Continuous Distributions

Random Variable of the Continuous Type

Consider a function X that assigns to each element s in a continuous interval S one and only one real number $X(s) = x$.

S is now a continuous interval.

Probability Density Function (pdf)

The pdf $f(x)$ of a continuous random variable X is a function that satisfies the following:

(A) $f(x) > 0 \quad \forall S$

(B) $\int_S f(x)\,dx = 1$ Normalized

(C) The probability of the event X belonging to a set A is

$$P(X \in A) = \int_A f(x)\,dx$$

Distribution Function

The distribution function of a random variable X is given by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$$

$F(x)$ accumulates all of the probability less than or equal to x

Mathematical Expectation

The expected value of X or the mean value of X is

$$\mu = E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

or

$$\langle x \rangle = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

Compare in QM, for example

$$\langle E \rangle = \langle \psi | \hat{H} | \psi \rangle = \int_{-\infty}^{\infty} \psi^* \hat{H} \psi\, dx$$
$$= \int_{-\infty}^{\infty} E |\psi|^2 dx \quad \Rightarrow |\psi|^2 \text{ is the pdf!}$$

Example: The Normal (or Gaussian) Distribution

The random variable X has a normal distribution if its PDF is defined as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2} \right] \qquad -\infty < x < \infty$$

In the homework, you will check to see that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\langle x \rangle = \mu$$

and the probability of lying within an interval from a to b is

$$P[a \le X \le b] = \int_a^b f(x)\,dx$$

$$= \int_a^b \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

Connection between Discrete and Continuous Distributions

We can unify the way we write down probability distributions by using the Dirac Delta Function.

Recall

$$\delta(x-\alpha) = \begin{cases} +\infty & x=\alpha \\ 0 & x \ne \alpha \end{cases}$$

such that

$$\int_a^b \delta(x-\alpha)\,dx = \begin{cases} 1 & \text{if } \alpha \text{ is within } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_a^b f(x)\,\delta(x-\alpha)\,dx = \begin{cases} f(\alpha) & \text{if } \alpha \text{ is within } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Dirac delta functions "Pick out" discrete values, but can be integrable.

- If a discrete variable can take n different values among real numbers, then we can write the probability density function as

$$f(t) = \sum_{i=1}^{n} p_i \delta(t - x_i)$$

where x_i are discrete values on X

- This allows us to calculate using all the standard methods of continuous distributions, even with a set of discrete values.

End Math!

Now we can do physics!

In general, we will see that both discrete and continuous distributions are important for understanding statistical ensembles.

Random Walks

Random walks, which show up in physics, ecology, psychology, chemistry, bio, finance..., can be described as a path that consists of a succession of random steps.