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Exercise 1

Let $a, b \in \mathbb{R}, a < b$. Consider the smooth function u on $[a, b]$. Given $\bar{x}$ and we want to approximate $u'(\bar{x})$ by $D_2u(\bar{x})$ of the form

$$D_2u(\bar{x}) = \frac{\alpha u(\bar{x}) + \beta u(\bar{x}-h) + \gamma u(\bar{x}-2h)}{h}.$$

We are required to find the coefficients α, β and γ to give the best possible accuracy. We are going to apply Taylor's series expansion.

$$\begin{aligned}
\alpha u(\bar{x}) &= \alpha u(\bar{x}) \\
\beta u(\bar{x}-h) &= \beta u(\bar{x}) - h\beta u'(\bar{x}) + \beta\frac{h^2u''(\bar{x})}{2} - \beta\frac{h^3u^3(\bar{x})}{3!} + \beta\frac{h^4u^4(\bar{x})}{4!} + O\left(h^5\right) \\
\gamma u(\bar{x}-2h) &= \gamma u(\bar{x}) - 2h\gamma u'(\bar{x}) + 2h^2\gamma u''(\bar{x}) - \frac{8}{6}h^3\gamma u^3(\bar{x}) + \frac{2}{3}h^4\gamma u^4(\bar{x}) + O\left(h^5\right) \\
D_2u(\bar{x}) &= \frac{(\alpha+\beta+\gamma)}{h}u(\bar{x}) - (\beta+2\gamma)u'(\bar{x}) + \frac{(\beta+4\gamma)}{2}hu''(\bar{x}) + \frac{h^2}{6}(\beta+8\gamma)u'''(\bar{x}) + O\left(h^4\right)
\end{aligned}$$

We now form the following system of equations;

$$\begin{aligned}
\alpha+\beta+\gamma &= 0 \\
\beta+2\gamma &= -1 \\
\beta+4\gamma &= 0
\end{aligned}$$

solving we obtain;

$$\begin{aligned}
\alpha &= \frac{3}{2} \\
\beta &= -2 \\
\gamma &= \frac{1}{2}
\end{aligned}$$

Exercise 2

Consider heat conduction in a rod with varying heat conduction properties, where the parameter $K(x)$ varies with x and is always positive. The steady-state heat-conduction problem is then

$$-\left(Ku'\right)'(x) = f(x) \quad x \in (a,b)$$

with the following boundary condition

$$u(a) = \alpha; \quad u(b) = \beta$$

Given a finite family of points (x_i) for $i = 0, \cdots N$ defined by $x_i = a + ih$ with $h = \frac{b-a}{N}$ and also the mid-points $x_{i+1/2} = x_i + h/2$ for all $0 \le i \le N-1$. Let $v(x) = K(x)u'(x)$.

1. We need to prove that $Du\left(x_{i+1/2}\right) = \frac{u(x_{i+1})-u(x_i)}{h}$ is a central approximation of $u'\left(x_{i+1/2}\right)$.
   We are given;

$$\begin{aligned} u(x_{i+1}) &= u(x_{i+\frac{1}{2}} + \frac{h}{2}) \\ u(x_i) &= u(x_{i+\frac{1}{2}} - \frac{h}{2}) \end{aligned}$$

   We use Taylor series expansion ;

$$\begin{aligned} u(x_{i+\frac{1}{2}} + \frac{h}{2}) &= u(x_{i+\frac{1}{2}}) + \frac{h}{2}u'(x_{i+\frac{1}{2}}) + \frac{(\frac{h}{2})^2 u''(x_{i+\frac{1}{2}})}{4} + \frac{(\frac{h}{2})^3 u'''(x_{i+\frac{1}{2}})}{6} + O\left(h^4\right) \\ u(x_{i+\frac{1}{2}} - \frac{h}{2}) &= u(x_{i+\frac{1}{2}}) - \frac{h}{2}u'(x_{i+\frac{1}{2}}) + \frac{(\frac{h}{2})^2 u''(x_{i+\frac{1}{2}})}{4} - \frac{(\frac{h}{2})^3 u'''(x_{i+\frac{1}{2}})}{6} + O\left(h^4\right) \\ u(x_{i+\frac{1}{2}} + \frac{h}{2}) - u(x_{i+\frac{1}{2}} - \frac{h}{2}) &= hu'(x_{i+\frac{1}{2}}) + \frac{1}{24}h^3 u'''(x_{i+\frac{1}{2}}) + O\left(h^4\right) \\ \frac{u(x_{i+\frac{1}{2}} + \frac{h}{2}) - u(x_{i+\frac{1}{2}} - \frac{h}{2})}{h} &= u'(x_{i+\frac{1}{2}}) + \frac{1}{24}h^2 u'''(x_{i+\frac{1}{2}}) + O\left(h^3\right) \end{aligned}$$

   Clearly we can deduce that;

$$Du\left(x_{i+1/2}\right) = u'(x_{i+\frac{1}{2}}) + \epsilon_{nj}$$

   Which implies that;

$$Du\left(x_{i+1/2}\right) \approx u'(x_{i+\frac{1}{2}})$$

   as required

2. We are required to deduce the approximation of $v(x_{i+\frac{1}{2}})$
   Given;

$$\begin{aligned} Du\left(x_{i+1/2}\right) &= \frac{u\left(x_{i+1}\right) - u\left(x_i\right)}{h} \\ v(x) &= K(x)u'(x) \end{aligned}$$

   We have;

$$\begin{aligned} v\left(x_{i+\frac{1}{2}}\right) &= k\left(x_{i+\frac{1}{2}}\right) u'\left(x_{i+\frac{1}{2}}\right) \\ v\left(x_{i+\frac{1}{2}}\right) &= k\left(x_{i+\frac{1}{2}}\right) \frac{u\left(x_{i+1}\right) - u\left(x_i\right)}{h} \end{aligned}$$

3. We need to prove that; $Dv\left(x_i\right)=\frac{v\left(x_{i+1/2}\right)-v\left(x_{i-1/2}\right)}{h}$ is a central approximation of $v'\left(x_i\right)$.
We are given;

$$\begin{aligned}
Du\left(x_{i+1/2}\right) &= \frac{u\left(x_{i+1}\right)-u\left(x_i\right)}{h} \\
v\left(x_{i+\frac{1}{2}}\right) &= v\left(x_i+\frac{h}{2}\right) \\
v\left(x_{i-\frac{1}{2}}\right) &= v\left(x_i-\frac{h}{2}\right)
\end{aligned}$$

We are going to use Taylor's expansion;

$$\begin{aligned}
v\left(x_i+\frac{h}{2}\right) &= v\left(x_i\right)+\frac{h}{2}v'\left(x_i\right)+\frac{h^2}{8}v''\left(x_i\right)+\frac{h^3}{48}v'''\left(x_i\right)+(O)\left(h^4\right) \\
v\left(x_i-\frac{h}{2}\right) &= v\left(x_i\right)-\frac{h}{2}v'\left(x_i\right)+\frac{h^2}{8}v''\left(x_i\right)-\frac{h^3}{48}v'''\left(x_i\right)+(O)\left(h^4\right) \\
v\left(x_i+\frac{h}{2}\right)-v\left(x_i-\frac{h}{2}\right) &= hv'\left(x_i\right)+\frac{h^3}{24}v'''\left(x_i\right)+(O)\left(h^4\right) \\
Dv\left(x_i\right) &= \frac{v\left(x_{i+1/2}\right)-v\left(x_{i-1/2}\right)}{h}=v'\left(x_i\right)+\frac{h^2}{24}v'''\left(x_i\right)+(O)\left(h^3\right) \\
\Longrightarrow Dv\left(x_i\right) &\approx v'\left(x_i\right) \\
\text{as required} &
\end{aligned}$$

4. We need to deduce that;

$$D^2u\left(x_i\right)=\frac{1}{h^2}\left(K\left(x_{i-1/2}\right)u\left(x_{i-1}\right)-\left(K\left(x_{i-1/2}\right)+K\left(x_{i+1/2}\right)\right)u\left(x_i\right)+K\left(x_{i+1/2}\right)u\left(x_{i+1}\right)\right)$$

is the approximation of $\left(Ku'\right)'\left(x_i\right)$.

We are given;

$$\begin{aligned}
v\left(x_{i+\frac{1}{2}}\right) &= k\left(x_{i+\frac{1}{2}}\right)\frac{u\left(x_{i+1}\right)-u\left(x_i\right)}{h} \\
Dv\left(x_i\right) &= \frac{v\left(x_{i+1/2}\right)-v\left(x_{i-1/2}\right)}{h}\approx v'\left(x_i\right)
\end{aligned}$$

Then ;

$$Dv\left(x_i\right) = k\left(x_{i+\frac{1}{2}}\right)\frac{u\left(x_{i+1}\right)-u\left(x_i\right)}{h^2}-k\left(x_{i-\frac{1}{2}}\right)\frac{u\left(x_i\right)-u\left(x_{i-1}\right)}{h^2}$$

Which simplifies to;

$$\frac{1}{h^2}\left(k\left(x_{i+\frac{1}{2}}\right)u\left(x_{i+1}\right)-k\left(x_{i+\frac{1}{2}}\right)u\left(x_i\right)-k\left(x_{i-\frac{1}{2}}\right)u\left(x_i\right)+k\left(x_{i-\frac{1}{2}}\right)u\left(x_{i-1}\right)\right)$$

And therefore have have shown;

$$D^2u\left(x_i\right)=\frac{1}{h^2}\left(K\left(x_{i-1/2}\right)u\left(x_{i-1}\right)-\left(K\left(x_{i-1/2}\right)+K\left(x_{i+1/2}\right)\right)u\left(x_i\right)+K\left(x_{i+1/2}\right)u\left(x_{i+1}\right)\right)$$
$$\approx\left(Ku'\right)'\left(x_i\right)$$

Exercise 3

We denote by $K_{i+1/2}$ the exact value of $K\left(x_{i+1/2}\right)$ and by U_i the approximate value of $u\left(x_i\right)$ for all $i = 0, \cdots N$. We consider the following scheme:

$$(PH) \quad \begin{cases} \frac{1}{h^2}\left(-K_{i-1/2}U_{i-1} + \left(K_{i-1/2} + K_{i+1/2}\right)U_i - K_{i+1/2}U_{i+1}\right) = f\left(x_i\right) \text{ for all } 1 \leq i \leq N-1 \\ U_0 = \alpha; \quad U_N = \beta \end{cases}$$

1. Write (PH) in the matrix form $AU = F$ where A, U and F have to be determined.

   We are going to replace for values i from 1 to N-1 obtain the following system of equations.

   For $i = 1$,

   $$(K_{\frac{1}{2}} + K_{\frac{3}{2}})U_1 - K_{\frac{3}{2}}U_2 \quad = \quad h^2 f(x_1) + \alpha K_{\frac{1}{2}}$$

   For $i = 2$,

   $$-K_{\frac{3}{2}}U_1 + (K_{\frac{3}{2}} + K_{\frac{5}{2}})U_2 - K_{\frac{5}{2}}U_3 \quad = \quad h^2 f(x_2)$$

   For $i = 3$,

   $$-K_{\frac{5}{2}}U_2 + (K_{\frac{5}{2}} + K_{\frac{7}{2}})U_7 - K_{\frac{7}{2}}U_4 \quad = \quad h^2 f(x_3)$$

   For $i = 4$,

   $$-K_{\frac{7}{2}}U_1 + (K_{\frac{7}{2}} + K_{\frac{9}{2}})U_2 - K_{\frac{9}{2}}U_5 \quad = \quad h^2 f(x_4)$$

   $\vdots$

   For $i = N-1$,

   $$-K_{N-\frac{3}{2}}U_{N-2} + (K_{N-\frac{3}{2}} + K_{N+\frac{1}{2}})U_{N-1} \quad = \quad \beta K_{N-\frac{1}{2}} + h^2 f(x_{N-1})$$

   We can write this system of equations in form of matrix, $AU = F$

$$\begin{pmatrix} K_{\frac{1}{2}} + K_{\frac{3}{2}} & -K_{\frac{3}{2}} & 0 & 0 & \cdots & \cdots & 0 \\ -K_{\frac{3}{2}} & K_{\frac{3}{2}} + K_{\frac{5}{2}} & -K_{\frac{5}{2}} & 0 & \cdots & \cdots & 0 \\ 0 & -K_{\frac{5}{2}} & K_{\frac{5}{2}} + K_{\frac{7}{2}} & -K_{\frac{7}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & & & -K_{\left(N-\frac{3}{2}\right)} & K_{\left(N-\frac{3}{2}\right)} + K_{\left(N-\frac{1}{2}\right)} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_{N-2} \\ U_{N-1} \end{pmatrix} = \begin{pmatrix} h^2 f(x_1) + \alpha K_{\frac{1}{2}} \\ h^2 f(x_2) \\ h^2 f(x_3) \\ \vdots \\ h^2 f(x_{N-2}) \\ \beta K_{N-\frac{1}{2}} + h^2 f(x_{N-1}) \end{pmatrix}$$

2. We are required to prove that matrix A is symmetric.
   It clear that
$$A = A^T$$
   This implies matrix A is symmetric.

   We want prove that matrix A is positive define;
   Let v;
$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{N-2} \\ v_{N-1} \end{pmatrix}$$
$$A = \begin{pmatrix} K_{\frac{1}{2}} + K_{\frac{3}{2}} & -K_{\frac{3}{2}} & 0 & 0 & \cdots & \cdots & 0 \\ -K_{\frac{3}{2}} & K_{\frac{3}{2}} + K_{\frac{5}{2}} & -K_{\frac{5}{2}} & 0 & \cdots & \cdots & 0 \\ 0 & -K_{\frac{5}{2}} & K_{\frac{5}{2}} + K_{\frac{7}{2}} & -K_{\frac{7}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & & & -K_{\left(N-\frac{3}{2}\right)} & K_{\left(N-\frac{3}{2}\right)} + K_{\left(N-\frac{1}{2}\right)} \end{pmatrix}$$
   Then
$$Av = \begin{pmatrix} (K_{\frac{1}{2}} + K_{\frac{3}{2}})v_1 - K_{\frac{3}{2}}v_2 \\ -K_{\frac{3}{2}}v_1 + (K_{\frac{3}{2}} + K_{\frac{5}{2}})v_2 - K_{\frac{5}{2}}v_3 \\ -K_{\frac{5}{2}}v_2 + (K_{\frac{5}{2}} + K_{\frac{7}{2}})v_3 - K_{\frac{7}{2}}v_4 \\ \vdots \\ \vdots \\ -K_{\left(N-\frac{3}{2}\right)}v_{N-2} + (K_{\left(N-\frac{3}{2}\right)} + K_{\left(N-\frac{1}{2}\right)})v_{N-1} \end{pmatrix}$$
   We want to show that the scalar product $(Av, v) \geq 0$ for all $v \in \mathbb{R}^N$
$$\begin{aligned} (Av, v) &= (K_{\frac{1}{2}} + K_{\frac{3}{2}})v_1^2 - K_{\frac{3}{2}})v_1v_2 + (K_{\frac{3}{2}} + K_{\frac{5}{2}})v_2^2 - K_{\frac{5}{2}}v_2v_3 - K_{\frac{5}{2}}v_2v_3 + (K_{\frac{5}{2}} + K_{\frac{7}{2}})v_3^2 \\ &\quad -K_{\frac{7}{2}}v_3v_4 + \cdots - K_{\left(N-\frac{3}{2}\right)}v_{N-1}v_{N-2} + (K_{\left(N-\frac{3}{2}\right)} + K_{\left(N-\frac{1}{2}\right)})v_{N-1}^2 \\ &= K_{\frac{1}{2}}v_1^2 + K_{\frac{1}{2}}(v_1 - v_2)^2 + K_{\frac{5}{2}}(v_2 - v_3)^2 + K_{\frac{7}{2}}(v_3 - v_4)^2 + \cdots + K_{\frac{N}{\frac{3}{2}}}(v_{N-2} - v_{N-1})^2 \\ &\quad +K_{N-\frac{1}{2}}v_{N-1}^2 \geq 0 \end{aligned}$$
   We conclude that matrix A is positive define as required.

Application

We are given;

$$-\left(Ku'\right)'(x) = f(x) \quad x \in (a, b)$$

with the following boundary condition

$$u(a) = \alpha; \quad u(b) = \beta$$

Given also the following;

$$a = 0, b = 1, K(x) = x^2 \text{ and } u(x) = x(1 + x)\ .$$

1. We are required to find the values of α, β and $f(x)$ such that $u(x) = x(1+x)$ is the exact solution of (P).

$$\begin{aligned} u(0) &= 0 \\ u(b) &= b(1+b) \\ \beta &= 2 \end{aligned}$$

   Taking differentials;

$$\begin{aligned} K'(x) = x^2, &\quad K''(x) = 2 \\ u'(x) = 1 + 2x, &\quad u''(x) = 2 \end{aligned}$$

$$\begin{aligned} f(x) \quad = \quad & -2x - 4x^2 - 2x^2 \\ & = -6x^2 - 2x \\ & = -2x(3x+1) \end{aligned}$$

2. Write a program to solve (PH) (that is to compute the approximated solution).

3. Plot in the same graph the approximated solution and the exact solution.

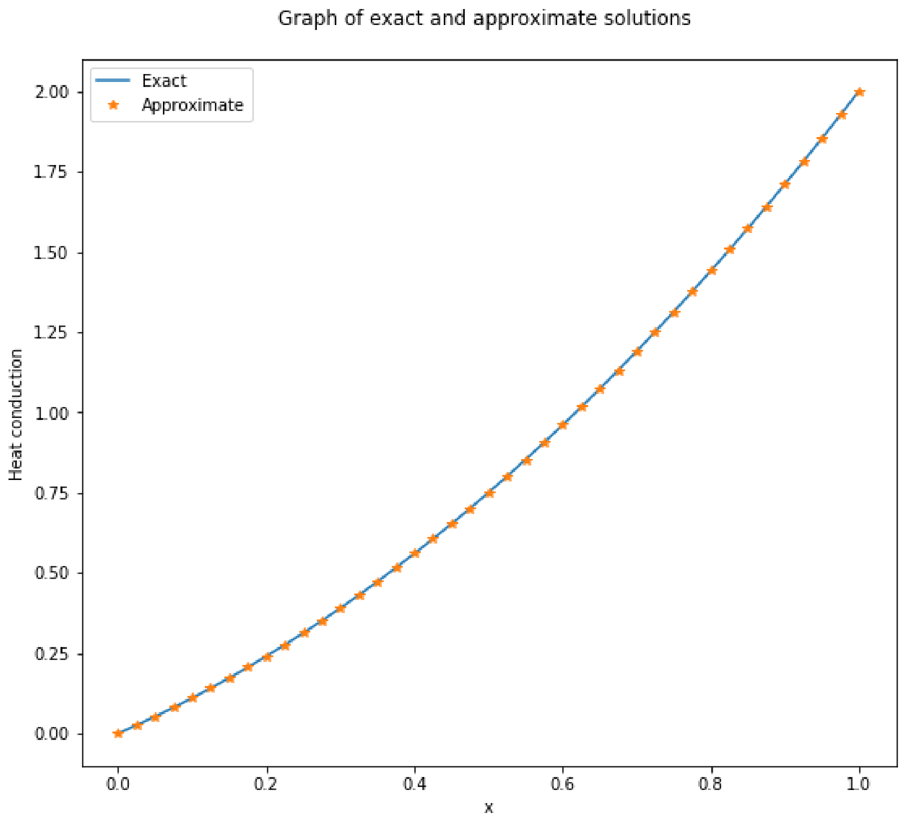

Figure 1: graph of exact vs approximate solutions