Statistical Mechanics

- Stat mech is a field of mathematical physics that studies the average behavior of a complex system for which the underlying processes are unknown, or unknowable

- In physics, we suppose that any given initial state of a system has a corresponding future state at a later time. The <u>equations of motion</u> are the physical laws that connect these states.

- Due to the complexity of finite, macroscopic systems, there is a disconnect between microscopic processes and the average behavior.
- Stat mech aims to connect the microscopic equations of motion and the macroscopic equations of state.

Consider a gas held in a container. We know from experience that this gas exerts pressure on its container and has some temperature, and that these quantities change as the volume of the gas' container changes. How are we to understand where these properties arise from? If we assume a monoatomic gas, we can think of each gas molecule as a ball with rotational symmetries. Hence, we only need to specify the position of each molecule, x_i, y_i, z_i. But we also need to know their velocities (or momenta), $p_{x_i}, p_{y_i}, p_{z_i}$. We're already up to six parameters for each molecule. To understand the properties of the gas, we need to keep track of these numbers for EVERY molecule; we're then talking $6 \times N_A$ ($N_A \sim 10^{23}$ molecules) parameters. If we consider interactions with the walls and with each other, we are talking even more parameters. This is impossible to simulate with Newton's laws purely from the numbers of the problem.

- Because of these large numbers of possible microscopic states, statistical mechanics depends strongly on <u>counting large numbers</u> and determining <u>number averages</u>.

- To do this, let's first review some math regarding <u>probability and statistics</u>

Probability

- Probability is a real-valued set function P that assigns to each event A in the sample space S a number $P(A)$, called the probability of the event A, such that the following properties are satisfied:

In math-speak $P: S \longrightarrow \mathbb{R}$, $A \in S$

(A) $P(A) \geq 0$

(B) $P(S) = 1$ (Normalized)

(C) If $A_1, A_2, A_3, \ldots$ are events and A_i and A_j are mutually exclusive $(i \neq j)$, then

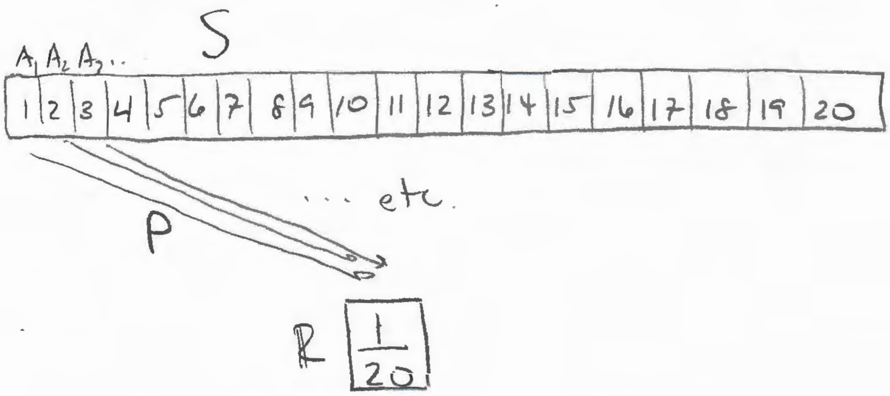
$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_k)$$

for each k

- Here "$\cup$" denotes the "union". I.e. $A_i \cup A_j$ means the collection of both A_i and A_j ... think "or"

- Also, $P(A_1 \cup A_2 \cup \ldots \cup_n) = P(A_1) + P(A_2) + \ldots$ for an infinite but countable number of events

Example: Your dungeon master demands that you roll 3 D20 (20-sided dice) and get a result of 15 or higher on each. For a given die, what is the probability of getting a 15? What is the probability of getting a 15 or higher? What is the probability of getting a 15 or higher on all 3 dice?



hence

$$P(A_1) = P(A_2) = \ldots = \frac{1}{20}$$

so

$$\boxed{P(15) = \frac{1}{20}}$$

Probability of 15+:

$$P(15+) = P(15) + P(16) + P(17) + P(18) + P(19) + P(20) = \frac{6}{20} = \boxed{\frac{3}{10} = P(15+)}$$

Probability of Independent Events

- Events A and B are independent if and only if (iff)

$$P(A \cap B) = P(A)\,P(B)$$

where "$\cap$" denotes the "intersection" of A, B ... think "and"

• Events A, B, and C are mutually independent iff the following conditions hold:

(A) They are pairwise independent:

$$P(A \cap B) = P(A)P(B)$$
$$P(A \cap C) = P(A)P(C)$$
$$P(B \cap C) = P(B)P(C)$$

and

(B) $P(A \cap B \cap C) = P(A)P(B)P(C)$

• In stat mech, we generally consider independent events.

Hence, the probability of a 15+ on all 3 dice is

$$P(15+ \cap 15+ \cap 15+) = \left(\frac{3}{10}\right)\left(\frac{3}{10}\right)\left(\frac{3}{10}\right) = \frac{27}{1000}$$

(Good luck!)