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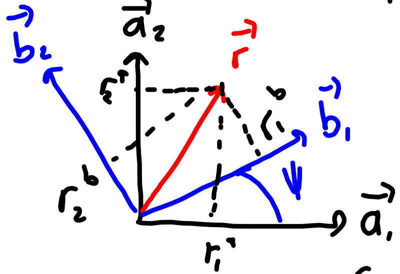
anders.rodningsby@ffi.no

Exam 15. Dec

A.2.6 Interpretation of the direction cosine matrix (DCM) (RKM)

1. C_b^a is a coordinate transformation matrix (CTM)

$$\underline{r}^a = C_b^a \underline{r}^b$$



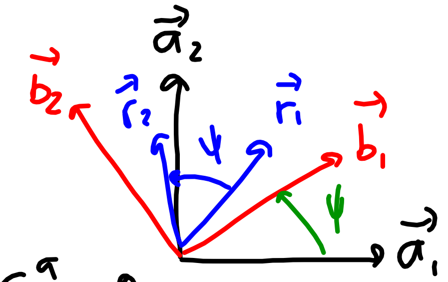
$$C_b^a \overset{an}{=} R_b^a = \begin{bmatrix} \underline{b}_1^a & \underline{b}_2^a \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$$

This is a passive operation

2. C_b^a is an attitude matrix (stillingsmatrise)

$$C_b^a = \left[\langle \vec{b}_j, \vec{a}_i \rangle \right] = \begin{bmatrix} \langle \vec{b}_1, \vec{a}_1 \rangle & \langle \vec{b}_2, \vec{a}_1 \rangle & \langle \vec{b}_3, \vec{a}_1 \rangle \\ \langle \vec{b}_1, \vec{a}_2 \rangle & \langle \vec{b}_2, \vec{a}_2 \rangle & \langle \vec{b}_3, \vec{a}_2 \rangle \\ \langle \vec{b}_1, \vec{a}_3 \rangle & \langle \vec{b}_2, \vec{a}_3 \rangle & \langle \vec{b}_3, \vec{a}_3 \rangle \end{bmatrix} = \begin{bmatrix} \underline{b}_1^a & \underline{b}_2^a & \underline{b}_3^a \end{bmatrix}$$

3. C_b^a is a rotation matrix (RM)



$$C_b^a = [\mathbb{R}_{ab}]^a = [\mathbb{R}_{ab}]^b$$

$$\vec{r}_2 = \mathbb{R}_{ab}\vec{r}_1 \iff \underline{r}_2^a = [\mathbb{R}_{ab}]^a \underline{r}_1^a = R_{ab}^a \underline{r}_1^a = C_b^a \underline{r}_1^a$$

$$\underline{r}_2^b = [\mathbb{R}_{ab}]^b \underline{r}_1^b = R_{ab}^b \underline{r}_1^b = C_b^a \underline{r}_1^b$$

This is an active operation. The vector is rotated.

The coordinate transformation matrix (CTM) C_b^a that transforms a vector in the b-fram to the a-frame acts as an rotationmatrix in one and the same frame, and rotates the vector in the same way as the a-frame needs to rotate to coinside with the b-frame.

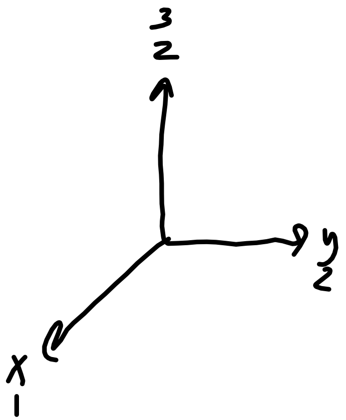
A.2.7 Representasjon av ortogonale RKM

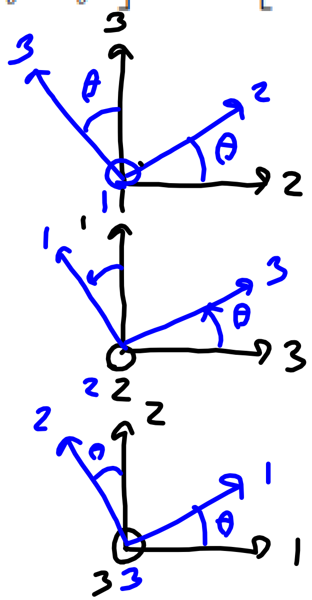
Vi skal i dette avsnittet se på ulike måter å representere ortogonale RKM.

Eulervinkelrepresentasjon av RKM

Elementære RKM. Gitt rammene q og p. Dersom en tenker seg at rammene opprinnelig var sammenfallende fås den endelige p-ramma ved å dreie den en vinkel θ om q_i-aksen. Vi har følgende elementære RKMer:

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix}, \; R_2 = \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix}, \; R_3 = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(A- 30)}$$





$$R_1(\theta) = R_{\text{blue}}^{\text{black}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_3(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Calculation rules for the elementary DCM

$$R_i(\theta_1+\theta_2) = R_i(\theta_1)R_i(\theta_2) = R_i(\theta_2)R_i(\theta_1)$$

$$\left(R_i(\theta)\right)^{-1} = \left(R_i(\theta)\right)^T = R_i(-\theta)$$

$$\Downarrow$$

$$\left(R_b^a\right)^T = R_a^b$$

Generally

$$R_a^c = R_b^c R_a^b \neq R_a^b R_b^c$$

DCM are non-comutative

Rotation sequences

3-2-1 Euler angles

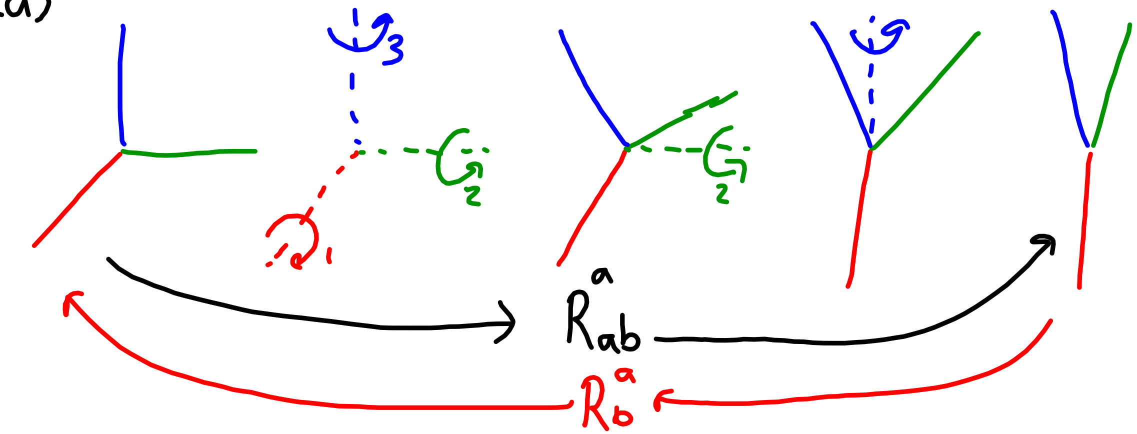
1. Rotation around new axis (Euler angles)

Rot. 1	1	1	2	2	3	3	1	1	2	2	3	3
Rot. 2	2	3	1	3	1	2	2	3	1	3	1	2
Rot. 3	3	2	3	1	2	1	1	1	2	2	3	3

} 12 sequences

2. Rotation around fixed axis (12 sequences)

$\{a\}$



$\{b\}$

R_{ab}^{a} : rot. $a \to b$

R_{b}^{a} : transforms $b \to a$

$$R_{b}^{a} = R_{ab}^{a} = R_{ab}^{b}$$

Example

What is R_b^a when rotating ψ arond axis 3, θ around axis 2 and ϕ arond axis 1?

Fixed axis: $R_b^a = R_1(\phi)\, R_2(\theta)\, R_3(\psi)$

New axis : $R_b^a = R_3(\psi)\, R_2(\theta)\, R_1(\phi)$

Teorem A.8 ***Sammenheng mellom rotasjon om nye og faste akser***
Tre rotasjoner om nye akser (eulervinkler) gir den samme endelige stilling som de samme rotasjoner tatt i omvendt rekkefølge om faste akser, dvs :

$$ {}^{E}R_p^q(\alpha_i, \beta_j, \gamma_k) = {}^{F}R_p^q(\gamma_k, \beta_j, \alpha_i) \qquad \text{(A- 35)} $$

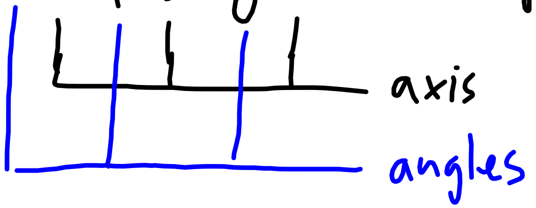
$$ {}^{E}R_p^q(\alpha_i, \beta_j, \gamma_k) = R_i(\alpha) R_j(\beta) R_k(\gamma) $$

$$ {}^{F}R_p^q(\gamma_k, \beta_j, \alpha_i) = R_i(\alpha) R_j(\beta) R_k(\gamma) $$

Direct problem

Given the rotation sequence and angles (fixed or new axis) find the DCM

$$\alpha_i \; \beta_j \; \gamma_k \rightarrow R_p^q$$



Gives always a clear R_p^q-matrix

Inverse problem

Given R_p^q-matrix find

α, β, γ for a given rotation sequence

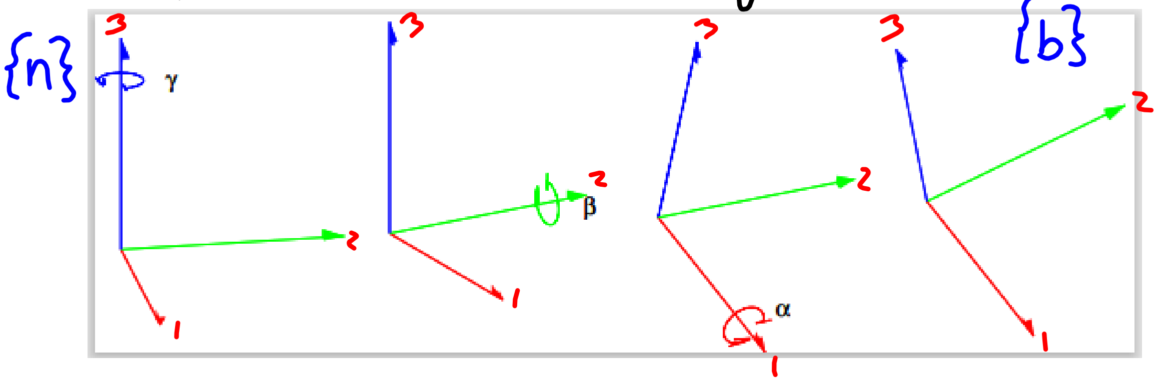
Cannot always find a unique solution $\Rightarrow$ we have singularities (for 3-2-1 euler angles if we rotate 90° around axis 2.)

Example A.6 3-2-1 Euler angles.

$R_b^n = R_3(\gamma)R_2(\beta)R_1(\alpha)$

b: body frame

n: navigation frame



Eksempel A.6 3-2-1 Eulervinkler.
Ved simulering av fly og båter bruker en ofte følgende stillingsmatrise:

$$R_b^n = {}^E R_b^n(\theta_3, \theta_2, \theta_1) = R_3(\theta_3)R_2(\theta_2)R_1(\theta_1) \qquad \text{(A- 36)}$$

Multipliseres de elementære RKM sammen får vi :

$$ {}^E R_b^n(\theta_3, \theta_2, \theta_1) = \begin{bmatrix} c_{\theta_3}c_{\theta_2} & c_{\theta_3}s_{\theta_2}s_{\theta_1} - s_{\theta_3}c_{\theta_1} & c_{\theta_3}s_{\theta_2}c_{\theta_1} + s_{\theta_3}s_{\theta_1} \\ s_{\theta_3}c_{\theta_2} & s_{\theta_3}s_{\theta_2}s_{\theta_1} + c_{\theta_3}c_{\theta_1} & s_{\theta_3}s_{\theta_2}c_{\theta_1} - c_{\theta_3}s_{\theta_1} \\ -s_{\theta_2} & c_{\theta_2}s_{\theta_1} & c_{\theta_2}c_{\theta_1} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad \text{(A- 37)}$$

$r_{31} = -\sin(\theta_2)$

Eksempel A.8 Det inverse problem for 3-2-1 Eulervinkler,:
Gitt R_b^n finn vinklene, har løsningen (Craig 1989, s. 47)

$$\begin{aligned} \theta_1 &= \operatorname{atan2}\left(r_{32}/c_{\theta_2}, r_{33}/c_{\theta_2}\right) \\ \theta_2 &= \operatorname{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right) \leftarrow \text{Solve first} \\ \theta_3 &= \operatorname{atan2}\left(r_{21}/c_{\theta_2}, r_{11}/c_{\theta_2}\right) \end{aligned} \tag{A- 39}$$

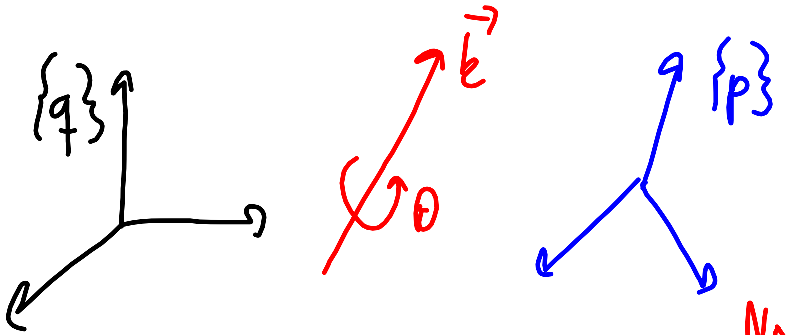
Vi løser først for θ_2. For $\theta_2 = \pm 90°$ har vi singularitet og bare summen av θ_1 og θ_3 kan beregnes.

Angle-axis representation of the direction cosine matrix (DCM)

Teorem A.9 ***Eulers rotasjonsteorem***
En vilkårlig retningskosinmatrise R_p^q kan fås ved å rotere p-systemet en vinkel θ om aksen $\underline{k}^q = [k_1^q, k_2^q, k_3^q]$ $(\|\underline{k}^q\| = 1)$. Dvs vi har :

$$R_p^q = R_{\underline{k}^q}(\theta) \tag{A- 40}$$

Denne representasjonen kalles for vinkel-akse representasjon.



Choose 1: $\|\vec{k}\| = \|\underline{k}^q\| = 1$, rot. θ

Choose 2: $\|\vec{k}\| = \|\underline{k}^q\| = \theta$

Note: R_p^q is represented by 3 parameters

Teorem A.10 ***Det direkte problem for vinkel-akse***

Når vinkel-akserepresentasjonen er gitt kan retningskosinmatrisa beregenes på følgende måte:

$$\left.\begin{aligned} R_p^q &= R_{\underline{k}}(\theta) = I + S(\underline{k}^q)\sin\theta + S^2(\underline{k}^q)(1-\cos\theta) && \text{(A- 41)} \\ &= I\cos\theta + S(\underline{k}^q)\sin\theta + \underline{k}^q\left(\underline{k}^q\right)^T(1-\cos\theta) && \text{(A- 42)} \end{aligned}\right\}$$

No proof

Fortegnet til θ bestemmes ut fra høgrehåndsregelen.

Teorem A.11 ***Det inverse problem for vinkel-akse***

Gitt retningskosinmatrise

$$R_p^q = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{221} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad \text{(A- 43)}$$

No proof

da er vinkel-akserepresentasjonen gitt ved

$$\theta = \arccos\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right); \quad \underline{k}^q = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix} \qquad \text{(A- 44)}$$

Vi får her en $\theta \in [0^\circ, 180^\circ\rangle$, det finns en annen løsning $(-\underline{k}^q, -\theta)$ som gir samme retningskosin-matrise. NB: for små vinkler θ kan de numeriske feilene ved bestemmelse av $\underline{k}^q$ bli store.