Stat 6021: In-Depth Explanation for Properties of Least-Squares Estimators

Read this after Section 2 in Guided Notes.

1 Derivation of Common Results

Consider $c_i = \frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2}$.

$$\begin{aligned} \sum c_i &= \sum \frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2} \\ &= \frac{1}{\sum (x_i - \bar{x})^2} \sum (x_i - \bar{x}) \\ &= 0, \end{aligned} \tag{1}$$

since $\bar{x} = \frac{\sum x_i}{n}$ and so $\sum (x_i - \bar{x}) = 0$.

$$\begin{aligned} \sum c_i x_i &= \sum \frac{(x_i - \bar{x}) x_i}{\sum (x_i - \bar{x})^2} \\ &= \frac{\sum x_i^2 - \bar{x} \sum x_i}{\sum x_i^2 - 2\bar{x} \sum x_i + n\bar{x}^2} \\ &= \frac{\sum x_i^2 - n\bar{x}^2}{\sum x_i^2 - 2n\bar{x}^2 + n\bar{x}^2} \\ &= 1. \end{aligned} \tag{2}$$

$$\begin{aligned} \sum c_i^2 &= \sum \left[\frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2} \right]^2 \\ &= \frac{1}{\left[\sum (x_i - \bar{x})^2\right]^2} \sum (x_i - \bar{x})^2 \\ &= \frac{1}{\sum (x_i - \bar{x})^2}. \end{aligned} \tag{3}$$

Please note: In a simple linear regression setting, we have $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ where $E(\epsilon) = 0$ and $Var(\epsilon) = Var(y_i) = \sigma^2$. Since β_0 and β_1 are parameters, they are fixed, hence

$E(\beta_0) = \beta_0$ and $E(\beta_1) = \beta_1$.

We will use results (1) to (3) to derive the properties of least-squares estimators.

2 Properties of $\hat{\beta}_1$

$$\begin{aligned}
\hat{\beta}_1 &= \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2} \\
&= \frac{\sum(x_i - \bar{x})y_i}{\sum(x_i - \bar{x})^2}, \text{ since } \sum(x_i - \bar{x}) = 0 \\
&= \sum c_i y_i \qquad (4)
\end{aligned}$$

$$\begin{aligned}
E(\hat{\beta}_1) &= E(\sum c_i y_i) \text{ using (4)} \\
&= \sum c_i E(\beta_0 + \beta_1 x_i + \epsilon_i) \\
&= \beta_0 \sum c_i + \beta_1 \sum c_i x_i + 0 \text{ since } E(\epsilon) = 0 \\
&= \beta_1 \text{ due to (1) and (2).}
\end{aligned}$$

$$\begin{aligned}
Var(\hat{\beta}_1) &= Var(\sum c_i y_i) \\
&= \sum c_i^2 Var(y_i) \\
&= \sigma^2 \sum c_i^2 \\
&= \frac{\sigma^2}{\sum(x_i - \bar{x})^2} \text{ using (3).}
\end{aligned}$$

3 Properties of $\hat{\beta}_0$

$$\begin{aligned}
E(\hat{\beta}_0) &= E(\bar{y} - \hat{\beta}_1 \bar{x}) \\
&= E(\frac{\sum y_i}{n}) - \bar{x} E(\hat{\beta}_1) \\
&= \frac{1}{n} \sum E(y_i) - \bar{x}\beta_1 \\
&= \frac{1}{n} \sum E(\beta_0 + \beta_1 x_i + \epsilon) - \bar{x}\beta_1 \\
&= \beta_0 + \beta_1 \bar{x} - \bar{x}\beta_1 \\
&= \beta_0.
\end{aligned}$$

Before deriving the variance of $\hat{\beta}_0$, we will show that $\bar{y}$ and $\hat{\beta}_1$ are uncorrelated.

$$
\begin{aligned}
Cov(\bar{y}, \hat{\beta}_1) &= Cov(\frac{1}{n}\sum y_i, \sum c_i y_i) \\
&= \sum \frac{c_i}{n}\sigma^2 \text{ since } Cov(y_i, y_i) = Var(y_i) = \sigma^2 \\
&= \frac{\sigma^2}{n}\sum c_i \\
&= 0 \text{ using (1).}
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
Var(\hat{\beta}_0) &= Var(\bar{y} - \hat{\beta}_1 \bar{x}) \\
&= Var(\frac{\sum y_i}{n}) + Var(\hat{\beta}_1 \bar{x}) - 2\bar{x} Cov(\bar{y}, \hat{\beta}_1) \\
&= \frac{1}{n^2}\sum Var(y_i) + \bar{x}^2 Var(\hat{\beta}_1) \text{ using (5)} \\
&= \frac{\sigma^2}{n} + \bar{x}^2 \frac{\sigma^2}{\sum (x_i - \bar{x})^2} \\
&= \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{\sum (x_i - \bar{x})^2}\right].
\end{aligned}
$$