Binomial Distribution

$$(p+q)^2 = (p+q)(p+q)$$

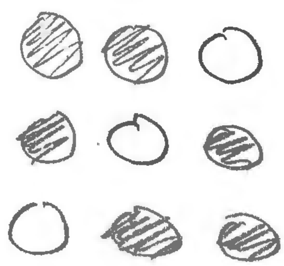
$$= p^2 + pq + qp + q^2$$

$$= \underset{\binom{2}{2}}{p^2} + \underset{\binom{2}{1}}{2pq} + \underset{\binom{2}{0}}{q^2}$$

$$(p+q)^3 = (p^2 + 2pq + q^2)(p+q)$$

$$= p^3 + p^2 q + 2pqp + 2pqq + qp^2 + q^3$$

$$= \underset{\binom{3}{3}}{p^3} + \underset{\binom{3}{2}}{3p^2 q} + \underset{\binom{3}{1}}{3pq^2} + \underset{\binom{3}{0}}{q^3}$$



$p^2 q$

$p^2 q$

$p^2 q$

In general,

$$(p+q)^N = \sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m}$$

$$= \sum_{m=0}^{N} \frac{N!}{m!(N-m)!} p^m (1-p)^{N-m} = 1 \quad \text{Normalized}$$

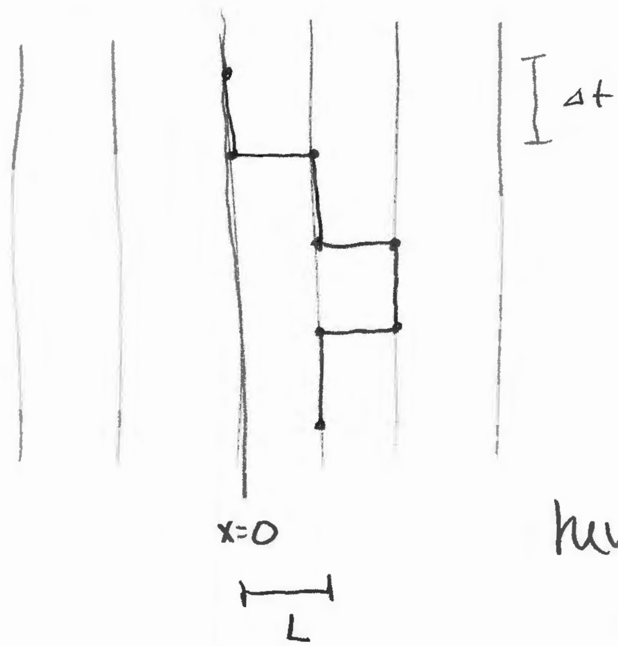
$$p + q = p + (1-p) = 1$$

1D Random Walk

Start at $x=0$

Every time interval Δt, flip a coin. If heads $(p=\frac{1}{2})$ go right; if tails $(q=1-p=\frac{1}{2}$ go left.

After N steps, how far have you gone?



Your displacement will be the number of steps right (m) minus the number of steps left (n).

$$N = m + n \Rightarrow n = N - m$$

hence,

$$X = m - n = m - (N-m)$$
$$= 2m - N$$

So we are interested in, on average, how many steps right do you take? i.e.

$$\langle x \rangle = \langle 2m - N \rangle = 2\langle m \rangle - N$$

Now, say for example, $N=3$. The possible routes are

$\rightarrow\rightarrow\rightarrow$ $\quad p^3 \quad m=3 \Rightarrow P(3) = p^3(1-p)^0$

$\rightarrow\rightarrow\leftarrow$ $\quad p^2q$

$\rightarrow\leftarrow\rightarrow$ $\quad pqp$

$\leftarrow\rightarrow\rightarrow$ $\quad qpp$

$$\left.\begin{matrix} p^2q \\ pqp \\ qpp \end{matrix}\right\} m=2 \Rightarrow P(2) = 3p^2q = 3p^2(1-p)^1$$

$\rightarrow\leftarrow\leftarrow$ $\quad pqq$

$\leftarrow\rightarrow\leftarrow$ $\quad qpq$

$\leftarrow\leftarrow\rightarrow$ $\quad qqp$

$$\left.\begin{matrix} pqq \\ pqp \\ qqp \end{matrix}\right\} m=1 \Rightarrow P(1) = 3pq^2 = 3p^1(1-p)^2$$

$\leftarrow\leftarrow\leftarrow$ $\quad qqq \quad m=0 \Rightarrow P(0) = q^3 = p^0(1-q)^3$

In general

$$P(m) = \binom{N}{m} p^m (1-p)^{N-m}$$

Now we want

$$\langle m \rangle = \sum_{m=0}^{N} m P(m) = \sum_{m=0}^{N} m \binom{N}{m} p^m q^{N-m}$$

Notice

$$mp^m = p \frac{d}{dp} p^m = p \cdot mp^{m-1} = mp^m$$

So we can write

$$\langle m \rangle = p \sum_{m=0}^{N} \binom{N}{m} \frac{d}{dp} p^m q^{N-m}$$

The sum does not depend on p, so

$$= p \frac{d}{dp} \underbrace{\sum_{m=0}^{N} \binom{N}{m} p^m q^{N-m}}_{=(p+q)^N}$$

$$= p \frac{d}{dp} (p+q)^N$$

$$= Np (p+q)^{N-1}$$

$$\boxed{\langle m \rangle = Np}$$

So

$$\langle x \rangle = 2\langle m \rangle - N$$

$$= 2Np - N$$

$$= (2p-1)N$$

If $p = \frac{1}{2}$,

$$\boxed{\langle x \rangle = 0}$$

On average, you go nowhere!

What about the standard deviation?

$$\sigma = \sqrt{\langle x^2 \rangle - \underbrace{\langle x \rangle^2}_{0}} = \langle (x - \langle x \rangle)^2 \rangle$$

$$\sigma = \sqrt{\langle x^2 \rangle}$$

So we want

$$\langle x^2 \rangle = \langle (2m-N)^2 \rangle$$

$$= \langle 4m^2 - 4mN + N^2 \rangle$$

$$= 4\langle m^2 \rangle - 4N\underbrace{\langle m \rangle}_{\text{we know this is } Np} + N^2$$

$$= 4\langle m^2 \rangle - 4N^2 p + N^2$$

We need

$$\langle m^2 \rangle = \sum_{m=0}^{N} m^2 P(m) = \sum_{m=0}^{N} m^2 \binom{N}{m} p^m q^{N-m}$$

Use the same trick!

$$= p \sum_{m=0}^{N} m \binom{N}{m} \frac{d}{dp}\left(p^m q^{N-m}\right)$$

$$= p \frac{d}{dp}\left(\sum_{m=0}^{N} m \binom{N}{m} p^m q^{N-m} \right)$$

$$= p \frac{d}{dp}\left[p \sum_{m=0}^{N} \binom{N}{m} \frac{d}{dp}\left(p^m q^{N-m}\right) \right]$$

$$= p\frac{d}{dp}\left[p\frac{d}{dp}\left(\sum_{m=0}^{N}\binom{N}{m}p^{m}q^{N-m}\right)\right]$$

$$= p\frac{d}{dp}\left[p\frac{d}{dp}(p+q)^{N}\right]$$

$$= p\frac{d}{dp}\left[p\cdot N(p+q)^{N-1}\right]$$

$$= p\left[\left(N(p+q)^{N-1}\right) + Np(N-1)(p+q)^{N-2}\right]$$

$$= Np + N(N-1)p^{2}$$

So

$$\langle x^{2}\rangle = 4\left[Np + N(N-1)p^{2}\right] - 4N^{2}p + N^{2}$$

If $p = \frac{1}{2}$,

$$\langle x^{2}\rangle = 2N + \cancel{N}^{2} - N - 2N^{2} + \cancel{N}^{2}$$

$$\langle x^{2}\rangle = N$$

and

$$\boxed{\sigma = \sqrt{\langle x^{2}\rangle} = \sqrt{N}}$$ $\blacksquare$