This is, fundamentally, due to the nature of phase transitions

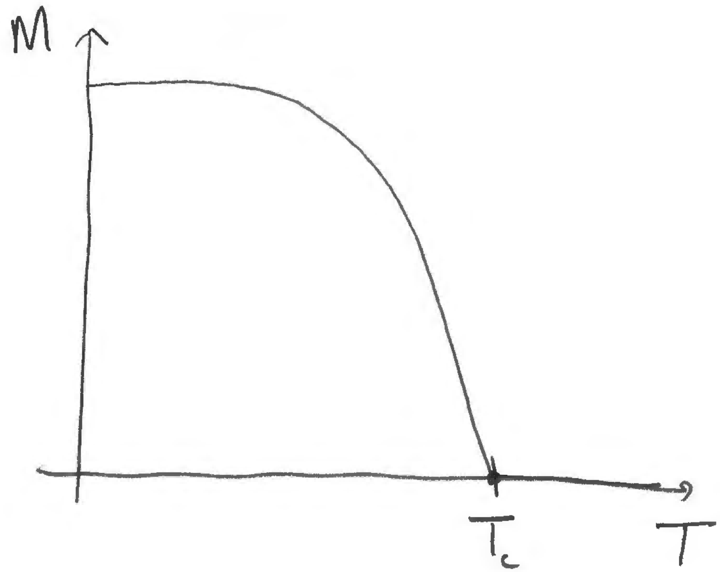
Magnetism and phase transitions

Consider the behavior of a real ferromagnet (e.g. a refrigerator magnet).

Even with zero applied field, there is still a spontaneous magnetization:

$$|M(T, h=0)| > 0$$

As the temperature increases, the magnetization decreases; in terms of the Ising model, this is due to decreased correlation between neighboring spins due to thermal fluctuations. At some finite temp, T_c, the magnetization hits zero:



T_c is called the "curie temperature"

This is a phase transition: a change from an ordered, magnetized state, to a disordered zero magnetization state

Phase transitions arise from singular behavior of the free energy
pathological
and/or its derivatives.

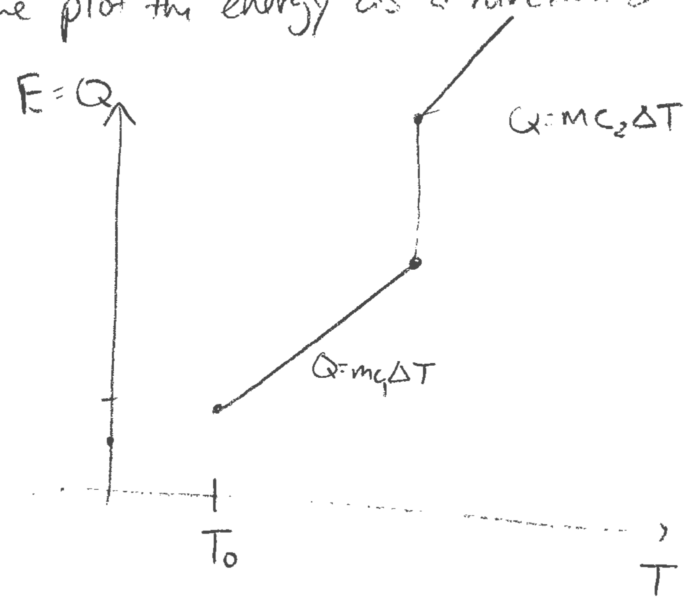
For example, consider the energy of water as you heat it. As you heat water, first it warms according to

$$Q = mc\Delta T$$

then it undergoes a phase change, characterized by a latent heat:

$$Q = m L_f$$

If we plot the energy as a function of T:



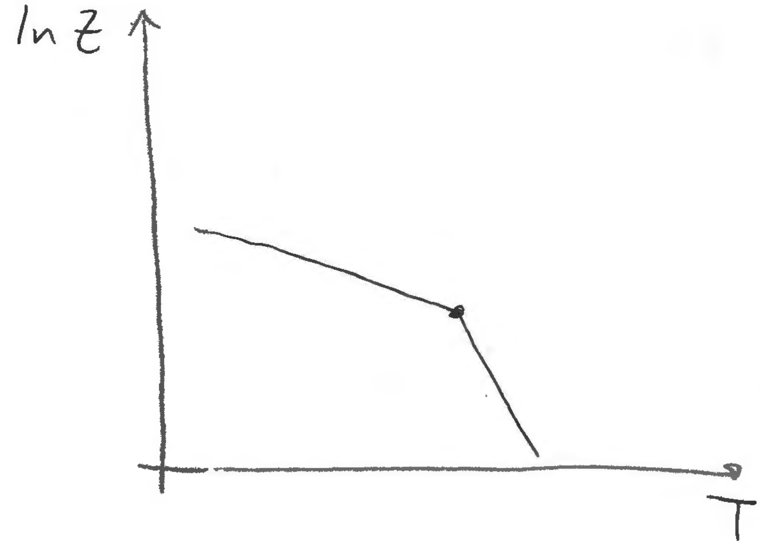
The latent heat yields a discontinuity in E(T).

Now E is the expectation value of the Hamiltonian:

$$E = \langle H \rangle = \frac{1}{Z} \sum H e^{-\beta H}$$

$$= -\beta^{-1} \frac{1}{Z} \frac{\partial}{\partial \beta} \sum e^{-\beta H} = -\beta \frac{\partial}{\partial \beta} (\ln Z)$$

The discontinuity of E arises from a "kink" in $\ln Z$, or the free energy:



This kink gives a discontinuity in the energy, or the first-order derivative of the free energy.

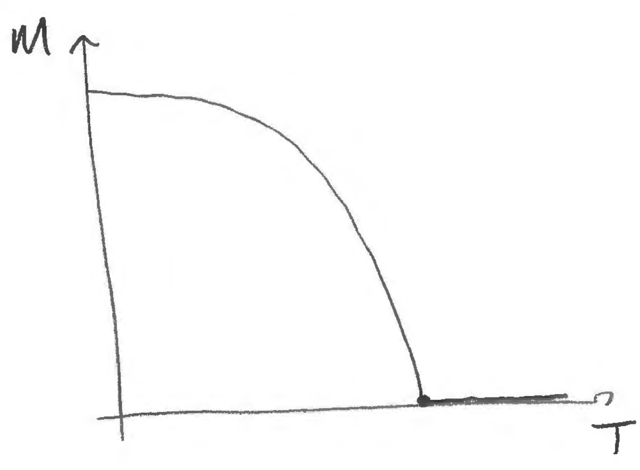
This is what is referred to as a "first order phase transition" because there is a discontinuity in the first order derivative of the free energy.

Now, back to magnetization. Recall that

$$M = -\frac{E}{h} = -\frac{1}{h}\frac{\partial \Phi}{\partial \beta}$$

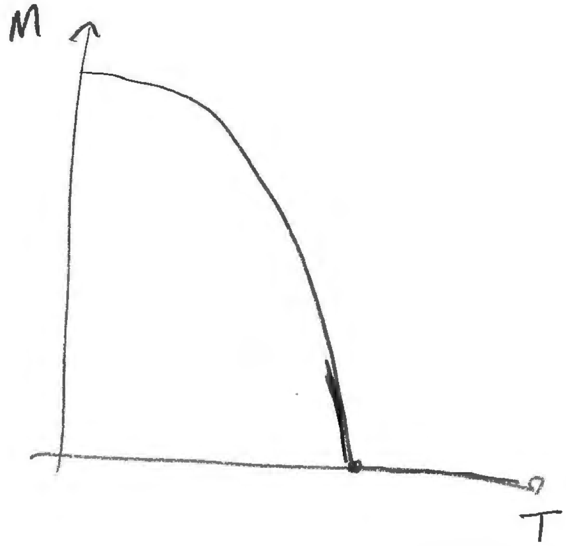
i.e the magnetization is given by the first order derivative of the free energy.

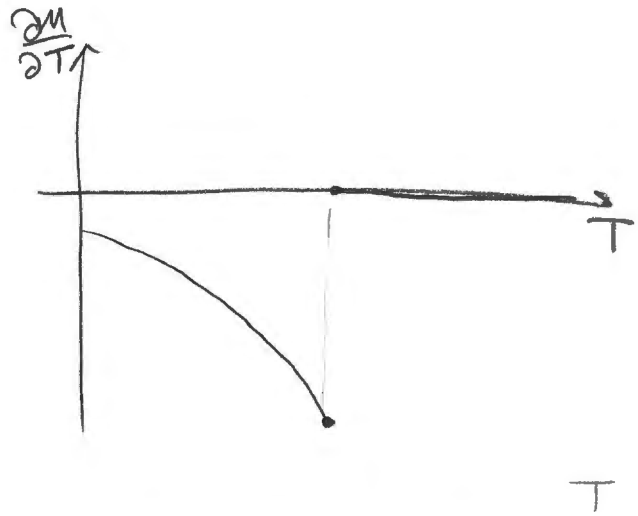
Now, it is this function which has a kink at T_c



Hence, the second order derivative of the free energy is going to have a discontinuity:

$$\frac{\partial M}{\partial \beta} = -\frac{1}{h}\frac{\partial^2 \Phi}{\partial \beta^2};$$





This is therefore called a "second-order phase transition".

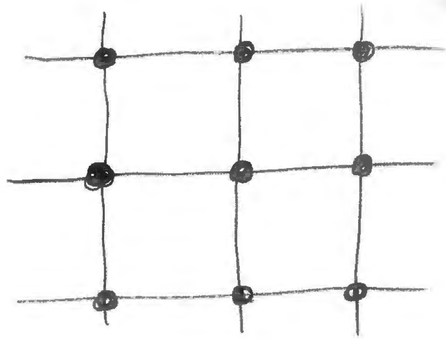
Now, our 1D free energy is well behaved and has no singularities (nor in its derivatives).

To get phase transitions, we have to go to 2 or higher dimensions.

Unfortunately, the Ising model in 2 or more dimensions is analytically... unpleasant. Hence we will make an approximation:

Mean Field Theory

Consider a cubic lattice in d dimensions:



each lattice site has $z = 2d$ nearest neighbors. z is called the "coordination number."

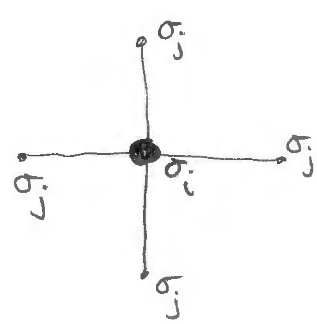
The Hamiltonian is

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i$$

Notice that this can be written

$$H = \sum_i \sigma_i \left[-J \sum_{\langle j \rangle} \sigma_j - h \right]$$

This first term is summing over all of i's nearest neighbors:



And

$$\frac{\sum_{\langle j \rangle} \sigma_j}{z} = \langle m \rangle$$

the mean field coming from the nearest neighbors:

$$H = \sum_i \sigma_i \left[-Jz\langle m \rangle - h \right]$$

Now, the assumption of mean field theory is that there is nothing special about a given σ_i... hence let's assume that $\langle m \rangle = M$, the magnetization of the entire material:

$$H = \sum_i \sigma_i \left[-JzM - h \right]$$

The partition function is then

$$Z = \sum_{\sigma_i} e^{\beta[JzM-h]\sigma_i}$$

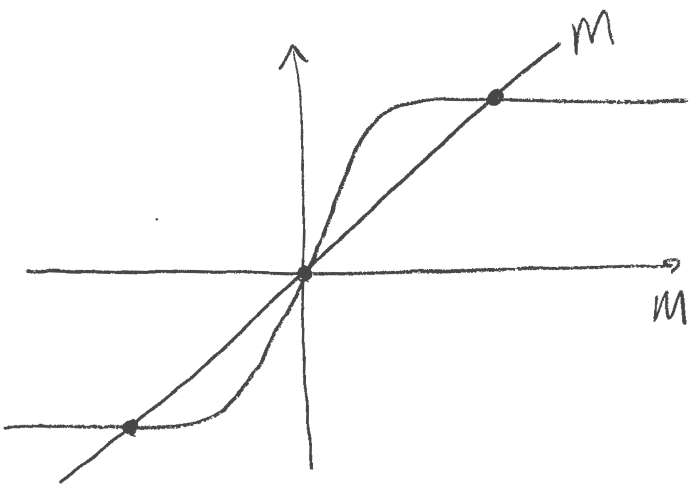
$$= e^{\beta[JzM-h]} + e^{-\beta[JzM-h]}$$

$$= 2\cosh[\beta(JzM-h)]$$

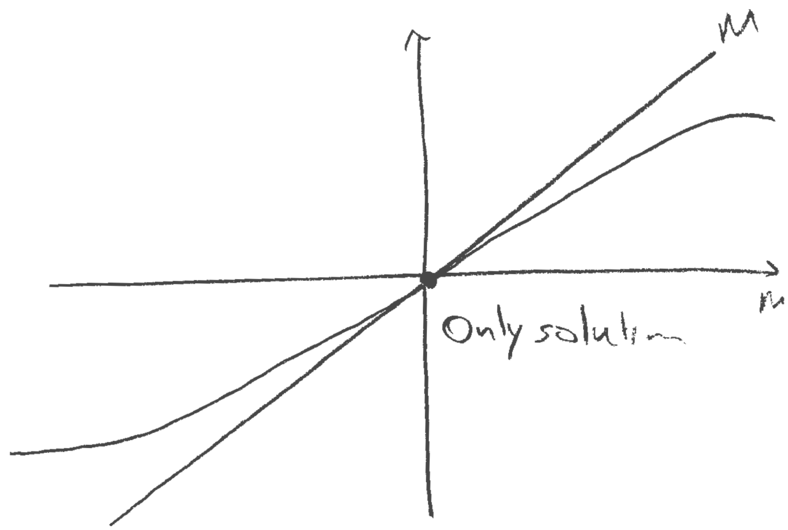
The magnetization thus is

$$M = -\frac{\partial \Phi}{\partial h} = \tanh[\beta(JzM - h)]$$

This equation is transcendental, and is most easily solved graphically. If we plot M and $\tanh[\beta(Jzm-h)]$ at different values of β (with $h=0$)



$T < T_c$
3 possible solutions!



$T \geq T_c$

At the critical temperature, M is tangent to $\tanh(\beta Jzm)$ at $m=0$, so

$$\left.\frac{\partial M}{\partial m}\right|_{m=0} = \left.\frac{\partial}{\partial m}\Big(\tanh(zJm\beta_c)\Big)\right|_{m=0}$$

$$1 = \left.\Big(1 - \tanh^2(\beta_c zJm)\Big)\beta_c Jz\right|_{m=0}$$

$$1 = \beta_c Jz = \frac{Jz}{k_B T_c} \Rightarrow \boxed{T_c = \frac{Jz}{k_B}}$$