CIS 511 Homework 5

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Problem 1

The NP-Completeness proof for SUBSET-SUM works because the input is encoded in $\log(n)$ bits, so solving the problem takes time exponential in the number of bits. Since complexity is measured against the size of the input, this places SUBSET-SUM $\in$ NP.

However, for UNARY-SSUM, each number occupies length equal to its value. Therefore, we can convert each number into binary which can be done in polynomial time via successive division by two and checking any remainder (which becomes the next significant digit). This runs in polynomial time. We do this for each number. Now, we have the classic SUBSET-SUM problem. We can solve this in a brute-force manner by simulating the non-deterministic TM.

This simulation takes time exponential in the length of the [created] input, which is $\log(n)$ of the original input. Therefore, this runs in time polynomial ($O(e^{log(n)}) = O(n)$) of the original input. Thus, UNARY-SSUM $\in$ P.

Problem 2

Part a

We know that $CNF_2 = \{\langle\phi\rangle \mid \phi$ is a satisfiable CNF formula with each variable appears at most 2 times$\}$ is in P since we can make an algorithm for it.

function $M(\phi)$
    Check that all variables x_i appear at most twice as
    Create ψ, a modifiable copy of ϕ
    Create bit vector b for the final assignment
    for all $i \in \{1, \ldots, n\}$ **do**
        if There is one clause left **then**
            Accept
        else if There is one variable left **then**
            if Clauses x_i and $\bar{x_i}$ both are in ϕ **then**
                Reject
            else
                Accept
            end if
        else if the literal x_i appears once **then**
            $\psi \leftarrow \psi(x_i = 1)$ (and simplify ψ)
        else if the literal $\bar{x_i}$ appears once **then**
            $\psi \leftarrow \psi(x_i = 0)$
        else if the literal x_i appears twice **then**
            $\psi \leftarrow \psi(x_i = 1)$
        else if the literal $barx_i$ appears twice **then**
            $\psi \leftarrow \psi(x_i = 0)$
        else
            Combine the clauses with x_i and $\bar{x_i}$ in ψ by creating a new clause with all the other literals in the two clauses together

end if
end for
end function

We need to show that this works, so we analyze each case in the if else chain in the loop. We show that each branch either makes the problem smaller, or solves the problem.

- If there is only one clause, we can set any literal in the clause to true and we have satisfied the problem, so we are done.
- If there is one variable x left, since we know each variable only appears twice, it is unsatisfiable if and only if the clauses left are x and $\bar{x}$, so we accept accordingly.
- If a literal x only appears once in the formula, we can just it to true and get rid of the clause that the the literal is in. Similarly, if the literal $\bar{x}$ is in only once, we can just set it to false to get rid of the clause
- If the literal x appears twice in the formula (or equivalently $\bar{x}$), then we can like before just set the literal to the appropriate value and get rid of the clauses that it occupies.
- The tricky case - if x and $\bar{x}$ both appear in different clauses. Note however that:

  $$(x + a)(\bar{x} + b) \iff \bar{x}a + xb + ab \iff a + b$$

  Since we can choose x accordingly. So it is satisifiable if and only if $a + b$ is true, so we combine the clauses and continue.

Since we reduce the problem by at least one variable each time, with simple boolean formula manipulations in each step, we can run this in polynomial time in the number of clauses and variables. Therefore, this is in P.

Part b

We show that $CNF_3 = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable CNF formula with each variable appears at most 3 times}\}$ is NP-Complete. We show how to change an instance of $3SAT$ to an instance in CNF_3.

Given a $3SAT$ we can increase the number of variables to make sure that no one variable shows more than 3 times. Let us say a variable x shows up in k clauses. Then we create new variables $y_1, \ldots, y_k$. We create a 'chain' of implications:

$$y_1 \implies y_2 \implies y_3 \implies \ldots \implies y_n \implies y_1$$

This means all these variables must have the same value. In CNF form, using $x \implies y \iff (\bar{x} + y)$, this would be

$$(\bar{y_1} + y_2)(\bar{y_2} + y_3) \ldots (\bar{y_n} + y_1)$$

Then for each clause that x is in, we use one of the y_i variables. Now each of the y_i variables appears twice in the 'chain' of implications, and once in the actual clauses, meaning it appears a total of 3 times.

We illustrate with an example, using the simple example with only the variable x:

$$(x + a_1 + b_1)(x + a_2 + b_2)(\bar{x} + a_2 + b_2)(\bar{x} + a_4 + b_4)$$

This becomes with these new rules:

$$(\bar{y_1} + y_2)(\bar{y_2} + y_3)(\bar{y_3} + y_4)(\bar{y_4} + y_1)$$
$$(y_1 + a_1 + b_1)(y_2 + a_2 + b_2)(\bar{y_3} + a_2 + b_2)(\bar{y_4} + a_4 + b_4)$$

Now we need to make sure this is still a polynomial time transformation. Say there are n variables and m clauses. In the worse case, a variable appears in every clause. This means we add m variables to the equation, and m clauses for the 'chain' of implications. If this happens for each variable we get mn new clauses, and mn new variables, given $O(mn)$ sized problem, which is just a quadratic blow up in this size of the formula, and therefore still a polynomial time transformation. Therefore if we can solve CNF_3 we can solve $3SAT$ which we know to be NP-complete. Since checking a boolean formula is in NP since we can use the assignment as a certificate, this is NP-complete

Problem 3

Let the Mine Consistency Problem (MCP) be as defined in the problem:
$\{G : G$ has a consistent assignment of mines$\}$.

This is clearly in NP: given a graph with mines and numbers, it is quick and easy to check if this it is a consistent and accurate labeling. Now we will show MCP $\in$ NP by reducing 3-SAT to MCP. This will show MCP is at least as hard as SAT and hence NP-complete.

Given a 3-SAT Boolean formula we can construct an instance of MCP as follows:

We create a node for each literal (both it and its negation) and for each clause. We connect each pair of literals (its positive and negation) to another node for the pair labeled with value 1. This is to ensure exactly one can be a mine (selected), corresponding to the value of the literal.

Each clause node has an edge to each node corresponding to literals it contains. We mark each clause node with a value of 3. However, this alone would force all literals to be true. To alleviate this, we create two extra nodes for each clause and attach them to the clause node. This allows a place for mines to sit if only 1 or 2 literals in the clause are true. Note that at least one literal must be true since only 2 mines can be occupied by these extra nodes.

Note that this construction is quick, requiring one pass of the input and creating a number of nodes linear in the number of literals and clauses.

We claim there is a consistent mine placement on this graph $\iff$ the Boolean formula is satisfiable.

- $\implies$ Given a consistent mine assignment, set each literal to true if its corresponding node has a mine. Now by construction a literal and its negation cannot be true at the same time, so this axiom holds. Also, since a clause node is neighbored by 3 mines, and only 2 can be held by non-literal nodes, at least one literal node in each clause has a mine (and is therefore true), so each clause is true. Thus, the Boolean formula used to create this graph is satisfiable.

- $\impliedby$ Given a satisfiable Boolean function, place a mine on each literal-node if is true in the satisfying assignment and false otherwise. Note that each node labeled "1" only has one mine next to it, since the equation is satisfiable and therefore either a literal or its negation must be true (exclusively). Also, each node marked with a "3" (for each clause) can have 3 mines next to it. Since the equation is satisfiable at least one literal node is a mine (and up to 3 can be). If there are only 1 or 2 true literals in the clause fill 2 or 1 of the "extra" nodes, respectively, for the clause with a mine. Now each clause node has 3 mines next to it. Therefore, this graph has a consistent mine placement.

Thus, MCP is NP-Complete.

Problem 4

If $P = NP$, then there exists a polynomial time decider D for SAT runs in polynomial time. From this we can create a new machine M that finds the assignment of the variables. The idea is that we assign true or false to each of the variables in order, and use D to determine if it is a valid assignment. If there are multiple assignments we just pick the one that works in order of the variables.

function $M(\phi)$
  Create ψ, a modifiable copy of ϕ
  Create bit vector b for the final assignment
  for all $i \in \{1, \ldots, n\}$ **do**
    Set $\psi' \leftarrow \psi(x_i = 1)$

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        if D(ψ') accepts then
            ψ ← ψ'
            b_i ← 1
        else
            ψ ← ψ(x_i = 0)
            b_i ← 0
        end if
    end for
    Output b
end function
```

In words, we greedily go through the variables assigning them. We initially assign each variable x_i to 1, and use D to check if that assignment is feasbile. If it is, then there must be a valid assignment where x_i is 1, and if not x_i must be 0. So we assign x_i appropriately. Then we update the formula to reflect this choice, and continue appropriately.

Now we show this runs in polynomial time. Let n be the number of variables, and m be the number of clauses in the original formula. On each iteration of the outer loop, we update the formula, which takes $O(m)$ time, and we run the decider D, which runs polynomial time for some polynomial $p(n+m)$. There are n iterations of the loop, so in total this takes $O(nmp(n+m))$ time, which is still a polynomial.

Problem 5

Let P = NP. We have seen that 3COL $\in$ NP, so by our assumption that P = NP there is a polynomial time algorithm that decides if a graph is 3-colorable. Call this L.

Note that by a similar construction as the original decidability proof, we can show that deciding if a graph is colorable given an already partially-colored graph (i.e., with some nodes given colors already) is NP-complete (and by our assumption, in P). The idea that for already assigned colors we can force their presence in the end solution by only allowing them to be assigned the color they have been given. The proof is otherwise the same.

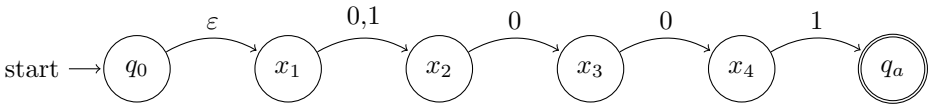
We will use L to 3-color a graph when given a 3-colorable graph. Given a graph, choose a node and give it a color. Ask L if the graph is 3-colorable. If if is, move on to another uncolored node. If it is not, try the another color. If any node fails for all colors, the graph cannot be 3-colored. Otherwise, all nodes will eventually be colored and the graph will have a valid 3-coloring.

Now, for each node we must try at most 3 colors, and once a node is visited it is not recolored (note we do not need to worry about recoloring since color assignments can be permuted between the node groups for each color given one valid assignment). So each node takes a constant amount of time to process. Since 3COL in in P by assumption, L runs in polynomial time. Finally, we visit each node once, so this is linear in the number of nodes. Therefore, this whole process runs in polynomial time.

Thus, if P = NP, we can 3-color a graph in polynomial time.

Problem 6

We want to, given a CNF formula ϕ, construct an NFA that accepts all nonaccepting configurations of ϕ. The idea is we create a sub-NFA for each clause that accepts if and only if the clause is false. To illustrate, we use a simple formula over 4 variables $\phi(x_1, x_2, x_3, x_4) = (x_2 + x_3 + \bar{x_4})$

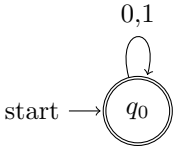


This takes in as input the bit string representing the assignment of all the variables. Each of the successive nodes represents the variables, and they transition to the next node on values that would allow the clause to reject. Notice that since x_1 does not appear in the clause, it can take whatever value it wants. So this accepts the non-satisfying strings $0, 1001$.

We can generalize this to more clauses, by making the initial state have ε-transitions to a chain for each clause, constructed in the exact same way as the above example. That would give, with m variables and c

clauses, cm nodes. The transition function would take then $O(cm)$ space to create. This means it can be created in $O(cm)$, or polynomial, time. This new NFA would accept on any rejecting input since it only takes one failing clause to make the formula reject.

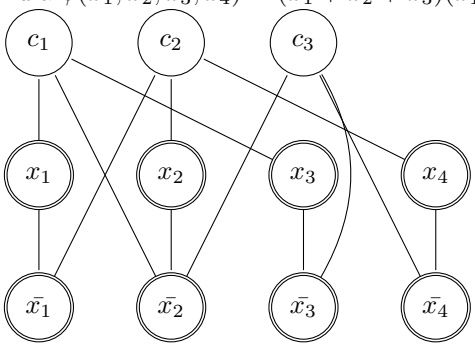
If this formula has no accepting input, this NFA would reduce to one that accepts $\{0,1\}^*$, or:



Therefore if we could reduce this NFA in polynomial, we would be able to solve SAT in polynomial time. In other words we just reduced SAT to minimizing an NFA. Therefore if $P \neq NP$ then there is no polynomial time solution for this.

Problem 7

We want to show that the problem of designating directions to edges of an undirected graph G so that a given subset C will have all nodes in it have either indegree 0 or outdegree 0, and all other nodes have indegree at least one is NP-complete. We reduce SAT to this. And example of the mapping is show for the formula $\phi(x_1, x_2, x_3, x_4) = (x_1 + \bar{x_2} + x_3)(\bar{x_1} + x_2 + x_4)(\bar{x_2} + \bar{x_3} + \bar{x_4})$



The highlighted nodes are the ones in C.

We want to create such a graph for any SAT formula ϕ. So in words the procedure is for every variable x_i, create nodes x_i and $\bar{x_i}$, and for every clause c_j create a node c_j. Connect the node x_i (or $\bar{x_i}$) to node c_j if the variable x_i (or $\bar{x_i}$) is in the clause c_j. Also create an edge between nodes x_i and $\bar{x_i}$. Place all the nodes x_i and $\bar{x_i}$ into C.

This graph has a solution the the direction problem if and only if the formula ϕ has a solution. If the formula ϕ has a solution, select the true literals $x_i/\bar{x_i}$ as the 'all outdegree' or 0 indegree nodes in C, and the complement pair as the 'all indegree' or 0 outdegree ones. Since only the pairs of literal nodes $x_i/\bar{x_i}$ are connected in C, this does not lead to an inconsistency. If this is the satisfying assignment then we should have at least one edge into each clause node c_j. Similarly, if we have such an assignment of edge directions, we can reconstruct the satisfying assignment, since only one in each of the pairs of literal nodes $x_i/\bar{x_i}$ has 'all outdegree' we can set those literals to true, and we know by the direction assignment that every clause will have one corresponding true literal inside it. Therefore, this graph has a solution to the problem if and only if the the formula has a satisfying assignment.