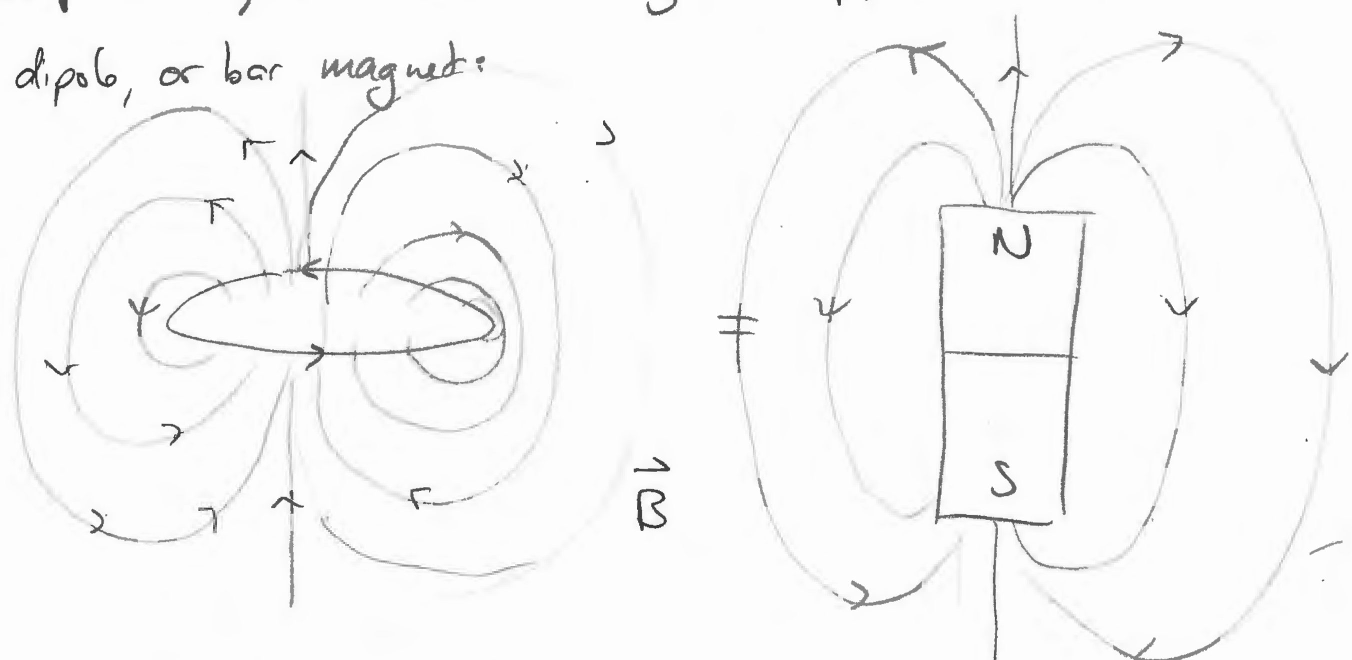
The Ising Model

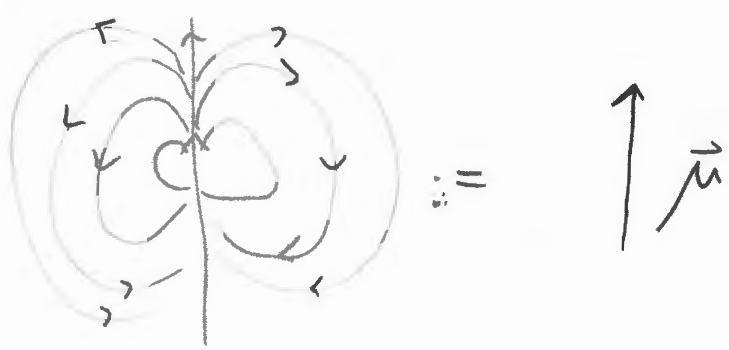
The Ising model is a model for the magnetization of materials as a result of the alignment of spins.

First, a little physics review.

Any time you have moving charged particles, they generate a magnetic field. In particular, for current travelling in a loop, the field looks like that of a dipole, or bar magnet:

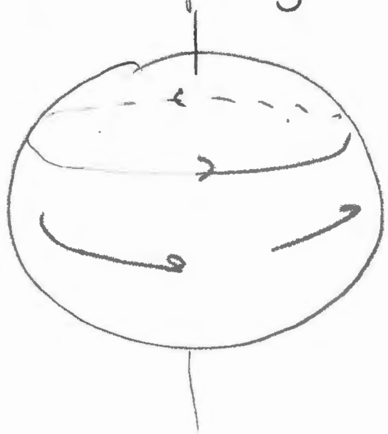


This field can be more easily represented by a vector:

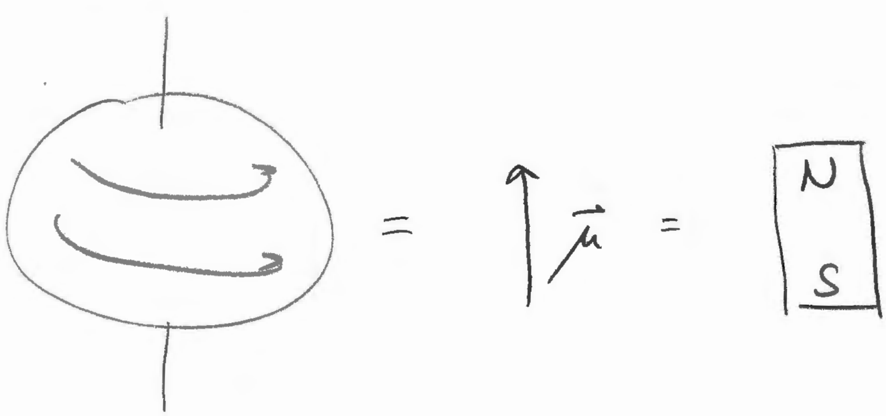


This vector is the magnetic dipole moment, $\vec{\mu}$. Its magnitude is related to the current + size of the loop, but we don't care about that right now. Its direction points S→N.

Now, charged elementary particles have spin, which, while not a really accurate picture, is analogous to a spinning charged ball. In other words, it looks like a collection of current loops.

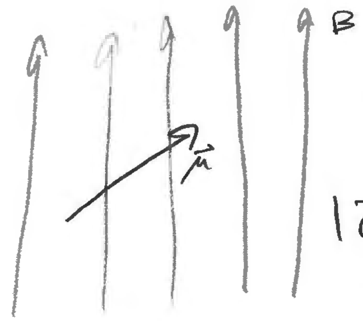


Hence, a charged particle with spin has a magnetic dipole moment, and acts like a little bar magnet:



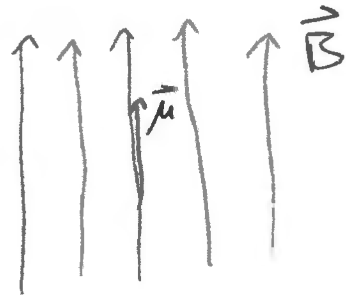
This dipole moment (of e^-, specifically) is primarily responsible for materials' magnetic properties

Now, when a magnetic dipole is placed within a magnetic field, it feels a torque causing it to align with the field:



$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

$$|\vec{\tau}| = \mu B \sin\theta$$



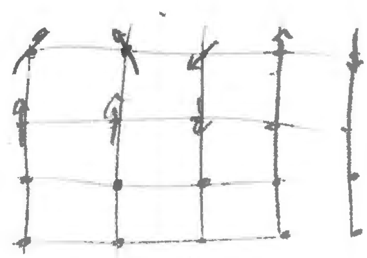
Hence, one can associate a potential energy with a dipole's orientation in a magnetic field

$$U(\theta) = -\vec{\mu} \cdot \vec{B} = -\mu B \cos\theta$$

The lowest energy state is aligned ($\theta = 0$), highest is anti-aligned ($\theta = 180°$) and the difference is

$$\Delta U = 2\mu B$$

Ok, now, the Ising model for a magnetic material is to consider particles w/ magnetic dipole moments in a lattice:



For simplicity, we'll assume these particles are electrons (hence two spin states) and in 1D:

We label each site with an index $i = 1, \dots, N$, and the spin state of each particle with a variable $\sigma_i = \pm 1$ for spin up (+) or down (−).

We can also place the entire lattice within an external magnetic field, B. Each of the spins σ_i will interact with the external field, and with each other. Hence the energy of the system is quite complex:

$$H = -\sum_{i,j} J_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j \quad - \quad \sum_i \vec{B} \cdot \vec{\mu}$$

$$= -\sum_{i,j} J_{ij} \sigma_i \sigma_j \quad - \mu B \sum_i \sigma_i$$

where the sum $\sum_{ij}$ is over all pairs of e^-, without double counting.

The interesting parameter here is J_{ij}, which is a matrix indicating how the spins interact.

If $J_{ij} > 0$, then having spins aligned is energetically favorable. Such materials are "ferromagnetic".

if $J_{ij} < 0$, then anti-aligned spins are energetically favorable. Such materials are called "antiferromagnetic"

If $J_{ij} = 0$, such materials are called "boring".

Now, in principle, the sum $\sum_{i,j}$ should be over <u>ALL</u> pairs σ_i, σ_j. However, to keep the problem tractable, we will only perform the sum over nearest neighbors:

$$H = -\sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j - \mu B \sum \sigma_i$$

$$H = -J_{12}\sigma_1\sigma_2 - J_{23}\sigma_2\sigma_3 - J_{34}\sigma_3\sigma_4 - \dots - \mu B \sum \sigma_i$$

What happens at the ends depends on the boundary conditions (periodic, infinite, etc.)

The collection of spin states of a given configuration

$$\{\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_N\}$$

is the spin configuration, $\{\sigma_i\}$.

Given

$$H = -\sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j - \mu B \sum_i \sigma_i$$