PHONONS AND PHOTONS

→ PHONONS ARE THE ELEMENTARY, HARMONIC OSCILLATIONS OF THE ELASTIC FIELD.

→ SIMILARLY, PHOTONS ARE THE ELEMENTARY, HARMONIC OSCILLATIONS OF THE EM FIELD.

→ WHY THEN DO WE THINK OF THESE HARMONIC EXCITATIONS AS PARTICLES?

→ TO ANSWER THIS, WE MUST EXAMINE THE STATISTICS OF THESE OBJECTS.

→ LET'S FIRST FIND THE CANONICAL PARTITION FUNCTION FOR A QUANTUM HARMONIC OSCILLATOR OF FREQUENCY ω

SINCE

$$\epsilon_n = \hbar\omega\left(n+\tfrac{1}{2}\right)$$

THEN

$$Z^{C}_{QHO} = \sum_{n=0}^{\infty} e^{-\epsilon_n/k_BT} = \sum_{n=0}^{\infty} e^{-\hbar\omega(n+\frac{1}{2})/k_BT}$$

→ WE CAN EASILY EVALUATE THIS SUM, SIMILAR TO THOSE IN LAST LECTURE

$$\boxed{Z^{C}_{QHO} = \frac{e^{-\hbar\omega/2k_BT}}{1-e^{-\hbar\omega/k_BT}}}$$

CANONICAL PARTITION FUNCTION OF THE Q.H.O.

→ THIS IS A VERY NEAT RESULT, AND LOOKS REMARKABLY SIMILAR TO THE GRAND CANONICAL PARTITION FUNCTION FOR BOSONS.

→ TO COMPARE, LET'S FIND THE GRAND CANONICAL PARTITION FUNCTION FOR BOSONS MULTIPLY FILLING A SINGLE STATE WITH ENERGY $\hbar\omega$ AND WITH $\mu = 0$:

$$Z_B^{GC} = \prod_{\alpha}\left(\sum_{n_\alpha} e^{-(\epsilon_\alpha-\mu)n_\alpha/k_BT}\right)$$

$$= \sum_{n=0}^{\infty} e^{-(\epsilon_\alpha-\cancelto{0}{\mu})n_\alpha/k_BT}$$

$$= \sum_{n=0}^{\infty} e^{-\hbar\omega/k_BT}$$

→ EVALUATING THIS SUM GIVES

$$\boxed{Z_B^{GC} = \frac{1}{1 - e^{-\hbar\omega/k_BT}}}$$

GRAND CANONICAL PARTITION FUNCTION OF MANY BOSONS FILLING A SINGLE STATE WITH $\epsilon = \hbar\omega$

→ NOW COMPARE Z^C_{QHO} TO Z_B^{GC} :

$$Z^C_{QHO} = e^{-\hbar\omega/2k_BT} \, Z_B^{GC}$$

→ THEY ARE NEARLY THE SAME, EXCEPT FOR A SHIFT OF THE TOTAL (AVERAGE) ENERGY BY $\frac{1}{2}\hbar\omega$.

→ THE <u>BOLTZMANN</u> STATISTICAL FILLING OF A HARMONIC OSCILLATOR IS <u>PRECISELY</u> THE SAME AS THE <u>BOSE</u>-<u>EINSTEIN</u> FILLING OF <u>BOSONS</u> INTO A SINGLE QUANTUM STATE, EXCEPT FOR AN EXTRA SHIFT IN THE ENERGY OF $\hbar\omega/2$.

→ THIS EXTRA SHIFT IS THE <u>ZERO</u> - <u>POINT</u> <u>ENERGY</u>.

→ <u>CONCLUSION</u>: THE EXCITATIONS WITHIN THE HARMONIC OSCILLATOR ARE THUS OFTEN CONSIDERED AS PARTICLES WITH BOSE STATISTICS:

→ THE n^{TH} EXCITATION IS n BOSONS OCCUPYING THE OSCILLATOR'S QUANTUM STATE.

Ideal Fermi Gas

Now let's see what happens when we fill our box with fermions.

As before, our density of k-states is

$$\rho(k) = \frac{L^3}{8\pi^3}$$

If the particle has spin s, then the total number of possible spin states is 2s+1 hence

$$\rho(k) = (2s+1) \cdot \frac{L^3}{8\pi^3}$$

Now, any time we want to sum over all quantum numbers, that is, over all states, we can

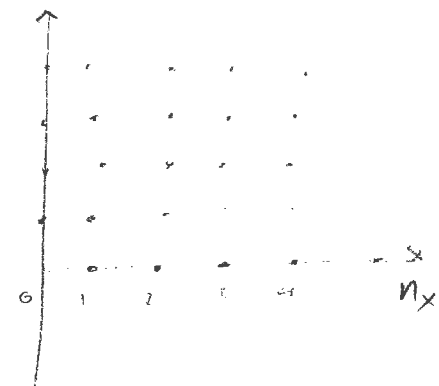
$$\sum_r \longrightarrow (2s+1) \cdot \rho(k) \int d^3k$$

$$= (2s+1) \cdot \frac{V}{(2\pi)^3} \int d^3k$$

$$= (2s+1) \frac{V}{h^3} \int d^3p \qquad (\text{from } p = \hbar k)$$

$$\frac{\hbar^2 k^2}{2m} \qquad \vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

Now, at $T=0$, the number of particles in the system is equal to the number of energy levels below the Fermi energy, ε_F:



$$N = 2 \cdot \frac{1}{8} \cdot \frac{4}{3}\pi n_F^{2}$$

where

$$\varepsilon_F = \frac{2\pi^2\hbar^2}{mL^2} n_F^2$$

This gives

$$n_F = \left(\frac{3N}{\pi}\right)^{1/3}$$

and hence

$$\varepsilon_F = \frac{2\pi^2\hbar^2}{mL^2}\left(\frac{3N}{\pi}\right)^{2/3}$$

The total energy of the gas is then

$$E_T = \sum_{N'=0}^{N} E_F(N') \simeq \int_0^N E_F \, dN' = \frac{3}{5}\varepsilon_F N$$

From thermo,

$$P = -\frac{\partial E}{\partial V} = \frac{2}{5}\frac{N}{V}\varepsilon_F =$$

So, even @ T=0, there is pressure. This is called the degeneracy pressure, due to the Pauli Exclusion Principle.