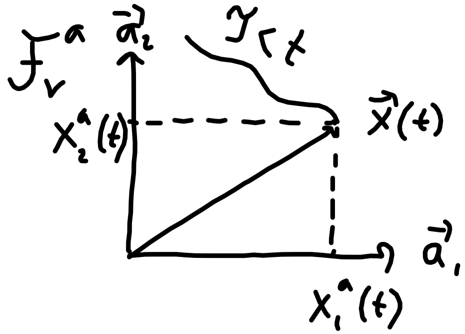
F.7 A.4 Time in vector space and affine space

Up to now we have looked at points and vectors as static. Given the vector space V and the frame $\mathcal{F}_V^a$ we can describe a time varying vector as:

$$\boxed{\vec{x}(t) = \sum_{i=1}^{n} x_i^a(t)\,\vec{a}_i}$$



Clear relation:

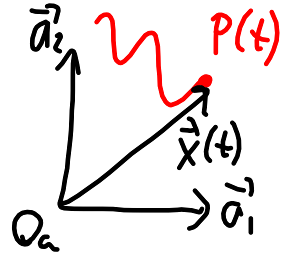
$\vec{x}(t)$ and $\underline{x}^a(t)$

Given the affine space A with frame $\mathcal{F}_A^a = \{O_a, \vec{a}_1, \vec{a}_2, \vec{a}_3\}$ a time varying point is defined by:

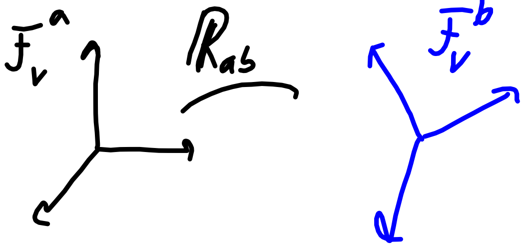
$$\boxed{P(t) = O_a + \sum_{i=1}^{n} x_i^a(t)\,\vec{a}_i}$$

Clear relation:

$P(t)$ and $\underline{x}^a(t)$



Relative attitude between two frames can be described using the rotation operator $\mathcal{R}_{ab}$



From before:

$$\underbrace{\left[\mathcal{R}_{ab}\right]^a}_{R_{ab}^a} = \underbrace{\left[\mathcal{R}_{ab}\right]^b}_{R_{ab}^b} = R_b^a$$

Representation of the rotation operator kan be made time variant:

$$\boxed{\left[\mathcal{R}_{ab}(t)\right]^a = R_b^a(t)}$$

For example using Euler angles (3-2-1)

$$R_b^a(t) = R_3(\psi(t))R_2(\theta(t))R_1(\phi(t))$$

To find the relation between the representation of a point $P(t)$ in the frames $\mathcal{F}_A^a$ and $\mathcal{F}_A^b$ we can use time varying transformation matrices:

$$\boxed{\begin{aligned} T_b^a(t) &= \begin{bmatrix} R_b^a(t) & \underline{r}_{ab}^a(t) \\ \underline{0}^T & 1 \end{bmatrix} \\ \tilde{\underline{r}}_p^a(t) &= T_b^a(t)\, \tilde{\underline{r}}_p^b(t) \end{aligned}}$$

If we have 3 frames: $\mathcal{F}_A^a, \mathcal{F}_A^b$ and $\mathcal{F}_A^c$

$$\tilde{\underline{r}}^a = T_b^a T_c^b \tilde{\underline{r}}^c \quad \text{(all function of } t)$$

NB! In classical mechanics we use Galileo transformations. We can add relative velocities.

NB! When calculating the distance between points position at two different times we need to choose <u>one</u> affine space:

$$\overset{A}{P}(t_2) - \overset{A}{P}(t_1) = \vec{r}_{\overset{A}{P}(t_2)\overset{A}{P}(t_1)}$$

A.5. Derivation in vector and affine space.

Notation need to take into accont two aspects:

1. In which frame do we represent the vector: $\underline{x}^{a}(t)$
2. In which frame do we see the derivation from: {b} : $\underline{\dot{x}}^{ba}(t)$

In math

Given $f(x,y)$ the partial derivation is:

$$\frac{\partial f(x,y)}{\partial x} = f_x(x,y) \qquad \frac{\partial f(x,y)}{\partial y} = f_y(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right) = f_{xy}(x,y) \qquad \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right) = f_{yx}(x,y)$$

Here we had two free variables and derivation w.r.t. these. We will only have one free variable (time-t), but we can se the time variations from different frames that are moving relative to each other.

Introduce notation:

$$\frac{d^a}{dt}\vec{x}(t) = \dot{\vec{x}}^a(t)$$ Derivation seen from $\mathcal{F}^a$

$$\frac{d^b}{dt}\vec{x}(t) = \dot{\vec{x}}^b(t)$$ —— " —— $\mathcal{F}^b$

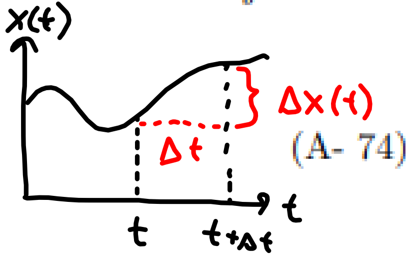
$$\frac{d^b}{dt}\left(\frac{d^a}{dt}\vec{x}(t)\right) = \ddot{\vec{x}}^{ab}(t)$$

Represent in $\mathcal{F}^c$

$$\left[\dot{\vec{x}}^a(t)\right]^c = \underline{\dot{x}}^{ac}(t), \quad \left[\ddot{\vec{x}}^{ab}(t)\right]^c = \underline{\ddot{x}}^{abc}(t)$$

A.5.1 Definisjon av deriverte i vektorrom og affine rom.

Når vi skal definere de deriverte av vektorer og punkter må vi starte med det vi kjenner fra matematikken nemlig derivasjon i $\mathbb{R}$ og så generalisere til $\mathbb{R}^n$, $\mathcal{V}$ og $\mathcal{A}$:



1. Derivasjon i $\mathbb{R}$:

$$\dot{x}(t) = \lim_{\Delta t \to 0}\left(\frac{1}{\Delta t}\overbrace{(x(t+\Delta t) - x(t))}^{\Delta x}\right) \tag{A- 74}$$

2. Derivasjon i $\mathbb{R}^n$:

$$\dot{\underline{x}}(t) = [\dot{x}_i(t)] \tag{A- 75}$$

3. Derivasjon i vektorrommet $\mathcal{V}$ sett fra en fast ramme $\mathcal{F}_{\mathcal{V}}^a$:

$$\dot{\vec{x}}^a = \sum_{i=1}^{n} \dot{x}_i^a(t)\,\vec{a}_i \qquad \vec{x}(t) = \sum_{i=1}^{n} x_i^a(t)\,\vec{a}_i \tag{A- 76}$$

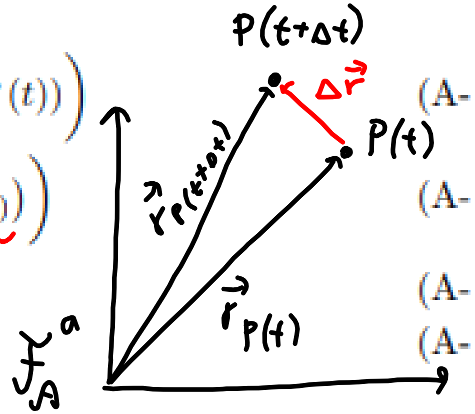
4. Derivasjon i det affine rom $\mathcal{A}$ sett fra en fast ramme $\mathcal{F}_{\mathcal{A}}^a$:

$$\dot{P}^a(t) = \lim_{\Delta t \to 0}\left(\frac{1}{\Delta t}(P(t+\Delta t) - P(t))\right) \tag{A- 77}$$

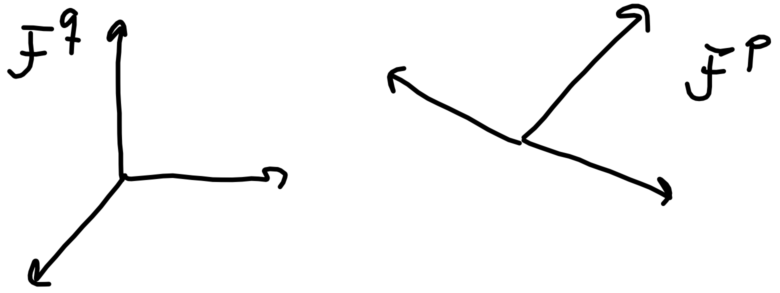
$$= \lim_{\Delta t \to 0}\left(\frac{1}{\Delta t}\underbrace{(\vec{r}_{P(t+\Delta t)} - \vec{r}_{P(t)})}_{\Delta\vec{r}}\right) \tag{A- 78}$$

$$= \dot{\vec{r}}_P^a \tag{A- 79}$$

$$= \vec{v}_P^a \tag{A- 80}$$



A5.2 Derivation of DCM



$R_p^q(t)$ gives the relative attitude

Assume R_p^q is a o.n. matrix $\Rightarrow (R_p^q)^T = (R_p^q)^{-1}$

Let $R = R_p^q \Rightarrow R \cdot R^T = I$

Derive on both sides:

$$\frac{d}{dt}\left(R(t)\, R^T(t)\right) = \frac{d}{dt} I$$

$$\dot{R}(t)\, R^T(t) + R(t) \frac{d}{dt}\left(R^T(t)\right) = 0$$

$$① \quad \dot{R}(t)\, R^T(t) + R(t) \left(\dot{R^T}(t)\right) = 0$$

Multiply from right side with $R(t)$

$$\dot{R} \underbrace{R^T R}_{I} + R(\dot{R^T})R = 0$$

$$② \quad \dot{R} + R(\dot{R^T})R = 0$$

Is $(\dot{R})^T = (\dot{R^T})$?

$$\frac{d}{dt}\left(R^T\right) = \frac{d}{dt}\left(\left[\underline{r}_1^q, \underline{r}_2^q, \underline{r}_3^q\right]^T\right)$$

$$= \frac{d}{dt}\begin{pmatrix} \underline{r}_1^{q\,T} \\ \underline{r}_2^{q\,T} \\ \underline{r}_3^{q\,T} \end{pmatrix} = \begin{bmatrix} \dot{\underline{r}}_1^{q\,T} \\ \dot{\underline{r}}_2^{q\,T} \\ \dot{\underline{r}}_3^{q\,T} \end{bmatrix} = \left(\frac{d}{dt}R\right)^T$$

I.e.

$$\frac{d}{dt}\left(R^T\right) = \frac{d}{dt}\left(R^{-1}\right) = \left(\frac{d}{dt}R\right)^T$$

Generally: $\frac{d}{dt}A^{-1} \neq \left(\frac{d}{dt}A\right)^T$

Equation ① can be written as:

$$\dot{R}R^T + \left(\dot{R}R^T\right)^T = 0$$

$$S := \dot{R}R^T \Rightarrow S + S^T = 0$$

I.e. S: Scew symmetrical matrix

$$S = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} = S(\underline{w})$$

$$\vec{w} \times \vec{a} \rightarrow S(\underline{w}^q)\,\underline{a}^q$$

Equation ②:

$$\dot{R} + \underbrace{R\dot{R}^T}_{S^T} R = 0$$

$$\dot{R} = -S^T R = S R$$

$$\boxed{\dot{R}_p^q = S(\underline{w}) R_p^q}, \quad R_p^q(t_0) \text{ given}$$

R_p^q is an attitude matrix

$$R_p^q = [\underline{p}_1^q, \underline{p}_2^q, \underline{p}_3^q]$$

$$\dot{R}_p^q = [\dot{\underline{p}}_1^q, \dot{\underline{p}}_2^q, \dot{\underline{p}}_3^q]$$

$$= S(\underline{w}) [\underline{p}_1^q, \underline{p}_2^q, \underline{p}_3^q]$$

$$= [S(\underline{w})\underline{p}_1^q, S(\underline{w})\underline{p}_2^q, S(\underline{w})\underline{p}_3^q]$$

$$\dot{\underline{p}}_i^q = S(\underline{w}^q)\,\underline{p}_i^q$$

$$= \underline{w}^q \times \underline{p}_i^q$$

We see that $\underline{w}$ is part of the calculation of the derivative of the rotating basis vectors ($\underline{p}_i^q$) seen from the q-system. We interpret therefore $\underline{w}$ as the angular velocity of the p-system seen from the q-system, and use the notation $\underline{w}_p^q = \underline{w}_p^{qq}$ because we derive seen from the q-system and represent in the q-system:

We therefore write : $\boxed{\dot{R}_p^q = S(\underline{w}_p^q)\,R_p^q}$

$S(\underline{\omega}_p^q) = S(\underline{\omega}_p^{qq})$ is the representation of the operator "$\vec{\omega}_p^q \times$" in the q-frame. But, linear operators can also be represented in other frames using the similarity transformation:

$$S(\underline{\omega}_p^{qq}) = R_p^q \, S(\underline{\omega}_p^{qp}) \, R_q^p$$

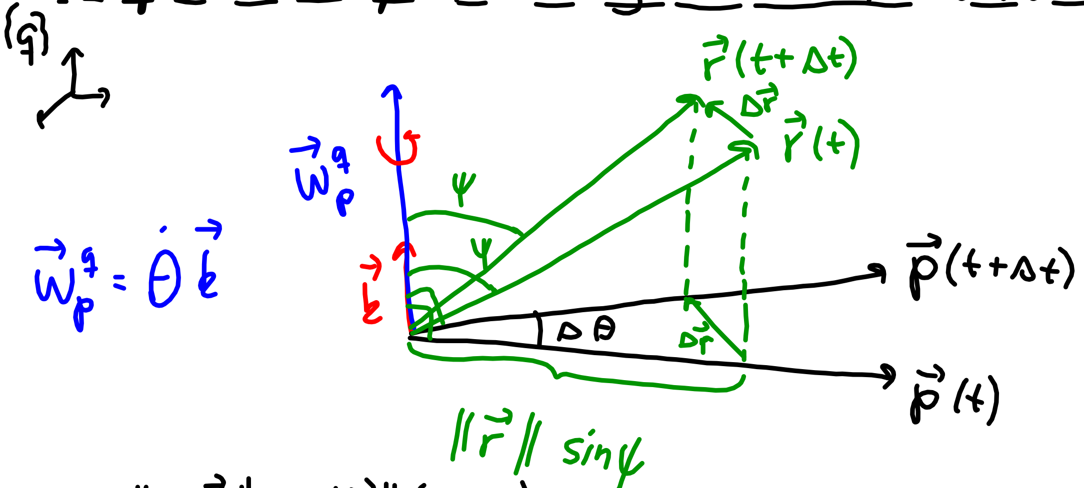
$$\dot{R}_p^q = S(\underline{\omega}_p^{qq}) R_p^q = R_p^q \, S(\underline{\omega}_p^{qp}) \underbrace{R_q^p \, R_p^q}_{I}$$

$$\boxed{\dot{R}_p^q = S(\underline{\omega}_p^q) R_p^q = R_p^q \, S(\underline{\omega}_p^{qp})}$$

Theorem A.15

$$\underline{\omega}_p^{qq} = R_p^q \, \underline{\omega}_p^{qp}$$

Angular velocity of rotating vectors with constant length



$$\vec{\omega}_p^q = \dot{\theta}\,\vec{k}$$

* $\vec{r}$ is fixed to $\{p\}$
* $\|\vec{r}\|$ is constant
* $\{p\}$ rotates relative to $\{q\}$ with $\vec{\omega}_p^q$
* $\|\vec{k}\| = 1$

$$\|\Delta\vec{r}\| = \|\vec{r}\|(\sin\psi)\cdot\Delta\theta$$

The direction og $\Delta\vec{r}$ shall be $\perp$ to both $\vec{r}$ and $\vec{k}$, and have the direction given by the r.-h.r. for rotation around $\vec{k}$. With unit length this becoms:

$$\frac{\vec{k}\times\vec{r}}{\|\vec{k}\times\vec{r}\|} \quad \text{where } \|\vec{k}\times\vec{r}\| = \|\vec{k}\|\,\|\vec{r}\|\,\sin\psi \;\Rightarrow\; \frac{\vec{k}\times\vec{r}}{\|\vec{r}\|\sin\psi}$$

$$\Rightarrow \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta\theta}{\Delta t}\sin\psi\,\|\vec{r}\|\frac{\vec{k}\times\vec{r}}{\|\vec{r}\|\sin\psi} = \frac{\Delta\theta}{\Delta t}\,\vec{k}\times\vec{r} \;\Rightarrow\; \boxed{\dot{\vec{r}}^{\,q} = \dot{\theta}\,\vec{k}\times\vec{r} = \vec{\omega}_p^q\times\vec{r}}$$

$$\underline{\dot{r}}^{qq} = \underline{\omega}_p^{qq}\times\underline{r}^q$$