Stat 6021: Simultaneous Confidence Intervals and the Bonferroni Method

Read this after Section 5 in Guided Notes.

1 Joint Estimation

Recall the significance level α is the probability of wrongly rejecting the null hypothesis if the null hypothesis is true. In the context of confidence intervals, α is the proportion of random samples that will result in a confidence interval that does not contain the true value of the population parameter.

Consider a simple linear regression equation: $E(y|x) = \beta_0 + \beta_1 x$. Suppose we want to construct $(1-\alpha)100\%$ confidence intervals for both coefficients β_0 and β_1, i.e.,

$$\hat{\beta}_0 \pm t_{1-\alpha/2;n-2} s\{\hat{\beta}_0\} \tag{1}$$

and

$$\hat{\beta}_1 \pm t_{1-\alpha/2;n-2} s\{\hat{\beta}_1\}. \tag{2}$$

respectively.

When conducting multiple inferences on our data, we are concerned whether we still have $(1-\alpha)100\%$ confidence that **both** (1) and (2) will contain the population parameter β_0 and β_1. Analysis of data frequently consists of a series of estimates where we want assurances about the correctness of the entire series of estimates.

2 Family Confidence

We want to maintain the same level of confidence for our entire series, or **family**, of estimates. This is the **family confidence coefficient**: the proportion of families of estimates that are entirely correct when repeated samples are selected and specified CIs for the entire family are calculated for each sample.

Consider two events A_1 and A_2, where

$$\begin{aligned} A_1 &= \{(1) \text{ does not cover } \beta_0\} \\ A_2 &= \{(2) \text{ does not cover } \beta_1\} \end{aligned}$$

and we know $P(A_1) = \alpha$ and $P(A_2) = \alpha$ based on the definition of α. Therefore, the probability of both intervals being correct, i.e., the family confidence, is denoted as $P\left(A_1^c \cap A_2^c\right)$.

$$
\begin{aligned}
P\left(A_1^c \cap A_2^c\right) &= 1 - P\left(A_1 \cup A_2\right) \\
&= 1 - P\left(A_1\right) - P\left(A_2\right) + P\left(A_1 \cap A_2\right).
\end{aligned} \tag{3}
$$

Since $P\left(A_1 \cap A_2\right) \geq 0$, (3) becomes

$$
\begin{aligned}
P\left(A_1^c \cap A_2^c\right) &\geq 1 - P\left(A_1\right) - P\left(A_2\right) \\
&= 1 - 2\alpha.
\end{aligned} \tag{4}
$$

We can see from (4) the family confidence coefficient is now **at least** $(1-2\alpha)100\%$ instead of being $(1-\alpha)100\%$.

3 Bonferroni Method

From (4), we can see that if we want the family confidence coefficient to be at least $(1-\alpha)100\%$ for our two confidence intervals, each confidence interval can be constructed at $(1-\alpha/2)100\%$ confidence. This means that (1) and (2) become

$$\hat{\beta}_0 \pm t_{1-\frac{\alpha}{2}/2;n-2} s\{\hat{\beta}_0\} \tag{5}$$

and

$$\hat{\beta}_1 \pm t_{1-\frac{\alpha}{2}/2;n-2} s\{\hat{\beta}_1\}. \tag{6}$$

In fact, (4), (5), and (6) can be generalized to any number, g, of confidence intervals we want in our family of estimates since

$$P\left(\bigcap_{i=1}^{g} A_i^c\right) \geq 1 - g\alpha.$$

Therefore, the Bonferroni method to ensure that we have at least $(1-\alpha)100\%$ confidence in our family of estimates is

$$\hat{\beta}_j \pm t_{1-\alpha/2g;n-k-1} s\{\hat{\beta}_j\}$$

for any $j = 0, 1, \cdots, k$. A few notes about the Bonferroni method:

- the method is conservative since we have at least $(1-\alpha)100\%$ confidence instead of exactly $(1-\alpha)100\%$ confidence
- when g gets large, the confidence intervals can get very wide

Other methods exist that are less conservative but are more complicated to implement. For the Bonferroni method, all we do is divide the significance level α by the number of confidence intervals we wish to construct, g.