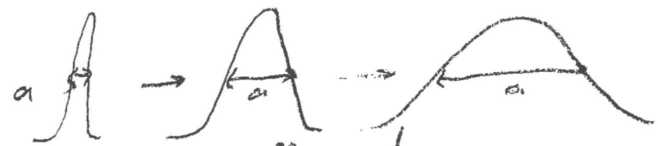
Finally,

$$\boxed{\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}}$$

1D Diffusion equation

with $D = \frac{a^2}{2\Delta t} \Rightarrow a = \sqrt{2D\Delta t}$

i.e. the width of the distribution increases w/ $\sqrt{\Delta t}$



Diffusion!

Drift Diffusion

Now, we ask what happens if we relax our first assumption of no drift, and hence a constant mean of zero.

Let's consider what happens when we add an external force F and allow for drift.

Similar to the diffusion case,

$$x(t+\Delta t) = x(t) + l(t) + \underbrace{F\gamma\Delta t}_{\text{External force term}}$$

γ: mobility $\Rightarrow$ units? $\frac{l}{Fs} = \frac{m}{\left(\frac{kg\,m}{s^{\cancel{2}}}\right)\cdot \cancel{s}} = \frac{s}{kg}$

Let's derive an expression for γ

• If the gas is dilute and the particles small, the particle trajectory under force F will be a free acceleration between collisions separated by Δt.

Then,

$$\bar{x} = \frac{1}{2} a \Delta t^2$$

$$= \frac{1}{2}\left(\frac{F}{m}\right)\Delta t^2$$

$$= F\Delta t\left(\frac{\Delta t}{2m}\right)$$

Compare to our expression for $x(t+\Delta t)$:

$$x(t+\Delta t) = x(t) + \ell(t) + F\Delta t\,\gamma$$

$$\Rightarrow \gamma = \frac{\Delta t}{2m}$$

so

$$x(t+\Delta t) = x(t) + \ell(t) + F\Delta t\left(\frac{\Delta t}{2m}\right)$$

• Using our expression for the diffusion coefficient, $D = \frac{a^2}{2\Delta t}$,

we can make a connection between unbiased + biased random walks:

$$\gamma = \frac{\Delta t}{2m} = \frac{\Delta t}{2m} \cdot \underbrace{\left(D \cdot \frac{2\Delta t}{a^2}\right)}_{=1} = \frac{D}{m(a/\Delta t)^2} = \frac{D}{m\bar{v}^2}$$

$$\therefore \boxed{\gamma = \frac{D}{m\bar{v}^2}}$$

where $\bar{v} = \frac{a}{\Delta t}$ is the velocity of the unbiased random walk.

Ok, now we can derive our diffusion equation with an external force.

Recall, with no external force,

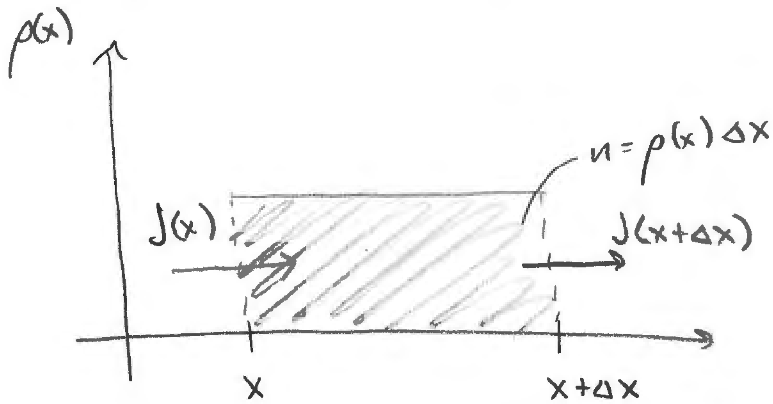
$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}$$

We wish to understand the space and time evolution of the particles in the drifting cloud of random walks.

- Let $\rho(x,t)$ be the density of some conserved quantity (e.g. number of particles) moving in 1D.
- Let $J(x)$ be the net rate at which the quantity is passing through a point x.

Then the amount of "stuff" in a small region $x, x+\Delta x$ is

$$n = \rho(x)\,\Delta x \quad \Rightarrow \quad \frac{\partial n}{\partial t} = \frac{\partial \rho}{\partial t}\,\Delta x$$



The flow into this region is $J(x)$, and the flow out is $J(x+\Delta x)$.

So

$$\frac{\partial n}{\partial t} = J(x) - J(x+\Delta x) = \frac{\partial \rho}{\partial t}\,\Delta x$$

or

$$\frac{\partial \rho}{\partial t} = -\frac{J(x+\Delta x) - J(x)}{\Delta x} = -\frac{\partial J}{\partial x}$$

Compare to

$$\frac{\partial \rho}{\partial t} = D\,\frac{\partial^2 \rho}{\partial x^2}$$

$$\Rightarrow \quad J_{\text{Diffusion}} = -D\,\frac{\partial \rho}{\partial x}$$

There will be an additional current due to the drift:

$$J_{Drift} = \gamma F \rho$$

So

$$J_{total} = J_{Diffusion} + J_{Drift}$$

and

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x} = -\frac{\partial}{\partial x}\left[J_{Diffusion} + J_{Drift}\right]$$

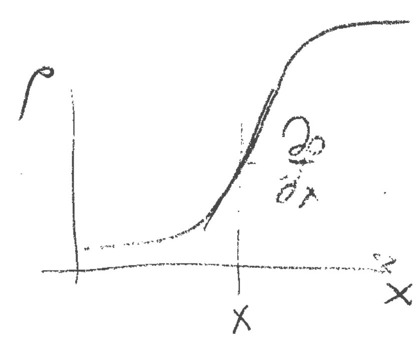
$$= -\gamma F \frac{\partial \rho}{\partial x} + D \frac{\partial^2 \rho}{\partial x^2}$$

$$\boxed{\frac{\partial \rho}{\partial t} = -\gamma F \frac{\partial \rho}{\partial x} + D \frac{\partial^2 \rho}{\partial x^2}}$$ 1 D drift-diffusion equation.

- In general, particles diffuse on average from region of high density towards regions of low density.

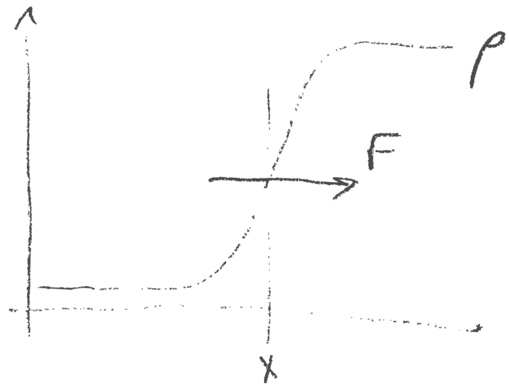
- Under an applied force, we get a term

$$-\gamma F \frac{\partial \rho}{\partial x}$$



This says: If ρ is increasing in space
$\frac{\partial \rho}{\partial x}$ is positive

and if the force is is positive $(F>0)$:



Then ρ decreases w/ time

$$\frac{\partial \rho}{\partial t} = \underbrace{-\gamma F \frac{\partial \rho}{\partial x}}_{\text{positive}} \Longrightarrow \frac{\partial \rho}{\partial t} \cdot \text{negative}$$

Because high density regions are moving away and low density region are moving in