Bosons and Fermions

All known particles have intrinsic spin. Particles with half integer spin are called fermions, while those with integer spin are called Bosons.

Relativistic QM (i.e. quantum field theory) shows that, in order for particles not to interact over space-like distances (i.e. signals travel faster than light), it is required that multiparticle wavefunctions be symmetric for bosons, and anti-symmetric for fermions:

$$|\Psi_{12}\rangle = \frac{1}{\sqrt{2}}\left[|\psi_1\rangle|\psi_2\rangle \overset{\text{Boson}}{\underset{\text{fermion}}{\pm}} |\psi_2\rangle|\psi_1\rangle\right]$$

As you can immediately see, if the two particles are in the same state, the fermion wavefunction vanishes. This is the Pauli exclusion principle: Two fermions cannot occupy the same state, while two bosons can.

The difference between bosons and fermions is only apparent at low energies (temps) and high densities → Densities high enough for particle wavefunctions to overlap. At high energy (temp) + low density, both just look like Boltzmann.

In classical stat mech, particles are "undistinguished," i.e. you can in principle tell them apart, we just have not done so.

Conversely, in QM, identical particles are not just hard to tell apart - it is impossible. Their wavefunctions must be identical (up to a phase change) when particles/coordinates are swapped.

For Bosons, the wavefunction is unchanged when particles are swapped:

$$\Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = \Psi(\vec{r}_2, \vec{r}_1, \dots, \vec{r}_N) = \Psi(\vec{r}_{P_1}, \vec{r}_{P_2}, \dots, \vec{r}_{P_N})$$

for any permutation P of the integers $1, 2, \dots, N$

This implies that particles can occupy the same state, i.e. the number of particles in any one state is not restricted:

$$n_\alpha \in \{0, 1, 2, \dots\} = \#\text{ of particles in Energy state } \alpha \text{ with } E_\alpha$$

For Fermions, swapping coordinates results in a phase change:

$$\Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = -\Psi(\vec{r}_2, \vec{r}_1, \dots, \vec{r}_N) = \sigma(P)\,\Psi(\vec{r}_{P_1}, \vec{r}_{P_2} \dots \vec{r}_{P_N})$$

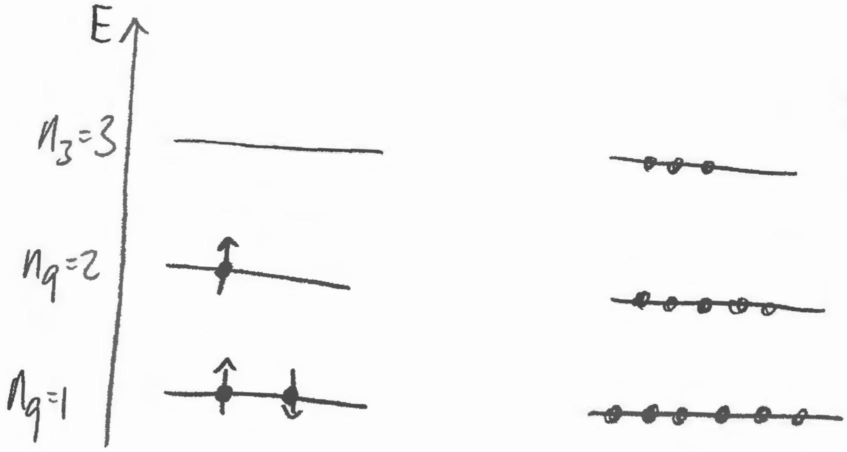
where $\sigma(P)$ is the sign of the permutation P.

As we showed earlier, this implies that each quantum state can only be occupied by a single particle:

$$n_\alpha \in \{0, 1\}$$

i.e. the number of fermions in a given state can only be 0 or 1.

Consider some system of energy levels labeled by n_q:



Fermions
$n_\alpha \in \{0,1\}$

Bosons
$n_\alpha \in \{0,1,2,\dots\}$

Recall that our goal is to find the ground state occupation numbers $\langle N \rangle$ and excited state $(T>0)$ occupation. Now we have everything we need!

First, the non-interacting many particle Hamiltonian can be written as

$$\hat{H} = \sum_\alpha \hat{n}_\alpha \varepsilon_\alpha$$

$\hat{n}_\alpha$ is the number of particles in state α with energy ε_α.

Any eigenstate of $\hat{H}$ may be labelled by the integer eigenvalues of the $\hat{n}_\alpha$ number operators, and written as $|\vec{n}\rangle = |n_1, n_2, n_3 \dots\rangle$.

We then have

$$\hat{n}_\alpha |\vec{n}\rangle = n_\alpha |\vec{n}\rangle$$

and

$$\hat{H}|\vec{n}\rangle = \sum_\alpha n_\alpha \varepsilon_\alpha |\vec{n}\rangle$$

For the Canonical formalism, we can write the N-particle partition function Z_N:

$$Z_N = \sum_{\{n_\alpha\}} e^{-\beta \sum_\alpha n_\alpha \varepsilon_\alpha} \delta_{N, \sum_\alpha n_\alpha}$$

The sum is over all allowed values of $\{n_\alpha\}$ which depends upon the type of particle

The Kronecker delta constrains the sum so that $N = \sum n_\alpha$ = total # of particles

We want to find $\langle n_\alpha \rangle$ as a function of E, T

From the Grand canonical free energy, we know

$$\langle n_\alpha \rangle = \frac{\partial \Phi}{\partial \mu} \qquad \Phi = -k_B T \ln \mathcal{Z}$$

The calculation therefore involves finding Z from our Schrodinger eq:

$$\hat{n}_\alpha |\vec{n}\rangle = n_\alpha |\vec{n}\rangle$$

$$\hat{H} |\vec{n}\rangle = \sum_\alpha n_\alpha \epsilon_\alpha |\vec{n}\rangle$$

From this, we can get Z:

$$Z_N^{GC} = \sum_N \exp\left[-\beta(E_N - \mu N_N)\right]$$

$$= \sum_{\{n_\alpha\}} e^{-\beta \sum_\alpha n_\alpha \epsilon_\alpha} e^{\beta\mu \sum_\alpha n_\alpha}$$

$$= \sum_{\{n_\alpha\}} e^{-\beta \sum_\alpha (\epsilon_\alpha - \mu) n_\alpha}$$

$$= \prod_\alpha \left(\sum_{n_\alpha} e^{-\beta(\epsilon_\alpha - \mu) n_\alpha} \right)$$

The Grand canonical partition function is a product over contributions from the individual particle states. We relax the constraint $\sum n_\alpha = N$, and the states can be factored.