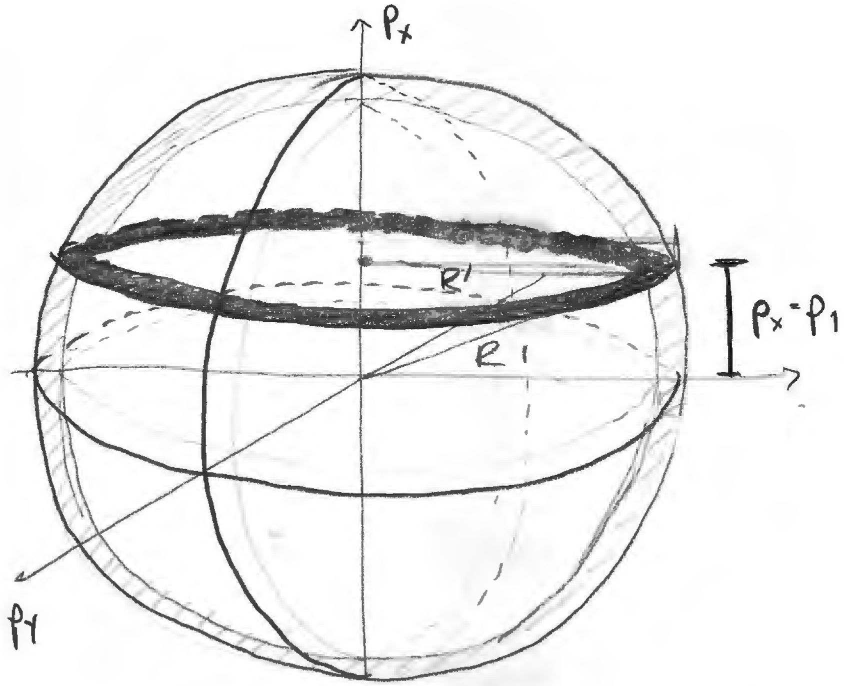

The probability density that this momentum is p_1 and in the energy range $E - E+\delta E$ is proportional to the area of the annulus divided by the total shell volume

The circle's radius is

$$R' = \sqrt{R^2 - p_1^2} = \sqrt{2mE - p_1^2}$$

We must find

Area for $E+\delta E$ - Area for E

$$\frac{\text{Annular Area}}{\delta E} = \frac{d}{dE}\left(S^{3N-2}_{\sqrt{2mE-p_1^2}} \right)$$

$$= \frac{d}{dE}\left(\frac{\pi^{(3N-1)/2}}{(3N-1)/2)!} \left(2mE-p_1^2\right)^{(3N-1)/2} \right)$$

$$= \frac{\pi^{(3N-1)/2}}{((3N-1)/2)!} \frac{(3N-1)}{2} \cdot 2m \left(2mE-p_1^2\right)^{(3N-3)/2}$$

$$= \frac{(3N-1)\, m\, \pi^{(3N-1)/2} R'^{\,3N-3}}{[(3N-1)/2]!} \qquad R' = \sqrt{2mE-p_1^2}$$

Hence,

$$\rho(p_1) = \frac{(3N-1)\, m\, \pi^{(3N-1)/2} R'^{\,3N-3} / (3N-1/2)!}{3Nm\pi^{3N/2} R^{3N-2} / (3N/2)!}$$

Let's think about the dependence on R/R':

$$\rho(p_1) \sim \frac{R^2}{R'^3}\left(\frac{R'}{R}\right)^{3N}$$

Now,

$$R'^2 = R^2 - p_1^2$$

$$\left(\frac{R'}{R}\right)^2 = 1 - \frac{p_1^2}{2mE}$$

$$\frac{R'}{R} = \left(1 - \frac{p_1^2}{2mE}\right)^{1/2}, \quad \left(\frac{R'}{R}\right)^{3N} = \left(1 - \frac{p_1^2}{2mE}\right)^{3N/2}$$

So,

$$\rho(p_1) \sim \left(\frac{R^2}{R'^3}\right)\left(1 - \frac{p^2}{2mE}\right)^{3N/2}$$

= The probability density of having one momentum coordinate equal to p_1

Now,

$$\rho(p_1) \sim \left(\frac{R^2}{R'^3}\right)\left(1 - \frac{p_1^2}{2mE}\right)^{3N/2}$$

Since N is very, very large, the second term will quickly go to zero unless

$$1 - \frac{p_1^2}{2mE} \sim 1$$

For this to be true, $\frac{p_1^2}{2mE}$ must be very, very small:

$$1 - \frac{p_1^2}{2mE} = 1 - \epsilon \simeq \exp(-\epsilon) = \exp\left(-\frac{p_1^2}{2mE}\right)$$

Also, as $\delta E \to 0$, and $R \simeq R'$. So

$$\frac{R^2}{R'^3} \simeq \frac{1}{R} = \frac{1}{\sqrt{2mE}}$$

Finally, then,

$$\rho(p_1) \sim \frac{1}{\sqrt{2mE}} \exp\left[-\frac{p_1^2}{2m} \cdot \frac{3N}{2E} \right]$$

This looks like a Gaussian with $\sigma = \sqrt{\frac{2mE}{3N}}$!

Let's Normalize :

$$\int_{-\infty}^{\infty} A \cdot \rho(p_1)\, dp_1 = 1$$

$$\Rightarrow \rho(p_1) = \frac{1}{\sqrt{2\pi m (2E/3N)}} \exp\left[-\frac{p_1^2}{2m} \frac{3N}{2E} \right]$$

This is the probability distribution for any momentum component of any of the particles.

Ok, now, this is pretty amazing. Simply by counting states, and without any knowledge of the trajectories of any particles, we are able to calculate the momentum distribution explicitly in terms of only E, N, and m!

This calculation shows several aspects we will discuss later on:

① Temperature: In this calculation, a single momentum component competed for some small fraction of the available energy of the entire gas. We'll study this type of competition in terms of temperature. For the ideal gas, we'll find

$$T = \frac{1}{k_B}\frac{2E}{3N}$$

So that

$$\rho(p_1) = \frac{1}{\sqrt{2\pi m k_B T}} \exp\left[-\frac{p^2}{2mk_BT}\right]$$

② The probability of some component of the momentum of a particle to have energy $K = \frac{p^2}{2m}$ is proportional to $\exp\left(-\frac{K}{k_BT}\right)$. This is the Boltzmann Distribution. The Boltzmann factor $\exp\left(-\frac{E}{k_BT}\right)$ is very general (and useful)!

③ The average kinetic energy $\left\langle \frac{p^2}{2m} \right\rangle$ is $\frac{1}{2}k_BT$. Every degree of freedom in an equilibrium classical system has this same average energy ⇒ "Equipartition Theorem"

④ This derivation, although dealing with monatomic gas of particle mass m, can be generalized to include various masses, and even interactions. The microcanonical ensemble approach works very well for classical momentum distributions. Our expression for $\rho(p)$ is correct for nearly <u>all</u> <u>classical equilibrium</u> systems!