The Microcanonical Ideal Gas

To calculate the specific properties of a collection of microscopic particles, we must choose a particular system.

The monatomic ideal gas is a simple system that gives rich behavior. By understanding the ideal gas, we can generalize to more complicated systems.

In the ideal gas, we assume a dilute collection of <u>non-interacting</u> particles

The key feature of the ideal gas: particles live in an infinite potential well defined by the bounds of some box of volume V. The hamiltonian is then dependent <u>only</u> on momentum, and does <u>not</u> depend on the spatial configuration $\mathbb{Q}$ of the particles. $\mathbb{Q}$ and $\mathbb{P}$ can be studied independently.

In the lectures, we'll work in momentum space, while the text treats the (simpler) configuration space.

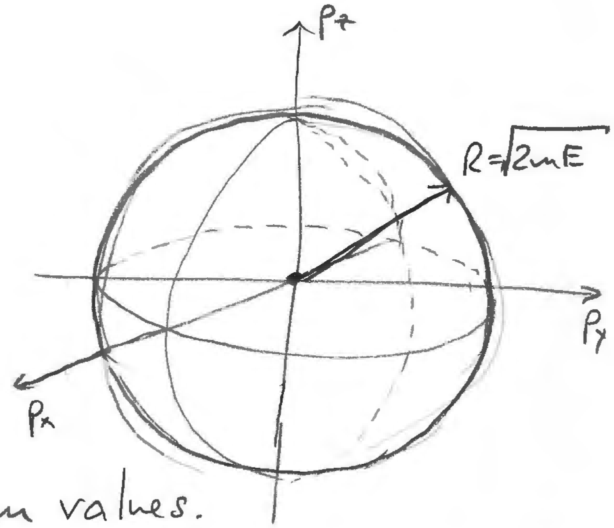
The total energy for non-interacting particles is

$$E = \sum_{\alpha=1}^{3N} \frac{1}{2} m_\alpha v_\alpha^2 = \sum_{\alpha=1}^{3N} \frac{p_\alpha^2}{2m_\alpha} = \underbrace{\frac{\mathbb{P}^2}{2m}}_{\text{assuming same mass } m}$$

If we know the system has an energy between E and $E+\delta E$, what is the corresponding volume in momentum space?

Start with one particle; there are three dimensions: p_x, p_y, and p_z

$E = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$ is the equation of a spherical shell of radius $R = \sqrt{2mE}$



The volume of a 3D sphere is $\frac{4}{3}\pi R^3$.

What about 3N dimensions?

- For N particles, we have 3N momentum values.

The condition that a system of equal mass particles has energy E is that the system lies on a spherical shell in 3N-dimensional space, $\mathcal{S}_R^{3N-1}$

- In the micro canonical ensemble, we must calculate the volume of a thin shell with energies between E and $E+\delta E$, then take the limit as $\delta E \to 0$.

The shell volume gives the number of states.

$$\frac{\delta N}{\delta E} = \frac{\text{Shell volume}}{\delta E} = \frac{\text{3N sphere volume }(E+\delta E) - \text{3N sphere volume }(E)}{\delta E}$$

In d dimensions, the volume of a sphere of radius $R=\sqrt{2mE}$ is

$$\mu_d(R) = \pi^{d/2} \frac{R^d}{(d/2)!}$$

$$\frac{\pi^{1/2} R}{}$$

e.g., for $d=3$,

$$\mu_3 = \pi^{3/2} \frac{R^3}{(3/2)!}$$

Using the gamma function,

$$\left(\frac{3}{2}\right)! = \frac{3\sqrt{\pi}}{4}$$ (see text problem 1.5)

$$\mu_3 = \frac{\pi^{3/2} R^3}{3\sqrt{\pi}/4} = \frac{4}{3}\pi R^3$$

So

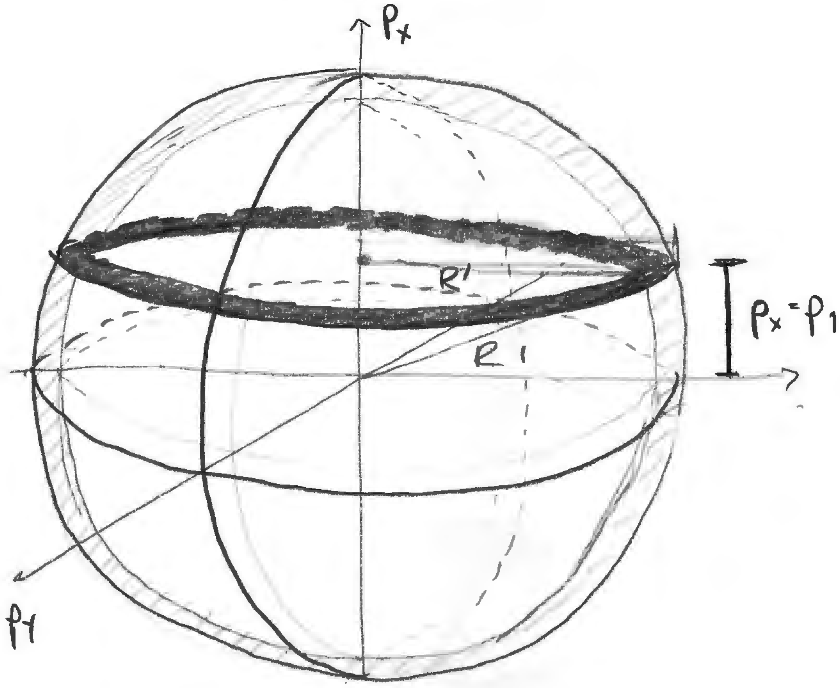
$$\frac{\text{Shell Volume}}{\delta E} = \frac{\mu(S^{3N-1}_{R E+\delta E}) - \mu(S^{3N-1}_{R E})}{\delta E}$$

as $\delta E \to 0$, this becomes a derivative:

$$\frac{d}{dE}\mu(S^{3N-1}_{\sqrt{2mE}})$$

$$= \frac{d}{dE}\left[\frac{\pi^{3N/2} R^{3N}}{(3N/2)!}\right] = \frac{d}{dE}\left[\frac{\pi^{3N/2}(2mE)^{3N/2}}{(3N/2)!}\right]$$

$$= \frac{\pi^{3N/2}\left(\frac{3N}{2}\right)(2m)(2mE)^{\frac{3N}{2}-1}}{(3N/2)!}$$



The probability density that this momentum is p_1 and in the energy range $E - E+\delta E$ is proportional to the area of the annulus divided by the total shell volume

The circle's radius is

$$R' = \sqrt{R^2 - p_1^2} = \sqrt{2mE - p_1^2}$$

We must find

Area for $E+\delta E$ - Area for E