F.12 / How to calculate the angular momentum $\vec{h}_c^{\,i} = \mathbb{J}_c \, \vec{\omega}_b^{\,i}$

If we represent $\vec{\rho}$ in $\mathcal{F}^i$ the matrix representation of $\mathbb{J}_c$ becomes tim variant, but represented in the body frame $\mathcal{F}^b$ it wil be time invariant.

We therefore choose to calculate $\vec{h}_c^{\,i}$ in $\mathcal{F}^b$.

Ang. mom:

$$\vec{h}_c^{\,i} = -\iiint_M \vec{\rho} \times (\vec{\rho} \times \vec{\omega}_b^{\,i}) \, dm = \mathbb{J}_c \, \vec{\omega}_b^{\,i}, \quad \text{where } O_b = A = C$$

$$\underline{h}_c^{ib} = -\iiint_M \underline{\rho}^b \times (\underline{\rho}^b \times \underline{\omega}_b^{ib}) \, dm = -\iiint S(\underline{\rho}^b) S(\underline{\rho}^b) \underline{\omega}_b^{ib} \, dm$$

$$= \underbrace{\left(-\iiint_M S(\underline{\rho}^b) S(\underline{\rho}^b) \, dm \right)}_{J_c^b = [\mathbb{J}_c]^b} \underline{\omega}_b^{ib} = J_c^b \, \underline{\omega}_b^{ib}$$

<u>Summary</u> Kinetic equations for the center of mass ($C = A = O_b$)

$$\vec{p}_c^{\,i} = m \cdot \vec{v}_c^{\,i} \quad , \quad \vec{h}_c^{\,i} = -\iiint_M \vec{\rho} \times (\vec{\rho} \times \vec{\omega}_b^{\,i})\, dm = \mathbb{J}_c \, \vec{\omega}_b^{\,i}$$

N.2 $\quad \vec{f} = \dot{\vec{p}}_c^{\,i} = \frac{d}{dt}\left(m \vec{v}_c^{\,i}\right) = \dot{\vec{p}}_c^{\,ib} + \vec{\omega}_b^{\,i} \times \vec{p}_c^{\,i}$ $\quad\Big|\quad$ $\underline{f}^{i} = m \dot{\underline{v}}_c^{i}$, $\underline{f}^{b} = \dot{\underline{p}}_c^{ibb} + \underbrace{\underline{\omega}_b^{ib}}_{S(\underline{\omega}_b^{ib})} \times \underline{p}^{ib}$

Ang. mom. $\quad \vec{n}_c = \dot{\vec{h}}_c^{\,i} = \frac{d}{dt}\left(\mathbb{J}_c \, \vec{\omega}_b^{\,i}\right) = \dot{\vec{h}}_c^{\,ib} + \vec{\omega}_b^{\,i} \times \vec{h}_c^{\,i}$ $\quad\Big|\quad$ $\underline{n}_c^{i} = \dot{\underline{h}}_c^{i}$, $\underline{n}_c^{b} = \dot{\underline{h}}_c^{ibb} + \underbrace{\underline{\omega}_b^{ib}}_{S(\underline{\omega}_b^{ib})} \times \underline{h}_c^{ib}$

$$\boxed{\begin{aligned}\underline{h}_c^{ib} &= J_c^b\, \underline{w}_b^{ib} \\ \underline{n}_c^b &= J_c^b\, \underline{\dot{w}}_b^{ibb} + S(\underline{w}_b^{ib})\, J_c^b\, \underline{w}_b^{ib}\end{aligned}}$$ — Gives B-149

J_c^b : Inertia matrix

$$J_c^b = -\iiint_M S(\underline{\rho}^b)\, S(\underline{\rho}^b)\, dm \quad , \quad \underline{\rho}^b = [\rho_1 ; \rho_2 ; \rho_3]$$

$$= -\iiint_M \begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix} dm$$

$$= + \iiint_M \begin{bmatrix} \rho_2^2 + \rho_3^2 & -\rho_1\rho_2 & -\rho_1\rho_3 \\ -\rho_2\rho_1 & \rho_1^2 + \rho_3^2 & -\rho_2\rho_3 \\ -\rho_3\rho_1 & -\rho_3\rho_2 & \rho_1^2 + \rho_2^2 \end{bmatrix} dm$$

$$J_c^b = [J_c^b]^T$$

Moment of inertia

$$[J_c^b]_{ii} = \iiint_M (\rho_j^2 + \rho_k^2)\, dm, \quad j \neq k,\ j \neq i,\ k \neq i$$

$$[J_c^b]_{ij} = -\iiint_M \rho_i \rho_j\, dm, \quad i \neq j$$

Product of inertia

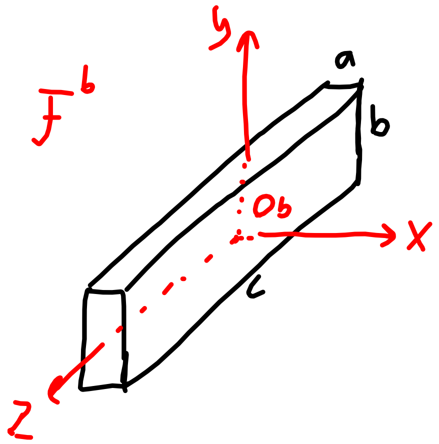
J_c^b is a real, symmetric, positive definite matrix ($\text{Det}(J_c^b) > 0$)

$\Rightarrow$ matrix has real eigenvalues and orthogonal eigenvectors

Eigenvectors are called the main axis of the body. I.e. if $\mathcal{F}^b$ has the basis vectors along the main axis :

$$J_c^b = \text{diag}(J_{xx}^b, J_{yy}^b, J_{zz}^b)$$

Example: Inertia matrix of a brickwall "murstein"



Assume: $a < b < c$

$\mathcal{F}^b$ has origo in the center of mass O_b

$$\underline{r}^b = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Crossterms:

$$\left[J_c^b\right]_{xy} = -\iiint_M xy\,dm = -\iiint_M xy\,k\,dx\,dy\,dz \quad , \; k: \text{mass density (const.)}$$

$$= -k \int_{-c/2}^{c/2}\int_{-b/2}^{b/2} y \left(\int_{-a/2}^{a/2} x\,dx\right) dy\,dz = 0 \quad \text{because of symmetry}$$

$$\left[J_c^b\right]_{xx} = k\iiint (y^2+z^2)\,dx\,dy\,dz = k\iint (y^2+z^2)\left(\int_{-a/2}^{a/2} 1\,dx\right) dy\,dz$$

$$= k\,a\iint (y^2+z^2)\,dy\,dz = ka\iint y^2\,dy\,dz + ka\iint z^2\,dy\,dz$$

$$= kac\int_{-b/2}^{b/2} y^2\,dy + kab\int_{-c/2}^{c/2} z^2\,dz$$

$$= akc\left[\frac{1}{3}y^3\right]_{-b/2}^{b/2} + akb\left[\frac{1}{3}z^3\right]_{-c/2}^{c/2} = akc\,\frac{2}{3}\,\frac{b^3}{8} + akb\,\frac{2}{3}\,\frac{c^3}{8}$$

$$= \underbrace{kabc}_{M}\,\frac{b^2}{12} + \underbrace{kabc}_{M}\,\frac{c^2}{12} = \frac{M}{12}\left(b^2+c^2\right)$$

$$J_c^b = \mathrm{diag}\left(\frac{M}{12}(b^2+c^2),\ \frac{M}{12}(a^2+c^2),\ \frac{M}{12}(a^2+b^2)\right)$$

We had $a < b < c \Rightarrow J_{xx}^b > J_{yy}^b > J_{zz}^b$

Euler equation

When we put A in the centre of mass C, $\mathcal{F}^b$ is fixed to the body and $O_b = C = A$, the law of angular momentum becomes:

$$① \quad \underline{n}_c^b = J_c^b \dot{\underline{w}}_b^{ibb} + S(\underline{w}_b^{ib}) J_c^b \underline{w}_b^{ib}$$

Assume $\mathcal{F}^b$ coinsides with the main axis, i.e. J_c^b is diagonal, and $\underline{n}_c^b = [n_x; n_y; n_z]$, $\underline{w}_b^{ib} = [w_x; w_y; w_z]$, $J_c^b = \mathrm{diag}(J_{xx}^b, J_{yy}^b, J_{zz}^b)$.

In this case we get the Euler equations.

Teorem B.9 ***Eulerlikningene***

Dersom k.s. b velges fast i legemet med origo i A, med akser langs hovedaksene for legemet og A i tillegg tilfredstiller 1 eller 2 :

1). A ligger i massesenteret.

2). A ligger i ro i treghetsrommet. $A = C = O_b$

kan spinnsatsen skrives på følgende enkle form :

$$\left.\begin{aligned} n_x &= J^b_{xx}\dot{\omega}_x + \omega_y\omega_z(J^b_{zz} - J^b_{yy}) \\ n_y &= J^b_{yy}\dot{\omega}_y + \omega_z\omega_x(J^b_{xx} - J^b_{zz}) \\ n_z &= J^b_{zz}\dot{\omega}_z + \omega_x\omega_y(J^b_{yy} - J^b_{xx}) \end{aligned}\right\}, \quad \underline{n}^b_A = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix}, \quad \underline{\omega}^{ib}_b = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad \text{(B- 152)}$$

Proof: Insert into eq. 1

$$\begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} J^b_{xx}\dot{w}_x \\ J^b_{yy}\dot{w}_y \\ J^b_{zz}\dot{w}_z \end{bmatrix} + \begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix} \begin{bmatrix} J^b_{xx} w_x \\ J^b_{yy} w_y \\ J^b_{zz} w_z \end{bmatrix} = \begin{bmatrix} J^b_{xx}\dot{w}_x + w_y w_z (J^b_{zz} - J^b_{yy}) \\ J^b_{yy}\dot{w}_y + w_x w_z (J^b_{xx} - J^b_{zz}) \\ J^b_{zz}\dot{w}_z + w_x w_y (J^b_{yy} - J^b_{xx}) \end{bmatrix}$$

Euler equations can be used in 2 ways:

1) Given the forces ($\underline{n}_c^b$) find the motion ($\underline{w}_b^{ib}$). Differential eq.

2) Given the motion ($\underline{w}_b^{ib}$) find the forces ($\underline{n}_c^b$). Algebraic eq.

Solution of 1)

(2)

$$\boxed{\begin{aligned}\dot{w}_x &= \frac{1}{J_{xx}}\left[(J_{yy}-J_{zz})w_y w_z + n_x\right]\\ \dot{w}_y &= \frac{1}{J_{yy}}\left[(J_{zz}-J_{xx})w_x w_z + n_y\right]\\ \dot{w}_z &= \frac{1}{J_{zz}}\left[(J_{xx}-J_{yy})w_x w_y + n_z\right]\end{aligned}}$$

Eq. (2) is on standard form:

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}), \quad \underline{x}(t_0) \text{ given}$$

For d.e. (2): $\underline{x} = \underline{w}_b^{ib}$, $\underline{u} = \underline{n}_c^b$

To find the orientation/attitude:

$$\boxed{\begin{aligned}\dot{R}_b^i &= R_b^i S(\underline{w}_b^{ib}), \quad R_b^i(t_0) \text{ given}\\ \dot{\underline{w}}_b^{ibb} &= \underline{f}(\underline{w}_b^{ib}, \underline{n}_c^b), \quad \underline{w}_b^{ib}(t_0) \text{ given}\end{aligned}}$$