Condensed Ricci Curvature of Paley Graphs and Their Generalizations

Vincent Bonini, Daniel Chamberlin, Stephen Cook, Parthiv Seetharaman, Tri Tran

California Polytechnic State University, San Luis Obispo

August 24, 2023

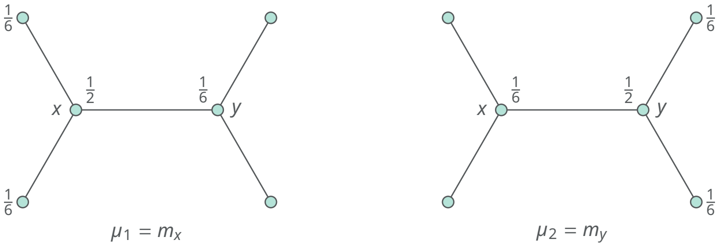
Condensed Ricci Curvature

Paley Graphs

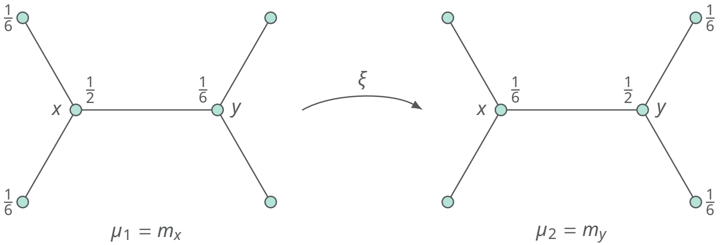
Results and Conjectures

Condensed Ricci Curvature

Transport Plan Example 1

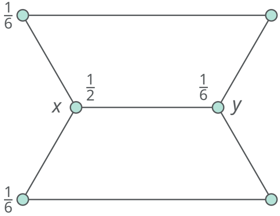


Transport Plan Example 1

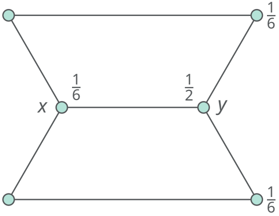


$$\xi(v,w) = \begin{cases} \frac{1}{2} - \frac{1}{6} & \text{if } v = x, w = y \\ \frac{1}{6} & \text{if } vw \text{ is a 3-step matching} \\ 0 & \text{otherwise} \end{cases}$$

Transport Plan Example 2

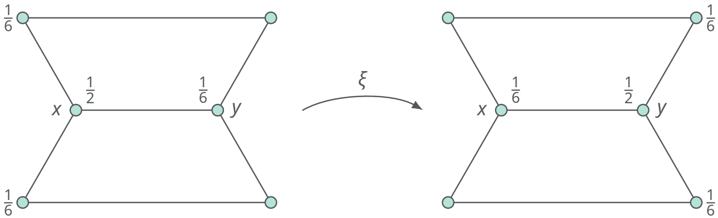


$\mu_1 = m_x$



$\mu_2 = m_y$

Transport Plan Example 2



$\mu_1 = m_x$ $\mu_2 = m_y$

$$\xi(v,w) = \begin{cases} \frac{1}{2} - \frac{1}{6} & \text{if } v = x, w = y \\ \frac{1}{6} & \text{if } vw \text{ is a 1-step matching} \\ 0 & \text{otherwise} \end{cases}$$

Mass Distributions and Transport Plans

Definition

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A ***mass distribution*** on G is a real-valued function $\mu : V \longrightarrow [0, 1]$ such that $\sum_{v \in V} \mu(v) = 1$.

Mass Distributions and Transport Plans

Definition

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A ***mass distribution*** on G is a real-valued function $\mu : V \longrightarrow [0, 1]$ such that $\sum_{v \in V} \mu(v) = 1$.

Definition

Let μ_1, μ_2 be mass distributions on a graph $G = (V, E)$. A ***transport plan*** between μ_1 and μ_2 is a function $\xi : V \times V \longrightarrow [0, 1]$ such that $\sum_{w \in V} \xi(v, w) = \mu_1(v)$ and $\sum_{v \in V} \xi(v, w) = \mu_2(w)$.

Condensed Ricci Curvature

- The condensed Ricci curvature is a graph invariant that measures *bottlenecking* and *connectivity* of a graph

Condensed Ricci Curvature

- The condensed Ricci curvature is a graph invariant that measures *bottlenecking* and *connectivity* of a graph

Definition

Let $G = (V, E)$ be a graph and suppose $xy \in E$. The ***condensed Ricci curvature*** between x and y is defined by

$$\Bbbk(x, y) = 2\big(1 - W(m_x, m_y)\big),$$

where $W(m_x, m_y)$ is the so-called *Wasserstein distance* and m_x, m_y are mass distributions of the form

$$m_z(v) = \begin{cases} \frac{1}{2} & \text{if } v = z, \\ \frac{1}{2\deg(z)} & \text{if } v \in \Gamma(z), \\ 0 & \text{otherwise.} \end{cases}$$

Here $\Gamma(z)$ is the set of vertices adjacent to z.

Wasserstein Distance

Definition

The ***Wasserstein distance*** between mass distributions μ_1 and μ_2 on a graph $G = (V, E)$ is defined by

$$W(\mu_1, \mu_2) = \inf_{\xi \in \chi(\mu_1, \mu_2)} \sum_{v \in V} \sum_{w \in V} \xi(v, w) \rho(v, w),$$

where $\chi(\mu_1, \mu_2)$ is the set of transport plans between μ_1 and μ_2.

Wasserstein Distance

Definition

The ***Wasserstein distance*** between mass distributions μ_1 and μ_2 on a graph $G = (V, E)$ is defined by

$$W(\mu_1, \mu_2) = \inf_{\xi \in \chi(\mu_1, \mu_2)} \sum_{v \in V} \sum_{w \in V} \xi(v, w)\rho(v, w),$$

where $\chi(\mu_1, \mu_2)$ is the set of transport plans between μ_1 and μ_2.

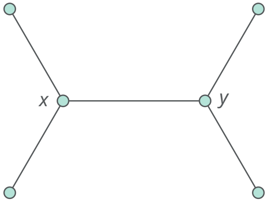
Theorem

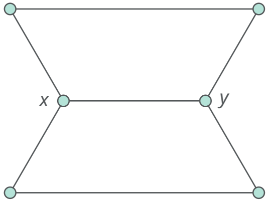
The Wasserstein distance between mass distributions μ_1 and μ_2 on a graph $G = (V, E)$ is given by

$$W(\mu_1, \mu_2) = \sup_{f \in \mathrm{Lip}(1)} \sum_{v \in V} f(v)\big(\mu_1(v) - \mu_2(v)\big),$$

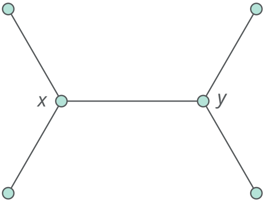
where $\mathrm{Lip}(1) = \big\{f : V \to \mathbb{R} : |f(v) - f(w)| \leq \rho(v, w) \text{ for all } v, w \in V\big\}$ *is the space of Lipschitz-*1 *functions on G.*

Condensed Ricci Curvature Example

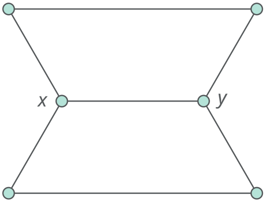




Condensed Ricci Curvature Example



$$\mathbb{k}(x,y) = -\frac{2}{3}$$

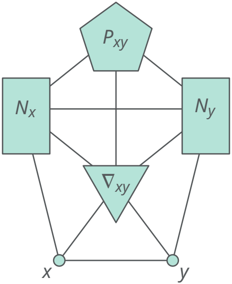


$$\mathbb{k}(x,y) = \frac{2}{3}$$

Core Neighborhoods

Definition

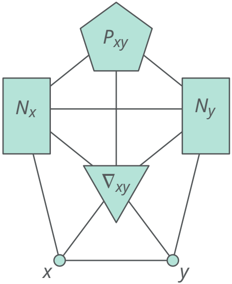
Suppose $G = (V, E)$ is a graph. The ***core neighborhood*** of $xy \in E$ is
$\mathcal{N}_{xy} = \{x\} \cup \{y\} \cup \nabla_{xy} \cup N_x \cup N_y \cup P_{xy}$.



Core Neighborhoods

Definition

Suppose $G = (V, E)$ is a graph. The ***core neighborhood*** of $xy \in E$ is $\mathcal{N}_{xy} = \{x\} \cup \{y\} \cup \nabla_{xy} \cup N_x \cup N_y \cup P_{xy}$.



- The condensed Ricci curvature of an edge $xy \in E$ is solely dependent on the core neighborhood about x and y

Paley Graphs

Residues

Definition

Suppose $\mathbb{F}_q$ is a field of order q. Then $x \in \mathbb{F}_q$ is a ***k-residue*** if $x = \alpha^k$ for some $\alpha \in \mathbb{F}_q$. Denote the set of non-zero k-residues over $\mathbb{F}_q$ by $(\mathbb{F}_q^\times)^k$.

Residues

Definition

Suppose $\mathbb{F}_q$ is a field of order q. Then $x \in \mathbb{F}_q$ is a ***k-residue*** if $x = \alpha^k$ for some $\alpha \in \mathbb{F}_q$. Denote the set of non-zero k-residues over $\mathbb{F}_q$ by $\left(\mathbb{F}_q^\times\right)^k$.

- Consider the field $\mathbb{F}_{13} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Residues

Definition

Suppose $\mathbb{F}_q$ is a field of order q. Then $x \in \mathbb{F}_q$ is a ***k-residue*** if $x = \alpha^k$ for some $\alpha \in \mathbb{F}_q$. Denote the set of non-zero k-residues over $\mathbb{F}_q$ by $(\mathbb{F}_q^\times)^k$.

- Consider the field $\mathbb{F}_{13} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
- Raising every element by 2 gives $\{0, 1, 4, 9, 3, 12, 10, 10, 12, 3, 9, 4, 1\}$

Residues

Definition

Suppose $\mathbb{F}_q$ is a field of order q. Then $x \in \mathbb{F}_q$ is a ***k-residue*** if $x = \alpha^k$ for some $\alpha \in \mathbb{F}_q$. Denote the set of non-zero k-residues over $\mathbb{F}_q$ by $\left(\mathbb{F}_q^\times\right)^k$.

- Consider the field $\mathbb{F}_{13} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
- Raising every element by 2 gives $\{0, 1, 4, 9, 3, 12, 10, 10, 12, 3, 9, 4, 1\}$
- Thus the set of non-zero 2-residues of $\mathbb{F}_{13}$ is $\left(\mathbb{F}_{13}^\times\right)^2 = \{1, 3, 4, 9, 10, 12\}$

Paley Graphs

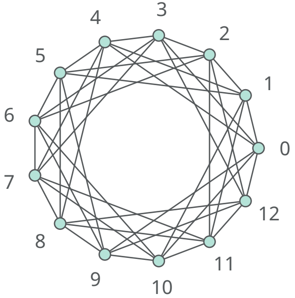
Definition

Let $k > 1$. Suppose $q = p^n$ is an odd prime power such that $q \equiv 1 \pmod{2k}$ and $\mathbb{F}_q$ be the field of order q. A ***generalized Paley graph*** of order q under k-residue is a graph $\mathcal{P}(q,k) = (V,E)$, where $V = \mathbb{F}_q$ and $E = \left\{xy : x - y \in \left(\mathbb{F}_q^\times\right)^k\right\}$.

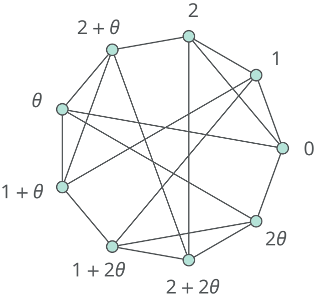
Paley Graphs

Definition

Let $k > 1$. Suppose $q = p^n$ is an odd prime power such that $q \equiv 1 \pmod{2k}$ and $\mathbb{F}_q$ be the field of order q. A ***generalized Paley graph*** of order q under k-residue is a graph $\mathcal{P}(q,k) = (V,E)$, where $V = \mathbb{F}_q$ and $E = \left\{xy : x - y \in \left(\mathbb{F}_q^\times\right)^k\right\}$.

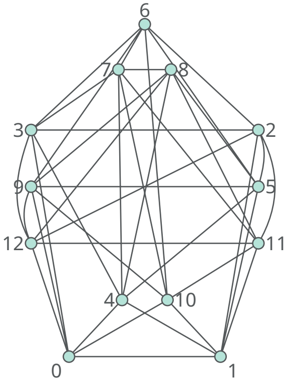


$\mathcal{P}(13,2)$

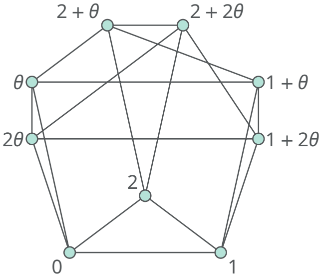


$\mathcal{P}(3^2,2)$

Core Neighborhood of Paley Graphs

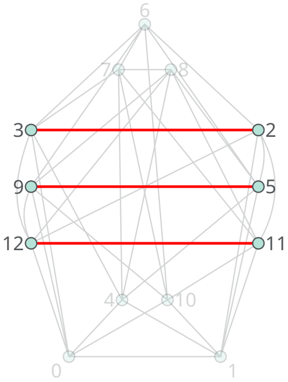


$\mathcal{P}(13,2)$

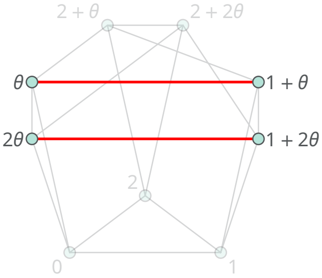


$\mathcal{P}(3^2,2)$

Core Neighborhood of Paley Graphs



$\mathcal{P}(13, 2), \quad \Bbbk = \frac{2}{3}$



$\mathcal{P}(3^2, 2), \quad \Bbbk = \frac{3}{4}$

Results and Conjectures

Main Question

Question. What is the condensed Ricci curvature of $\mathcal{P}(q, k)$?

Main Question

Question. What is the condensed Ricci curvature of $\mathcal{P}(q,k)$?

Conjecture

Suppose $\mathcal{P}(q = p^n, k)$ is a connected generalized Paley graph of sufficient size. Then the condensed Ricci curvature of an edge $xy \in E$ is

$$\Bbbk(x,y) = \frac{k\left(2 + |\nabla_{xy}|\right)}{q-1}.$$

Main Question

Question. What is the condensed Ricci curvature of $\mathcal{P}(q,k)$?

Conjecture

Suppose $\mathcal{P}(q = p^n, k)$ is a connected generalized Paley graph of sufficient size. Then the condensed Ricci curvature of an edge $xy \in E$ is

$$\Bbbk(x,y) = \frac{k\left(2 + |\nabla_{xy}|\right)}{q-1}.$$

- Equivalently, every connected generalized Paley graph has a perfect matching between the neighbor sets of any adjacent vertices

Result 1: A Condition for Perfect Matching

Theorem

Let $\mathcal{P}(q = p^n, k)$ be a generalized Paley graph. If $k \mid \frac{q-1}{p-1}$, then

$$\Bbbk(0,1) = \frac{k\left(2 + |\nabla_{0,1}|\right)}{q-1}.$$

Result 1: A Condition for Perfect Matching

Theorem

Let $\mathcal{P}(q = p^n, k)$ be a generalized Paley graph. If $k \mid \frac{q-1}{p-1}$, then

$$\Bbbk(0,1) = \frac{k\left(2 + |\nabla_{0,1}|\right)}{q-1}.$$

- Approximately 55.186% of all Paley graphs for the first 100000 primes, powers 1–100, and residues 2–100 satisfy this condition

Result 1: A Condition for Perfect Matching

Theorem

Let $\mathcal{P}(q = p^n, k)$ be a generalized Paley graph. If $k \mid \frac{q-1}{p-1}$, then

$$\Bbbk(0,1) = \frac{k\left(2 + |\nabla_{0,1}|\right)}{q-1}.$$

- Approximately 55.186% of all Paley graphs for the first 100000 primes, powers 1–100, and residues 2–100 satisfy this condition
- This includes, but not limited to, Paley graphs of the following forms:
  1. $\mathcal{P}\left(p^{nk}, k\right)$
  2. $\mathcal{P}\left(p^n, k\right)$, for prime k and $k \nmid p-1$

Result 2: Characterization of Disconnect Paley Graphs

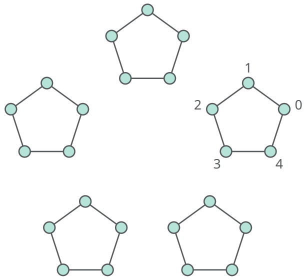
Theorem

All Paley graphs of the from $\mathcal{P}(q = p^{nk}, k)$ are connected. If they are not of this form, then they are either connected, or disconnected with each disjoint subgraph isomorphic to some smaller order Paley graph. Moreover, every connected Paley graph is itself isomorphic to the connected components of some larger Paley graph.

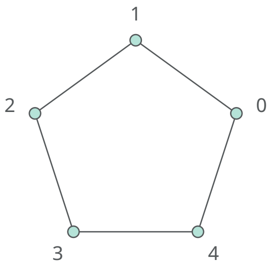
Result 2: Characterization of Disconnect Paley Graphs

Theorem

All Paley graphs of the from $\mathcal{P}(q = p^{nk}, k)$ are connected. If they are not of this form, then they are either connected, or disconnected with each disjoint subgraph isomorphic to some smaller order Paley graph. Moreover, every connected Paley graph is itself isomorphic to the connected components of some larger Paley graph.



$\mathcal{P}(5^2, 12)$



$\mathcal{P}(5, 2)$

Future Work

- Is there always a perfect matching for Paley graphs of the from $\mathcal{P}(p, 2)$?

Future Work

- Is there always a perfect matching for Paley graphs of the from $\mathcal{P}(p,2)$?
- Given a generalized Paley graph $\mathcal{P}(q,k)$, what is $|\nabla_{0,1}|$?

Future Work

- Is there always a perfect matching for Paley graphs of the from $\mathcal{P}(p, 2)$?
- Given a generalized Paley graph $\mathcal{P}(q, k)$, what is $|\nabla_{0,1}|$?
- When are generalized Paley graph strongly regular?

Questions?