- If a discrete variable can take n different values among real numbers, then we can write the probability density function as

$$f(t) = \sum_{i=1}^{n} p_i \delta(t - x_i)$$

where x_i are discrete values on X

- This allows us to calculate using all the standard methods of continuous distributions, even with a set of discrete values.

End Math!

Now we can do physics!

In general, we will see that both discrete and continuous distributions are important for understanding statistical ensembles.

Random Walks

Random walks, which show up in physics, ecology, psychology, chemistry, bio, finance..., can be described as a path that consists of a succession of random steps.

- We'll discuss the Random Walk and show that it introduces two kinds of EMERGENT BEHAVIOR. We define emergence as the way in which complex systems and patterns arise out of a multiplicity of simple interactions.

- First, we'll discuss SCALE INVARIANCE which emerges from considering the random walk of a single particle

- Second we'll discuss CONTINUUM DIFFUSION, which emerges from considering the probability distributions of the end points of many particles undergoing random walks.

- Consider flipping a coin and recording the difference S_N between the number of heads and tails found. Each flip contributes $l_i = \pm 1$ to the total:

$$S_N = \sum_{i=1}^{N} l_i = \text{HEADS} - \text{TAILS}$$

- For a fair coin, $\langle S_N \rangle = 0$ for large N, so the expected value is not a useful quantity.

- Instead, we generally measure the root mean square (RMS) sum $\sqrt{\langle S_N^2 \rangle}$. This quantity is always positive, which allows us to assign a quantity to the statistical distribution

- For one coin flip,

$$\langle S_1^2 \rangle = P(\text{heads})(l_{HEADS})^2 + P(\text{tails})(l_{TAILS})^2$$
$$= \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2$$
$$= 1$$

- For two flips:

$$\langle S_2^2 \rangle = P(1)P(1)(1+1)^2 + P(-1)P(-1)(-1-1)^2 + P(1)P(-1)(1-1)^2 + P(-1)P(1)(-1+1)^2$$
$$= \frac{1}{2}\cdot\frac{1}{2}\cdot 2^2 + \frac{1}{2}\cdot\frac{1}{2}(-2)^2 + \frac{1}{2}\cdot\frac{1}{2}\cdot 0^2 + \frac{1}{2}\cdot\frac{1}{2}\cdot 0^2$$
$$= 2$$

- We can generalize this to N flips by writing the RMS after N steps in terms of the RMS after N-1 steps plus the last step:

$$\langle S_N^2 \rangle = \langle (S_{N-1} + l_N)^2 \rangle$$
$$= \langle (S_{N-1} + 2 S_{N-1} l_N + l_N^2) \rangle$$
$$= \langle S_{N-1} \rangle + 2 \langle S_{N-1} l_N \rangle + \langle l_N^2 \rangle$$

- For fair coin flips, $l = \pm 1$ with equal probability, INDEPENDENT of any other coin flip. So,

$$\langle S_{N-1} l_N \rangle = P(1) S_{N-1}(1) + P(-1) S_{N-1}(-1)$$

$$= \frac{1}{2} S_{N-1} \cdot 1 - \frac{1}{2} S_{N-1} \cdot 1$$

$$= 0$$

- So

$$\langle S_N^2 \rangle = \langle S_{N-1}^2 \rangle + \langle l_N^2 \rangle$$

$$= \langle S_{N-1}^2 \rangle + 1$$

- We found $\langle S_1^2 \rangle = 1$, $\langle S_2^2 \rangle = 2$, ... so $\langle S_{N-1}^2 \rangle = N-1$, and

$$\langle S_N^2 \rangle = \langle S_{N-1}^2 \rangle + 1$$

$$= (N-1) + 1$$

$$= N$$

- So

$$\sqrt{\langle S_N^2 \rangle} = \sqrt{N}$$

- The STANDARD DEVIATION of the distribution is defined as:

$$\sigma = \sqrt{\langle S_N^2 \rangle - \underbrace{\langle S_N \rangle}_{0}^2}$$

$$\therefore \boxed{\sigma = \sqrt{\langle S_N^2 \rangle} = \sqrt{N}}$$

- We can now extend this idea to random motion of particles by introducing a displacement.

- We call this the RANDOM WALK, or DRUNKARD's WALK

RANDOM WALK

- First let's consider the random motion of a single particle, like a gas molecule in air, a colloid in suspension, or a drunkard wandering UCR.

- Because these systems are dilute and interactions are short ranged, collisions occur infrequently, and the particle moves in directed lines between each collision.

- After several collisions, the particle's velocity will be uncorrelated with its original velocity

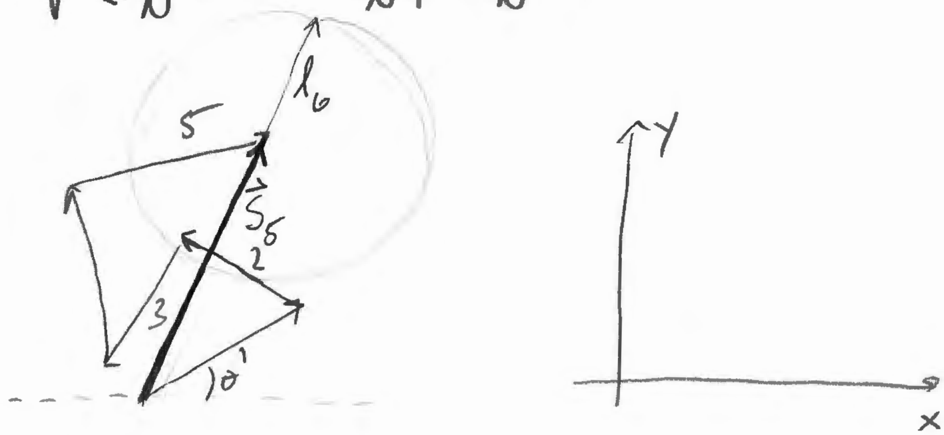
- The path taken will be a jagged, random walk.

- For the random walk, we assume the particle takes a step at regular time intervals Δt. We avoid random step sizes and velocities, and randomize <u>only direction</u>.

• Begin at $x=0, y=0$. Then at regular intervals Δt, take steps $\vec{l}_N$ of length L. $\vec{l}_N$ is the random variable

• We want to find the RMS displacement after N steps

$$\sqrt{\langle S_N^2 \rangle} = \langle (\vec{S}_{N-1} + \vec{l}_N)^2 \rangle$$



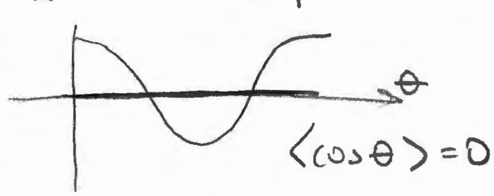
$$\langle S_N^2 \rangle = \langle (\vec{S}_{N-1} + \vec{l}_N)^2 \rangle$$

$$= \langle (\vec{S}_{N-1} + \vec{l}_N) \cdot (\vec{S}_{N-1} + \vec{l}_N) \rangle$$

$$= \langle S_{N-1}^2 \rangle + \langle l_N^2 \rangle + 2 \langle \vec{S}_{N-1} \cdot \vec{l}_N \rangle$$

• To find $\langle \vec{S}_{N-1} \cdot \vec{l}_N \rangle$, we have to recognize that steps are random and uncorrelated. Then

$$\langle \vec{S}_{N-1} \cdot \vec{l}_N \rangle = \langle S_{N-1} L \cos\theta \rangle = S_{N-1} L \langle \cos\theta \rangle$$



Hence,

$$\langle S_N^2 \rangle = \langle S_{N-1}^2 \rangle + \langle \ell_N^2 \rangle$$
$$= (N-1)L^2 + L^2$$
$$= N L^2$$

and

$$\boxed{\sigma = \sqrt{\langle S_N^2 \rangle} = L\sqrt{N}}$$

RMS displacement of a random walk after N steps.