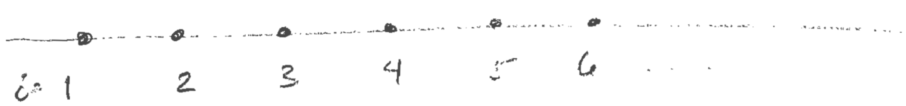
if $J_{ij} < 0$, then anti-aligned spins are energetically favorable. Such materials are called "antiferromagnetic"

If $J_{ij} = 0$, such materials are called "boring".

Now, in principle, the sum $\sum_{ij}$ should be over ALL pairs $\sigma_i, \sigma_j$. However, to keep the problem tractable, we will only perform the sum over nearest neighbors:

$$H = -\sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j - \mu B \sum \sigma_i$$




$$H = -J_{12}\sigma_1\sigma_2 - J_{23}\sigma_2\sigma_3 - J_{34}\sigma_3\sigma_4 - \dots - \mu B \sum \sigma_i$$

What happens at the ends depends on the boundary conditions (periodic, infinite, etc.)

The collection of spin states of a given configuration

$$\{\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_N\}$$

is the spin configuration, $\{\sigma_i\}$.

Given

$$H = -\sum_{\langle i,j \rangle} J_{ij} \sigma_i \sigma_j - \mu B \sum_i \sigma_i$$

The partition function is

$$Z_N(B,T) = \sum_{\sigma_1} \sum_{\sigma_2} \cdots \sum_{\sigma_N} e^{-\beta H\{\sigma_i\}}$$

$$= \sum_{\sigma_1} \sum_{\sigma_2} \cdots \sum_{\sigma_N} \exp\left[\beta \sum_{\langle i,j\rangle} J_{ij}\,\sigma_i\sigma_j + \beta\mu B \sum_i \sigma_i\right]$$

From now on, we will assume that all interactions are identical, i.e. $J_{ij} = J$ for allowed sets of $i,j$:

$$Z_N(B,T) = \sum_{\sigma_1} \sum_{\sigma_2} \cdots \sum_{\sigma_N} \exp\left[\beta J \sum_{\langle i,j\rangle} \sigma_i\sigma_j + \beta\mu B \sum_i \sigma_i\right]$$

The free energy is thus

$$\Phi = -\beta^{-1} \ln Z_N$$

and the <u>MAGNETIZATION</u> is related to the average spin state of the material:

$$M(B,T) = \mu\langle\{\sigma\}\rangle = \langle\mu\rangle$$

$$= \mu\left\langle \sum \sigma_i \right\rangle$$

$$= -\frac{\partial\Phi}{\partial B} = \frac{1}{\beta}\frac{1}{Z_N}\frac{\partial Z_N}{\partial B}$$

$$= \frac{1}{Z_N} \sum_{\sigma_1} \sum_{\sigma_2} \cdots \sum_{\sigma_N} \left(\mu \sum \sigma_i\right) \exp\left[\beta J \sum_{\langle i,j\rangle} \sigma_i\sigma_j + \beta\mu B \sum_i \sigma_i\right]$$

The quantity $M(0,T)$ is the spontaneous magnetization of the material. At low temps $(T<T_c)$, the spins align and the material is ferromagnetic. At high temps $(T>T_c)$, thermal fluctuations overcome the spin-spin interactions, and the material is only weakly magnetic (paramagnetic). Hence, there is a phase transition at $T=T_c$ ... for the Ising model in 2D. This does not occur in 1D, but we'll work through it as warm-up.

## Non Interacting Model

For $J=0$, the spins are non-interacting. Hence

$$Z_N = \sum_{\{\sigma_i\}} e^{\beta\mu B\sum_i \sigma_i} = \sum_{\{\sigma_i\}} \prod_{i=1}^{N} e^{\beta\mu B\sigma_i} = \prod_{i=1}^{N} \sum_{\sigma_i=\pm1} e^{\beta\mu B\sigma_i}$$

$$= \prod_{i=1}^{N} \left(e^{\beta\mu B} + e^{-\beta\mu B}\right)$$

Recall

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \& \quad \cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$$

So

$$Z_N = \prod_{i=1}^{N} \left(2\cosh(\beta\mu B)\right) = 2^N \cosh^N(\beta\mu B)$$

Hence, In general

$$\Phi = -\frac{1}{\beta}\ln Z_N$$

$$= -\frac{N}{\beta}\ln[2\cosh\beta\mu B]$$

The average energy is

$$E = \langle H\rangle = \frac{\sum_i H e^{-\beta H}}{Z_N}$$

$$= -\frac{\partial}{\partial\beta}\ln Z_N$$

$$= -\frac{\partial}{\partial\beta}\ln[2^N\cosh^N(\beta\mu B)]$$

$$= -\frac{1}{2^N\cosh^N(\beta\mu B)}\frac{\partial}{\partial\beta}(2^N\cosh^N(\beta\mu B))$$

$$= -\frac{1}{\cosh^N(\beta\mu B)}\cdot N\cosh^{N-1}(\beta\mu B)(\sinh(\beta\mu B))\cdot(\mu B)$$

$$= -N\mu B\frac{\sinh(\beta\mu B)}{\cosh(\beta\mu B)}$$

$$= -N\mu B\tanh(\beta\mu B)$$

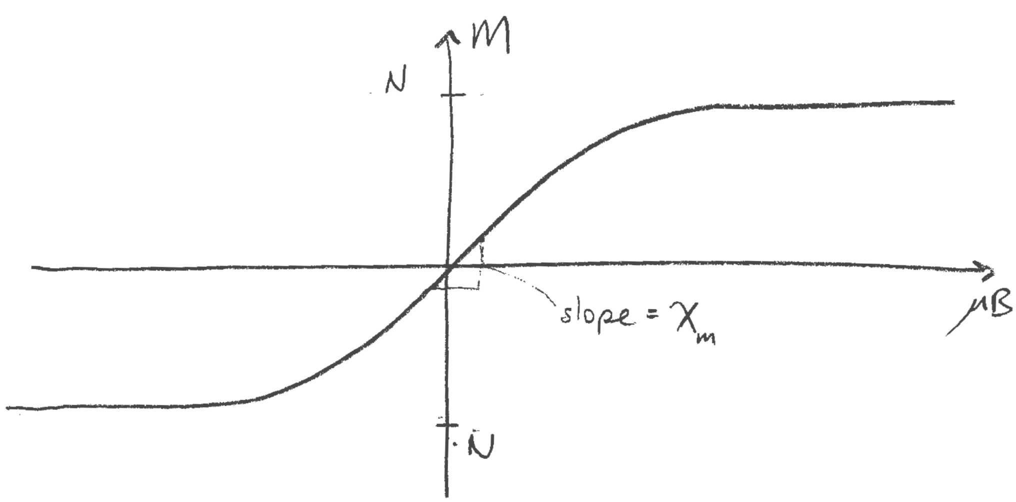
In this case $J=0$, so

$$E = \langle H \rangle = -\mu B M$$

$$\Rightarrow M = -\frac{E}{\mu B} = N \tanh(\beta\mu B)$$

The magnetic susceptibility, $\chi_m$, is

$$\chi_m = \frac{1}{N}\left(\frac{\partial M}{\partial(\mu B)}\right) = \beta\left(1-\tanh^2(\beta\mu B)\right)$$




The magnetization smoothly transitions from $-N$ to $N$ as the field changes direction