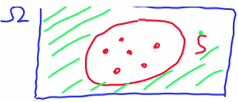
**Unit 1: Probability models and axioms**

**Lecture 2: Mathematical background: Sets; sequences, limits, and series; (un)countable sets**

**Sets**

* **Set =** collection of distinct elements:
* **Finite set 🡺** {a, b, c, d}
* **Infinite se** 🡺 {ℜ} (where ℜ = all real numbers)
* X is an element of set S = 
* X is NOT an element of set S =
* Can specify a set by saying we have a set of all real numbers whose Cos > ½ = {x ε ℜ : cos(x) > 1/2}
* Restricting a set to a particular property
* **Universal set** {Ω} = fix a collection of all possible objects we might want to consider
* Can then consider smaller sets that lie inside the universal set, or say all elements that belong to Ω but not to subset S (green) = S(c) = **complement** of S



* Can formally say this as x is part of the complement of S if it’s in the universal set but not in S



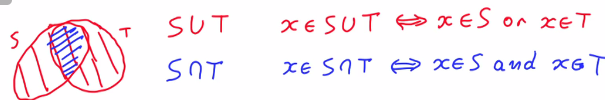
* Complement of the complement is the set itself 🡪 (S(c))(c) = S
* **Empty set φ =** set of no elements (complement of universal set)
* Suppose we have 2 sets where S > T 🡪 S is a subset of set T



* This means if x is an element of S, it is also an element of T
* There’s also a possibility that subset S = the larger set T



* **Union** of 2 sets = all elements that belong to 1 set, the other, or both
* Some element x belongs to (S U T) if + only if x belongs to 1 of the 2 sets
* **Intersection** = collection of elements that belong to BOTH sets
* Some element x belongs to (S intersect T) if + only if x belongs both sets



* Can define unions + intersections of infinitely many sets
* S(n) = infinite collection of sets



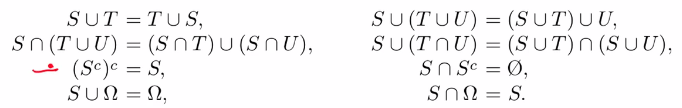
* Can still define their union to be the set of all elements that belong to 1 of those sets S(n) that we started w/
* X belongs to that union if + only if X belongs to some set we started w/



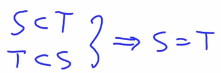
* X belongs to the intersection of all these sets if + only if X belongs to S(n) *for ALL n.*



* **Set operations** satisfy certain basic properties.

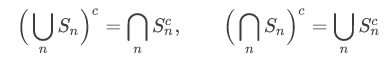


* The union of a set w/ another set = union of the 2 sets in different order.
* If you take the union of 3 sets, can do it by forming:
* the union of 2 sets + then the union w/ 3, or in any alternative order.
* The same would be true for intersections 🡪 intersection of 3 sets is the same no matter how you put parentheses around different sets.
* If you take a union of a set w/ a universal set, you cannot get anything bigger than the universal set, so you just get the universal set.
* On the other hand, if you take the intersection of a set w/ universal set, you get just the set itself
* Perhaps the more complicated properties out of this are a distributive property of intersections + unions.  
  
* The way that you verify them is by proceeding logically.
* If X is an element of (T U U), X must be an element of S + must also be an element of either T or U.
* Therefore, it's going to belong either to (S intersect T) or to (S intersect U)
* Fact: If S is a subset of T, and T is a subset of S, this implies that the 2 sets are equal

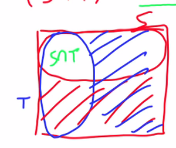


* You can use a similar argument to convince yourselves about the 2nd equality

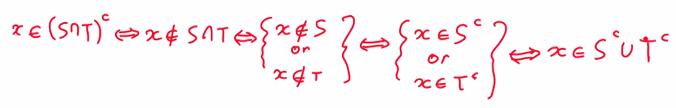
**De Morgan's laws**



* **De Morgan's laws** are some very useful relations between sets + their complements
* If we take the intersection of 2 sets + then take the complement of this intersection, we obtain the union of the complements of the 2 sets.



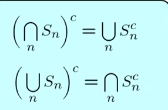
* If X belongs to the complement of (S intersect T), this is the same as saying X does NOT belong to (S intersect T)
* Since it’s not in the intersection, it’s the same as saying X does not belong to S nor to T.
* This is the same as saying X belongs to S(c) or x belongs to T(c).
* This is equivalent to saying that X belongs to the union of the 2 complements
* This establishes this 1st De Morgan's law.



* There's another De Morgan's law, obtained from this 1st one by syntactic substitution.
* Wherever we see an S, replace it w/ S(c) + vice versa + then do the same for T
* Then take the complement of both sides (complement of complement = set itself)

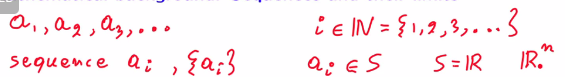


* This 2nd De Morgan's law tells us the intersection of the complements = the complement of a union
* It turns out De Morgan's laws are valid when we take unions or intersections of more than 2 sets as well in a more general form.



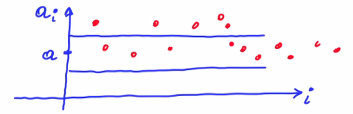
* If we have a collection of sets, Sn, perhaps an infinite collection, + we take the intersection of those sets + then the complement, that result is the union of the complements.
* If we have the union of certain sets + we take the complement, we obtain the intersection of the complements

**Sequences and their limits**

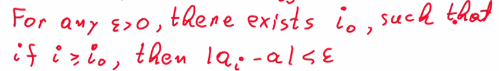
* **Sequence = some collection of elements from some set that is indexed by natural numbers**
* Sequence **a(i)** or sequence **{a(i)}** 🡪 index i trends over the natural numbers (set of real positive integers) + each a(i) is an element of some set
* In many cases the set is the real line lR in which we’re dealing w/ a sequence of real numbers
* Set could also be over the Euclidean space (n-dimensional) 🡪 this is a sequence of vectors
* 
* Formally, a sequence = a function that associates an element of set s to any natural number
* 
* Evaluate f() at some index i 🡪 f(i) 🡪 we get the i-th element of the sequence
* Typically care if sequence converges to some number a 🡺 a(i) 🡪 a as i converges to infinity (limits)

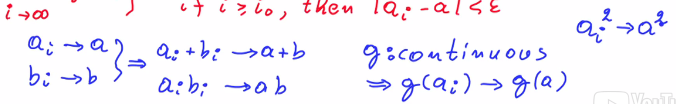
 

* Plot sequence as function of i
* For a sequence to converge to a certain number a, we need the elements of the sequence to get closer to a as i increases



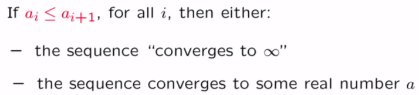
* For some positive number Epsilon, we define a band around a and there exists a time, i(0), after which a(i) will get inside the band + stay there (is w/in Epsilon of a)
* Convergence of a sequence to a certain number a:



* Can derive this in other things 🡪 say a(i) converges to a and b(i) converges to b, therefore we can write a new sequence a(i) + b(i) converges to a + b or a(i)\*b(i) = ab
* Also, if g is a continuous function, g(a(i)) converges to g(a)
* 

**When does a Sequence Converge?**

* How can we tells whether a given sequence converges or not?
* 2 common criteria to determine this:



* 1) case w/ a sequence of #’s that keep increasing (or at least never decrease)
* In this case these #’s may go up forever w/ no bound
* If you look at any particular value, there's going to be a time at which the sequence has exceeded that value
* Here, the **sequence converges to infinity**
* 2) if not, the entire of the sequence are bounded (don’t grow arbitrarily large)
* Then it’s guaranteed the **sequence converges to a certain number**
* Can also derive some bound on the distance of the sequence from the number we suspect to be the limit, a
* If the distance gets smaller + smaller + we bound that distance by a certain number which goes to 0, it’s guaranteed that sequence a(i) would converge to a
* Variation of this argument = **sandwich argument**
* We have a certain sequence that converges to a certain number a + we have another sequence that converges to that same # a,
* If our sequence is somewhere in-between, our sequence must also converge to a



**Infinity Series**

* **Infinite Series** = limit as n goes to infinity of the *finite* series of which we sum the 1st n terms in the series



* Only makes sense so long as limit exists, so *when* does this limit exist?
* Does when all terms, a(i), all non-negative 🡪 partial sum gets bigger as we increase n
* Sequence of partial sums = **monotonic sequence** 🡺 always converges to a finite number or infinity (exists in either case)
* But if all a(i) terms don’t have same sign
* Limit might not exist (series is NOT well-defined)
* Could exist, but if we rearrange the terms, we may get a different limit
* Can avoid this if the sum of the *absolute values* of the terms sums to a finite number
* 

**Geometric Series**

* Given some # alpha + want to sum up all powers of alpha to get an infinite series



* In order for this series to converge, we need subsequent terms to get smaller + smaller
* Therefore we make the assumption that alpha < 1 to imply consecutive terms go to 0
* Can evaluate this series by starting from an algebraic identity:
* Take 1 - alpha + multiply it by the terms in the series, only up to the alpha^ = a finite series.
* Do this multiplication, get terms, do cancellations, + what is left at the end = **1 – α^(n + 1)**



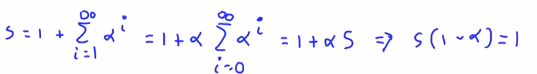
* Take the limit of these 3 terms as n goes to infinity.
* On the left, 1st term = 1 - α, +the 2nd term is the infinite series so the limit = 1.
* On the right, since α < 1, **α^(n + 1)** converges to 0 as α goes to infinity, so the whole term = 0

 = 0

* Solve this relation + obtain **s** = 1 / (1 – α), the formula for the **infinite geometric series**



* Alternative way of deriving the same result.



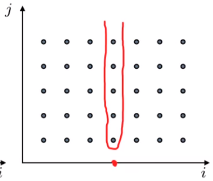
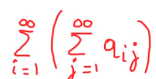
* Word of caution 🡪 subtracting α\*s from both sides is only possible if we take for granted that s is a finite number.

**About the order of summation in series with multiple indices**

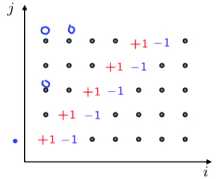
* Sometimes we have to deal w/ series where the terms being added are indexed by *multiple* indices



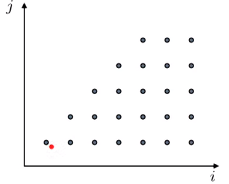
* Think of this as a 2D grid that corresponds to all pairs (i, j) 🡺 each point corresponds to a term we want to add in some arbitrary linear order.
* As long as this sum converges as we keep adding more terms, this series will be well defined.
* Can add terms in many different orders + in principle, different orders might give different results
* As long as the sum of the absolute values of all the terms turns out to be finite, the particular order in which we're adding terms doesn't matter.
* Another way that we can add the terms together: Consider fixing a particular choice of i + adding all terms associated w/ this particular i as j ranges from 1 to infinity.
* Summation from j = 1 to infinity, while keeping i fixed.

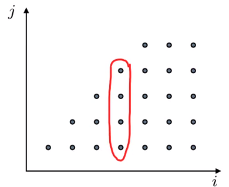
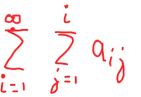
* For every possible i, get a particular number + add those together.
* Could also carry out the summation by fixing a particular choice of j + summing over all i's
* These are going to give us the same result as long as the series is **well-defined** (we have a guarantee that the sum of the absolute values of those numbers is finite, no matter which way we add them)
* A word of caution: this condition is not always satisfied + in those cases, strange things can happen
* Suppose we're dealing w/ the a(i,j)'s w/ the particular values below + all remaining terms are 0's.



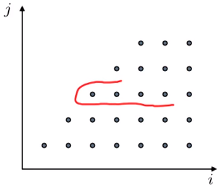
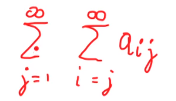
* If we carry out the summation by fixing j + adding over all i's, we get all 0’s for all rows b/c in each row we have a 1 and a minus 1, which cancel out and give us 0's
* But if we fix I + add over all j's, the 1st term = 1 + in the remaining columns we get 0's
* This is an example that shows you that *the order of summation actually may matter.*
* Consider the case of adding terms of a double sequence over a *limited* range of 2 indices 🡪 only for those (i, j) pairs for which j <= i



* Can carry out this summation in 2 ways
* 1) Fix a value of i + consider all terms to corresponding choices of j.
* But we only go up to the point where i = j, the largest term.

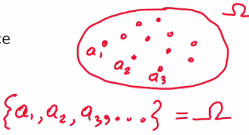
* 2) Fix a value of j + sum over all choices of i.

* So this corresponds to the sum over all choices of i
* These 2 ways of approaching this problem should give us the same answer + are going to be, again, subject to the usual qualification:
* *As long as the sum of the absolute values of the terms we're trying to add is < infinity, the summations are equal, just 2 different methods*

**Countable And Uncountable Sets**

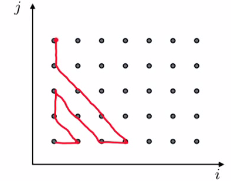
* Probability models often involve infinite SS’s = **infinite sets**
* Not all sets are of the same kind 🡪 Some are **discrete** (**countable**), some **continuous** (**uncountable**)
* **Countable set** = elements can be put into a 1-to-1 correspondence w/ the positive integers = means we look at the elements of that set, take 1 element+ call it the 1st element, take another + call it the 2nd, and so on.
* Eventually we exhaust all elements of a set + each element corresponds to a particular positive integer = its **index**



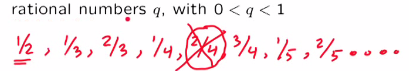
* Formally = take elements of a set *that are arranged in a sequence* 🡪 look at the set (the entire range of values of that sequence), + want that sequence to exhaust the entire set Ω.
* In simpler terms, want to be able to arrange all of the elements of Ω in a sequence.
* In a trivial sense, positive integers themselves are countable, b/c we can arrange them in a sequence (almost tautological/by definition)
* For a more interesting example, let's look at the set of ALL integers.
* Can we arrange them in a sequence? Yes, we can 🡪 alternate between positive + negative #’s.
* This way we cover all integers + have arranged them in a sequence.



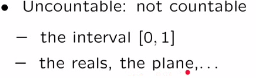
* How about the set of all pairs of positive integers?
* This is less clear 🡪 look at a picture = the set of all pairs of positive integers, which we understand to continue indefinitely.
* Can arrange this set in a sequence by tracing a path to cover the entire set of all pairs of positive integers.



* We have managed to arrange the pairs of positive integers in a sequence such that the set of all pairs is indeed a countable set
* The same argument can be extended to argue for the set of all triples of positive integers, or set of all quadruples of positive integers, + so on.
* This is actually not just a trivial mathematical point, b/c we will often have SS’s of this kind + it's important to know they're countable.
* Now for a more subtle example 🡪 Look at all **rational numbers** (#’s that can be expressed as a ratio of 2 integers) w/in the range between 0-1.
* 1st look at rational numbers w/ a denominator term of 2, then, the rational numbers w/ a denominator term of 3 + so on
* This way, we exhaust all of the rational numbers (also, 2/4 = 1/2, so we don’t need to include it in the sequence)
* Whenever we see a rational # that has already been encountered before in a simplified form, just delete it.
* In the end, we end up w/ a sequence that goes over all of the possible rational numbers + conclude that the set of all rational numbers is itself a countable set.



* So what kind of set would be **uncountable**?
* **An uncountable set**, by definition, = a set that is not countable.
* Example of uncountable set = continuous subsets of the real line (most prominent).
* Whenever we have an interval/the **unit interval** **[0-1]**, or any other interval w/ positive length, *that* interval is an uncountable set.
* Same is true if, instead of an interval, we look at the entire real line, or the 2D plane or 3D space, + so on.
* All the usual sets that we think of as continuous sets turn out to be uncountable.



**Proof That The Set Of Real Numbers Is Uncountable**

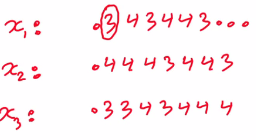
* **Cantor's Diagonalization Argument**.
* Look at the set of all numbers, x, that belong to the **open unit interval** = numbers between 0-1, such that their decimal expansion involves only 3’s and 4’s.



* The choice of 3 and 4 is somewhat arbitrary + it doesn't matter.
* What *really* matters = we do not have *long strings of nines*.
* Suppose this set was countable + if so, then that set could be written as equal to a set of this form:



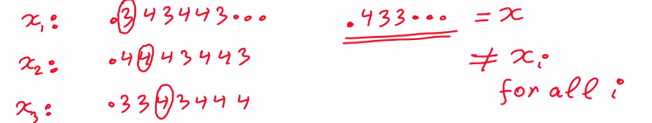
* W/ x1, x2, x3 + so on, each one of these is a real number inside that set.
* Now, suppose we take those numbers + write them down in decimal notation.



* Since we're talking about real numbers, their decimal expansion will go on forever
* We have assumed our set is countable + therefore, the set is equal to the sequence {x1, x2, …}
* This sequence exhausts all the numbers in that unit set.
* Can it do that?
* Construct a new number in the following fashion🡪 looks at the 1st digit in x1, the 2nd digit in x2, the 3rd digit in x3, + does something different to each + we continue this way.



* This constructed number differs from the 1st number in the 1st digit, from the 2nd number in the 2nd digit, and so on.
* So this is a number which is different from x(i) for all i.



* So, we have an element of the unit set which does NOT belong to this sequence {x1, x2, …}
* Therefore, it CANNOT be true that the unit set = to the set formed by the sequence {x1, x2, …}
* This is a contradiction to the initial assumption that the unit set could be written in this form {x1, x2, …}
* This contradiction establishes that since this is not possible, the unit set is an **uncountable set**.
* The unit set is a subset of the set of real numbers + since it is uncountable, it is not hard to show that the set of real numbers, a bigger set, will also be uncountable.