Thermal de Broglie Wavelength

Consider a particle in a box. The canonical partition function of the momentum states is

$$Z_1^p = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta\varepsilon}\, dp_x\, dp_y\, dp_z$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\beta\frac{(p_x^2+p_y^2+p_z^2)}{2m}}\, dp_x\, dp_y\, dp_z$$

$$= (2\pi m k_B T)^{3/2}$$

This has units of momentum cubed. This suggest we define a characteristic thermal momentum of

$$p_T = Z_1^{1/3} = \sqrt{2\pi m k_B T}$$

The de Broglie Wavelength associated with this momentum...

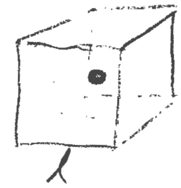
$$\lambda_T = \frac{h}{p_T} = \frac{h}{\sqrt{2\pi m k_B T}}$$

This is kind of like the expected "size" of quantum particles at a temperature T.

Now, we pack in many particles in a box of volume V. There will be a volume per particle of

$$\frac{V}{N}$$

This defines a length of



$$\ell = \left(\frac{V}{N}\right)^{1/3}$$

when this size becomes of order of the thermal de Broglie wavelength, we expect quantum effects to become important

$$\ell = \left(\frac{V}{N}\right)^{1/3} = \lambda_T = \frac{h}{\sqrt{2\pi m k_B T}}$$

or when

$$\left(\frac{N}{V}\right)^{1/3} = \frac{\sqrt{2\pi m k_B T}}{h} = \frac{2\pi}{h} \cdot \frac{\sqrt{2\pi m k_B T}}{2\pi} = \sqrt{\frac{m k_B T}{2\pi \hbar^2}}$$

This defines a density, called the "quantum concentration":

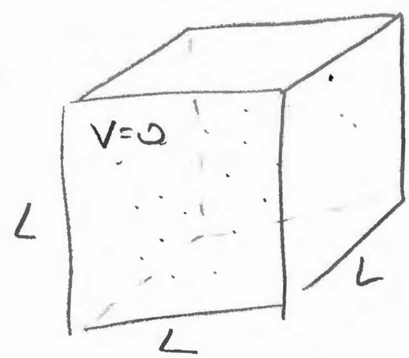
$$\boxed{n_Q = \frac{N}{V} = \left(\frac{m k_B T}{2\pi \hbar^2}\right)^{3/2}}$$

Again, if our density of massive particles exceeds n_Q, then quantum effects are going to become relevant.

Also Boltzmann statistics. So at high energy and low density both look like Boltzmann, as advertised.

Free Particles in a Box

Now, we are going to make a gas of particles contained within a box of volume $L^3 = V$. We're again going to assume non-interacting particles (i.e. $V=0$).



We know from QM class that such a particle in a box has plane wave wave functions, e.g.

$$\psi = \left(\frac{2}{L}\right)^{3/2} \sin\left(\frac{2\pi n_1}{L} x\right) \sin\left(\frac{2\pi n_2}{L} y\right) \sin\left(\frac{2\pi n_3}{L} z\right) \quad n_1, n_2, n_3 = 1, 2, 3$$

or cos...

These can be compactly written

$$\psi = \left(\frac{1}{L}\right)^{3/2} e^{i\vec{k}\cdot\vec{r}}$$

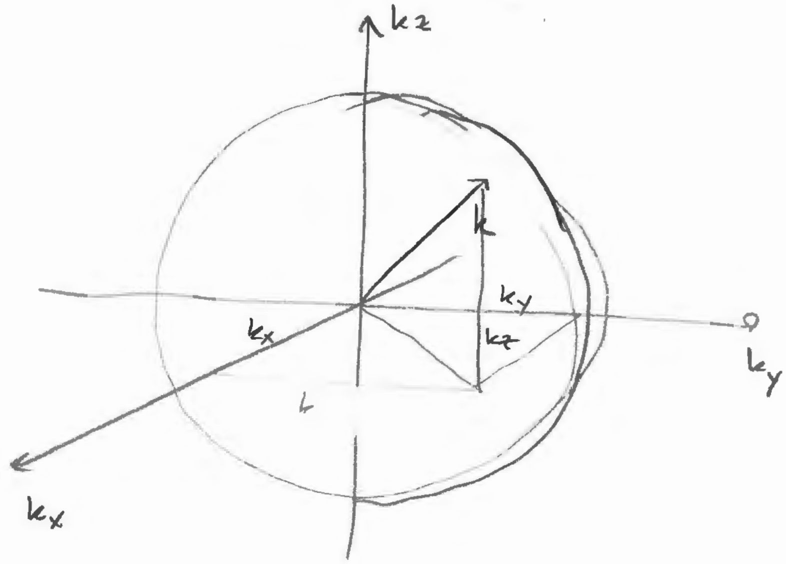
where

$$\vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z) = (k_x, k_y, k_z)$$

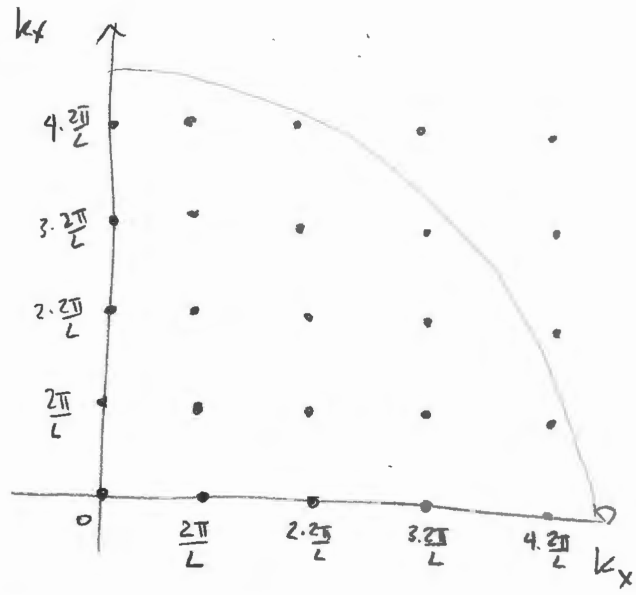
is the wave number vector and n_x, n_y, n_z can be any integer. The energies are

$$E = \frac{\hbar^2 k^2}{2m} \quad \text{where } k^2 = k_x^2 + k_y^2 + k_z^2$$

i.e., all k vectors with the same length have the same energy. In k-space this describes a sphere:



all points lying on the surface of this sphere have the same energy. However, k can only take on certain values:



the allowed k states therefore form a regular grid with average density

$$\rho = \frac{1 \text{ state}}{\left(\frac{2\pi}{L}\right)^3} = \left(\frac{L}{2\pi}\right)^3 = \frac{V}{8\pi^3}$$

If the box is large, the grid is extremely fine, and we can use the approximation that ρ is continuous, i.e

$$\#_{\text{states}} = \rho \cdot V$$

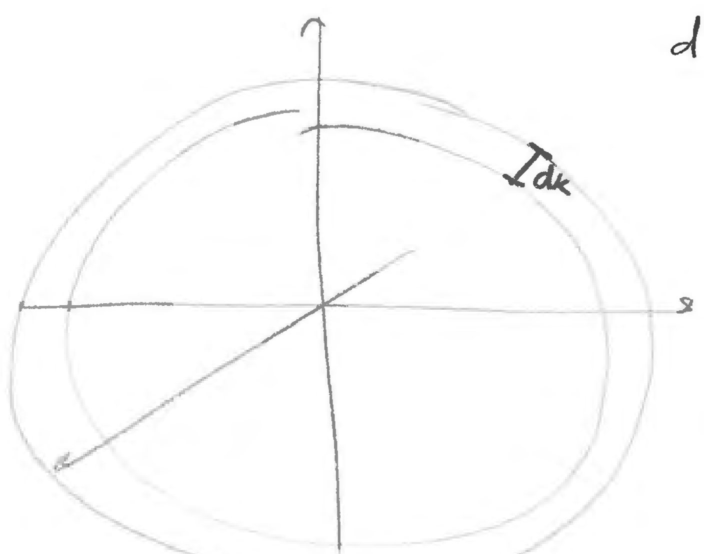
Next we're going to see what happens when these states are filled with Bosons.

First, let's try photons. Photons have $m=0$, and $E = hf = \hbar\omega$. Also

$$c = \lambda f = \frac{\omega}{k}$$

$$\Rightarrow \omega = ck \Rightarrow dk = \frac{1}{c} d\omega$$

For a given, frequency (energy), how many possible states are there?



$$dV = (4\pi k^2)\cdot(dk)$$

$$= 4\pi\left(\frac{\omega^2}{c^2}\right)\cdot\left(\frac{dk}{d\omega}d\omega\right)$$

$$= \frac{4\pi\omega^2}{c^3}d\omega$$

So the density of states is

$$g(\omega)\,d\omega = dV \cdot \rho$$

$$= \left(\frac{V}{8\pi^3}\right) \cdot \left(\frac{4\pi\omega^2}{c^3}\,d\omega\right)$$

$$= \frac{V\omega^2}{2\pi^2 c^3}\,d\omega$$

Now, we also know that photons can have 2 polarization states (left + right handed), so the total number of states is

$$\boxed{g(\omega)\,d\omega = \frac{V\omega^2}{\pi^2 c^3}\,d\omega}$$

Photons can be freely created or destroyed (i.e., absorption does not cost energy). So $\mu = 0$.

Next, photons are bosons $(s=1)$ and hence the average number of bosons per state is

$$\langle n_{BE} \rangle = \frac{1}{e^{\beta\varepsilon} - 1}$$

So that the total number of photons

(number of photons) $d\omega$ = (# of states) $\cdot$ (bosons per state)

$$= \frac{g(\omega)}{e^{\beta\varepsilon} - 1}\,d\omega$$

Finally, the energy per photon is $\hbar\omega$, and hence

$$E(\omega) = \frac{\hbar\omega\, g(\omega)\, d\omega}{e^{\hbar\omega} - 1}.$$

$$\boxed{E(\omega) = \frac{V\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\beta\hbar\omega} - 1}}$$

This is Planck's formula for blackbody radiation! At low frequencies

$$e^{\beta\hbar\omega} - 1 \simeq 1 + \beta\hbar\omega - 1$$
$$= \beta\hbar\omega$$

and

$$E(\omega) \simeq V\left(\frac{k_B T}{\pi^2 c^3}\right)\omega^2$$
$$= k_B T \cdot g(\omega)$$

which is the old Rayleigh-Jeans formula for blackbody radiation.