Entropy and Thermodynamics of Black Holes

Black holes are the end state of massive stars, once the outward pressures of fusion + degeneracy are insufficient to counteract gravity

As matter falls inward and concentrates, the escape velocity at the "Event Horizon" reaches the speed of light. A simple calculation (which surprisingly yields the correct result):

$$E_0 = \frac{1}{2} m v_e^2 - \frac{GMm}{R_s}$$

where M is the star mass, v is escape velocity, and R_s is the Event Horizon radius (Schwarzschild Radius)

$$E_f = 0$$

$$G = 6.67 \times 10^{-8}\ cm^3/g\,s^2$$

$$\Rightarrow \boxed{R_s = \frac{2GM}{c^2}} \quad \text{when } v_e = c$$

$$c = 2.998 \times 10^{10}\ cm/s$$

After crossing the event horizon nothing can escape, since $v_e < v$.
No information can be communicated back to the outside since not even light can escape.

You can, then, envision a black hole as the fundamental bound on memory storage: A high density of information goes in, and is stored for a long time.

Can we calculate the maximum number of bits that can be stored in a sphere of radius 1cm?

First, let's understand the basic thermodynamics of a black hole

Let's calculate a black hole's specific heat.

Recall

$$Q = c\Delta T = E$$

$$\rightarrow \frac{1}{c} = \frac{\partial T}{\partial E}$$

We'll assume that energy that falls into a black hole contributes to its mass,

$$E = Mc^2$$

$$\rightarrow dE = c^2 dM$$

So

$$\frac{1}{c} = \frac{\partial T}{c^2 \partial M}$$

Now, Stephen Hawking showed, by combining methods from QM and General Relativity, that black holes radiate.

The spectrum of emitted radiation is a perfect blackbody, with temperature

$$T_{BH} = \frac{\hbar c^3}{8\pi G M k_B}$$

Notice this expression has fundamental constants relevant to all fundamental theories

Now

$$\frac{1}{C} = \frac{1}{c^2}\frac{\partial T}{\partial M} = -\frac{\hbar c}{8\pi G k_B} \cdot \frac{1}{M^2}$$

So

$$C = -\frac{8\pi k_B}{\hbar c} M^2 = -\frac{\hbar c^5}{8\pi G k_B}\frac{1}{T^2}$$

This is weird, the specific heat is NEGATIVE
→ a black hole gets colder as you add energy to it

In a bulk material, this would lead to instability. Heat flows from hot to cold, making cold regions colder, and drawing more heat...

Indeed a population of black holes IS unstable. They coalesce to form supermassive black holes.

Now let's calculate the ENTROPY of a black hole.

We know

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

and let $S(M=0)=0$ since you cannot have configurations if there is nothing to configure!

Integrate to find

$$S(M) - S(0) = \int_0^{Mc^2} \frac{1}{T}\, dE$$

$$= \int_0^{Mc^2} \frac{8\pi G k_B}{\hbar c^5} \underbrace{(Mc^2)}_{=E} dE$$

$$= \frac{4\pi k_B G (Mc^2)^2}{\hbar c^5}$$

$$\therefore \boxed{S_{BH} = \frac{4\pi k_B G M^2}{\hbar c}}$$

Thermodynamic Entropy of a Black Hole

We can use 2 facts to simplify

① $A_{BH} = 4\pi R_s^2$ is the surface area

② $L^* = \sqrt{\frac{\hbar G}{c}}$ is the Planck length

So

$$S = \frac{4\pi k_B G}{\hbar c}\left(\frac{R_s^2 c^4}{4G^2}\right) = k_B \frac{A}{4} \qquad \text{where } A = 4\pi\left(\frac{R_s}{L^*}\right)^2$$

This entropy represents the inaccessibility of all information about what it was built of.

We can now calculate the fundamental bound on information storage (in Bits).

A bit is some system that can have two states: "0" and "1" (or "ON" and "OFF"

Hence

$\Omega_{Bit} \cdot 2$

and

$$S_{Bit} = k_B \ln \Omega_{Bit}$$
$$= k_B \ln 2$$

The most information we can have in a r=1cm sphere before it collapses into a black hole is

$$S_{max} = 4 k_B \pi \left(\frac{c^3}{\hbar G} \right) (1 cm)^2$$

Then,

$$\boxed{\frac{S_{max}}{S_{Bit}} = 6.96 \times 10^{66} \text{ Bits}}$$

Information storage limit for 1 cm radius

This is called the "Covariant Entropy Bound"

Compare this to the state of the art hard drive. Currently we're at ~ 1.34 Tbit/in^2, expected in the near future to hit 5 Tbit/in^2:

$$\left(\frac{5 \text{ Tb}}{in^2}\right) \times \left(\frac{(1024)^6 \text{ Bytes}}{1 \text{ Tbyte}}\right) \left(\frac{8 \text{ bits}}{1 \text{ byte}}\right) \left(\frac{1 \text{ in}^2}{6.45 \text{ cm}^2}\right) = 6.82 \times 10^{12} \frac{\text{bits}}{\text{cm}^2}$$