F.11 / B.2.2 Law of angular momentum ("Spinn satsen")

We want to define the law of angular momentum of a rigid body that gives us the relation between outer torque, the inertia matrix and angular accelleration. We do this in three steps.

1. Law of angular momentum for one particle.
2. —"— n particles.
3. —"— rigid body

Teorem B.6 ***Spinnsatsen for en partikkel***
Gitt en partikkel, P, som er utsatt for en kraft, F . La A være et vilkårlig punkt i treghetssystemet **i** $(\vec{r} = \vec{r}_A + \vec{\rho}_A)$. *Da er sammenhengen mellom momentet og spinnet om punktet A:*

($\triangleq$ Definition)

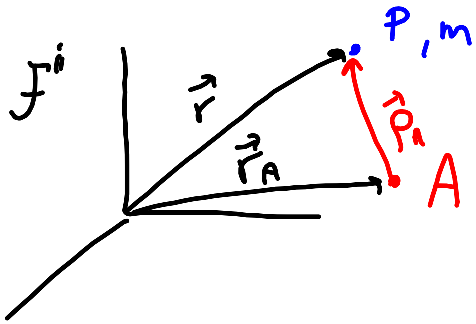
$$\vec{n}_A = \dot{\vec{h}}^{\mathbf{i}}_A + \vec{\rho}_A \times (m\ddot{\vec{r}}^{\mathbf{i}}_A) \quad hvor$$

$$\vec{n}_A \triangleq \vec{\rho}_A \times \vec{F} \qquad \text{(B- 141)}$$

$$\vec{h}^{\mathbf{i}}_A \triangleq \vec{\rho}_A \times (m\dot{\vec{\rho}}^{\mathbf{i}}_A) \qquad \text{(B- 142)}$$

Dersom $\vec{\rho}_A \times (m\ddot{\vec{r}}^{\mathbf{i}}_A) = \vec{0}$, *dvs bla når A oppfyller 1 eller 2 :*
1). $\ddot{\vec{r}}^{\mathbf{i}}_A = \vec{0}$ *: A har konstant hastighet i treghetssystemet (ligger f.eks i ro).*
2). $\vec{\rho}_A \parallel \ddot{\vec{r}}^{\mathbf{i}}_A$ *: A akselererer mot/fra partikkel P*
så kan spinnsatsen skrives :

$$\boxed{\vec{n}_A = \dot{\vec{h}}^{\mathbf{i}}_A} \qquad \text{(B- 143)}$$



$$\vec{r} = \vec{r}_A + \vec{\rho}_A$$

Def:

Torque (moment) around A: $\vec{n}_A = \vec{\rho} \times \vec{f}$

Angular momentum (spin) –"– : $\vec{h}_A^i = \vec{\rho} \times (m \dot{\vec{\rho}}^i)$

Note: Torque and angular momentum can be defined differently in different text books.

What is the relationship between torque $\vec{n}_A$ and the angular momentum $\vec{h}_A^i$?

Answer: $\vec{n}_A = \dot{\vec{h}}_A^i + \underbrace{\vec{\rho}_A \times (m \ddot{\vec{r}}_A^i)}_{\text{When} = 0?\ \text{How to choose } A.}$

Proof:

$$\vec{n}_A \underset{\text{Def.}}{\overset{\Delta}{=}} \vec{\rho} \times \vec{f} \underset{\text{N.2}}{=} \vec{\rho} \times \left(m\, \ddot{\vec{r}}^{\,i}\right) = \vec{\rho} \times \left(m \frac{d^{i\,2}}{dt^2}\left(\vec{r}_A + \vec{\rho}\right)\right)$$

$$= \vec{\rho} \times m\, \ddot{\vec{r}}_A^{\,i} + \underbrace{\vec{\rho} \times m\, \ddot{\vec{\rho}}^{\,i}}_{= \dot{\vec{h}}_A^{\,i}\ ?}$$

Yes, they are.

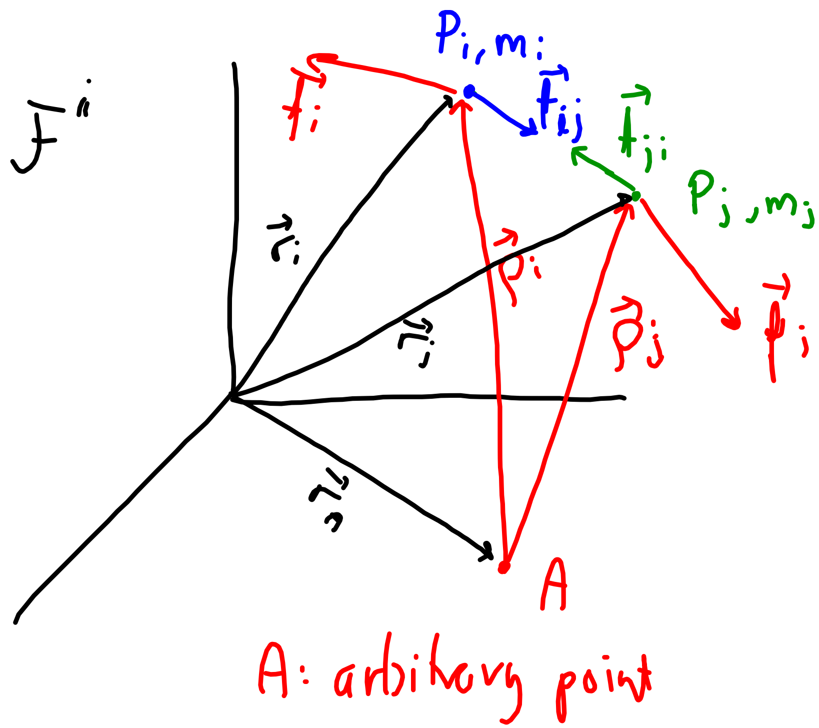
$$\vec{h}_A^{\,i} \overset{\Delta}{=} \vec{\rho} \times \left(m \dot{\vec{\rho}}^{\,i}\right)$$

$$\dot{\vec{h}}_A^{\,i} = \underbrace{\dot{\vec{\rho}}^{\,i} \times \left(m \dot{\vec{\rho}}^{\,i}\right)}_{=0} + \vec{\rho} \times \left(m\, \ddot{\vec{\rho}}^{\,i}\right)$$

$$\Rightarrow \boxed{\vec{n}_A = \dot{\vec{h}}_A^{\,i} + \vec{\rho} \times m\, \ddot{\vec{r}}_A^{\,i}}$$

q.e.d

Law of ang. mom. for a n-particle system



$\mathcal{F}^i$: inertial frame

P_i : particle nr. i

m_i : mass of P_i

$\vec{f}_i$: outer force on P_i

$\vec{f}_{ij}$: force P_j acts on P_i

Assume that $\vec{f}_{ij} = -\vec{f}_{ji}$ and

$\vec{f}_{ij} \parallel \vec{r}_i - \vec{r}_j = \vec{r}_{ij}$, i.e.

central forces

For P_i we have:

$$\vec{n}_{Ai} = \vec{\rho}_i \times \left(\vec{f}_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} \vec{f}_{ij} \right)$$

$$\dot{\vec{h}}_{Ai} = \vec{\rho}_i \times \left(m \dot{\vec{\rho}}_i' \right)$$

For n-particles:

$$\vec{n}_A = \sum_{i=1}^{n} \vec{n}_{Ai} \quad : \text{Total torque around } A$$

$$\dot{\vec{h}}_A = \sum_{i=1}^{n} \dot{\vec{h}}_{Ai} \quad : \text{Total angular momentum (spin) around } A$$

What is the relationship between $\vec{n}_A$ and $\dot{\vec{h}}_A$?

Teorem B.7 ***Spinnsatsen for et n-partikkel system***
Gitt et system av n partikler hvor partikkel i er utsatt for den ytre kraft $\vec{F}_i$ og krafta $\vec{f}_{ij}$ fra partikkel j ($j = 1, \ldots, n$) antas å oppfylle Newtons 3.lov ($\vec{f}_{ij} = -\vec{f}_{ji}$) og i tillegg være en sentralkraft (ligger langs $\vec{r}_i - \vec{r}_j$). Da vil for et vilkårlig punkt A i treghetssystemet **i**:

$$\vec{n}_A = \dot{\vec{h}}^{\mathbf{i}}_A + \sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{r}}^{\mathbf{i}}_A \quad \textit{hvor} \tag{B- 144}$$

$$\vec{n}_A = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \vec{F}_i \quad \textit{totalt ytre moment om A} \tag{B- 145}$$

$$\vec{h}_A = \sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \dot{\vec{\rho}}^{\mathbf{i}}_{Ai} \quad \textit{totalt spinn om A} \tag{B- 146}$$

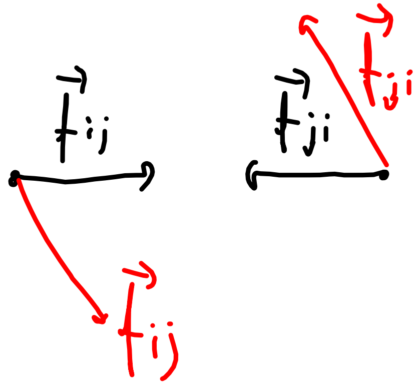
Dersom $\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{r}}^{\mathbf{i}}_A = 0$, *dvs bl.a. når A oppfyller 1, 2 eller 3 :*
1). $\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} = \vec{0}$ *: A er i massesenteret.* normal thing to do
2). $\ddot{\vec{r}}^{\mathbf{i}}_A = \vec{0}$ *: A har konstant hastighet i treghetsrommet (f.eks. i ro).*
3). $\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \parallel \ddot{\vec{r}}^{\mathbf{i}}_A$*: A akselererer mot massesenteret.*
så kan spinnsatset for et n-partikkel system skrives :

$$\vec{n}_A = \dot{\vec{h}}^{\mathbf{i}}_A \tag{B- 147}$$

Proof:

Sum $\vec{h}_{Ai}^{\,i}$ over all i and do the same with $\vec{n}_{Ai}$.

Use Newtons. 2nd law and see that all cross terms dissapears because $\vec{f}_{ij} + \vec{f}_{ji} = 0$ and they are central forces.



$\vec{f}_{ij} = -\vec{f}_{ji}$ are central forces.

$\vec{f}_{ij} = -\vec{f}_{ji}$ are not central forces

Proof of B.7

$$\vec{n}_A = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \Big(\vec{f}_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} \vec{f}_{ij} \Big)$$

$$= \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \vec{f}_i + \underbrace{\begin{aligned} &\vec{\rho}_{A1} \times (0 + \vec{f}_{12} + \dots + \vec{f}_{1n}) \\ &+ \vec{\rho}_{A2} \times (\vec{f}_{21} + 0 + \dots + \vec{f}_{2n}) \\ &\vdots \\ &+ \vec{\rho}_{An} \times (\vec{f}_{n1} + \vec{f}_{n2} + \dots + 0) \end{aligned}}_{= \vec{0}}$$

$$\boxed{\vec{n}_A = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \vec{f}_i}$$

$= \vec{0}$ because:

$$\vec{\rho}_{Ai} \times \vec{f}_{ij} + \vec{\rho}_{Aj} \times \vec{f}_{ji}$$

$$= \vec{\rho}_{Ai} \times \vec{f}_{ij} - \vec{\rho}_{Aj} \times \vec{f}_{ij}$$

$$= (\vec{\rho}_{Ai} - \vec{\rho}_{Aj}) \times \vec{f}_{ij}$$

$$= \vec{0}$$

because $\vec{\rho}_{Ai} - \vec{\rho}_{Aj} \parallel \vec{f}_{ij}$

(central forces)

$$\vec{n}_A = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \vec{f}_i = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \left(m_i \ddot{\vec{r}}_i^{\,i}\right) = \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \left(m_i \frac{d^{2\,i}}{dt^2}\left(\vec{r}_i\right)\right)$$

$$= \sum_{i=1}^{n} \vec{\rho}_{Ai} \times \left(m_i \frac{d^{2\,i}}{dt^2}\left(\vec{r}_A + \vec{\rho}_{Ai}\right)\right)$$

$$= \sum_{i=1}^{n} \vec{\rho}_{Ai} \times m_i \ddot{\vec{r}}_A^{\,i} + \underbrace{\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{\rho}}_{Ai}^{\,i}}_{= \dot{\vec{h}}_A^{\,i}\ ?}$$

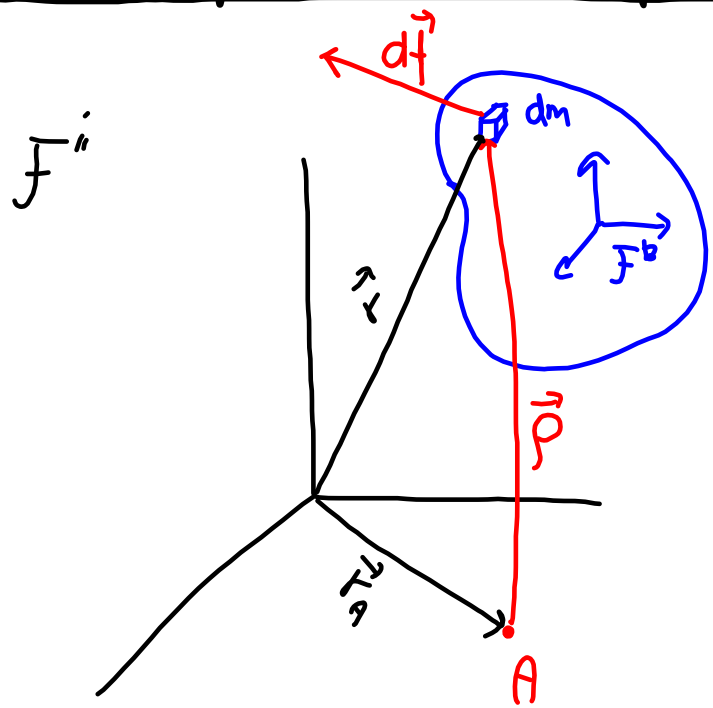
$$\dot{\vec{h}}_A^{\,i} = \frac{d^{\,i}}{dt}\left(\sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \dot{\vec{\rho}}_{Ai}^{\,i}\right)$$

Yes, they are!

$$= \underbrace{\sum_{i=1}^{n} m_i \dot{\vec{\rho}}_{Ai}^{\,i} \times \dot{\vec{\rho}}_{Ai}^{\,i}}_{= \vec{0}} + \sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{\rho}}_{Ai}^{\,i}$$

$$\Rightarrow \boxed{\vec{n}_A = \dot{\vec{h}}_A^{\,i} + \sum_{i=1}^{n} m_i \vec{\rho}_{Ai} \times \ddot{\vec{r}}_A^{\,i}}$$ q.e.d.

Law of angular momentum (spinnsatzen) for a rigid body



F^i : inertial frame

F^b : frame fixed to the body

$d\vec{f}$: outer force acting on the mass element dm

$dm = k(\vec{r})\, dV$ where

$k(\vec{r})$ is the mass density.

b- body

The body is rigid, i.e. the distance between the molecyls (particles) is const.
We have already found the law of angular momentum for a n-particle system.
We want to use integrals instead of sums (Σ)

We let $n \to \infty$: $\Sigma \to \iiint_M dm$ or $\iiint_V k(\vec{r})\, dV$

M: total mass of body
V: volume of body

Particles $\to$ mass differentials

$$\vec{h}_{Ai}^{\,i} = \vec{\rho}_i \times \left(m_i \dot{\vec{\rho}}_i^{\,i}\right) \Rightarrow d\vec{h}_A^{\,i} = \vec{\rho} \times \left(\dot{\vec{\rho}}^{\,i}\, dm\right)$$

$$\vec{h}_A^{\,i} = \iiint d\vec{h}_A^{\,i} = \iiint_M \vec{\rho} \times \dot{\vec{\rho}}^{\,i}\, dm$$

$$= \iiint_M \vec{\rho} \times \left(\dot{\vec{\rho}}^{\,b} + \vec{\omega}_b^{\,i} \times \vec{\rho}\right) dm$$

If A is fixed to the body:

$$\dot{\vec{h}}_A^i = \iiint_M \vec{\rho} \times (\dot{\vec{\omega}}_b^i \times \vec{\rho})\, dm = \overbrace{-\iiint_M dm\, \vec{\rho} \times (\vec{\rho} \times \dot{\vec{\omega}}_b^i)}^{\mathbb{J}_A} = \mathbb{J}_A\, \dot{\vec{\omega}}_b^i$$

Total outer torque:

one-particle: $\vec{n}_{Ai} = \vec{\rho}_i \times \vec{f}_i$

n-particles: $\vec{n}_A = \sum \vec{\rho}_i \times \vec{f}_i \Longrightarrow \vec{n}_A = \iiint_M \vec{\rho} \times d\vec{f}$

What is the relationship between $\vec{n}_A$ and $\dot{\vec{h}}_A^i$?

Teorem B.8 ***Spinnsatsen for stive legemer***
Gitt treghetssystemet **i** *og et k.s. b som ligger fast i legemet og har sitt origo i A. Dersom A tilfredstiller 1 eller 2 :*
1). A ligger i massesenteret.
2). A ligger i ro i treghetsrommet.
er spinnsatsen på en koordinatuavhengig form :

Theorem A.19

$$\vec{n}_A = \dot{\vec{h}}^{\mathbf{i}}_A = \dot{\vec{h}}^{\mathbf{ib}}_A + \vec{\omega}^{\mathbf{i}}_b \times \vec{h}^{\mathbf{i}}_A \tag{B- 148}$$

eller representert i b-systemet :

Theorem A.18

$$\begin{aligned} \underline{n}^b_A &= \dot{\underline{h}}^{\mathbf{i}bb}_A + \underline{\omega}^{\mathbf{i}b}_b \times \underline{h}^{\mathbf{i}b}_A \\ &= J^b \dot{\underline{\omega}}^{\mathbf{i}bb}_b + \underline{\omega}^{\mathbf{i}b}_b \times (J^b \underline{\omega}^{\mathbf{i}b}_b) \\ \dot{\underline{\omega}}^{\mathbf{i}bb}_b &= R^b_{\mathbf{i}} \dot{\underline{\omega}}^{\mathbf{i}}_b \end{aligned} \tag{B- 149}$$

hvor spinnet er definert ved :

$$\underline{h}^{\mathbf{i}b}_A = J^b \underline{\omega}^{\mathbf{i}b}_b$$

$$\vec{n}_A = \iiint_M \vec{\rho} \times d\vec{f} \quad \text{(only outer forces)}$$

$$\vec{h}^{i}_A = -\iiint_M \vec{\rho} \times (\vec{\rho} \times \vec{w}^{i}_b)\, dm = \mathbb{J}_A \vec{w}^{i}_b$$

$$J^b = [\mathbb{J}_A]^b$$

Treghetsmatrisa J^b *beregnes via treghetsmomenta,* J^b_{ii}*, og treghetsprodukta,* J^b_{ij} *:*

$$J^b = \begin{bmatrix} J^b_{xx} & -J^b_{xy} & -J^b_{xz} \\ -J^b_{yx} & J^b_{yy} & -J^b_{yz} \\ -J^b_{zx} & -J^b_{zy} & J^b_{zz} \end{bmatrix} \tag{B- 150}$$

$$\begin{bmatrix} J^b_{xx} & -J^b_{xy} & -J^b_{xz} \\ -J^b_{yx} & J^b_{yy} & -J^b_{yz} \\ -J^b_{zx} & -J^b_{zy} & J^b_{zz} \end{bmatrix} = \begin{bmatrix} \int_M \left(y^2+z^2\right) dm & -\int_M xydm & -\int_M xzdm \\ -\int_M xydm & \int_M \left(x^2+z^2\right) dm & -\int_M yzdm \\ -\int_M xzdm & -\int_M yzdm & \int_M \left(x^2+y^2\right) dm \end{bmatrix} \tag{B- 151}$$

Dvs treghetsmatrisa er symmetrisk.

NB! Here is $\rho^b = \begin{bmatrix} \rho_1^b \\ \rho_2^b \\ \rho_3^b \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$