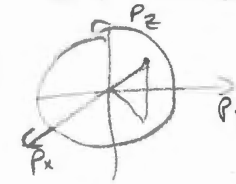
Bose-Einstein Condensation

We've seen what happens when our particle are massless and $\mu = 0$ (photons). What happens with massive bosons?

We assume we have spin-0 bosons of mass m that are non-relativistic, i.e.

$$\varepsilon = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2 \Rightarrow \text{sphere in } p \text{ space}$$



$$= \frac{\hbar^2 k^2}{2m} = \frac{2\pi^2\hbar^2}{mL^2}(n_x^2+n_y^2+n_z^2) \qquad n_{x,y,z} = 0, 1, 2, \ldots$$

Let's put them in our box of volume $V = (L)^3$. As before for the photons, the number of plane wave states per unit volume in k space is $\frac{V}{8\pi^3}$. So in p space ($\vec{p} = \hbar\vec{k}$),

$$\rho(k) = \frac{V}{(2\pi)^3} \Rightarrow \rho(p) = \frac{V}{(2\pi\hbar)^3}$$

Again making the continuum approximation, the number of plane wave states in a volume of width $d\varepsilon$ is

$$g(\varepsilon)d\varepsilon = (\text{surface area of sphere})(\text{width of sphere})\cdot(\text{density of states})$$

$$= (4\pi p^2)\cdot dp \cdot \left(\frac{V}{(2\pi\hbar)^3}\right) = (4\pi p^2)\left(\frac{dp}{d\varepsilon}d\varepsilon\right)\left(\frac{V}{(2\pi\hbar)^3}\right)$$

$$\varepsilon = \frac{p^2}{2m} \Rightarrow d\varepsilon = \frac{|\vec{p}|}{m}dp, \quad p = \sqrt{2m\varepsilon}$$

$$\therefore \frac{dp}{d\varepsilon} = \frac{m}{p} = \sqrt{\frac{m}{2\varepsilon}}$$

$$\Rightarrow g(\varepsilon)d\varepsilon = \left[4\pi(2m\varepsilon)\right]\left[\sqrt{\frac{m}{2\varepsilon}}d\varepsilon\right]\left[\frac{V}{(2\pi\hbar)^3}\right]$$

$$\boxed{g(\varepsilon)d\varepsilon = \frac{Vm^{3/2}}{\sqrt{2}\pi^2\hbar^3}\sqrt{\varepsilon}\,d\varepsilon}$$

Now we fill the box up with bosons. The total number of particles is

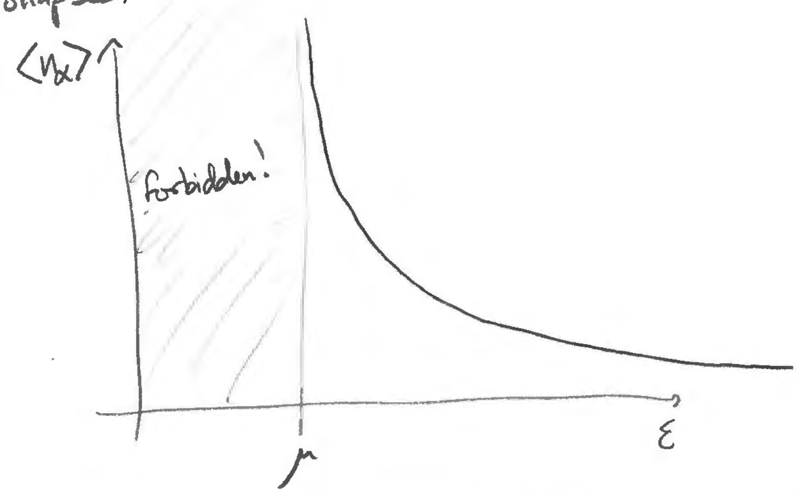
$$\langle N \rangle = \int \langle n_\varepsilon \rangle g(\varepsilon)\, d\varepsilon$$

$$= \int \frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}-1}\, d\varepsilon$$

Ok, now an aside. Remember we derived $\langle n_\alpha^{BE} \rangle$ as a sum of a geometric series:

$$Z_\alpha = \sum_{n_\alpha=0}^{\infty} e^{-\beta(\varepsilon_\alpha-\mu)\, n_\alpha} = \frac{1}{1-e^{-\beta(\varepsilon_\alpha-\mu)}}$$

This only converges if $\varepsilon_\alpha > \mu$, i.e the lowest energy level ε_0 must be higher than μ. Otherwise we can lower the energy of the system by adding more particles and everything collapses.



Now, for a given μ, $\langle N \rangle$ particles will fill the system at equilibrium. If we forcefully add another particle, we must supply energy μ to add it. This effectively raises μ for the next particle, and so on. Also, as we add more particles, the density $\frac{\langle N \rangle}{V}$ goes up

If we keep adding particles until $\mu = \varepsilon_0 = 0$. Here $\langle n_0 \rangle$ diverges and the density heads to infinity.

This is Bose-Einstein condensation.

Interestingly, our integral for $\langle N \rangle$ (assuming a continuum of states) still converges for $\mu=0$:

$$\langle N \rangle = \int \frac{g(\varepsilon)}{e^{\beta\varepsilon}-1}\, d\varepsilon$$

$$= \frac{V m^{3/2}}{\sqrt{2}\,\pi^2 \hbar^3} \int_0^{\varepsilon} d\varepsilon \frac{\sqrt{\varepsilon}}{e^{\beta\varepsilon}-1}$$

$$= V \underbrace{\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2}}_{1/\lambda_T^3} \cdot \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{\sqrt{z}}{e^z - 1}\, dz \qquad z = \beta\varepsilon$$

Now,

$$\frac{2}{\sqrt{\pi}} = \frac{1}{(1/2)!} = \frac{1}{\Gamma(\frac{3}{2})}$$

So we have

$$\frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{z^{1/2}}{e^z - 1}\, dz = \frac{1}{\Gamma(\frac{3}{2})} \cdot \int \frac{z^{\frac{3}{2}-1}}{e^z - 1}\, dz$$

$$\equiv \zeta(\tfrac{3}{2})$$

So

$$\langle N \rangle = V \lambda_T^{-3} \zeta(\tfrac{3}{2})$$

Hence when the density gets higher than

$$\frac{\langle N \rangle}{V} = \frac{\zeta(\frac{3}{2})}{\lambda_T^3} = \frac{2.612 \text{ particles}}{\text{de Broglie volume}}$$

Usually, the way this is done is not to add particles at fixed temp, but to lower the temp at fixed N. We massage our equation to yield

$$\boxed{T_c^{BEC} = \frac{2\pi\hbar^2}{m k_B}\left(\frac{N}{V\,\zeta(\frac{3}{2})}\right)^{2/3}}$$

This is the critical temperature at which BEC occurs.