AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES

(AIMS RWANDA, KIGALI)

Name: Stephen Kiilu

Course: Fluid Dynamics

Assignment Number: FD2

Date: February 27, 2021

Question 1

We are given the following dynamical systems;

(1)

$$\frac{dx}{dt} = \begin{pmatrix} 5 & 1 \\ 3 & 1 \end{pmatrix} x$$

We calculate eigenvalues and the corresponding eigenvalues. The eigenvalues are ;

$$\lambda_1 = 3 + \sqrt{7}$$
$$\lambda_2 = 3 - \sqrt{7}$$

And corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 2 + \sqrt{7} \\ 3 \end{pmatrix}$$
$$\mathbf{v}_2 = \begin{pmatrix} -\sqrt{7} + 2 \\ 3 \end{pmatrix}$$

The general solution becomes

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 + \sqrt{7} \\ 3 \end{pmatrix} e^{(3+\sqrt{7})t} + c_2 \begin{pmatrix} -\sqrt{7} + 2 \\ 3 \end{pmatrix} e^{(3-\sqrt{7})t}$$

The following figure 1 shows the corresponding phase portrait. The eigenvalues are all positive and real. We have **improper node** and it is **unstable**.

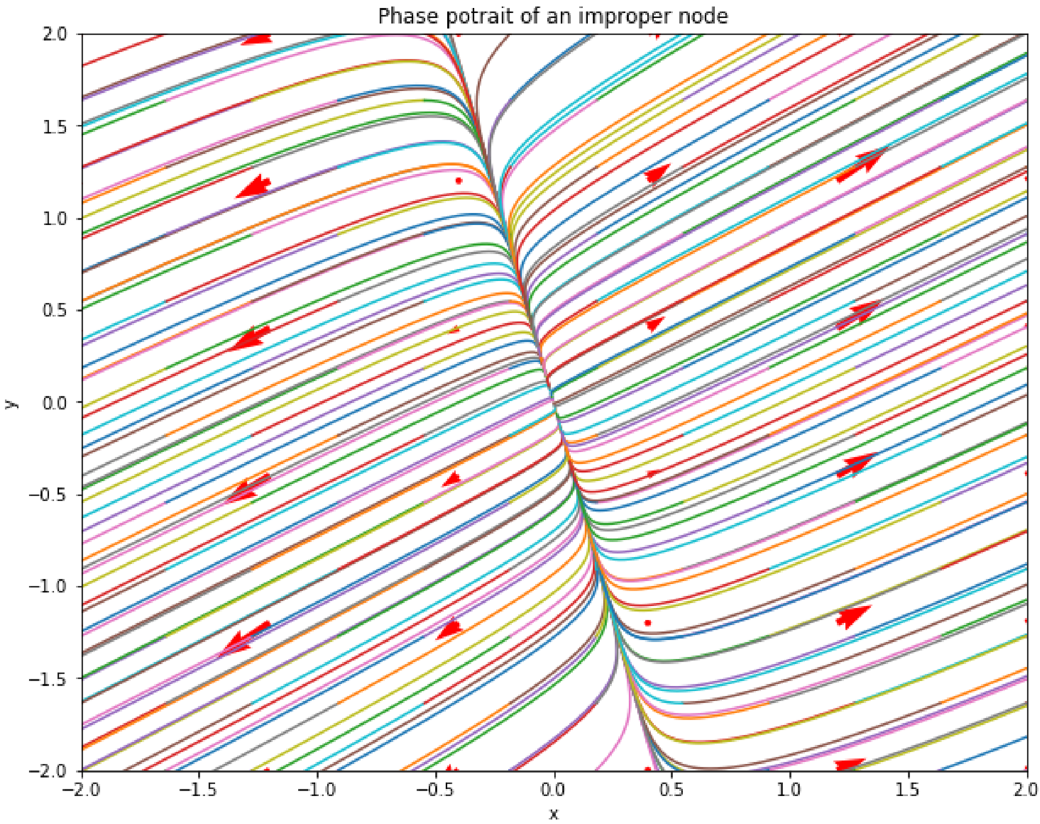

Figure 1: improper node

(2)

$$\frac{dx}{dt} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x$$

We calculate eigenvalues and the corresponding eigenvalues. The eigenvalues are ;

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

And corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$
$$\mathbf{v}_2 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$$

The general solution becomes

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos 2t - 2\sin 2t \\ \cos 2t \end{pmatrix} + c_2 \begin{pmatrix} \sin 2t + 2\cos 2t \\ \sin 2t \end{pmatrix}$$

The following figure 2 shows the phase portrait. We have complex eigenvalues and the real part is 0. This means that we have **center** and it is **stable.**

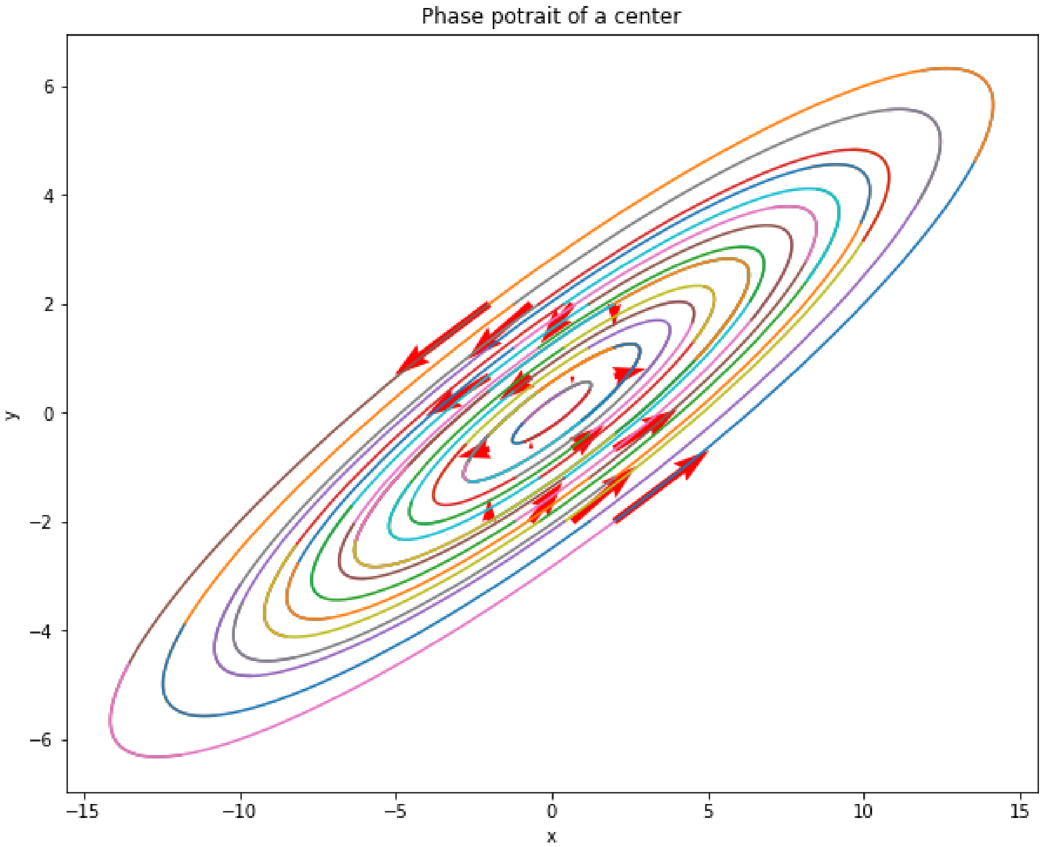

Figure 2: center

(3)

$$\frac{dx}{dt} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} x$$

We calculate eigenvalues and the corresponding eigenvalues. The eigenvalues are ;

$$\lambda_1 = -1 + i$$
$$\lambda_2 = -1 - i$$

And corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$$
$$\mathbf{v}_2 = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$$

The general solution becomes

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin t\right) + c_2 e^{-t}\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos t\right)$$

The following figure 3 shows the corresponding phase portrait. We have complex eigenvalues and the real part is negative. This means that we a **stable spiral** and it is **asymptotically stable.**

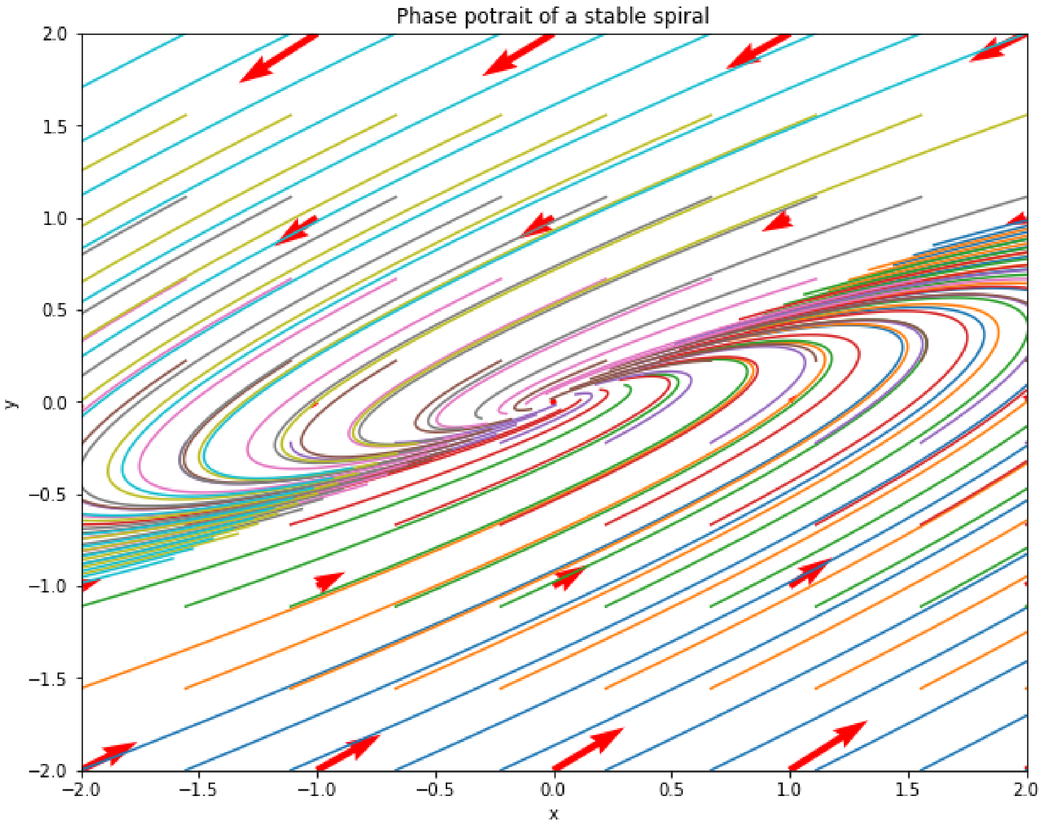

Figure 3: stable spiral

(4)

$$\frac{dx}{dt} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} x$$

We calculate eigenvalues and the corresponding eigenvalues. The eigenvalues are ;

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

And corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The general solution becomes

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$$

The following figure 4 shows the corresponding phase portrait. The eigenvalues are all real and of opposite sign, that is , we have a negative and positive eigenvalues. This implies that we have a **saddle point** and it is **unstable.**

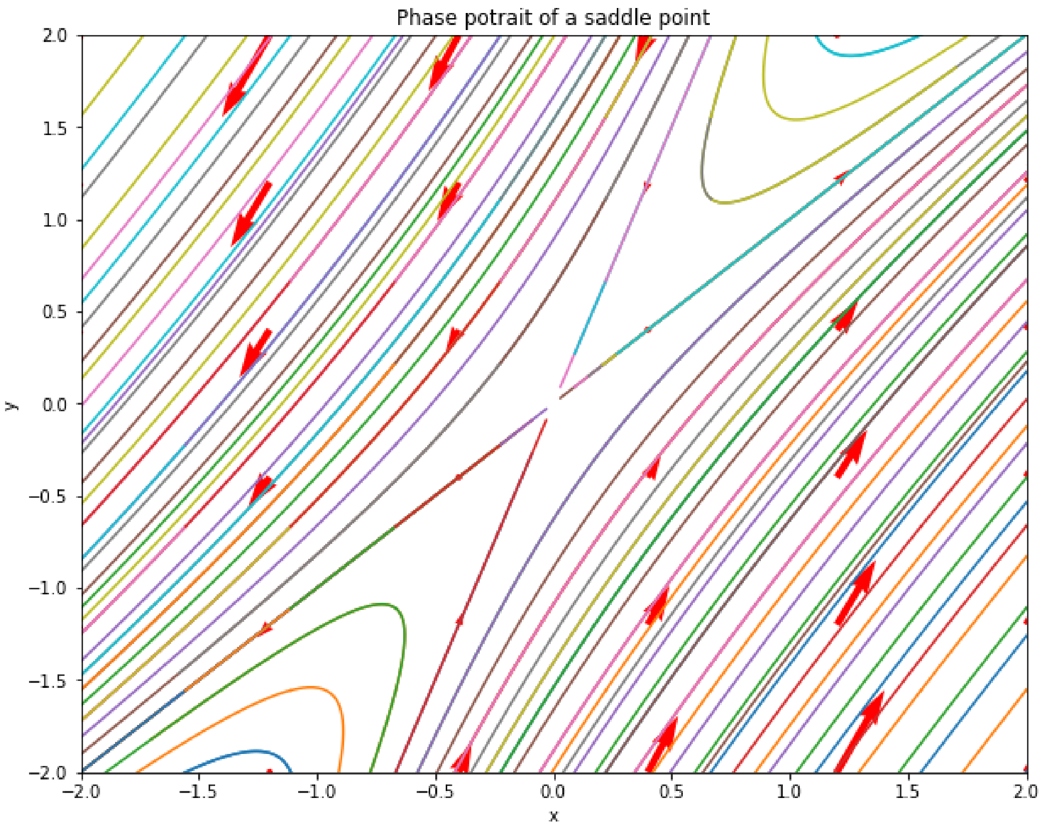

Figure 4: saddle point

Question 2

We are given the following dynamical system;

$$\begin{aligned} \frac{dx}{dt} &= y - x^2 \\ \frac{dy}{dt} &= x - 2 \end{aligned}$$

(1) Calculate the equilibrium point. We form to equations and solve to obtain equilibrium points

$$\begin{aligned} y &= x^2 \\ x &= 2 \end{aligned}$$

Solving we obtain;

$$x = 2 \text{ and; } y = 4$$

Our equilibrium points becomes $(2, 4)$.

(2) Linearize the dynamical system around the equilibrium points. We first find the Jacobian, J of our system . Let;

$$\begin{aligned} f_1 &= y - x^2 \\ f_2 &= x - 2 \\ J &= \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \end{aligned}$$

We apply Jacobian transformation to our system and obtain;

$$A = \begin{pmatrix} -2x & 1 \\ 1 & 0 \end{pmatrix}$$

But $x = 2$, we obtain A, as;

$$A = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix}$$

(3) We now obtain characteristic equation and solve to obtain the eigenvalues;

$$\begin{aligned} \det(A - \lambda \mathbb{I}) &= 0 \\ \lambda^2 + 4\lambda - 1 &= 0 \end{aligned}$$

we solve to obtain the eigenvalues as

$$\lambda_1 = -2 + \sqrt{5}$$
$$\lambda_2 = -2 - \sqrt{5}$$

We solve to obtain the corresponding eigenvectors

$$\text{for } \lambda_1 = -2+\sqrt{5}$$

$$\begin{pmatrix} -2-\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$0$$

$$(-2-\sqrt{5})x + y = 0$$

$$x + (2-\sqrt{5})y = 0$$

$$\text{solving ;}$$

$$x = (-2+\sqrt{5})y$$

$$\text{let } y = 1$$

$$x = -2+\sqrt{5}$$

Our eigenvector becomes;

$$\mathbf{v}_1 = \begin{pmatrix} -2+\sqrt{5} \\ 1 \end{pmatrix}$$

Similarly we find the other eigenvector;

$$\text{for } \lambda_2 = -2-\sqrt{5}$$

$$\begin{pmatrix} -2+\sqrt{5} & 1 \\ 1 & 2+\sqrt{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$0$$

$$(-2+\sqrt{5})x + y = 0$$

$$x + (2+\sqrt{5})y = 0$$

$$\text{solving ;}$$

$$x = (-2-\sqrt{5})y$$

$$\text{let } y = 1$$

$$x = -2-\sqrt{5}$$

Our eigenvector becomes;

$$\mathbf{v}_2 = \begin{pmatrix} -2-\sqrt{5} \\ 1 \end{pmatrix}$$

Having solved the eigenvalues and the corresponding eigenvectors, we notice that, the eigenvalues are real and of opposite sign. This implies that we have **saddle point** and it is **unstable**. We can now write the general solution as;

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} -2+\sqrt{5} \\ 1 \end{pmatrix} e^{(-2+\sqrt{5})t} + c_2 \begin{pmatrix} -2-\sqrt{5} \\ 1 \end{pmatrix} e^{(2-\sqrt{5})t}$$

The following figure 5 shows the corresponding phase portrait.

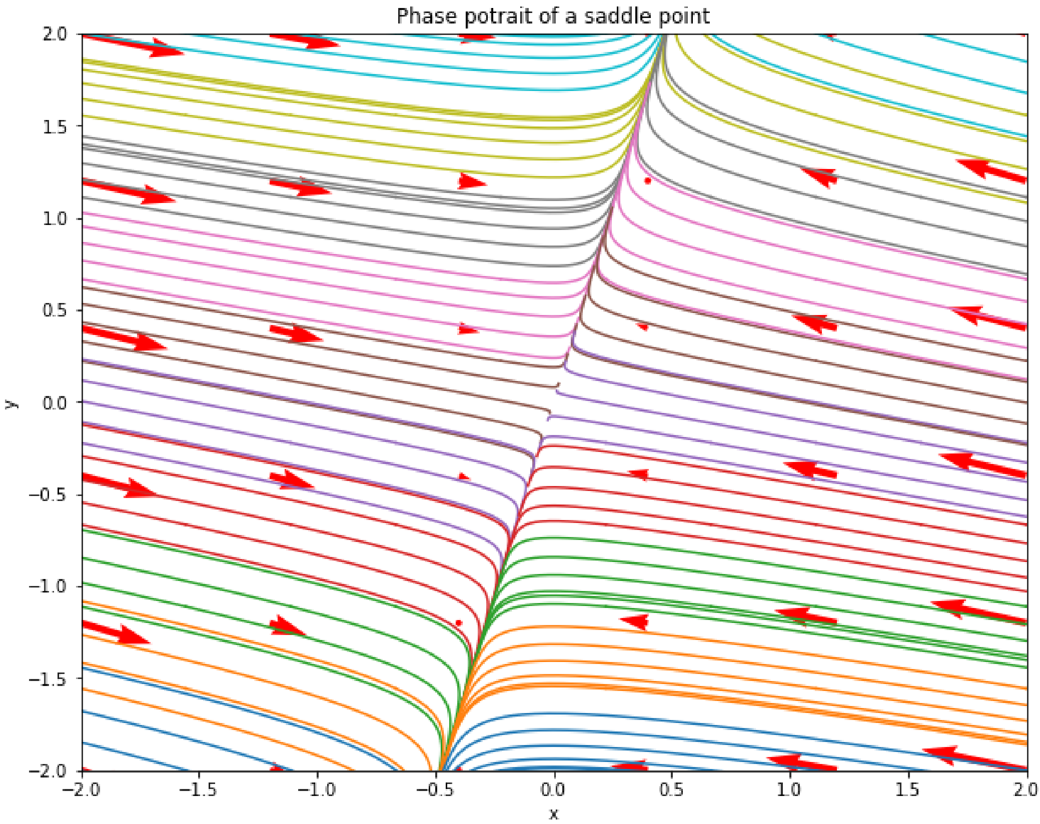

Figure 5: saddle point

References

[1] Prof.Yoshifumi Kimura ,*Lecture notes,* 2021.

[2] https://courses.lumenlearning.com/boundless-physics/chapter/damped-and-driven-oscillations/