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Exercise 1

1. We are given ;

$$(S) \quad \begin{cases} v^{'''}(t) = v(t)v^{'} - s(t)(v^{'''}(t))^2 \quad \text{for all} \quad t > 0 \\ v(0) = 1,\ v^{'}(0) = 0,\ v^{'''}(0) = 2 \\ \end{cases}$$

We are required to write (S) in the following form ;

$$\begin{aligned} Y^{'}(t) &= F(t, Y) \\ Y(0) &= Y_0 \end{aligned}$$

Now we let;

$$\begin{aligned} w(t) &= v^{'}(t) \\ w^{'}(t) &= v^{''}(t) \\ z(t) &= v^{''}(t) \\ z^{'}(t) &= v^{'''}(t) = v(t)w(t) - s(t)(z(t))^2 \end{aligned}$$

We now have the following systems of equations;

$$\begin{cases} v^{'}(t) = w(t) \\ w^{'}(t) = z(t) \\ z^{'}(t) = v(t)w(t) - s(t)(z(t))^2 \\ v(0) = 1,\ w(0) = 0,\ z(0) = 2 \end{cases}$$

Which can be written in matrix form;

$$\begin{pmatrix} v^{'}(t) \\ w^{'}(t) \\ z^{'}(t) \end{pmatrix} = \begin{pmatrix} w(t) \\ z(t) \\ v(t)w(t) - s(t)(z(t))^2 \end{pmatrix}$$

Where;

$$Y^{'}(t) = \begin{pmatrix} v^{'}(t) \\ w^{'}(t) \\ z^{'}(t) \end{pmatrix}$$

$$F(t, Y(t)) = \begin{pmatrix} w(t) \\ z(t) \\ v(t)w(t) - s(t)(z(t))^2 \end{pmatrix}$$

We need also to identify;

$$Y = \begin{pmatrix} v \\ w \\ z \end{pmatrix}$$

$$F(t, w, v, z) = \begin{pmatrix} w \\ z \\ vw - s(t)z^2 \end{pmatrix}$$

$$Y_0 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

2. We need to write explicitly mid-point scheme using uniform mesh;

$$\begin{cases} Y(0) = Y_0 \\ t_{n+\frac{1}{2}} = t_n + \frac{1}{2}\Delta t \\ Y_{n+\frac{1}{2}} = Y_n + \frac{1}{2}\Delta t f(t_n, Y_n) \\ Y_{n+1} = Y_n + \Delta t f(t_{n+\frac{1}{2}}, Y_{n+\frac{1}{2}}) \end{cases}$$

Which we can write in matrix form as;

$$\begin{cases} Y(0) = Y_0 \\ t_{n+\frac{1}{2}} = t_n + \frac{1}{2}\Delta t \\ \begin{pmatrix} v_{n+\frac{1}{2}} \\ w_{n+\frac{1}{2}} \\ z_{n+\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} v_n \\ w_n \\ z_n \end{pmatrix} + \frac{\Delta t}{2} \begin{pmatrix} w_n \\ z_n \\ v_n w_n - s(t_n) z_n^2 \end{pmatrix} \\ \begin{pmatrix} v_{n+1} \\ w_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} v_n \\ w_n \\ z_n \end{pmatrix} + \Delta t \begin{pmatrix} w_{n+\frac{1}{2}} \\ z_{n+\frac{1}{2}} \\ v_{n+\frac{1}{2}} w_{n+\frac{1}{2}} + \frac{1}{2} - s(t_{n+\frac{1}{2}}) z_{n+\frac{1}{2}}^2 \end{pmatrix} \end{cases}$$

3. Taking t_0=0, Δ=0.5 and s(t)=$1 - t^2$. We calculate $v_1 \approx v(t_1)$. When $n = 0$

$$\begin{aligned} t_{\frac{1}{2}} &= 0.25 \\ \begin{pmatrix} v_{\frac{1}{2}} \\ w_{\frac{1}{2}} \\ z_{\frac{1}{2}} \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 0.25 \begin{pmatrix} 0 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ 1 \end{pmatrix} \\ \begin{pmatrix} v_1 \\ w_1 \\ z_1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 0.5 \begin{pmatrix} 0.5 \\ 1 \\ 0.5 - (1 + (\frac{1}{4})^2) \end{pmatrix} = \begin{pmatrix} 1.25 \\ 0.5 \\ 1.72 \end{pmatrix} \\ v_1 \approx v(t_1) &= 1.25 \end{aligned}$$

We need also to calculate $v_2 \approx v(t_2)$. When $n = 1$

$$\begin{aligned} t_{\frac{3}{2}} &= 0.75 \\ \begin{pmatrix} v_{\frac{3}{2}} \\ w_{\frac{3}{2}} \\ z_{\frac{3}{2}} \end{pmatrix} &= \begin{pmatrix} 1.25 \\ 0.50 \\ 1.72 \end{pmatrix} + 0.25 \begin{pmatrix} 0 \\ 1.72 \\ 0.625 - (1 + (0.5)^2)1.72 \end{pmatrix} = \begin{pmatrix} 1.38 \\ 0.93 \\ 0.96 \end{pmatrix} \\ \begin{pmatrix} v_2 \\ w_3 \\ z_3 \end{pmatrix} &= \begin{pmatrix} 1.25 \\ 0.5 \\ 1.72 \end{pmatrix} + 0.5 \begin{pmatrix} 0.93 \\ 0.96 \\ 1.38 - (1 + (\frac{1}{4})^2)0.96^2 \end{pmatrix} = \begin{pmatrix} 1.72 \\ 0.98 \\ 1.69 \end{pmatrix} \\ v_2 \approx v(t_2) &= 1.72 \end{aligned}$$

Exercise 2

The model of atmosphere is given by;

$$C\frac{dT}{dt} = \frac{(1-\alpha)}{4}S_0 - \epsilon\sigma T^4$$

1) Find the equilibrium temperature T_{eq} At equilibrium;

$$\frac{dT}{dt} = 0$$

So we have the following expression;

$$\frac{(1-\alpha)}{4}S_0 = \epsilon\sigma T^4$$

We make T the subject of formula and obtain as T_{eq}

$$T_{eq} = \sqrt[4]{\frac{(1-\alpha)S_0}{4\epsilon\sigma}}$$

We now plug in the given constants to get the value of T_{eq}

$$\begin{aligned} T_{eq} &= \sqrt[4]{\frac{0.7 \times 1367}{4 \times 0.6 \times 5.67 \times 10^{-8}}} \\ T_{eq} &= 289.5796\,K \end{aligned}$$

2) We need to prove that;

$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_0}{4} - \epsilon\sigma(\tilde{T} + T_{eq})^4$$

We are given;

$$T(t) = T_{eq} + \tilde{T}(t)$$

Taking derivatives w.r.t t;

$$\frac{dT(t)}{dt} = \frac{dT_{eq}}{dt} + \frac{d\tilde{T}(t)}{dt}$$

But

$$\frac{dT_{eq}}{dt}$$

is constant so we have;

$$\frac{dT(t)}{dt} = \frac{d\tilde{T}(t)}{dt}$$

Replacing this expression to our equation

$$C\frac{dT}{dt} = \frac{(1-\alpha)}{4}S_0 - \epsilon\sigma T^4$$

we now obtain

$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_0}{4} - \epsilon\sigma(\tilde{T} + T_{eq})^4$$

as required.

3) We assume that

$$\left(1+\frac{\tilde{T}}{T_{eq}}\right)^4 = 1 + 4\frac{\tilde{T}}{T_{eq}}$$

We are required to prove that $\tilde{T}$ satisfies;

$$\frac{d\tilde{T}}{dt} = -\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right)\tilde{T}$$

We have

$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_0}{4} - \epsilon\sigma T_{eq}^4\left(1+\frac{\tilde{T}}{T_{eq}}\right)^4$$

Which we can write as;

$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_0}{4} - \epsilon\sigma T_{eq}^4\left(1+4\frac{\tilde{T}}{T_{eq}}\right)$$

We open up this expression and write it as;

$$C\frac{d\tilde{T}}{dt} = \frac{(1-\alpha)S_0}{4} - \epsilon\sigma T_{eq}^4 - 4\epsilon\sigma T_{eq}^3\,\tilde{T}$$

But

$$\frac{(1-\alpha)S_0}{4} - \epsilon\sigma T_{eq}^4 = 0$$

And therefore we obtain;

$$\frac{d\tilde{T}}{dt} = -\left(\frac{4\epsilon\sigma T_{eq}^3}{C}\right)\tilde{T}$$

as required.

4) We are given $T(0) = 10$ we need to find the exact solution.We integrate on both sides;

$$\begin{aligned}\int\frac{d\tilde{T}}{dt} &= -\frac{4\epsilon\sigma T_{eq}^3}{C}\int dt\\ \tilde{T} &= A\exp\left(\frac{-4\epsilon\sigma T_{eq}^3}{C}\right)t\end{aligned}$$

We plug in the given constants and to obtain;

$$\tilde{T} = A\,e^{-0.038876\,t}$$

5) a) We are given;

$$(P) \qquad \begin{cases} y_0 = y(t_0) \\ K_1 = f(t_n, y_n) \\ t_{n+\frac{3}{4}} = t_n + \frac{3}{4}\Delta t \\ y_{n+\frac{3}{4}} = y_n + \frac{3}{4}\Delta t K_1 \\ K_2 = f(t_{n+\frac{3}{4}}, y_{n+\frac{3}{4}}) \\ y_{n+1} = y_n + \frac{1}{3}\Delta t(K_1 + 2K_2) \end{cases}$$

We are required to prove that (P) given is a one step method. In this case we need to show that;

$$y_{n+1} = y_n + \Delta t \phi(t_n, y_n, \Delta t)$$

where ϕ is a continuous function.

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{3}\Delta t(K_1 + 2K_2) \\ y_{n+1} &= y_n + \frac{1}{3}\Delta t \left(f(t_n, y_n), 2\, f(t_{n+\frac{3}{4}}, y_{n+\frac{3}{4}}) \right) \\ &= y_n + \frac{1}{3}\Delta t \left(f(t_n, y_n) + 2\, f(t_n + \frac{3}{4}\Delta t), y_n + \frac{3}{4}\Delta t\, f(t_n, y_n)) \right) \end{aligned}$$

and hence we have

$$y_{n+1} = y_n + \Delta t \phi(t_n, y_n, \Delta t)$$

where

$$\phi(t_n, y_n, \Delta t) = \frac{1}{3} \left(f(t_n, y_n) + 2\, f(t_n + \frac{3}{4}\Delta t), y_n + \frac{3}{4}\Delta t\, f(t_n, y_n)) \right)$$

is a continuous function.

b) The figure 1 below show a graph of exact solution versus the approximate solution for the energy conservation model given by $T(t) = T_{eq} + \tilde{T}(t)$.

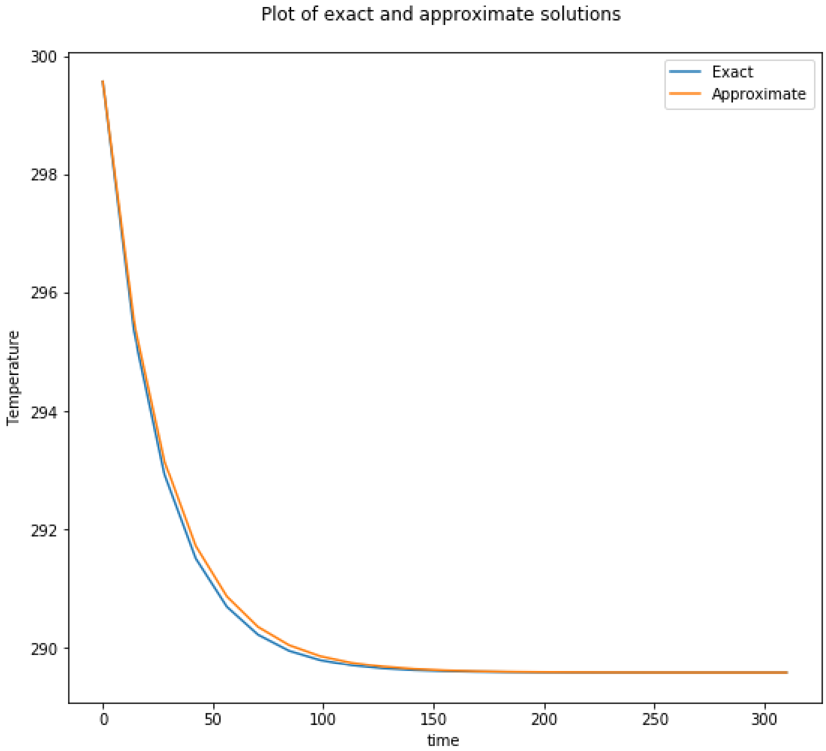

Figure 1: graph of exact vs approximate solutions

From our graph it is clear that the approximate method given by our scheme almost matches accurately with the actual values implying that we can use this method to do numerical approximations of exact solutions.