We want to find $\langle n_\alpha \rangle$ as a function of E, T

From the Grand canonical free energy, we know

$$\langle n_\alpha \rangle = \frac{\partial \Phi}{\partial \mu} \qquad \Phi = -k_B T \ln \mathcal{Z}$$

The calculation therefore involves finding Z from our Schrödinger eq

$$\hat{n}_\alpha |\vec{n}\rangle = n_\alpha |\vec{n}\rangle$$

$$\hat{H} |\vec{n}\rangle = \sum_\alpha n_\alpha \epsilon_\alpha |\vec{n}\rangle$$

From this, we can get $\mathcal{Z}$:

$$\mathcal{Z}_N^{GC} = \sum_N \exp\left[-\beta(E_N - \mu N_N)\right]$$

$$= \sum_{\{n_\alpha\}} e^{-\beta \sum_\alpha n_\alpha \epsilon_\alpha} e^{\beta\mu \sum_\alpha n_\alpha}$$

$$= \sum_{\{n_\alpha\}} e^{-\beta \sum_\alpha (\epsilon_\alpha - \mu) n_\alpha}$$

$$= \prod_\alpha \left(\sum_{n_\alpha} e^{-\beta(\epsilon_\alpha - \mu) n_\alpha} \right)$$

The Grand canonical partition function is a product over contributions from the individual particle states. We relax the constraint $\sum n_\alpha = N$, and the states can be factored.

Let's peform the sums for bosons and fermions,

$$\sum e^{-\beta(\varepsilon-\mu)n}$$

for bosons,

$$n \in \{0, 1, 2, \ldots\}$$

hence

$$\sum_{n=0}^{\infty} e^{-\beta(\varepsilon-\mu)n} = 1 + e^{-\beta(\varepsilon-\mu)} + e^{-2\beta(\varepsilon-\mu)} + \ldots$$

$$= 1 + x + x^2 + \ldots \qquad x = e^{-\beta(\varepsilon-\mu)}$$

$$= \sum_{n=0}^{\infty} x^n$$

$$= \frac{1}{1-x}$$

$$\boxed{= \frac{1}{1-e^{-\beta(\varepsilon-\mu)}} \qquad \text{Bosons}} \Rightarrow \Phi_{\alpha_B} = k_B T \ln(1-e^{-\beta(\varepsilon_\alpha-\mu)}$$

for fermions

$$n \in \{0, 1\}$$

so

$$\sum_{n=0}^{1} e^{-\beta(\varepsilon-\mu)n} = \boxed{1 + e^{-\beta(\varepsilon-\mu)} \qquad \text{Fermions}}$$

$$\Rightarrow \Phi_{\alpha_F} = -k_B T \ln\left[1 + e^{-\beta(\varepsilon_\alpha-\mu)}\right]$$

The average occupancy of each single particle state α is then

$$\langle \hat{n}_\alpha \rangle = \frac{\partial \Phi_\alpha}{\partial \varepsilon_\alpha}$$

$$= \frac{\partial}{\partial \varepsilon_\alpha}\left[\pm \frac{1}{\beta} \ln\left[1 \mp e^{-\beta(\varepsilon_\alpha - \mu)}\right]\right]$$

$$\pm \frac{1}{\beta}\left[\frac{1}{1 \mp e^{-\beta(\varepsilon_\alpha - \mu)}} \cdot (-\beta)\, e^{-\beta(\varepsilon_\alpha - \mu)}\right]$$

$$= \frac{-}{+} \frac{e^{-\beta(\varepsilon_\alpha - \mu)}}{1 \mp e^{-\beta(\varepsilon_\alpha - \mu)}}$$

$$\boxed{\langle n_\alpha \rangle = \frac{1}{e^{\beta(\varepsilon_\alpha - \mu)} \mp 1}}$$

And the total particle number is

$$\boxed{\langle N \rangle = \frac{\partial \Phi}{\partial \varepsilon_\alpha} = \sum_\alpha \frac{1}{e^{\beta(\varepsilon_\alpha - \mu)} \mp 1}}$$

Let's look at each one more closely.

$$\langle n_{\alpha_{FD}} \rangle = \frac{1}{e^{\beta(\varepsilon_\alpha - \mu)}+1} \qquad \text{(Fermions)}$$

is called the "Fermi-Dirac Distribution", and gives the occupancy of each state ε_α when dealing with fermions (half integer spin particles)

$$\langle n_{\alpha_{BE}} \rangle = \frac{1}{e^{\beta(\varepsilon_\alpha - \mu)}-1}$$

is called the "Bose-Einstein Distribution," and does similarly for bosons (Integer spin particles).

Let's get a sense for how each behaves.

Examine the FD distribution as $T \to 0$:

$$\langle n_{FD} \rangle = \frac{1}{e^{(\varepsilon_\alpha - \mu)/k_B T}+1}$$

If $\varepsilon < \mu$, then the exponent is negative. As $T \to 0$ the exponent $\to -\infty$ and $e^{(\varepsilon - \mu)/k_B T} \to 0$

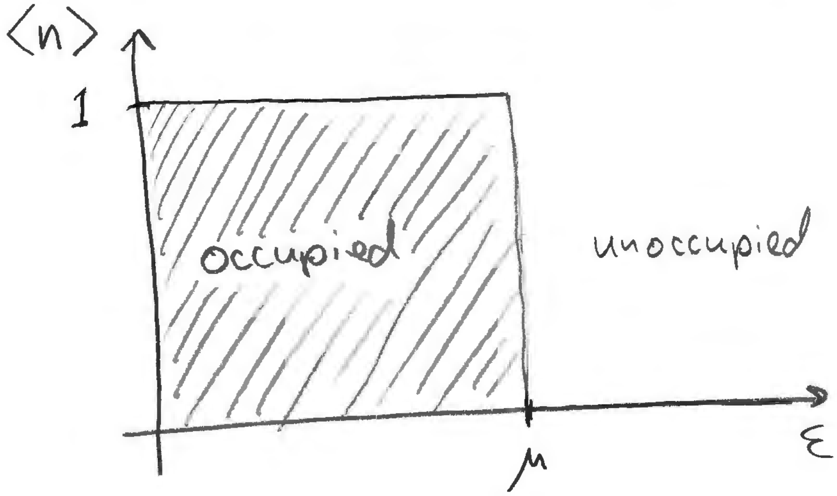
Hence,

$$\langle n_{FD} \rangle = 1 \qquad \varepsilon < \mu,\ T \to 0$$

As soon as ε becomes greater than μ, the exponent becomes positive. Hence as $T \to 0$, $(\varepsilon-\mu)/k_BT \to \infty$, $e^{(\varepsilon-\mu)/k_BT} \to \infty$ and

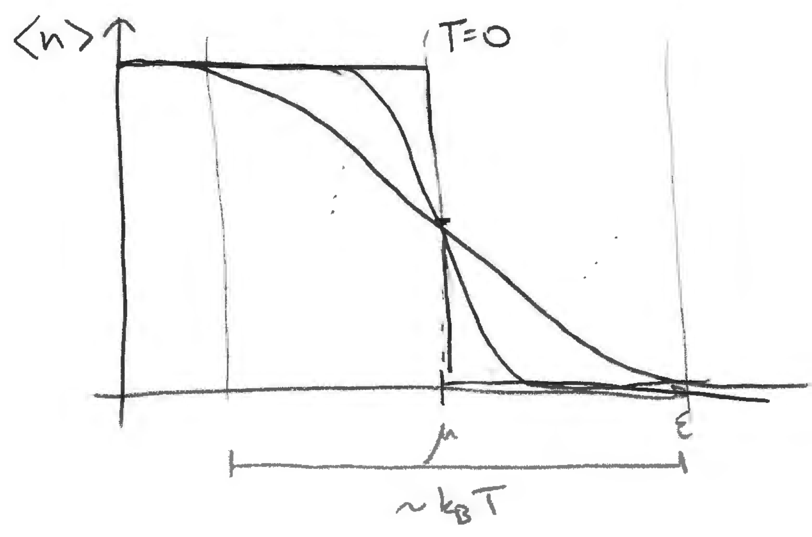
$$\langle n_{FD} \rangle = 0 \qquad \varepsilon > \mu, \; T \to 0$$

So the FD distribution looks like (at $T=0$)



For fermions, you can see that any energy levels less than μ are fully occupied at $T=0$, and any $\varepsilon > \mu$ are completely unoccupied. μ in this context is hence a cutoff called the "fermi energy", $\varepsilon_F = \mu$.

As T increases from zero, it looks like



That is, the distribution becomes more smooth. The width of the region composed only partially filled states is $\sim k_B T$

The Bose Einstein distribution is

$$\langle n_{BE} \rangle = \frac{1}{e^{\beta(\varepsilon-\mu)}-1} = \frac{1}{e^{(\varepsilon-\mu)/k_B T}-1}$$

and is ... much more weird. We will discuss it at length. Notice that for low occupancy (i.e high ε),

$$\langle n_{BE} \rangle \ll 1$$

$$\Rightarrow \frac{1}{e^{\beta(\varepsilon-\mu)}-1} \ll 1$$

suggest that $e^{\beta(\varepsilon-\mu)} \gg 1$, i.e.

$$\simeq \frac{1}{e^{\beta(\varepsilon-\mu)}}$$

$$= e^{-\beta(\varepsilon-\mu)}$$

which looks like Boltzmann statistics. Similarly,

$$\langle n_{FD} \rangle \ll 1$$

$$\Rightarrow \langle n_{FD} \rangle = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} \simeq e^{-\beta(\varepsilon-\mu)}$$