Stat 6021: Mean-Squared Error

Read this after Section 2 from Guided Notes

1 Mean-Squared Error: Bias and Variance Decomposition

The mean-squared error (MSE) of an estimator $\hat{\theta}$ with respect to an unknown parameter θ is defined as

$$MSE(\hat{\theta}) = E\left[(\hat{\theta} - \theta)^2\right]. \tag{1}$$

The MSE of an estimator can be decomposed into the variance and bias (or squared bias) of the estimator. From (1),

$$\begin{aligned}
MSE(\hat{\theta}) &= E\left[(\hat{\theta} - \theta)^2\right] \\
&= E\left[\left(\hat{\theta} - E(\hat{\theta}) + E(\hat{\theta}) - \theta\right)^2\right] \\
&= E\left[\left(\hat{\theta} - E(\hat{\theta})\right)^2 + 2\left(\hat{\theta} - E(\hat{\theta})\right)\left(E(\hat{\theta}) - \theta\right) + \left(E(\hat{\theta}) - \theta\right)^2\right] \\
&= E\left[\left(\hat{\theta} - E(\hat{\theta})\right)^2\right] + E\left[2\left(\hat{\theta} - E(\hat{\theta})\right)\left(E(\hat{\theta}) - \theta\right)\right] + E\left[\left(E(\hat{\theta}) - \theta\right)^2\right] \\
&= E\left[\left(\hat{\theta} - E(\hat{\theta})\right)^2\right] + 2\left(E(\hat{\theta}) - \theta\right)E\left[\hat{\theta} - E(\hat{\theta})\right] + \left(E(\hat{\theta}) - \theta\right)^2 \text{ since } E(\hat{\theta}) - \theta \text{ is constant} \\
&= E\left[\left(\hat{\theta} - E(\hat{\theta})\right)^2\right] + 2\left(E(\hat{\theta}) - \theta\right)\left(E(\hat{\theta}) - E(\hat{\theta})\right) + \left(E(\hat{\theta}) - \theta\right)^2 \text{ since } E(\hat{\theta}) \text{ is constant} \\
&= E\left[\left(\hat{\theta} - E(\hat{\theta})\right)^2\right] + \left(E(\hat{\theta}) - \theta\right)^2 \\
&= Var(\hat{\theta}) + Bias(\hat{\theta})^2
\end{aligned}$$