Stat 6021: In-Depth Explanation for Properties of Least-Squares Estimators

Read this after Section 3 in Guided Notes.
In the previous set of notes, we have shown that

$$Var(\hat{\beta}_1) = \frac{\sigma^2}{\sum(x_i - \bar{x})^2} \tag{1}$$

and that

$$Cov(\bar{y}, \hat{\beta}_1) = 0 \tag{2}$$

1 Properties of the Estimated Mean Response for Given Predictor

Let x_0 denote the value of the predictor for which we want to estimate the mean response for, which is in turn denoted by $\mu_{y|x_0}$. Then we have the following for the expected value and variance for the estimate as:

$$\begin{aligned} E(\hat{\mu}_{y|x_0}) &= E(\hat{\beta}_0 + \hat{\beta}_1 x_0) \\ &= \beta_0 + \beta_1 x_0. \end{aligned}$$

$$\begin{aligned} Var(\hat{\mu}_{y|x_0}) &= Var(\hat{\beta}_0 + \hat{\beta}_1 x_0) \\ &= Var(\bar{y} + \hat{\beta}_1 (x_0 - \bar{x})) \\ &= Var(\bar{y}) + (x_0 - \bar{x})^2 Var(\hat{\beta}_1) + 2Cov(\bar{y}, \hat{\beta}_1) \\ &= \frac{\sigma^2}{n} + (x_0 - \bar{x})^2 \frac{\sigma^2}{\sum(x_i - \bar{x})^2} \text{ using (1) and (2)} \\ &= \sigma^2 \left[\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\sum(x_i - \bar{x})^2} \right] \end{aligned}$$