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Question 1

We are given

$$\bar{R}_1 = 0.01,\ \bar{R}_2 = 0.18,\ \sigma_1^2 = 0.0016,\ \sigma_{12} = 0.0016,\ \sigma_2^2 = 0.01$$

We need to find opportunity set $(\bar{R}, \sigma)$ space .

$$\sigma_\Pi = \frac{1}{400}(210000\bar{R}_\Pi^2 - 42000\bar{R}_\Pi + 2356)^{\frac{1}{2}}$$

Mean return of the portfolio is given by;

$$\begin{aligned}
\bar{R}_\Pi &= \lambda\bar{R}_1 + (1-\lambda)\bar{R}_2 \\
\bar{R}_\Pi &= \frac{1}{10}\lambda_1 + \frac{18}{100}(1-\lambda) \\
&= \frac{18-8\lambda}{100}
\end{aligned}$$

We also find λ

$$\lambda = \frac{18 - 100\bar{R}_\Pi}{8}$$

Portfolio variance is given by;

$$\begin{aligned}
\sigma_\Pi^2 &= \lambda^2\sigma_1^2 + 2\lambda_1\lambda_2\sigma_{12} + \lambda_2^2 \\
&= \frac{16}{10000}\lambda^2 + 2\lambda(1-\lambda)\frac{16}{10000} + (1-\lambda)^2\frac{1}{100} \\
&= \frac{1}{10000}\left((16\lambda^2 + 32\lambda - 32\lambda^2 + 100(1-\lambda)^2\right)\frac{1}{10000}(84\lambda^2 - 168\lambda + 100)
\end{aligned}$$

We now substitute for

$$\lambda = \frac{18 - 100\bar{R}_\Pi}{8}$$

$$
\begin{aligned}
\sigma_\Pi^2 &= \frac{1}{10000}\left((\frac{18-100\bar{R}_\Pi}{8}) - \frac{168}{8}(18-100\bar{R}_\Pi^2) + 100 \right) \\
&= \frac{1}{160000}(6804 - 75600\bar{R}_\Pi + 210000\bar{R}_\Pi^2 + 33600\bar{R}_\Pi - 4448) \\
&= \frac{1}{160000}(210000\bar{R}_\Pi^2 - 4200\bar{R}_\Pi + 2356)
\end{aligned}
$$

Taking square root on both sides

$$
\sigma_\Pi = \frac{1}{400}(210000\bar{R}_\Pi^2 - 4200\bar{R}_\Pi + 2356)^{\frac{1}{2}}
$$

as required.

The Sharpe ratio θ is given by;

$$
\begin{aligned}
\theta &= \frac{\bar{R}_\Pi^2 - R_0}{\sigma_\Pi} \\
&= \frac{\frac{18-8\lambda}{100} - \frac{6}{100}}{\frac{1}{100}(84\lambda^2 - 168\lambda + 100)^{\frac{1}{2}}} \\
&= \frac{12-8\lambda}{(84\lambda^2 - 168\lambda + 100)^{\frac{1}{2}}}
\end{aligned}
$$

To minimize, we differentiate with respect to λ and equate to 0.

$$
\begin{aligned}
\frac{d\theta}{d\lambda} &= -8(84\lambda^2 - 168\lambda + 100)^{-\frac{1}{2}} - \frac{1}{2}(12-8\lambda)(168\lambda - 168)(84\lambda^2 - 168\lambda + 100)^{-\frac{3}{2}} = 0 \\
&= (84\lambda^2 - 168\lambda + 100)^{-\frac{1}{2}}\left(-8 - (-6-4\lambda)(16\lambda - 168)(84\lambda^2 - 168\lambda + 100)^{-1}\right) = 0 \\
&= \frac{-8(-6-4\lambda)(168\lambda - 168)}{(84\lambda^2 - 168\lambda + 100)^{-1}} = 0 \\
&= -672\lambda^2 + 1344 - 800 - 1008\lambda + 1008 + 672\lambda + 672\lambda^2 = 0 \\
208 &= 336\lambda \\
\lambda &= \frac{13}{21}
\end{aligned}
$$

This implies that;

$$
\begin{aligned}
\lambda_1 &= \frac{13}{21} \\
\lambda_2 &= \frac{8}{21}
\end{aligned}
$$

under these circumstances, the market price of risk is given by;

$$
\begin{aligned}
\theta &= \frac{12 - \frac{13}{21} \times 8}{\left(84 \times (\frac{13}{21}) - 168(\frac{13}{21} + 100)^{\frac{1}{2}}\right)} \\
&= \frac{7.0476}{5.3095} \\
&= 1.32736.
\end{aligned}
$$

The figure 1 below shows efficient frontier and Capital Market Line(CML).

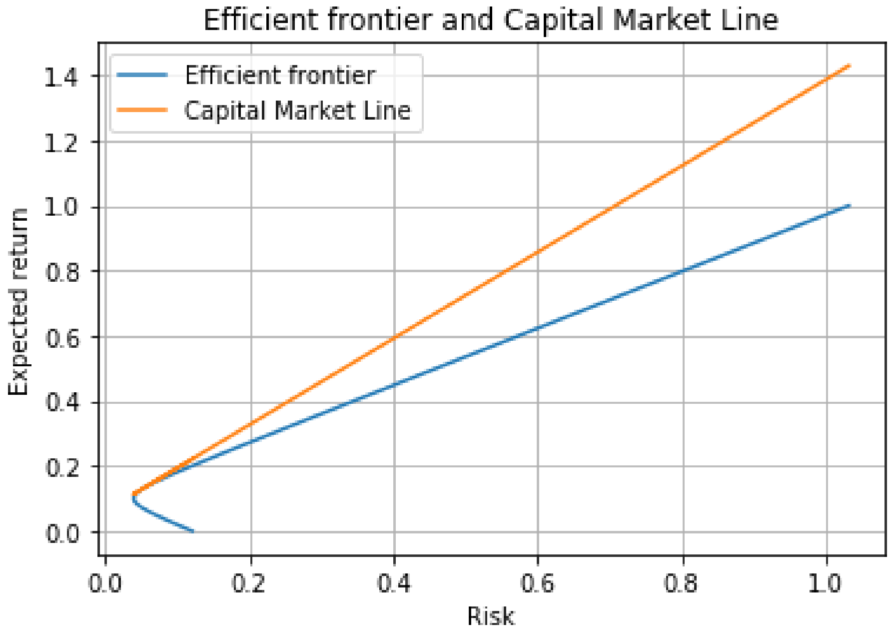

Figure 1: CML and efficient frontier

All investors are faced with the same efficient frontier of risky investment which are all subject to risk less rate of interest. From the above figure 1, the efficient frontier collapses to a straight line in the opportunity set($\bar{R}, \sigma$) which cuts across risk free rate of return R_0. Any rational investor would always be interested in investing in combination of risk and risk less portfolio at tangent point along the maximal slope. This line is the CML as illustrated above. The tangent point shows the optimum portfolio, from our plot the optimum portfolio is one with expected return, $\bar{R}$ above 10% and risk, σ around 4%.

Question 2

We are given

$\bar{R}_1 = 0.09$, $\bar{R}_2 = 0.11$, $\bar{R}_3 = 0.17$, $\sigma_1^2 = 0.0016$, $\sigma_{12} = 0.0016$, $\sigma_2^2 = 0.01$, $\sigma_{13} = 0$, $\sigma_{23} = 0.0012$, $R_0 = 0.05$.

Short selling and borrowing are allowed. We to Show that under these circumstances, the market price of risk is iven by ;

$$\theta = \frac{167}{\sqrt{13861}} \sim 1.418.$$

We also required to and that the optimal portfolio of risky assets consists of the following proportions of total wealth invested in S1, S2 and S3, respectively,

$$\frac{237}{331} \sim 0.716,\ \frac{12}{331} \sim 0.36,\ \frac{82}{331} \sim 0.248,$$

We first create the covariance matrix;

$$\begin{aligned}
\sigma_{ij} &= \frac{1}{10^4}\begin{pmatrix} 16 & 16 & 0 \\ 16 & 100 & 12 \\ 0 & 12 & 144 \end{pmatrix} \\
&= \begin{pmatrix} 16 & 16 & 0 \\ 16 & 100 & 12 \\ 0 & 12 & 144 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}\begin{pmatrix} \bar{R}_1 - R_0 \\ \bar{R}_2 - R_0 \\ \bar{R}_3 - R_0 \end{pmatrix}
\end{aligned}$$

We get the following system of equations;

$$\begin{aligned}
16Z_1 + 16Z_2 + 0Z_3 &= 4 \\
16Z_1 + 100Z_2 + 12Z_3 &= 6 \\
0Z_1 + 12Z_2 + 144Z_3 &= 12
\end{aligned}$$

We solve to obtain the values of z_i as follows;

$$\begin{aligned}
Z_1 &= \frac{79}{332} \\
Z_2 &= \frac{1}{83} \\
Z_3 &= \frac{41}{498}
\end{aligned}$$

We now find the value of λ.

$$\begin{aligned}
Z_1 + Z_2 + Z_3 &= \lambda \\
\lambda &= \frac{79}{332} + \frac{1}{83} + \frac{41}{498} = \frac{331}{996}
\end{aligned}$$

We now calculate the values of x_i the optimal portfolio. This is the value of total wealth invested in S_1 , S_2 and S_1 respectively.

$$\begin{aligned}
X_1 &= \frac{Z_1}{\lambda} = \frac{79 \times 996}{332 \times 331} = \frac{237}{331} \\
X_2 &= \frac{Z_2}{\lambda} = \frac{996}{83 \times 331} = \frac{12}{331} \\
X_3 &= \frac{Z_3}{\lambda} = \frac{41 \times 996}{498 \times 331} = \frac{82}{331}
\end{aligned}$$

and we can we see

$$X_1 + X_2 + X_3 = 1$$

The mean return of the portfolio is given by;

$$\begin{aligned}
\bar{R}_\Pi &= X_1\bar{R}_1 + X_2\bar{R}_2 + X_3\bar{R}_3 \\
&= \frac{2133}{33100} + \frac{132}{33100} + \frac{1394}{33100} \\
&= 0.1105 \\
&= 11.05\%
\end{aligned}$$

We can also calculate the portfolio risk , σ_Π

$$\begin{aligned}
\sigma_\Pi^2 &= X_1^2\sigma_1^2 + 2X_1X_2\sigma_{12} + X_2^2\sigma_2^2 + 2X_2X_3\sigma_{23} + X_3^2\sigma_3^2 \\
\sigma_\Pi^2 &= 0.00082 + 0.000013 + 0.000022 + 0.00088
\end{aligned}$$

$$= 0.001822$$

$$\begin{aligned}
\sigma_\Pi &= \sqrt{0.001822} \\
&= 0.0427 \\
&= 4.27\%.
\end{aligned}$$

θ, the market price risk is given by;

$$\begin{aligned}
\theta &= \frac{\bar{R}_\Pi^2 - R_0}{\sigma_\Pi} \\
&= \frac{0.1105 - 0.05}{0.0427} \\
&= 1.418
\end{aligned}$$

Question 3

We know that

$$\lambda_1 + \lambda_2 = 1$$

We first calculate portfolio return given by;

$$\bar{R}_\Pi = \lambda_1\bar{R}_1 + \lambda_2\bar{R}_2$$

We are given $\bar{R}_1 = 6\%$ and $\bar{R}_2 = 8\%$
The return portfolio is given by;

$$\bar{R}_\Pi = 6\lambda_1 + 8\lambda_2$$
$$6\lambda_1 + 8\lambda_2 - \bar{R}_\Pi = 0$$

The portfolio risk is given

$$\begin{aligned}
\sigma_\Pi^2 &= (\lambda_1 \ \lambda_2)\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \\
&= \lambda_1^2 + 4\lambda_1\lambda_2 + 2\lambda_2^2
\end{aligned}$$

The Lagrangian L is thus given by;

$$L = \sigma_\Pi^2 + \alpha(\bar{R}_\Pi - R) + \beta(\lambda_1 + \lambda_2 - 1)$$

We find the first two minimisation equations are

$$\begin{aligned}
0 = \frac{\partial L}{\partial \lambda_1} &= 2\lambda_1 + 4\lambda_2 + 6\alpha + \beta = 0 \\
0 = \frac{\partial L}{\partial \lambda_2} &= 4\lambda_1 + 4\lambda_2 + 8\alpha + \beta = 0
\end{aligned}$$

Rearranging gives

$$\begin{aligned}
2\lambda_1 + 4\lambda_2 &= -6\alpha - \beta \\
4\lambda_1 + 4\lambda_2 &= -8\alpha - \beta
\end{aligned}$$

We solve for λ_1 and λ_2 to obtain;

$$\begin{aligned}
\lambda_1 &= -\alpha \\
\lambda_2 &= -\frac{1}{4}(-4\alpha - \beta)
\end{aligned}$$

The other two minimisation equations give;

$$\begin{aligned}
\frac{\partial L}{\partial \alpha} &= (\bar{R}_\Pi - R) = -14\alpha - 2\beta - R = 0 \\
\frac{\partial L}{\partial \beta} &= (\lambda_1 + \lambda_2 - 1) = \frac{1}{4}(-4 - 8\alpha - \beta) = 0
\end{aligned}$$

Solving these for α and β in terms of R we get

$$\alpha = \frac{1}{2}(R - 8) \ \text{ and } \beta = 4(7 - R)$$

Substituting back into the equations for λ_i gives

$$\begin{aligned}
\lambda_1 &= -\alpha = 4 - \frac{R}{2} \\
\lambda_2 &= -\frac{1}{4}(-4\alpha - \beta) = \frac{R}{2} - 3
\end{aligned}$$

Finally putting these values into the risk portfolio, we find the risk as a function of the return;

$$\sigma_\Pi^2 = 4R - \frac{R^2}{4} - 14$$

If short selling is forbidden then we need all λ_i to be grater than zero. Since $\lambda_1 = 4 - \frac{R}{2}$ and $\lambda_2 = \frac{R}{2} - 3$ we notice that if these are all possible we need

$$6 < R < 8$$

so that if short selling is forbidden the maximum return we can achieve is 8%.

Question 4

(a) Let return on an investment be $R = Q \times X$ where Q is size of investment.
The Value at Risk (VaR) at confidence level C is the amount t such that

$$P(R \leq -t) = 1 - c$$

. We need to show that

$$t = Q\left(\sigma\Phi^{-1}(c) - \mu\right)$$

Where Φ^{-1} is the inverse of cumulative distribution function for $N[0,1]$.
X is a normally distributed random variable with mean μ and σ^2.

$$X \sim N[\mu, \sigma^2]$$

. We can write ;

$$P(QX \leq -t) \quad = \quad 1 - c$$

We know that;

$$N[a,b] = N[0,b] + a$$

and therefore

$$X = \mu + \sigma N[0,1]$$

$$\begin{aligned}
P\left(Q(\mu + \sigma N[0,1]) \leq -t\right) &= 1 - c \\
P\left(\mu + \sigma N[0,1] \leq \frac{-t}{Q}\right) &= 1 - c \\
P\left(N[0,1] \leq -\frac{t}{\sigma Q} - \frac{\mu}{\sigma}\right) &= 1 - c \\
\Phi\left(-\left(\frac{t}{Q\sigma} + \frac{\mu}{\sigma}\right)\right) &= 1 - c
\end{aligned}$$

But we know that;

$$\Phi(-t) = 1 - \Phi(t)$$

$$\begin{aligned}
1 - \Phi\left(\frac{t}{Q\sigma} + \frac{\mu}{\sigma}\right) &= 1 - c \\
\Phi\left(\frac{t}{Q\sigma} + \frac{\mu}{\sigma}\right) &= c
\end{aligned}$$

We introduce Φ^{-1} on both sides and make t the subject of the formula to obtain;

$$t = \sigma Q\left(\Phi^{-1}(c) - \frac{\mu}{\sigma}\right)$$

which simplifies to;

$$t = Q\left(\sigma\Phi^{-1}(c) - \mu\right)$$

as required.

(b) **(Python code)**
We need to calculate mean and variance for "Netflix" over period (2017,1,1) to (2020,1,1) and use it to calculate 1-day Value at Risk (VaR) at 95% confidence interval of an investment of 1000$ in Netflix. VaR is a measure of how the market value of a portfolio is likely

to decrease over a certain period, usually 1 day under usual conditions. For instance, VaR of 1000 at 95% confidence level would simply mean that there is a 95% chance of losing at most 1000$, which we can write as;

$$P(L \leq VaR) = 0.95.$$

Here we assume that the portfolio is normally distributed and the unit of currency is USD.
From our analysis we found that;

$$P(L \leq 36.8789\$) = 0.95.$$

This implies that, provided that the usual conditions will prevail over a day, we expect with probability of 95% that the rate of the portfolio will decrease by at most 36.8789$.

References

[1] Prof. James Vickers , *Lecture notes,* 2021.

[2] https://www.quantstart.com/articles/Value-at-Risk-VaR-for-Algorithmic-Trading-Risk-