Quantum Statistics

Recall that we derived the equilibrium probability distribution $\rho(p)$ of a momentum component equal to p in the microcanonical ensemble.

We found that the probability of the momentum component having energy $E = \frac{p^2}{2m}$ is proportional to $e^{-\frac{E}{k_B T}}$: the Boltzmann Distribution.

What were the assumptions that lead us to this?

1. Particles do not interact
2. Particles are like billiard balls (i.e. classical)
3. Particles are in thermal equilibrium

The second assumption is important, as it implies that each particle's state can be determined independently from the other particle states

Quantum Statistics relaxes this assumption.

Before we proceed, let's write down the Boltzmann Distribution

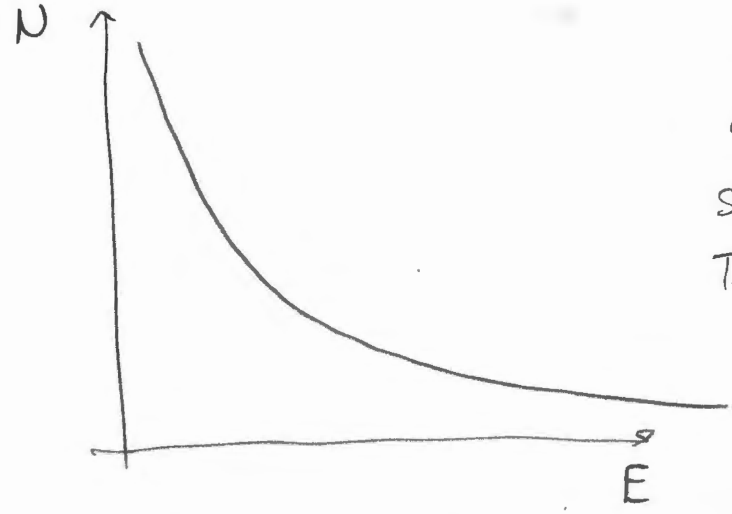
$$\rho(E_i) = \frac{N_i}{N} = \frac{e^{-\frac{E_i}{k_B T}}}{\sum_i e^{-E_i/k_B T}} \Longrightarrow$$ Give the average fraction (Number) of particles found in a given single-particle microstate.

The denominator is a normalization factor so that N_i/N sums to 1.

This factor is called the PARTITION FUNCTION

$$Z = \sum_i e^{-E_i/k_B T}$$

At sufficiently high temps, the Boltzmann Distribution looks exponential at decays at high energies



It's useful for studying gasses at high temperatures, but not so useful for low energy / low Temp behavior where interactions quantum effects become important.

In QM, we are often interested in the ground state Energy, E_0, and the ground state occupation $\langle N_0 \rangle$

Recall from QM that we could find the energy of the harmonic oscillator:

$$E_n = \left(\tfrac{1}{2}+n\right)\hbar\omega$$

At zero temp, the particle will occupy the ground state $n=0$ w/ $E_0 = \frac{1}{2}\hbar\omega$.

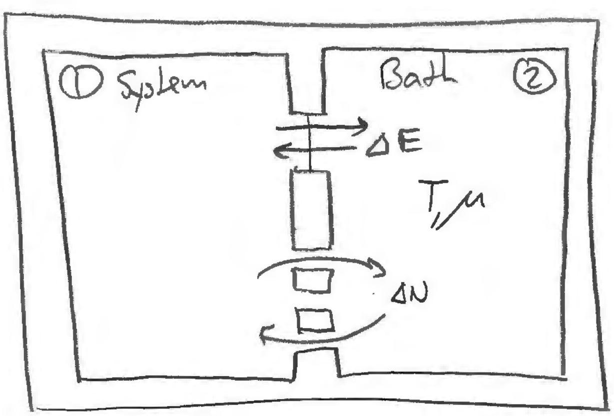
We could guess that with a gas (collection of particles) at zero temp, all particles will live in low energy states. But surprisingly, as we will see, exactly what levels are populated depends on the nature of the particles and their interactions

In a QM statistical system, this is the question that drives us: What is the ground state and Excited state ($T>0$) occupation of a large number of Quantum particles?

First, let's introduce the GRAND CANONICAL ENSEMBLE

Grand Canonical Ensemble

Consider an equilibrium system which can exchange energy and particles with a heat bath (A second subsystem)



The probability density that the system will be in a state s is

$$\rho(s) \propto \Omega_2(E-E_s, N-N_s)$$
$$\propto \Omega_1(E_s, N_s)$$

Since
$$E = E_{bath} + E_s \Rightarrow E_{Bath} = E - E_s$$
$$N = N_{Bath} + N_s \Rightarrow N_{Bath} = E - N_s$$

This is completely analogous to the 2 subsystem model we used previously.

Recall
$$S = k_B \log \Omega \Rightarrow \Omega = e^{S/k_B}$$