Ideal Fermi Gas

Now let's see what happens when we fill our box with fermions.

As before, our density of k-states is

$$\rho(k) = \frac{L^3}{8\pi^3}$$

If the particle has spin s, then the total number of possible spin states is $2s+1$ hence

$$\rho(k) = (2s+1) \cdot \frac{L^3}{8\pi^3}$$

Now, any time we want to sum over all quantum numbers that.. over all states, we can

$$\sum_r \longrightarrow (2s+1) \cdot \rho(k) \int d^3k$$

$$= (2s+1) \cdot \frac{V}{(2\pi)^3} \int d^3k$$

$$= (2s+1) \frac{V}{h^3} \int d^3p \quad (\text{from } p = \hbar k)$$

Let's calculate the Grand Free Energy, Φ, which you'll recall is for fermions

$$\Phi = -k_B T \sum_r \ln\left(1 + e^{-\beta(\varepsilon_r - \mu)}\right)$$

We replace $\sum_r$ as above to get

$$-\beta\Phi = (2s+1)\frac{V}{(2\pi)^3}\cdot\int d^3k \ \ln\left[1 + e^{-\beta(\varepsilon_r - \mu)}\right]$$

Now let's use spherical coordinates:

$$\int d^3k = \int_0^\infty \int_0^\pi \int_0^{2\pi} k^2 \sin\theta \, dk \, d\theta \, d\phi$$

$$= 4\pi \int_0^\infty k^2 dk$$

With the identifications

$$\varepsilon_r = \frac{\hbar^2 k^2}{2m}, \quad z = e^{\beta\mu}$$

Then

$$-\beta\Phi = (2s+1)\frac{V}{(2\pi)^3} 4\pi \int dk \, k^2 \ln\left[1 + z e^{-\beta\frac{\hbar^2 k^2}{2m}}\right]$$

Now let's change variables:

$$x = \hbar k\sqrt{\frac{\beta}{2m}} \Rightarrow k^2 dk = \left(\frac{2m}{\beta\hbar^2}\right)^{3/2} x^2 dx$$

We get

$$-\beta\Phi = (2s+1)\,\frac{4V}{\sqrt{\pi}}\underbrace{\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2}}_{n_Q!}\int_0^\infty dx\; x^2 \ln\left[1+ze^{-x^2}\right]$$

$$-\beta\Phi = (2s+1)\,\frac{4V}{\sqrt{\pi}}\,n_Q\int_0^\infty dx\; x^2 \ln\left[1+ze^{-x^2}\right]$$

Now we can integrate term by term using

$$\ln(1+y) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{y^n}{n}$$

So

$$\int_0^\infty x^2\ln\left[1+ze^{-x^2}\right] = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n}\int_0^\infty dx\; x^2 e^{-nx^2}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n}\left(-\frac{d}{dn}\int_0^\infty dx\; e^{-nx^2}\right)$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n}\left(-\frac{d}{dn}\,\frac{1}{2}\sqrt{\frac{\pi}{n}}\right)$$

$$= \frac{\sqrt{\pi}}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^n}{n^{5/2}}$$

I don't want to keep writing that, so define

$$f_{5/2}(z) \equiv \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n^{5/2}}$$

Hence,

$$\beta \Phi = -(2s+1) \cdot n_Q V f_{5/2}(z)$$

Now,

$$P = -\left(\frac{\partial \Phi}{\partial V}\right)_{T,\mu}$$

So at non zero temp,

$$\boxed{P = \frac{(2s+1)\, n_Q}{\beta} f_{5/2}(z)}$$