Condensed Ricci Curvature on Paley Graphs and Their Generalizations

Vincent Bonini, Daniel Chamberlin, Stephen Cook, Parthiv Seetharaman, Tri Tran

California Polytechnic State University, San Luis Obispo

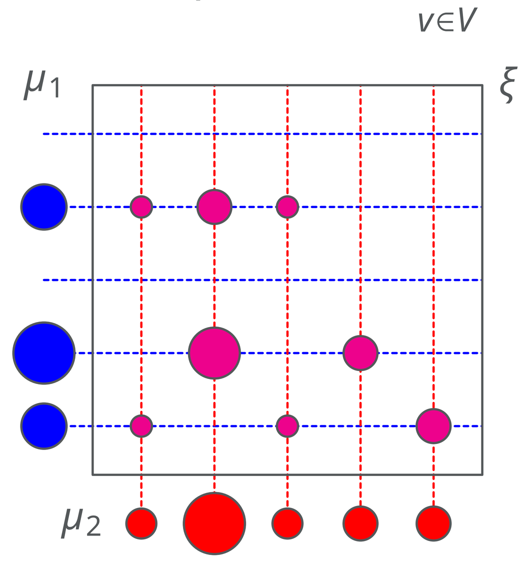
Introduction

Condensed Ricci curvature, first introduced by Ollivier [6] on general metric spaces, is a graph invariant that measures *curvature* on graphs and quantifies local connectivity of edges. We provide a general formula for the Ricci curvature of edges on certain Paley graphs and their generalizations in terms of the size of the Paley graph, it's residue class, and the size of the so-called triangle set.

Transport Plans

Definition: Let $G = (V, E)$ be a graph with vertex set V and edge set E. A *probability measure* (or *mass distribution*) on G is a function $\mu : V \longrightarrow [0,1]$ such that $\sum_{v \in V} \mu(v) = 1$.

Definition: Let μ_1, μ_2 be mass distributions on a graph $G = (V, E)$. A *coupling* (or *transport plan*) between μ_1 and μ_2 is a function $\xi : V \times V \longrightarrow [0,1]$ such that $\sum_{w \in V} \xi(v,w) = \mu_1(v)$ and $\sum_{v \in V} \xi(v,w) = \mu_2(w)$.



Wasserstein Distance

Definition: The *Wasserstein distance* is a metric that measures the "minimum cost" of a transport plan between mass distributions μ_1 and μ_2.

For a graph $G = (V, E)$, the Wasserstein distance is formally defined

$$W(\mu_1, \mu_2) = \inf_{\xi \in \chi(\mu_1, \mu_2)} \sum_{v \in V} \sum_{w \in V} \xi(v,w) \rho(v,w)$$

where $\chi(\mu_1, \mu_2)$ is the set of all transport plans between μ_1 and μ_2. On the other hand, by the Kantorovich duality theorem

$$W(\mu_1, \mu_2) = \sup_{f \in \mathrm{Lip}(1)} \sum_{v \in V} f(v)\big(\mu_1(v) - \mu_2(v)\big)$$

where

$$\mathrm{Lip}(1) = \{f : V \to \mathbb{R} : |f(v) - f(w)| \leq \rho(v,w) \;\; \forall\, v,w \in V\}$$

is the space of 1-Lipschitz functions on G.

Condensed Ricci Curvature

Definition: Let $G = (V, E)$ be a graph. The *condensed Ricci curvature* along an edge $xy \in E$ is defined

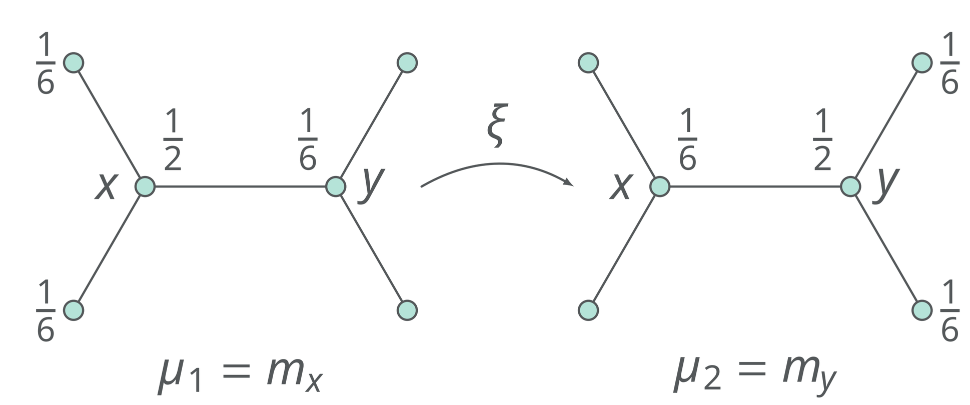
$$\Bbbk(x,y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(1 - W(m_x^\varepsilon, m_y^\varepsilon)\right),$$

where $m_x^\varepsilon, m_y^\varepsilon$ are mass distributions of the form

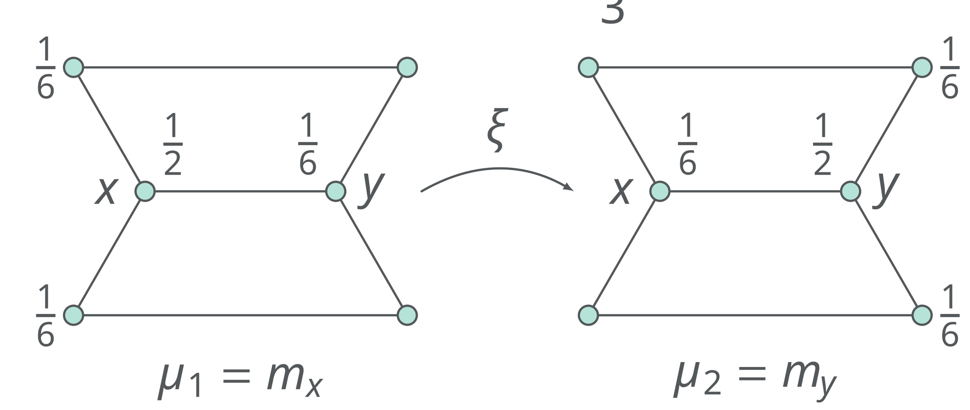
$$m_z^\varepsilon(v) = \begin{cases} 1 - \varepsilon & \text{if } v = z, \\ \varepsilon/\deg(z) & \text{if } v \in \Gamma(z), \\ 0 & \text{otherwise.} \end{cases}$$

Here $\Gamma(z)$ is the set of vertices adjacent to z.

Examples



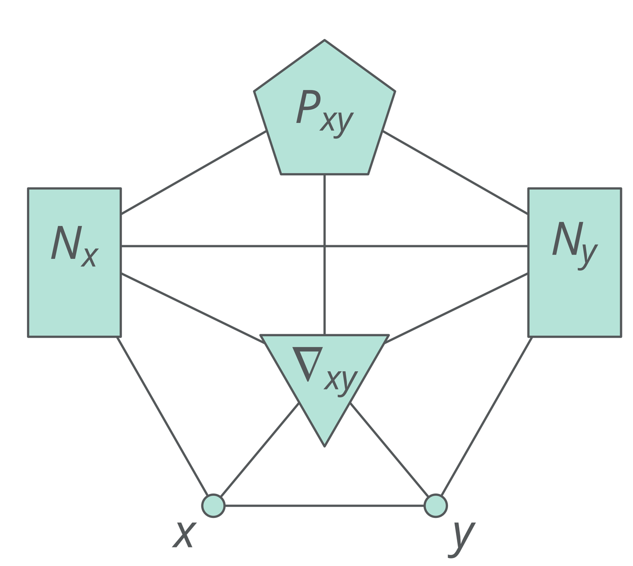
$$\Bbbk(x,y) = -\frac{2}{3}$$



$$\Bbbk(x,y) = \frac{2}{3}$$

Core Neighborhoods

Definition: Suppose $G = (V, E)$ is a graph. The *core neighborhood* of $xy \in E$ is $N_{xy} = \{x\} \cup \{y\} \cup \nabla_{xy} \cup N_x \cup N_y \cup P_{xy}$.

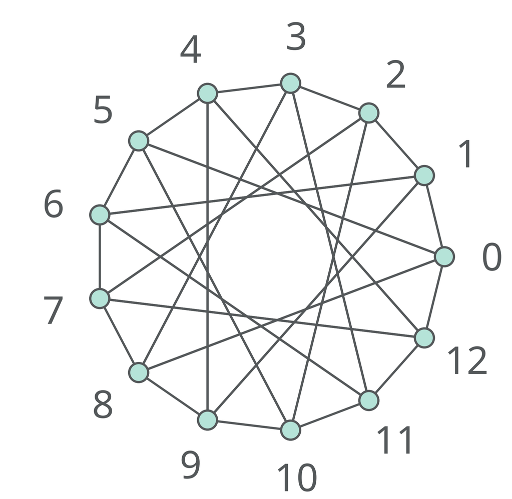


The Ricci curvature of an edge $xy \in E$ only depends on its core neighborhood [1].

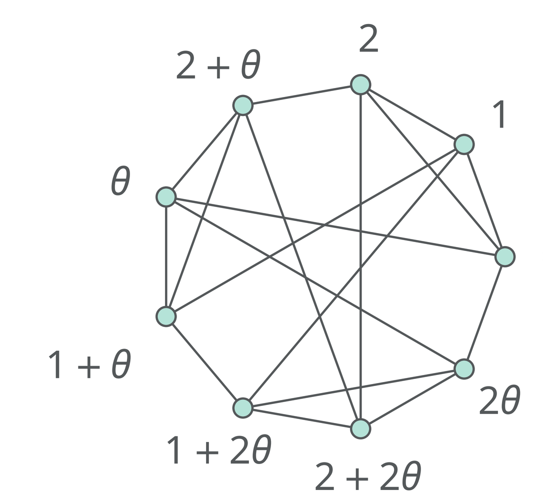
Paley Graphs

Definition: Suppose $\mathbb{F}_q$ is the field of order q. Then $x \in \mathbb{F}_q$ is a *k-residue* if $x = \alpha^k$ for some $\alpha \in \mathbb{F}_q$. The set of nonzero k-residues over $\mathbb{F}_q$ is denoted by $(\mathbb{F}_q^\times)^k$.

Definition: Let $k > 1$. Suppose $q = p^n$ is an odd prime power such that $q \equiv 1 \pmod{2k}$ and let $\mathbb{F}_q$ be the field of order q. A *generalized Paley graph* of order q with k-residues is the graph $\mathcal{P}(q,k) = (V, E)$, where $V = \mathbb{F}_q$ and $E = \{xy : x - y \in (\mathbb{F}_q^\times)^k\}$.



$\mathcal{P}(13,3)$



$\mathcal{P}(3^2,2)$

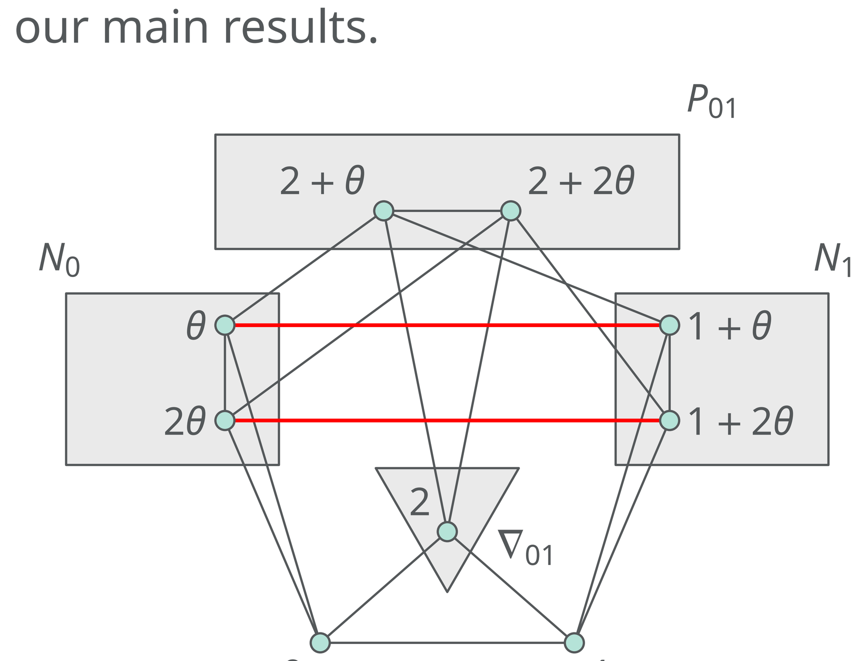
Main Question

For an odd prime power q and $k > 1$, what is the condensed Ricci curvature of a general edge in $\mathcal{P}(q,k)$?

Paley Graphs in terms of the Core Neighborhood

Lemma: *Generalized Paley graphs $\mathcal{P}(q,k)$ are symmetric and $\left(\frac{q-1}{k}\right)$-regular.*

The next figure shows a perfect matching between N_0 and N_1 of $\mathcal{P}(3^2,2)$ and demonstrates a simple case of the sorting method which is used to prove our main results.



Results

Theorem: *All Paley graphs of the from $\mathcal{P}(q = p^{nk}, k)$ are connected.*

Theorem: *Every connected Paley graph is itself isomorphic to the connected components of some larger disconnected Paley graph.*

Main Theorem: *Let $\mathcal{P}(q,k)$ be a generalized Paley graph. If $k \mid \frac{q-1}{p-1}$ then, for any edge $xy \in E$,*

$$\Bbbk(x,y) = \frac{k}{q-1}\left(2 + |\nabla_{xy}|\right).$$

Future Work

Conjecture: *Suppose $\mathcal{P}(q = p^n, k)$ is a connected generalized Paley graph. Then the condensed Ricci curvature of an edge $xy \in E$ is*

$$\Bbbk(x,y) = \frac{k}{q-1}\left(2 + |\nabla_{xy}|\right).$$

Further research questions include:

- Is there always a perfect matching for Paley graphs of the form $\mathcal{P}(p,2)$?
- When are generalized Paley graphs strongly regular?

Acknowledgments

This research was generously supported by the William and Linda Frost Fund in the Cal Poly Bailey College of Science and Mathematics.

This material is based upon work supported by the National Science Foundation under Grant No. 2015553.

References

[1] B. Bhattacharya and S. Mukherjee. "Exact and Asymptotic Results on Coarse Ricci Curvature of Graphs." *Discrete Mathematics* 338, 2015.

[2] V. Bonini et al. "Condensed Ricci Curvature of Complete and Strongly Regular Graphs." *Involve, a Journal of Mathematics* 13, 2020.

[3] A. Elsawy. "Paley Graphs and Their Generalizations." MA Thesis. Heinrich Heine University, 2009.

[4] T. Lim and C. Praeger. "On Generalized Paley Graphs and Their Automorphism Groups." *Michigan Math. J.* 58, 2009.

[5] Y. Lin, L. Lu, and S. T. Yau. "Ricci Curvature of Graphs." *Tohoku Math. J.* 63, 2011.

[6] Y. Ollivier. "Ricci Curvature of Markov Chains on Metric Spaces." *Journal of Functional Analysis* 256, 2009.