The General Ising Model ($J, B \neq 0$)

For the general Ising Model, the partition function is usually expressed in terms of matrices.

Consider two spins, σ_1 and σ_2. They each have two possible values, ± 1. Arrange them like:

$$P = \begin{array}{c c} & \begin{array}{cc} \sigma_2 = +1 & \sigma_2 = -1 \end{array} \\ \begin{array}{c} \sigma_1 = +1 \\ \sigma_1 = -1 \end{array} & \begin{pmatrix} e^{-\beta H_{++}} & e^{-\beta H_{+-}} \\ e^{-\beta H_{-+}} & e^{-\beta H_{--}} \end{pmatrix} \end{array}$$

$$\left.\begin{aligned} H_{++} &= -J(1)(1) - \tfrac{1}{2}\mu B(1) - \tfrac{1}{2}\mu B(1) \\ H_{--} &= -J(-1)(-1) - \tfrac{1}{2}\mu B(-1) - \tfrac{1}{2}\mu B(-1) \\ H_{+-} &= -J(+1)(-1) - \tfrac{1}{2}\mu B(1) - \tfrac{1}{2}\mu B(-1) \\ H_{-+} &= -J(-1)(+1) - \tfrac{1}{2}\mu B(-1) - \tfrac{1}{2}\mu B(1) \end{aligned}\right\} \quad P_{\sigma_1 \sigma_2} = e^{\beta[J\sigma_1\sigma_2 + \frac{\mu B}{2}\sigma_1 + \frac{\mu B}{2}\sigma_2]}$$

So

$$P = \begin{pmatrix} e^{\beta(J+\mu B)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-\mu B)} \end{pmatrix}$$

Now we take the TRACE, defined as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdot \\ a_{21} & a_{22} & \\ & & \end{pmatrix}$$

$$\operatorname{Tr} A = \sum_i a_{ii} = a_{11} + a_{22} + a_{33} + \cdots$$

Take

$$\operatorname{Tr}[P \cdot P] = \sum_{\sigma_1} (P \cdot P)_{\sigma_1, \sigma_1} = \sum_{\sigma_1, \sigma_2} P_{\sigma_1 \sigma_2} P_{\sigma_2 \sigma_1}$$

$$= \sum_{\sigma_1, \sigma_2} e^{\beta[J\sigma_1\sigma_2 + \frac{\mu B}{2}\sigma_1 + \frac{\mu B}{2}\sigma_2]} \cdot e^{\beta[J\sigma_2\sigma_1 + \frac{\mu B}{2}\sigma_2 + \frac{\mu B}{2}\sigma_1]}$$

$$= \sum_{\sigma_1, \sigma_2} e^{\beta H(\sigma_1, \sigma_2)}$$

$$= Z$$

In general, for N spins forming a linear chain

$$Z = \sum_{\{\sigma_i\}} e^{-\beta H(\{\sigma_i\})} = \sum_{\{\sigma_i\}} e^{\beta[J\sigma_1\sigma_2 + \frac{\mu B}{2}\sigma_1 + \frac{\mu B}{2}\sigma_2]} e^{\beta[J\sigma_2\sigma_3 + \frac{\mu B}{2}\sigma_2 + \frac{\mu B}{2}\sigma_3]}$$

$$\cdots$$

$$= \operatorname{Tr}[P^N]$$

Now we need to know some matrix math.

1. Every real symmetric matrix P can be diagonalized, D:

$$P = U \cdot D \cdot U^T$$

where U is unitary, $(U \cdot U^T = \mathbb{1})$

For, e.g. a 2×2 matrix define $\lambda_+ \equiv D_{11}$, $\lambda_- \equiv D_{22}$, $D_{12} = D_{21} = 0$

$\lambda_\pm$ are the eigenvalues of P

So $$D = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

2. The trace is unchanged after diagonalization:

$$\mathrm{Tr}\, P = \mathrm{Tr}\, D = \lambda_+ + \lambda_-$$

i.e. the trace gives the sum of the eigenvalues.

3. Note

$$P^N = P \cdot P \cdot \cdots P$$

$$= (U D U^T)(U D U^T) \cdots (U D U^T)$$

$$= U [D \cdot D \cdots D] U^T$$

$$= U D^N U^T$$

This is, fundamentally, due to the nature of phase transitions

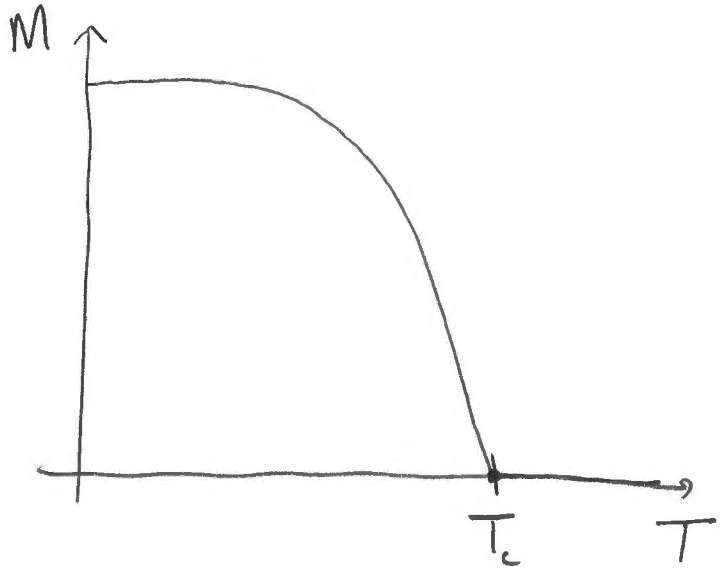
Magnetism and phase transitions

Consider the behavior of a real ferromagnet (e.g. a refrigerator magnet).

Even with zero applied field, there is still a spontaneous magnetization:

$$|M(T, h=0)| > 0$$

As the temperature increases, the magnetization decreases; in terms of the Ising model, this is due to decreased correlation between neighboring spins due to thermal fluctuations. At some finite temp, T_c, the magnetization hits zero:



T_c is called the "curie temperature"

This is a phase transition: a change from an ordered, magnetized state, to a disordered zero magnetization state

Phase transitions arise from singular behavior of the free energy
pathological
and/or its derivatives.

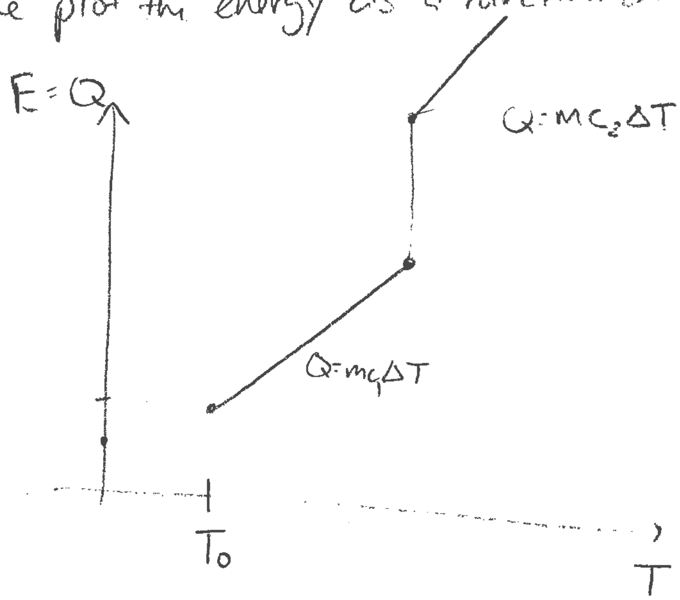
For example, consider the energy of water as you heat it. As you heat water, first it warms according to

$$Q = mc\Delta T$$

then it undergoes a phase change, characterized by a latent heat:

$$Q = mL_f$$

If we plot the energy as a function of T:



The latent heat yields a discontinuity in E(T).

Now E is the expectation value of the Hamiltonian:

$$E = \langle H \rangle = \frac{1}{Z} \sum H e^{-\beta H}$$

$$= -\beta^{-1} \frac{1}{Z} \frac{\partial}{\partial \beta} \sum e^{-\beta H} = -\beta \frac{\partial}{\partial \beta} (\ln Z)$$