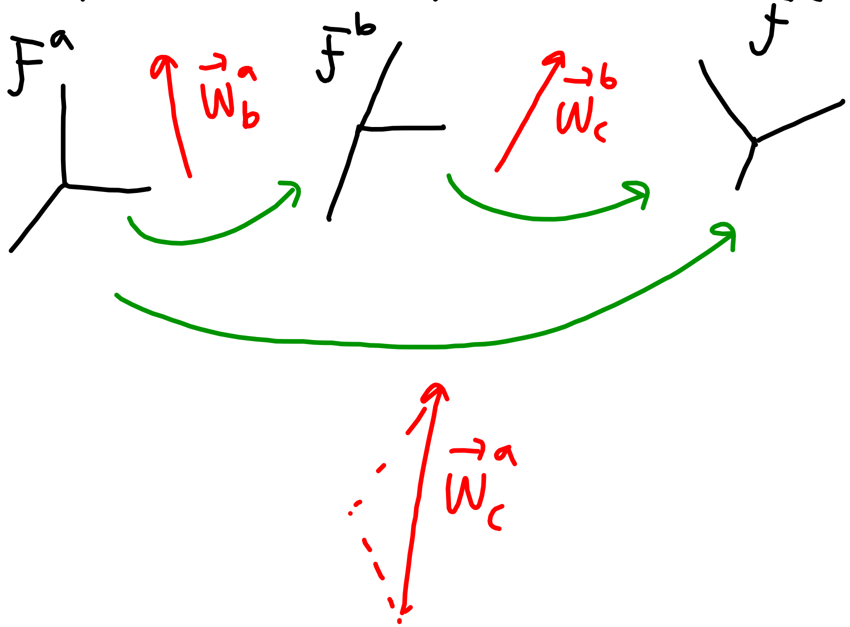
$$\boxed{\underline{x}_{k+1} = \underline{x}_k + \Delta t \cdot \underline{f}(\underline{x}_k), \quad \underline{x}_0 \text{ given}}$$

c) Matrix $\dot{R}(t) = S(\underline{w}(t))R(t)$, $R(t_0)$ given

$$\boxed{R_{k+1} = R_k + \Delta t\, S(\underline{w}_k)R_k, \quad R_0 \text{ given}}$$



Angular velocity and ang. accelleration in different basis systems



$$\vec{\omega}_c^a = \vec{\omega}_b^a + \vec{\omega}_c^b$$

Derivation seen from $\mathcal{F}^a$

$$\dot{\vec{\omega}}_c^{aa} = \dot{\vec{\omega}}_b^{aa} + \dot{\vec{\omega}}_c^{ba}$$

What is $\dot{\vec{\omega}}_c^{ba}$ expressed with $\dot{\vec{\omega}}_c^{bb}$

$$\vec{\omega}_c^b = \sum_{i=1}^{3} \omega_{ci}^{bb} \vec{b}_i$$

$$\frac{d^a}{dt}\vec{\omega}_c^b = \frac{d^a}{dt}\left(\sum \omega_{ci}^{bb} \vec{b}_i\right)$$

$$= \sum \dot{\omega}_{ci}^{bbb} \vec{b}_i + \underbrace{\sum \omega_{ci}^{bb} \overbrace{\vec{\omega}_b^a \times \vec{b}_i}^{\dot{\vec{b}}_i^a}}_{\vec{\omega}_b^a \times \sum \omega_{ci}^{bb} \vec{b}_i}$$

$$\dot{\vec{\omega}}_c^{ba} = \dot{\vec{\omega}}_c^{bb} + \vec{\omega}_b^a \times \vec{\omega}_c^b$$

$$\boxed{\dot{\vec{\omega}}_c^{aa} = \dot{\vec{\omega}}_b^{aa} + \dot{\vec{\omega}}_c^{bb} + \vec{\omega}_b^{a} \times \vec{\omega}_c^{b}}$$

Natural to represent also the derivation (actually 2. derivation) in the same frame as the 1. derivation

Angular velocities and their derivations in case of algebraic vectors can either be find by representing the equations for the geometrical vectors in a desired frame or by derivating:

$$\underline{\omega}_c^{aa} = \underline{\omega}_b^{aa} + \underline{\omega}_c^{ba}$$

$$= \underline{\omega}_b^{aa} + R_b^a\, \underline{\omega}_c^{bb}$$

: Easier to derivate and represent in the same frame

We had:

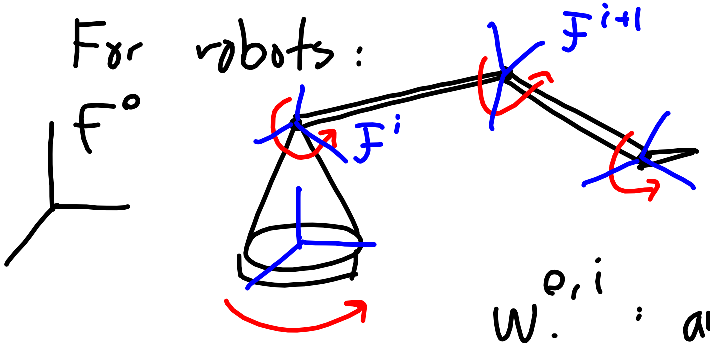
$$\underline{w}_c^{aa} = \underline{w}_b^{aa} + R_b^a \underline{w}_c^{bb}$$

Derivate:

$$\boxed{\dot{\underline{w}}_c^{aaa} = \dot{\underline{w}}_b^{aaa} + S(\underline{w}_b^{aa}) R_b^a \underline{w}_c^{bb} + R_b^a \dot{\underline{w}}_c^{bbb}}$$

$$\dot{\underline{w}}_c^{aaa} = \dot{\underline{w}}_b^{aaa} + R_b^a S(\underbrace{\underline{w}_b^{ab}}_{R_a^b \underline{w}_b^{aa}}) \underline{w}_c^{bb} + R_b^a \dot{\underline{w}}_c^{bbb}$$

For robots:



$\underline{\omega}_i^{0,i}$: angular velocity for link i (frame i) seen from the link 0 (frame 0) represented in link i (frame i)

$$\underline{\omega}_{i+1}^{0,i+1} = \underline{\omega}_i^{0,i+1} + \underline{\omega}_{i+1}^{i,i+1}$$

$$\boxed{\underline{\omega}_{i+1}^{0,i+1} = R_i^{i+1}\,\underline{\omega}_i^{0,i} + \underline{\omega}_{i+1}^{i,i+1}}$$

$$\boxed{\dot{\underline{\omega}}_{i+1}^{0,i+1,i+1} = S\left(\underline{\omega}_i^{i+1,i+1}\right) R_i^{i+1}\,\underline{\omega}_i^{0,i} + R_i^{i+1}\,\dot{\underline{\omega}}_i^{0,i,i} + \dot{\underline{\omega}}_{i+1}^{i,i+1,i+1}}$$

Theorem A.18 Derivation of angular velocities

$$\boxed{\dot{\underline{w}}_b^{abb} = R_a^b \, \dot{\underline{w}}_b^{aaa}}$$

Proof: $\underline{w}_b^{ab} = R_a^b \, \underline{w}_b^{aa}$

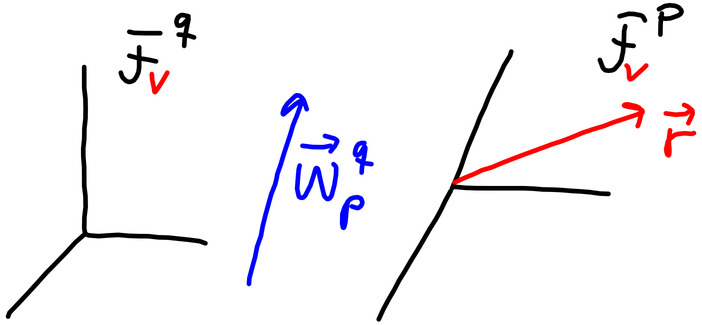
$$\dot{\underline{w}}_b^{abb} = \underbrace{\dot{R}_a^b \, \underline{w}_b^{aa}}_{0\,?} + R_a^b \, \dot{\underline{w}}_b^{aaa}$$

$\vec{w}_a^{\,b} = -\vec{w}_b^{\,a}$

$$\dot{R}_a^b \, \underline{w}_b^{aa} = S(\underline{w}_a^{bb}) R_a^b \, \underline{w}_b^{aa} = S(\underline{w}_a^{bb}) \, \underline{w}_b^{ab} = \underline{w}_a^{bb} \times \underline{w}_b^{ab}$$

$$= -\underline{w}_a^{bb} \times \underline{w}_a^{bb} = 0$$

A 5.3 Derivation of vectors



Assume $\|\vec{r}\| = \text{constant}$ and fixed to the p-frame.

Proved earlier:

$$\boxed{\dot{\vec{r}}^{\,q} = \vec{\omega}_p^{\,q} \times \vec{r}}$$

Assume $\vec{r}$ varies seen from $\mathcal{F}_v^p$

$$\vec{r} = \sum r_i^p \vec{p}_i$$

$$\dot{\vec{r}}^{\,q} = \sum \dot{r}_i^{pp} \vec{p}_i + \sum r_i^p \dot{\vec{p}}_i^{\,q}$$

$$= \sum \dot{r}_i^p \vec{p}_i + \sum r_i^p \vec{\omega}_p^{\,q} \times \vec{p}_i$$

$\vec{p}_i$ const. length.

$$\boxed{\dot{\vec{r}}^{\,q} = \dot{\vec{r}}^{\,p} + \vec{\omega}_p^{\,q} \times \vec{r}}$$