Topiarism: The Kernel Embedding of Distributions Applied to Modern Portfolio Theory

Stephen Cook

California Polytechnic State University, San Luis Obispo

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Motivation

What is a Portfolio?

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Definition

Given a set of assets/stocks $X_1, X_2, \cdots X_n$, a ***portfolio*** is a series of weights assigned to each asset that corresponds to what proportion of your total funds you wish to invest in it.

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Question: What portfolio SHOULD you invest in?

Modern Portfolio Theory

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$$\lambda \psi^\intercal w - \frac{1}{2} w^\intercal C w$$

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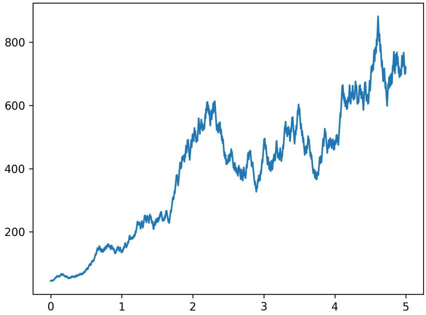
$$\lambda \psi^\top w - \frac{1}{2} w^\top C w$$

This is what stocks look like



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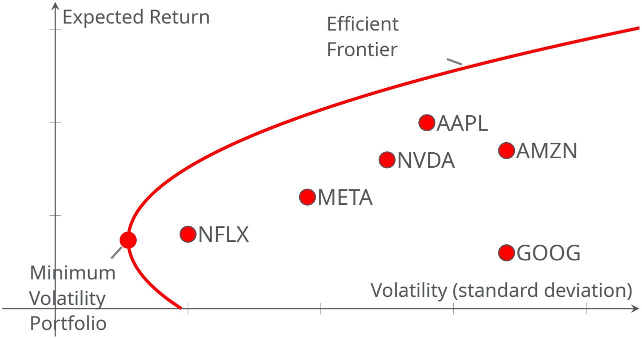

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- It is sometimes hard to find the best portfolio analytically
- True optimizers are impossible to observe
- The best portfolios use too many assets

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- ♤ Finding/estimating the best portfolios should be efficient.
- ♦ The model should match the trend of the market while using less stocks.
- $ The model should preserve some of the efficient frontier’s structure.

Measure Theory

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Other Examples of Measures

The ***Dirac*** measure

$$d\delta_a = \delta_a(x)dx$$

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- How do you measure area under a curve:

The ***Gaussian*** measure

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What other structure does $\mathcal{M}(\Omega)$ have?

Weak* Topology

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Definition

Let V be a vector space a functions on a set Ω, and let $\mathcal{M}(\Omega)$ be the vector space of measures on Ω. We say a sequence of measures $\mu_n \subseteq \mathcal{M}(\Omega)$ approaches a measure μ ***weak*-ly*** or ***vaguely*** if

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- Under this topology, the space of nonnegative probability measures $\mathcal{M}_+^1(\Omega)$ is compact

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$$\text{Portfolio } w \iff \text{Probability Measure } \mu_w$$

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Individual Assets $X \iff$ Dirac Measures δ_X

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$$\Omega \xrightarrow{X \mapsto \delta_X} \mathcal{M}(\Omega)$$

Reproducing Kernel Hilbert Spaces

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Let $\mathcal{H}$ be a vector space over $\mathbb{F}$. We call $\mathcal{H}$ a ***Hilbert space*** if

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Examples include:

- $\mathbb{C}^n$ with inner product $\left\langle \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \right\rangle = \sum_{k=1}^{n} a_k \overline{b_k}$

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- $\ell^2 = \left\{ (x_n)_{n \in \mathbb{N}} \subseteq \mathbb{C} : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}$ with inner product $\left\langle \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ \vdots \end{bmatrix} \right\rangle_{\ell^2} = \sum_{k=1}^{\infty} a_k \overline{b_k}.$

The Hardy Space

The **Hardy space** is the space of analytic power series on the complex unit disk, and is formally defined by

$$H^2 = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}.$$

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The Szegő Function in the Hardy Space

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- For all $|w| < 1$ and for all $f \in H^2$,

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  so we say k_w has a ***reproducing property*** for w and call it the **reproducing kernel** at the point w.

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- We call the function $K(z, w) = k_w(z)$ a ***kernel function*** for our Hilbert space.

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Let Ω be a set and let $\mathcal{H} \subseteq \mathbb{F}^{\Omega}$ be a set of functions from Ω to $\mathbb{F}$. We call $\mathcal{H}$ a ***reproducing kernel Hilbert space*** (often shortened to be ***RKHS***) on Ω if

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- Example: H^2 with kernel function $K(z, w) = \dfrac{1}{1 - \overline{w}z}$

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- Probability theory
- Image and signal processing
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- Support vector machines
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They are used in a variety of different fields:

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- They're just cool!

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- We can approximate elements of $\mathcal{H}$ with kernel functions
- Any cool properties of kernel functions will translate over to all elements of $\mathcal{H}$

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Theorem (Moore-Aronszajn)

Let Ω be a set and let $K \colon \Omega \times \Omega \longrightarrow \mathbb{F}$. Then the following are equivalent.

1. *K is a positive semidefinite kernel function.*
2. *There exists a RKHS $\mathcal{H}$ such that K is its reproducing kernel.*

Covariance is a Positive Semidefinite Kernel Function

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This gives us a unique RKHS $\mathcal{H}$ on Ω with the following properties:

- $\left\langle k_{X_i}, k_{X_j} \right\rangle = \mathrm{Cov}(r_i, r_j)$
- $\left\| k_{X_i} \right\|^2 = \mathrm{Var}(r_i)$

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$$\Omega \xrightarrow{x \longmapsto k_x} \mathcal{H}$$

The Kernel Embedding of Distributions

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$$\begin{tikzcd}
\Omega \arrow[r, "x \mapsto \delta_x"] \arrow[dr, "x \mapsto k_x"'] & \mathcal{M}(\Omega) \\
& \mathcal{H}
\end{tikzcd}$$

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\Omega \arrow[r, "x \mapsto \delta_x"] \arrow[dr, "x \mapsto k_x"'] & \mathcal{M}(\Omega) \arrow[d] \\
& \mathcal{H}
\end{tikzcd}$$

Definition

Let Ω be a set, let K be a continuous positive semidefinite kernel function on Ω, and let $\mathcal{H}$ be the induced RKHS. Then the ***kernel embedding of distributions*** is the map $\hat{\phi} : \mathcal{M}(\Omega) \to \mathcal{H}$ defined by $\hat{\phi}(\mu) = \hat{\mu}$ where

$$\hat{\mu}(x) = \int k_x(z)\, d\mu(z).$$

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What Our Abstract Stuff Got Us

For a given $\lambda \in \mathbb{R}$, find a probability measure μ_w that maximizes

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Topiarism

Reminder of Our Goals

- ♥ Optimizing should be done over positive portfolios.
- ♧ Approximation over subsets of stocks should get close to the ”true” maximum.
- ♤ Finding/estimating the best portfolios should be efficient.
- ♦ The model should match the trend of the market while using less stocks.
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What is Topiarism?

Topiarism is just the application of measure theory and reproducing kernel Hilbert spaces to modern portfolio theory.

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We will assume Ω is a sufficiently "nice" space, and $\mathcal{H}$ is some RKHS on this space with a continuous kernel function K.

The Objective Function

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Definition

Let ψ be a continuous function on Ω (not necessarily in $\mathcal{H}$). Define $\eta : \mathcal{M}_+^1(\Omega) \longrightarrow \mathbb{R}$ as

$$\eta(\mu) = \int \psi \, d\mu - \frac{1}{2} ||\hat{\mu}||^2$$

where $\hat{\mu}$ is the kernel embedding of μ. We call ψ the ***payout function*** and η the ***objective function***.

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A nonnegative probability measure $\mu \in \mathcal{M}_+^1(\Omega)$ is called ***topiaric*** (with respect to Ω) if

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Proof: Recall that $\mathcal{M}_+^1(\Omega)$ is a compact set. Additionally, it can be shown that η is continuous. Thus η exhibits a maximizer! $\square$

The Invisible Index Theorem

Theorem (The Invisible Index Theorem)

Suppose $\Omega \subseteq \mathcal{W}$. Let ν be topiaric with respect to $\mathcal{W}$. Then any topiaric measure with respect to Ω minimizes

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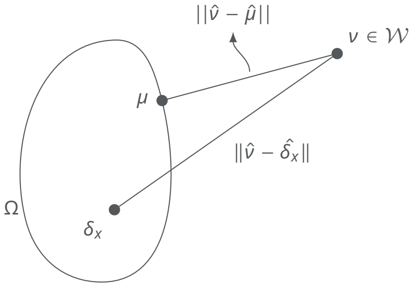

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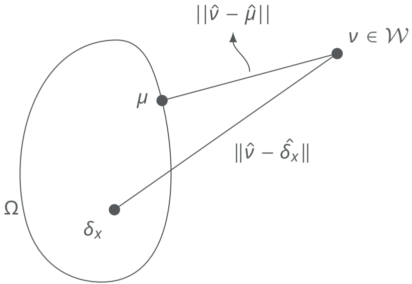

Neat Theorems We Get

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*Let μ be topiaric and let μ_n be a topiaric sequence. Then $\hat{\mu_n} \longrightarrow \hat{\mu}$ in $\mathcal{H}$ and $\mu_n \longrightarrow \mu$ weak*ly as measures.*

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Theorem

Let $\mu \in \mathcal{M}_+^1(\Omega)$. Then μ is topiaric if and only if $D_\eta(\mu, \delta_x - \mu) \leq 0$ for all $x \in \mathcal{K}$.

Approximating Using Gradient Ascent

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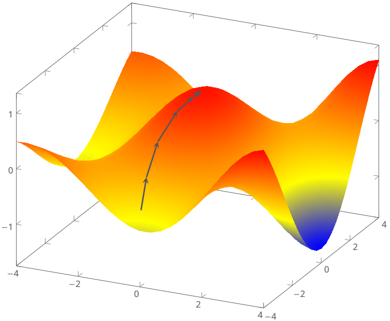

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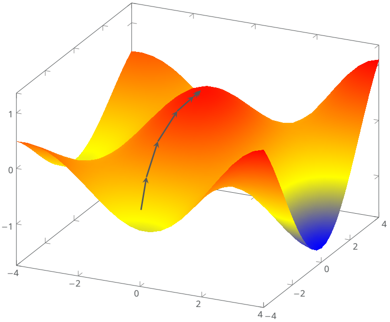

Recall that that η has no local maximums, only global ones. This makes it perfect for gradient ascent!

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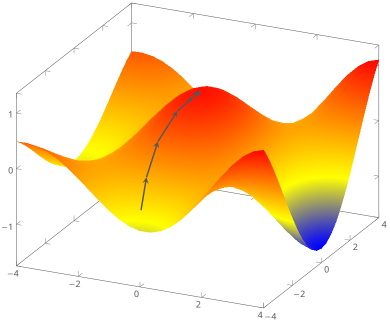

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Theorem

The sequence of measures obtained by performing gradient ascent on the objective function is a topiaric sequence.

Testing our Algorithm

		Time Elapsed (s)		η		**# of assets used**	
# of available assets	λ	Topiary	SciPy	Topiary	SciPy	Topiary	SciPy
10	1	$5.740 \cdot 10^{-5}$	$1.388 \cdot 10^{-2}$	$5.021 \cdot 10^{-4}$	$5.021 \cdot 10^{-4}$	1	1
	.1	$9.150 \cdot 10^{-3}$	$1.607 \cdot 10^{-2}$	$2.969 \cdot 10^{-5}$	$2.968 \cdot 10^{-5}$	1	1
	.03	$6.492 \cdot 10^{-3}$	$1.774 \cdot 10^{-2}$	$-1.764 \cdot 10^{-6}$	$-1.764 \cdot 10^{-6}$	4	4
	0	$1.408 \cdot 10^{-2}$	$1.849 \cdot 10^{-2}$	$-9.936 \cdot 10^{-6}$	$-9.936 \cdot 10^{-6}$	5	5
100	1	$1.021 \cdot 10^{-3}$	.5298	$1.034 \cdot 10^{-3}$	$1.034 \cdot 10^{-3}$	1	1
	.1	$2.681 \cdot 10^{-2}$	.9956	$4.3973 \cdot 10^{-5}$	$3.870 \cdot 10^{-5}$	5	19
	.03	$9.981 \cdot 10^{-2}$	.9639	$5.280 \cdot 10^{-6}$	$1.477 \cdot 10^{-6}$	6	41
	0	$8.213 \cdot 10^{-2}$	.9534	$-3.638 \cdot 10^{-6}$	$-6.013 \cdot 10^{-6}$	7	45
488	1	$2.807 \cdot 10^{-4}$	33.82	$1.968 \cdot 10^{-3}$	$1.968 \cdot 10^{-3}$	3	3
	.1	$2.262 \cdot 10^{-2}$	44.34	$1.443 \cdot 10^{-4}$	$1.441 \cdot 10^{-4}$	4	4
	.03	$2.739 \cdot 10^{-2}$	46.87	$1.990 \cdot 10^{-5}$	$1.803 \cdot 10^{-5}$	11	55
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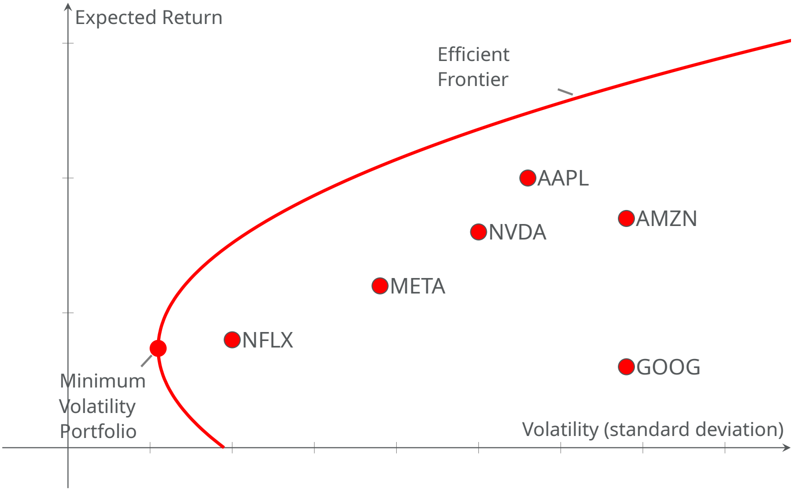

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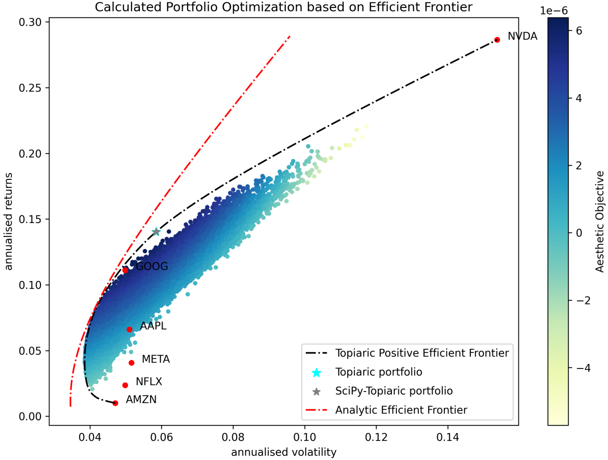

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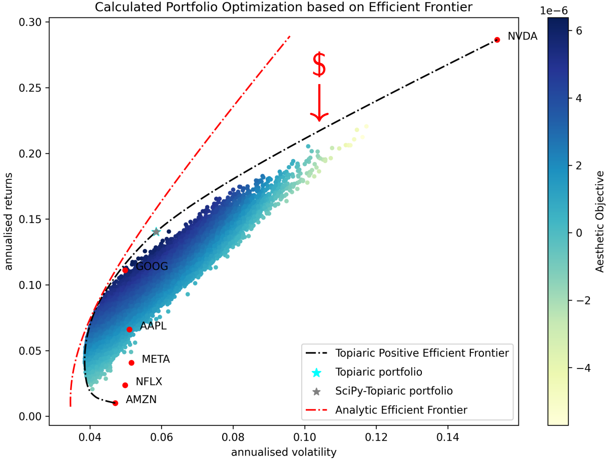

Questions?

Appendix

Problems with Modern Portfolio Theory

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- The model is entirely based upon historical data, which is well known to not be a good indicator of future asset performance.
- Using variance as a stand-in for risk does not differentiate between upside and downside risk.
- The model allows for short selling assets, but does not treat positive weights different than negative weights. In reality, short selling is inherently more risky because there is no cap to losses, whereas when buying longs, one can only lose as much as they invest.
- The market portfolio must be a portfolio that has investment in every asset possible, and it has been proved to be unobservable. This implies the model is unverifiable, as any empirical tests of the model can only use substitutes for the market portfolio. This is known as **Roll's critique**.
- The model also relies on the **efficient market hypothesis**, which states that asset prices reflect all available information instantaneously and that all investors act completely rationally.
- Many investors believe that it is near impossible to beat an index in the long run since if one was able to, there would exist an arbitrage opportunity. Such investors go back to methods used before the invention of modern portfolio theory, in particular focusing on dedicated research into a smaller subset of "growth stocks", i.e. assets that seem to be "on the rise".
- The model assumes that potential returns follow a normal distribution, when in

Duality

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This allows us to think of measures as a "subspace" of V^*.

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If V is a vector space, then so is V^*. So we can define linear operators on V^*. In particular, define $\Psi_x : V^* \longrightarrow \mathbb{F}$ by

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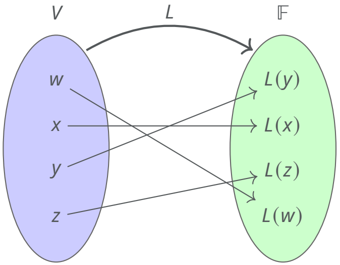

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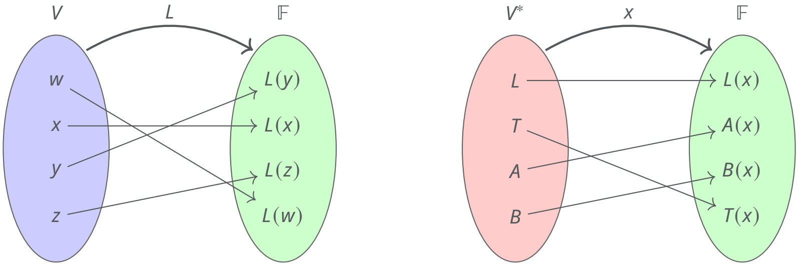

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Definition

Let V be a vector space a functions on a set Ω, and let $\mathcal{M}(\Omega)$ be the vector space of measures on Ω. We say a sequence of measures $\mu_n \subseteq \mathcal{M}(\Omega)$ approaches a measure μ ***weak*-ly*** or ***vaguely*** if

$$\int f \, d\mu_n \rightarrow \int f \, d\mu$$

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Recall that we can think of $\mathcal{M}(\Omega)$ as a subspace of V^*, so to put structure on it, we steal this notion of convergence from that of the weak* topology!

Definition

Let V be a vector space a functions on a set Ω, and let $\mathcal{M}(\Omega)$ be the vector space of measures on Ω. We say a sequence of measures $\mu_n \subseteq \mathcal{M}(\Omega)$ approaches a measure μ ***weak*-ly*** or ***vaguely*** if

$$\int f d\mu_n \to \int f d\mu$$

for all $f \in V$.

- Under this topology, the space of nonnegative probability measures $\mathcal{M}_+^1(\Omega)$ is compact

Further Applications of Topiarism

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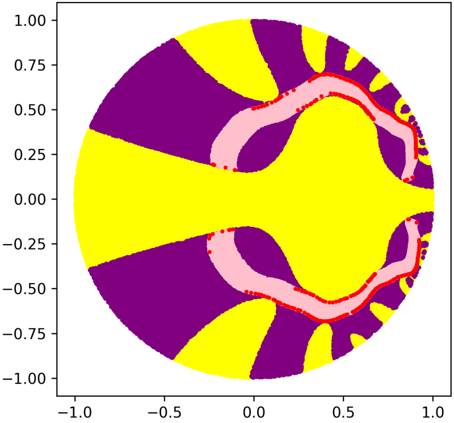
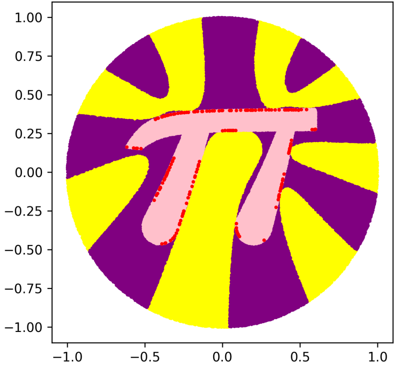
$$h^2_{\mathbb{R}}(\mathbb{D}) = \left\{ f \in h^2(\mathbb{D}) : f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta},\ \sup_{0 \le r < 1} \left(\int_{\partial \mathbb{D}} |f(rw)|\ dw \right)^{\frac{1}{2}} < \infty,\ c_n = \overline{c_{-n}} \right\},$$

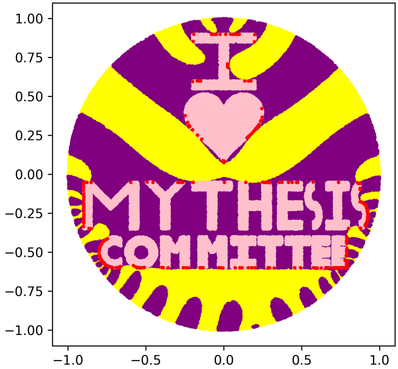
also known as the ***real harmonic Hardy space*** on the complex unit disk.

Example of Topiarism on the Harmonic Hardy Space

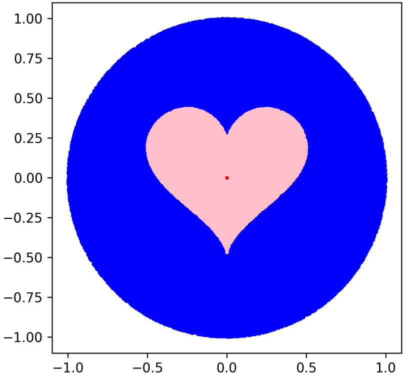


More Examples of Topiarism on the Harminc Hardy Space





Weird Edge Case for Topiary Algorithm on Harmonic Hardy Space



Questions?