Ising Model at Zero Field ($B=0$)

Now

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$$

$$= -J \sum_i \sigma_i \sigma_{i+1}$$

Now let's choose some specific boundary condition... Let's do "free-end" BC

Here

$$H = -J\left[\sigma_1\sigma_2 + \sigma_2\sigma_3 + \dots + \sigma_{N-1}\sigma_N\right]$$

We can evaluate this using a coordinate transformation:

$$\{\sigma_1, \sigma_2, \dots, \sigma_N\} \longrightarrow \{\sigma_1, p_2, \dots, p_N\}$$

where

$$p_2 = \sigma_1\sigma_2 \ , \ p_3 = \sigma_2\sigma_3 \ , \ \dots \ , p_N = \sigma_{N-1}\sigma_N$$

Since $\sigma_i = \pm 1$, $p_i = \pm 1$ indicates whether the spin flips from i to $i+1$.

The Hamiltonian is then

$$H = -J \sum_{i=2}^{N} p_i$$

and

$$Z = \sum e^{-\beta H}$$

$$= \sum_{\{p_i\}} e^{\beta J [p_2 + p_3 + \dots p_N]}$$

$$= \sum_{\{p_2\}} e^{\beta J p_2} e^{\beta J p_3} \dots$$

$$= 2 \prod_{i=2}^{N} \sum_{p_i = \pm 1} e^{\beta J p_i}$$

(2 since there are two ways to get each $e^{\beta J p_i}$: $e^{\beta J (1)(1)} + e^{\beta J (-1)(-1)}$)

$$= 2 \prod_{i=2}^{N} \left(e^{\beta J} + e^{-\beta J}\right)$$

$$= 2 \left[2 \cosh \beta J\right]^{N-1}$$

This looks extremely similar to Z for $J=0$, just with J taking the place of μB!

The General Ising Model ($J, B \neq 0$)

For the general Ising Model, the partition function is usually expressed in terms of matrices.

Consider two spins, σ_1 and σ_2. They each have two possible values, ± 1.
Arrange them like:

$$P = \begin{array}{cc} & \begin{array}{cc} \sigma_2 = +1 & \sigma_2 = -1 \end{array} \\ \begin{array}{c} \sigma_1 = +1 \\ \sigma_1 = -1 \end{array} & \begin{pmatrix} e^{-\beta H_{++}} & e^{-\beta H_{+-}} \\ e^{-\beta H_{-+}} & e^{-\beta H_{--}} \end{pmatrix} \end{array},$$

$$\left.\begin{aligned}
H_{++} &= -J(1)(1) - \tfrac{1}{2}\mu B(1) - \tfrac{1}{2}\mu B(1) \\
H_{--} &= -J(-1)(-1) - \tfrac{1}{2}\mu B(-1) - \tfrac{1}{2}\mu B(-1) \\
H_{+-} &= -J(+1)(-1) - \tfrac{1}{2}\mu B(1) - \tfrac{1}{2}\mu B(-1) \\
H_{-+} &= -J(-1)(+1) - \tfrac{1}{2}\mu B(-1) - \tfrac{1}{2}\mu B(1)
\end{aligned}\right\} \quad P_{\sigma_1\sigma_2} = e^{\beta[J\sigma_1\sigma_2 + \frac{\mu B}{2}\sigma_1 + \frac{\mu B}{2}\sigma_2]}$$

So

$$P = \begin{pmatrix} e^{\beta(J+\mu B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-\mu B)} \end{pmatrix}$$

Now we take the TRACE, defined as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot \\ a_{21} & a_{22} & \\ & & \end{pmatrix}$$

$$\mathrm{Tr}\, A = \sum a_{ii} = a_{11} + a_{22} + a_{33} + \cdot$$

Take

$$\mathrm{Tr}[P \cdot P] = \sum_{\sigma_1} (P \cdot P)_{\sigma_1, \sigma_1} = \sum_{\sigma_1, \sigma_2} P_{\sigma_1 \sigma_2} P_{\sigma_2 \sigma_1}$$

$$= \sum_{\sigma_1, \sigma_2} e^{\beta[J\sigma_1\sigma_2 + \frac{\mu B}{2}\sigma_1 + \frac{\mu B}{2}\sigma_2]} \cdot e^{\beta[J\sigma_2\sigma_1 + \frac{\mu B}{2}\sigma_2 + \frac{\mu B}{2}\sigma_1]}$$

$$= \sum_{\sigma_1, \sigma_2} e^{\beta H(\sigma_1, \sigma_2)}$$

$$= Z$$

In general, for N spins forming a linear chain

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H(\{\sigma_i\})} \cdot \sum_{\{\sigma_i\}} e^{\beta[J\sigma_1\sigma_2 + \frac{\mu B}{2}\sigma_1 + \frac{\mu B}{2}\sigma_2]} e^{\beta[J\sigma_2\sigma_3 + \frac{\mu B}{2}\sigma_2 + \frac{\mu B}{2}\sigma_3]}$$

$$\cdots$$

$$= \mathrm{Tr}[P^N]$$

Now we need to know some matrix math.

1. Every real symmetric matrix P can be diagonalized, D:

$$P = U \cdot D \cdot U^T$$

where U is unitary ($U \cdot U^T = \mathbb{1}$)

For, e.g. a 2×2 matrix, define $\lambda_+ \equiv D_{11}$, $\lambda_- \equiv D_{22}$, $D_{12} = D_{21} = 0$

$\lambda_\pm$ are the eigenvalues of P

So $$D = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

2. The trace is unchanged after diagonalization:

$$\mathrm{Tr}\, P = \mathrm{Tr}\, D = \lambda_+ + \lambda_-$$

i.e. the trace gives the sum of the eigenvalues.

3. Note

$$P^N = P \cdot P \cdot \dots P$$

$$= (UDU^T)(UDU^T) \cdot \quad (UDU^T)$$

$$= U[D \cdot D \cdots D]U^T$$

$$= UD^NU^T$$

4. And

$$D^N = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}^N = \begin{pmatrix} \lambda_+^N & 0 \\ 0 & \lambda_-^N \end{pmatrix}$$

So that

$$\mathrm{Tr}(P^N) = \mathrm{Tr}(D^N) = \lambda_+^N + \lambda_-^N$$

The problem then boils down to diagonalizing P and finding .. eigenvalues.

Skipping all the algebra, you get:

$$\lambda_\pm = e^{\beta J}\left[\cosh(\beta\mu B) \pm \sqrt{\sinh^2(\beta\mu B) + e^{-4\beta J}}\right]$$

$$U = \begin{pmatrix} e^{-\beta J}\left[e^{\beta(J-\mu B)} - \lambda_+\right] & 1 \\ 1 & e^{-\beta J}\left[e^{\beta(J+h)} - \lambda_-\right] \end{pmatrix}$$

Hence

$$Z = \mathrm{Tr}(P^N) = \lambda_+^N - \lambda_-^N$$

$$= e^{N\beta J}\left[\left[\cosh(\beta\mu B) + \sqrt{\sinh^2(\beta\mu B) + e^{-4\beta J}}\right]^N + \left[\cosh(\beta\mu B) - \sqrt{\sinh^2(\beta\mu B) + e^{-4\beta J}}\right]^N\right]$$