Leaving Certificate Honours Level 2003 Paper 2 Solutions

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Introduction

The following are the solutions to Paper 2 of the 2003 Leaving Certificate Honours course.

Question 1

(a)
If it's true for values $t \in R$ then it's true for $t = 0$. Thus the point $((3-0)/(1+0), 6(0)/(1+0))$ is on the circle, i.e the point $(3, 0)$.

$$\begin{aligned} 3^2 + 0^2 &= r^2 \\ 9 &= r^2 \\ r &= 3 \end{aligned}$$

(b)(i)
If the two circles touch externally then the sum of their radii is equal to the distance between their centers. The center of C_1 is $(-1, 1)$ and the center of C_2 is $(7, 1)$. The distance between the centers is thus 8. The radius of C_1 is $\sqrt{(-1)^2 + (1)^2 - (-23)} = 5$ and the radius of C_2 is $\sqrt{(7)^2 + (1)^2 - 41} = 3$. The sum of the radii is $3+5 = 8$ which is equal to the distance between their centers and thus they touch externally.

(b)(ii)
From figure 1 we can see that all 3 circle centers lie on the line $y = 1$. The diameter of the circle is equal to the sum of the diameters of the other two circles, i.e. 16 and thus the radius of the circle is 8. The x co-ordinate of the center is located at $x = 2 = -g$. The y co-ordinate is $-f = -1$. To find the value of c.

$$\begin{aligned} r &= \sqrt{g^2 + f^2 - c} \\ 8 &= \sqrt{(-2)^2 + (-1)^2 - c} \end{aligned}$$

$$64 = 4 + 1 - c$$
$$c = -59$$

Thus, the equation of the circle is

$$\begin{aligned} x^2 + y^2 + 2gx + 2fy + c &= x^2 + y^2 + 2(-2)x + 2(-1)y - 59 \\ &= x^2 + y^2 - 4x - 2y - 59 \end{aligned}$$

(c)(i)

Figure 2 shows the circle $x^2+y^2-12x+6y+9=0$ and the tangent $ax+by=0$. Note that the line goes through $(0,0)$. Now, if $ax+by=0$ then $y=(-a/b)x$. Substituting this into the circle equation we get

$$\begin{aligned} x^2 + y^2 - 12x + 6y + 9 &= 0 \\ x^2 + \left(\frac{-a}{b}x\right)^2 - 12x + 6\left(\frac{-a}{b}\right) + 9 &= 0 \\ x^2\left[\frac{a^2+b^2}{b^2}\right] - 6x\left[\frac{2b+a}{b}\right] + 9 &= 0 \end{aligned}$$

The general solution to a quadratic equation is

$$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

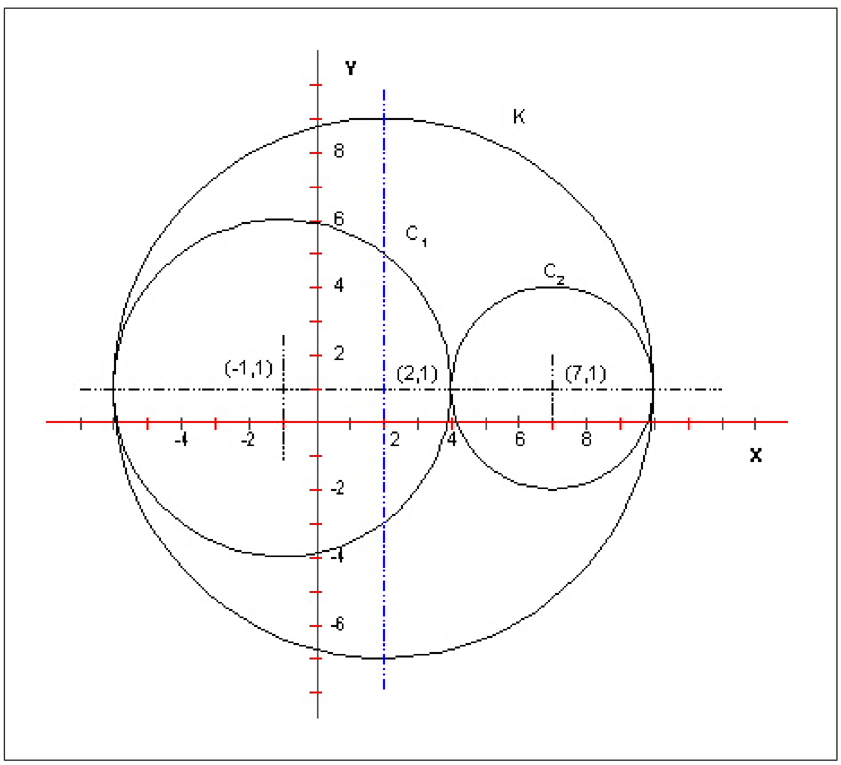


Figure 1: Circles C_1,C_2 and K

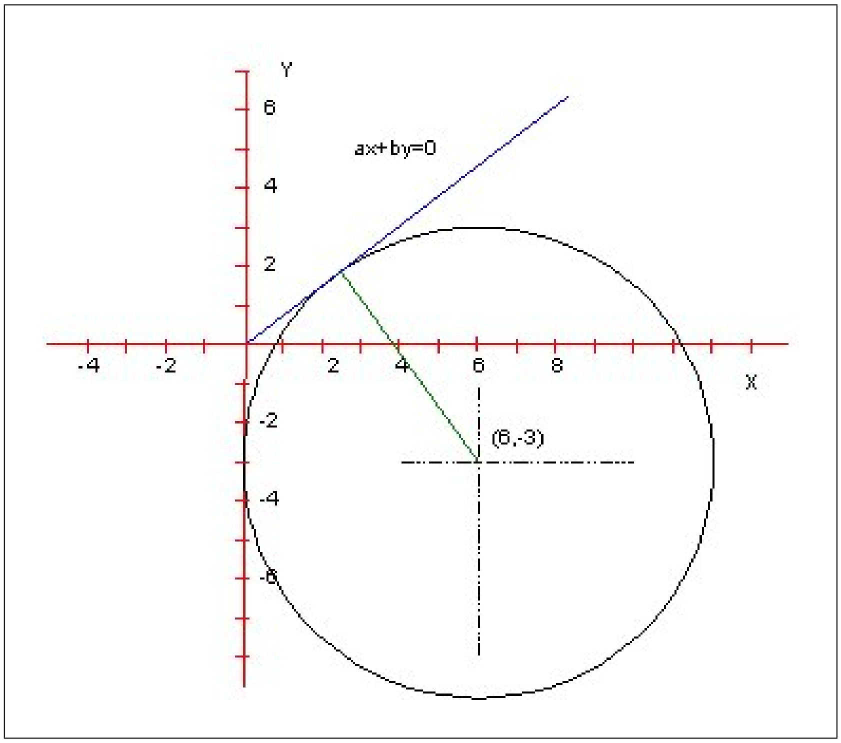


Figure 2: Tangent and Circle

Now if the line is a tangent then this equation must only have one solution, i.e. the line only touches the circle. Thus $b^2 - 4ac = 0$. Hence

$$
\begin{aligned}
\left(-6\left[\frac{2b+a}{b}\right]\right)^2 - 4\left[\frac{a^2+b^2}{b^2}\right]\times 9 &= 0 \\
\frac{36}{b^2}\left[4b^2+4ab+a^2-a^2-b^2\right] &= 0 \\
3b^2+4ab &= 0 \\
b(3b+4a) &= 0 \\
3b &= -4a \\
\frac{a}{b} &= -\frac{3}{4}
\end{aligned}
$$

(c)(ii)

The slope of the line perpendicular to the tangent is $-4/3$ and thus the equation of the line through the center of the circle is

$$
\begin{aligned}
y-(-3) &= \frac{-4}{3}(x-6)) \\
3(y+3) &= -4(x-6) \\
3y+9+4x-24 &= 0 \\
4x+3y-15 &= 0
\end{aligned}
$$

The equation of the tangent is $y = -(a/b)x = -(-3/4)x$ so $4y = 3x$ or $3x - 4y = 0$. Solving between them we obtain

$$\begin{aligned} 4x + 3y &= 15 \\ 3x - 4y &= 0 \end{aligned}$$

$$\begin{aligned} 12x + 9y &= 45 \\ 12x - 16y &= 0 \end{aligned}$$

$$\begin{aligned} 25y &= 45 \\ y &= \frac{9}{5} \\ 3x &= 4 \times \frac{9}{5} \\ x &= \frac{12}{5} \end{aligned}$$

So the point is $(12/5, 9/5)$.

Question 2

(a)
$\overrightarrow{a} = 3\overrightarrow{i} - \overrightarrow{j}$ and $\overrightarrow{b} = 4\overrightarrow{i} + 3\overrightarrow{j}$. Now

$$\begin{aligned} \overrightarrow{c} &= \overrightarrow{ab} \\ &= \overrightarrow{b} - \overrightarrow{a} \\ &= 4\overrightarrow{i} + 3\overrightarrow{j} - (3\overrightarrow{i} - \overrightarrow{j}) \\ &= \overrightarrow{i} + 2\overrightarrow{j} \end{aligned}$$

(b(i))
Since $\overrightarrow{p} \perp \overrightarrow{q}, \overrightarrow{p}.\overrightarrow{q} = 0$. Thus

$$\begin{aligned} \overrightarrow{p}.\overrightarrow{q} &= (2\overrightarrow{i} + 3\overrightarrow{j})(3\overrightarrow{i} + k\overrightarrow{j}) \\ 0 &= (2 \times 3) + (3 \times k) \\ 0 &= 6 + 3k \\ k &= -2 \end{aligned}$$

(b)(ii)
The angle θ between $\overrightarrow{p}$ and $\overrightarrow{r}$ is given by

$$\boxed{\theta = \cos^{-1}\left(\frac{\overrightarrow{p}.\overrightarrow{r}}{|\overrightarrow{p}||\overrightarrow{r}|}\right)}$$

Thus

$$\begin{aligned}
\theta &= \cos^{-1}\left(\frac{(2\overrightarrow{i}+\overrightarrow{j}).(3\overrightarrow{i}+t\overrightarrow{j})}{|\overrightarrow{2\overrightarrow{i}+\overrightarrow{j}}||3\overrightarrow{i}+t\overrightarrow{j}|}\right)\\
\cos 45 &= \left(\frac{6+t}{\sqrt{2^2+1^2}\times\sqrt{3^2+t^2}}\right)\\
\frac{1}{\sqrt{2}} &= \left(\frac{6+t}{\sqrt{5}\times\sqrt{9+t^2}}\right)\\
\frac{\sqrt{5}\times\sqrt{9+t^2}}{\sqrt{2}} &= 6+t\\
\frac{5(9+t^2)}{2} &= (6+t)^2\\
&= t^2+12t+36\\
45+5t^2 &= 2t^2+24t+72\\
3t^2-24t-27 &= 0\\
t^2-8t-9 &= 0\\
(t-9)(t+1) &= 0\\
t &= 9 \qquad \text{or}\\
t &= -1
\end{aligned}$$

(c)(i)
$\overrightarrow{x} = \overrightarrow{a} + \overrightarrow{ax}$. Now $\overrightarrow{ax} = \frac{1}{4}\overrightarrow{ab} = \frac{1}{4}(\overrightarrow{b} - \overrightarrow{a})$.

Thus $\overrightarrow{x} = \overrightarrow{a} + \frac{1}{4}(\overrightarrow{b} - \overrightarrow{a}) = \frac{3}{4}\overrightarrow{a} + \frac{1}{4}\overrightarrow{b}$

(c)(ii)
$\overrightarrow{by} = \overrightarrow{y} - \overrightarrow{b}$. Now $\overrightarrow{y} = \frac{2}{3}\overrightarrow{a}$.Thus $\overrightarrow{by} = \frac{2}{3}\overrightarrow{a} - \overrightarrow{b}$

(c)(iii)

$$\begin{aligned}
\overrightarrow{g} &= m\overrightarrow{x}\\
&= m\left(\frac{3}{4}\overrightarrow{a} + \frac{1}{4}\overrightarrow{b}\right)
\end{aligned}$$

$$
\begin{aligned}
\overrightarrow{bg} &= n\overrightarrow{by} \\
&= n\left(\frac{2}{3}\overrightarrow{a} - \overrightarrow{b}\right) \\
\overrightarrow{g} - \overrightarrow{b} &= n\left(\frac{2}{3}\overrightarrow{a} - \overrightarrow{b}\right) \\
\overrightarrow{g} &= \frac{2n}{3}\overrightarrow{a} + (1-n)\overrightarrow{b} \\
m\left(\frac{3}{4}\overrightarrow{a} + \frac{1}{4}\overrightarrow{b}\right) &= \frac{2n}{3}\overrightarrow{a} + (1-n)\overrightarrow{b} \\
\frac{1}{4} &= 1-n \\
n &= \frac{3}{4} \\
\frac{3m}{4} &= \frac{2n}{3} \\
m &= \frac{2}{3}
\end{aligned}
$$

Question 3

(a)

$$
\begin{aligned}
x' &= x+y \\
y' &= x-y
\end{aligned}
$$

$$
\begin{aligned}
x'+y' &= 2x \\
\frac{x'+y'}{2} &= x
\end{aligned}
$$

$$
\begin{aligned}
x'-y' &= 2y \\
\frac{x'-y'}{2} &= y
\end{aligned}
$$

Then

$$
\begin{aligned}
f(L) &= f(4x-2y-1) \\
&= 4\left(\frac{x'+y'}{2}\right) - 2\left(\frac{x'-y'}{2}\right) - 1 \\
&= 2x' + 2y' - x' + y' - 1 \\
&= x' + 3y' - 1 \\
x + 3y - 1 &= 0
\end{aligned}
$$

(b)(i)

The equation of the line M is $3x - 4y + c = 0$ as it has the same slope as K. The perpendicular distance d from p to K is given by

$$\begin{aligned} d &= \frac{|3(2) + (-4)(-1) + 9|}{\sqrt{3^2 + 4^2}} \\ &= \frac{19}{5} \end{aligned}$$

This is the same distance to the line M. Thus

$$\begin{aligned} \frac{19}{5} &= \frac{|3(2) + (-4)(-1) + c|}{\sqrt{3^2 + 4^2}} \\ &= \frac{|10 + c|}{5} \\ 19 &= |10 + c| \\ \pm 19 &= 10 + c \\ c &= -29 \end{aligned}$$

Thus the equation of M is $3x - 4y - 29 = 0$.

(b)(ii)

The perpendicular distance between K and M is simply twice the distance from K to p, i.e. $2 \times 19/5 = 38/5$.

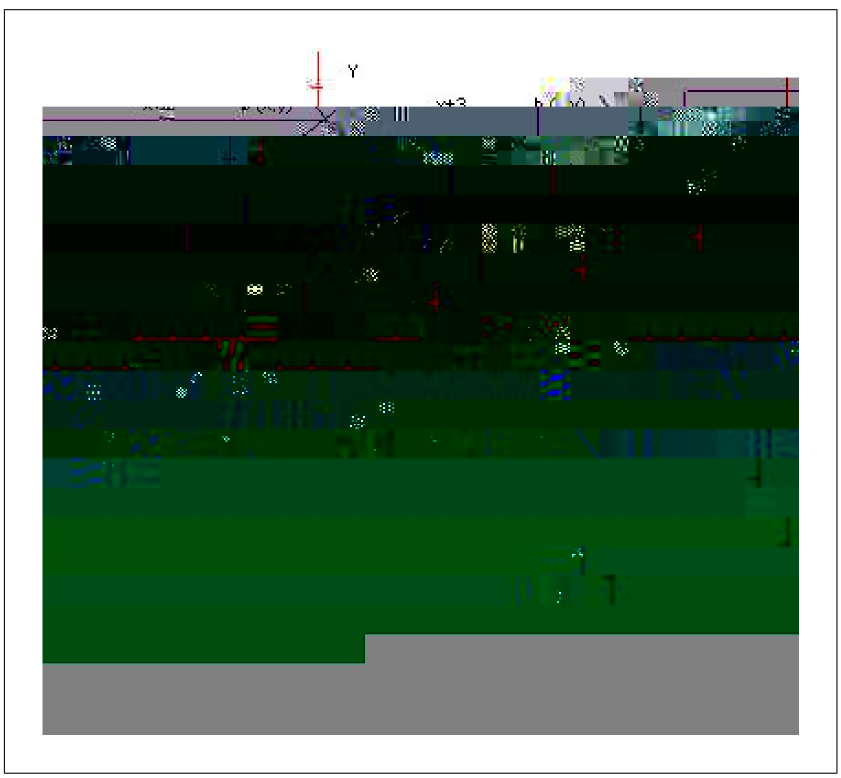


Figure 3: Lines K and M

(b)(iii)
The slope of the line through a and p is

$$\begin{aligned} m_1 &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{0 - (-1)}{-3 - 2} \\ &= -\frac{1}{5} \end{aligned}$$

The slope of the line m_K is $\frac{3}{4}$. Let θ be the angle. Then the angle between the two lines is given by

$$\begin{aligned} \tan\theta &= \pm\frac{m_1 - m_K}{1 + m_1 m_K} \\ &= \pm\frac{\frac{-1}{5} - \frac{3}{4}}{1 + (\frac{-1}{5})(\frac{3}{4})} \\ &= \pm\frac{-19}{17} \\ \theta &= 48 \text{ degrees} \end{aligned}$$

(b)(iv)
Since $b(x, y)$ is in the line K it satisfies the equation of K, i.e. $3x - 4y + 9 = 0$ or $y = (3x + 9)/4$. By Pythagoras's theorem

$$\begin{aligned} (x+3)^2 + y^2 &= 15^2 \\ &= 225 \\ (x+3)^2 + \left(\frac{3x+9}{4}\right)^2 &= 225 \\ (x+3)^2\left(\frac{25}{16}\right) &= 225 \\ x + 3 &= 12 \\ x &= 9 \\ y &= \frac{3(9)+9}{4} \\ y &= 9 \end{aligned}$$

Question 4

(a)
The circumference of a circle is

$$\begin{aligned} 2\pi r &= 30\pi \\ r &= 15 \text{ cm} \end{aligned}$$

The area of a sector is

$$\begin{aligned}
\frac{1}{2}r^2\theta &= 75 \\
\theta &= \frac{150}{225} \\
&= 0.67 \text{ radians}
\end{aligned}$$

(b)

$$\begin{aligned}
\sin 2x + \sin x &= 0 \\
2\sin x\cos x + \sin x &= 0 \\
\sin x(2\cos x + 1) &= 0 \\
\sin x &= 0 \\
x &= 0^\circ \text{ and } 180^\circ \\
\text{or} & \\
2\cos x + 1 &= 0 \\
\cos x &= -\frac{1}{2} \\
x &= 120^\circ \text{ and } 240^\circ
\end{aligned}$$

(c)(i)
As can be seen from figure 4 the triangle kab is equilateral as the distance from a to b is r and so is the distance from a to k and b to k. Thus the angle kab is $60°$. Similarly the angle apb is also $60°$. Thus the angle kap is $120°$ or $2\pi/3$ radians.

(c)(ii)
The shaded area is equal to the area of the sector $akp = A_1$ plus the area

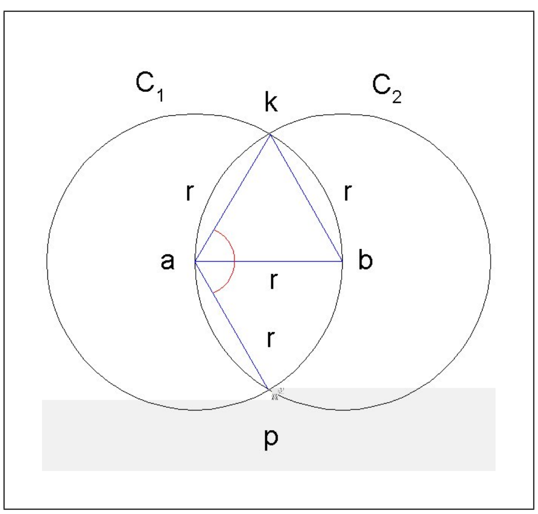


Figure 4: Circles $C1_1$ and C_2

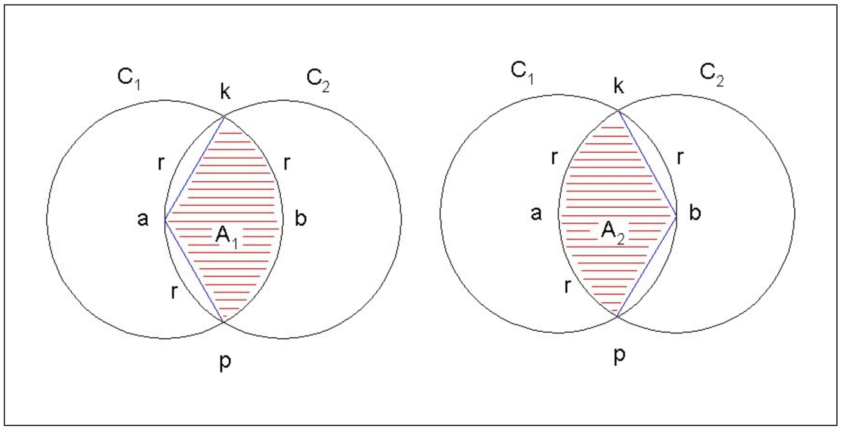


Figure 5: Areas A_1 and A_2

of the sector $bkp = A_2$ minus the area $kapb = 2A_{kab}$ as this area is counted twice when the sectors are added. Thus

$$\begin{aligned}
A_{shaded} &= A_1 + A_2 - 2(A_{kab}) \\
&= \frac{1}{2}r^2\frac{2\pi}{3} + \frac{1}{2}r^2\frac{2\pi}{3} - 2\left(\frac{1}{2}r \times r \times \sin 60\right) \\
&= \frac{2\pi r^2}{3} - r^2\frac{\sqrt{3}}{2} \\
&= r^2\left(\frac{4\pi - 3\sqrt{3}}{6}\right)
\end{aligned}$$

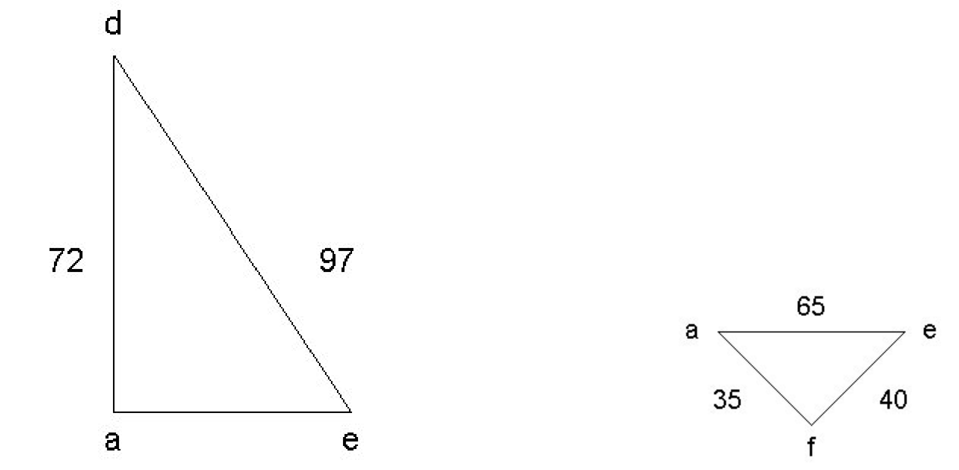
Question 5

(a)

$$\begin{aligned}
\sin 15 &= \sin(45 - 30) \\
&= \sin 45 \cos 30 - \cos 45 \sin 30 \\
&= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \\
&= \frac{\sqrt{3} - 1}{2\sqrt{2}}
\end{aligned}$$

(b)(i)

The solution to this problem is to re-draw it. We are told that d is directly above a and that a and e are on horizontal ground. Thus, this can be re-drawn as shown in figure 6(a). Now the problem is quite simple and, using Pythagoras's theorem



(a) Side view (b) Bottom view

Figure 6: Two views of the problem

$$
\begin{aligned}
|ae|^2 + 72^2 &= 97^2 \\
|ae| &= 65 \text{ m}
\end{aligned}
$$

(b)(ii)

To find the angle $\angle afe$ we use the cosine rule.

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos A \\
\text{Here} \quad |ae|^2 &= |af|^2 + |ef|^2 - 2|af||ef|\cos\angle afe \\
65^2 &= 35^2 + 40^2 - 2(35)(40)\cos\angle afe \\
4225 - 1225 - 1600 &= -2800\cos\angle afe \\
-0.5 &= \cos\angle afe \\
\angle afe &= 120\,^\circ
\end{aligned}
$$

(c)(i)

$$
\begin{aligned}
\cos(A - B) &= \cos A\cos B + \sin A\sin B \quad \text{(given)} \\
\cos\left(\left[\frac{\pi}{2} - A\right] - B\right) &= \cos\left[\frac{\pi}{2} - A\right]\cos B + \sin\left[\frac{\pi}{2} - A\right]\sin B \\
\cos\left(\left[\frac{\pi}{2}\right] - (A + B)\right) &= \sin A\cos B + \cos A\sin B \\
\sin(A + B) &= \sin A\cos B + \cos A\sin B
\end{aligned}
$$

(c)(ii)

$$
\sin(A + B)\sin(A - B) \quad = \quad [\sin A\cos B + \cos A\sin B][\sin A\cos B - \cos A\sin B]
$$

$$
\begin{aligned}
&= \sin^2 A\cos^2 B - \sin A\cos B\cos A\sin B + \sin A\cos B\cos A\sin B \\
&\quad - \cos^2 A\sin^2 B \\
&= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B \\
&= \sin^2 A(1-\sin^2 B) - \cos^2 A\sin^2 B \\
&= \sin^2 A - \sin^2 A\sin^2 B - \cos^2 A\sin^2 B \\
&= \sin^2 A - \sin^2 B(\sin^2 A + \cos^2 A) \\
&= \sin^2 A - \sin^2 B \\
&= (\sin A + \sin B)(\sin A - \sin B) \quad \text{(difference of 2 squares)}
\end{aligned}
$$

Question 6

(a)(i)
If Kieron and Anne are both chosen then this leaves 6 other people with 3 to be chosen. Thus

$$\binom{6}{3} = 20$$

(a)(ii)
If neither Kieron nor Anne are chosen then this leaves 6 other people with 5 to be chosen. Thus

$$\binom{6}{5} = 6$$

(b)(i)

$$
\begin{aligned}
u_{n+2} - 4u_{n+1} + 3U_n &= 0 \quad n \geq 0 \\
x^2 - 4x + 3 &= 0 \quad \text{(characteristic equation)} \\
(x-3)(x-1) &= 0 \\
x &= 3 \quad \text{and} \\
x &= 1 \\
\text{So} \quad u_n &= l(3)^n + m(1)^n \\
u_0 &= l(3)^0 + m(1)^0 \\
-2 &= l + m \\
u_1 &= l(3)^1 + m(1)^1 \\
4 &= 3l + m \\
2l &= 6 \\
l &= 3
\end{aligned}
$$

$$\begin{aligned} m &= -5 \\ u_n &= 3(3)^n - 5(1)^n \\ &= 3(3)^n - 5 \end{aligned}$$

(b)(ii)
Check it for u_0 and u_1.

$$\begin{aligned} u_n &= 3(3)^n - 5 \\ u_0 &= 3(3)^0 - 5 \\ &= 3 - 5 \\ &= -2 \\ u_1 &= 3(3)^1 - 5 \\ &= 9 - 5 \\ &= 4 \end{aligned}$$

(c)(i) This can occur as (7xx) or (x7x) or (xx7) which is

$$\begin{aligned} P(7) &= \frac{1}{10} + \frac{1}{10} + \frac{1}{10} \\ &= \frac{3}{10} \end{aligned}$$

(c)(ii)

$$\begin{aligned} P(\text{odd}) &= \left(\frac{5}{10}\right) \times \left(\frac{4}{9}\right) \times \left(\frac{3}{8}\right) \\ &= \frac{1}{12} \end{aligned}$$

(c)(iii)
This will happen if the numbers drawn are (Odd,Odd,Even) or (Odd,Even,Odd) or (Even,Odd,Odd) or (Even,Even,Even). Thus

$$\begin{aligned} P(\text{even product}) &= \left(\frac{5}{10}\right) \times \left(\frac{4}{9}\right) \times \left(\frac{5}{8}\right) \\ &\quad + \left(\frac{5}{10}\right) \times \left(\frac{5}{9}\right) \times \left(\frac{4}{8}\right) \\ &\quad + \left(\frac{5}{10}\right) \times \left(\frac{5}{9}\right) \times \left(\frac{4}{8}\right) \\ &\quad + \left(\frac{5}{10}\right) \times \left(\frac{4}{9}\right) \times \left(\frac{3}{8}\right) \\ &= \frac{100}{720} + \frac{100}{720} + \frac{100}{720} + \frac{60}{720} \end{aligned}$$

$$= \frac{360}{720}$$
$$= \frac{1}{2}$$

(c)(iv)
This will occur if the numbers are (4,a,b) or (a,4,b) or (a,b,4) where a and b are greater than 4. There are 6 numbers greater than 4. Thus the answer is

$$\begin{aligned} P(\text{smallest no. is 4}) &= 3\left(\frac{1}{10}\right)\times\left(\frac{6}{9}\right)\times\left(\frac{5}{8}\right) \\ &= 3\left(\frac{30}{720}\right) \\ &= \frac{1}{8} \end{aligned}$$

Question 7

(a)(i)
There are $5! = 120$ ways to arrange the cars.

(a)(ii)
There are $\binom{5}{3}$ ways to arrange the group of cars and 3! ways to arrange the cars within the group, thus

$$\begin{aligned} \binom{5}{3}\times 3! &= 10\times 6 \\ &= 60 \end{aligned}$$

(b)(i)
Assuming that you cannot choose 3 points on one line as a triangle you can select 2 of the top 4 and 1 of the bottom 3, or 1 of the top 4 and 2 from the bottom 3. Thus

$$\begin{aligned} \binom{4}{2}\times\binom{3}{1}+\binom{4}{1}\times\binom{3}{2} &= (6\times 3)+(4\times 3) \\ &= 18+12 \\ &= 30 \end{aligned}$$

(b)(ii)
Again, assuming that you cannot choose 3 points on one line as part of a quadrilateral then you can only select 2 from the top line and 2 from the bottom line. Thus

$$\binom{4}{2}\times\binom{3}{2} = (6\times 3)$$

$$= \quad 18$$

(b)(iii)
Looking at the graphic the possible parallelograms are $axyb$,$axzc$,$ayzb$,$byzc$,$bxyc$,$bxzd$,$cyzd$ and $dyxc$. Thus there are 8 possible parallelograms.

(b)(iii)
The probability that it is not a parallelogram is 1 minus the probability that it is a parallelogram. Hence

$$\begin{aligned} P(\text{Not a parallelogram}) &= 1 - \frac{8}{18} \\ &= \frac{5}{9} \end{aligned}$$

(c)(i)

$$\begin{aligned} \overline{x} &= \frac{a+b}{2} \\ \sigma &\triangleq \sqrt{\sum \frac{(x-\overline{x})^2}{n}} \\ &= \sqrt{\frac{(a-\overline{x})^2 + (b-\overline{x})^2}{2}} \end{aligned}$$

(c)(ii)

$$\begin{aligned} a - \overline{x} &= a - \frac{a+b}{2} \\ &= \frac{a-b}{2} \end{aligned}$$

$$\begin{aligned} b - \overline{x} &= b - \frac{a+b}{2} \\ &= \frac{b-a}{2} \end{aligned}$$

$$\begin{aligned} \sigma &= \sqrt{\frac{\left(\frac{a-b}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2}{2}} \\ &= \sqrt{\frac{\left(\frac{a^2-2ab+b^2}{4}\right) + \left(\frac{b^2-2ab+a^2}{4}\right)}{2}} \\ &= \sqrt{\frac{\left(\frac{a^2-2ab+b^2}{2}\right)}{2}} \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\left(\frac{a-b}{2}\right)^2} \\
&= \frac{a-b}{2}
\end{aligned}
$$

(c)(ii)

$$
\begin{aligned}
\overline{x}^2 - \sigma^2 &= \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 \\
&= \frac{a^2 + 2ab + b^2 - (a^2 - 2ab + b^2)}{4} \\
&= \frac{4ab}{4} \\
&= ab
\end{aligned}
$$

Question 8

(a)

$$
\boxed{\int u dv = uv - \int v du}
$$

Let $u = x$ and $dv = e^{-5x}dx$. Then $du = dx$ and $v = (-1/5)e^{-5x}$. So

$$
\begin{aligned}
\int xe^{-5x}dx &= -\frac{xe^{-5x}}{5} - \int \left(\frac{-e^{-5x}}{5}\right) dx \\
&= -\frac{xe^{-5x}}{5} - \frac{e^{-5x}}{25} + C
\end{aligned}
$$

(b)(i)

$$
\begin{aligned}
f(x) &= \ln(1+x) \\
f(0) &= \ln(1+0) \\
&= 0 \\
f'(x) &= \frac{1}{1+x} \\
f'(0) &= \frac{1}{1+0} \\
&= 1 \\
f''(x) &= (-1)(1+x)^{-2} \\
&= -\frac{1}{(1+x)^2}
\end{aligned}
$$

$$
\begin{aligned}
f''(0) &= -1 \\
f'''(x) &= (-1)(-2)(1+x)^{-3} \\
&= \frac{2}{(1+x)^3} \\
f'''(0) &= 2 \\
f''''(x) &= (2)(-3)(1+x)^{-4} \\
&= -\frac{6}{(1+x)^4} \\
f''''(0) &= -6 \\
\text{Hence} \quad f(x) &= 0 + \frac{1x}{1!} + \frac{-1x^2}{2!} + \frac{2x^3}{3!} + \frac{-6x^4}{4!} + \ldots \\
&= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots
\end{aligned}
$$

(b)(i)

$$
u_r = \frac{(-1)^{r+1}x^r}{r}
$$

The Ratio Test checks for

$$
\begin{aligned}
\lim_{r\to\infty}\left(\frac{u_{r+1}}{u_r}\right) &= \lim_{r\to\infty}\left[\frac{\left(\frac{(-1)^{r+2}x^{r+1}}{r+1}\right)}{\left(\frac{(-1)^{r+1}x^r}{r}\right)}\right] \\
&= \lim_{r\to\infty}\left[\frac{(-1)xr}{r+1}\right] \\
&= \lim_{r\to\infty}\left[\frac{-x}{1+\frac{1}{r}}\right] \\
&= -x
\end{aligned}
$$

This will converge if $-1 < x < 1$ as per the Ratio Test Criteria.

(c)(i)

A travels at $12km/h = 0.2km/min$ while B travels at $6km/h = 0.1km/min$. Thus, after x minutes A will have travelled $0.2x$ km and B will have travelled $0.1x$ km.

(c)(ii)

As can be seen from figure 7, using Pythagoras's theorem, the distance l from B to A is

$$
l = \sqrt{(0.2x)^2 + (4 - 0.1x)^2}
$$

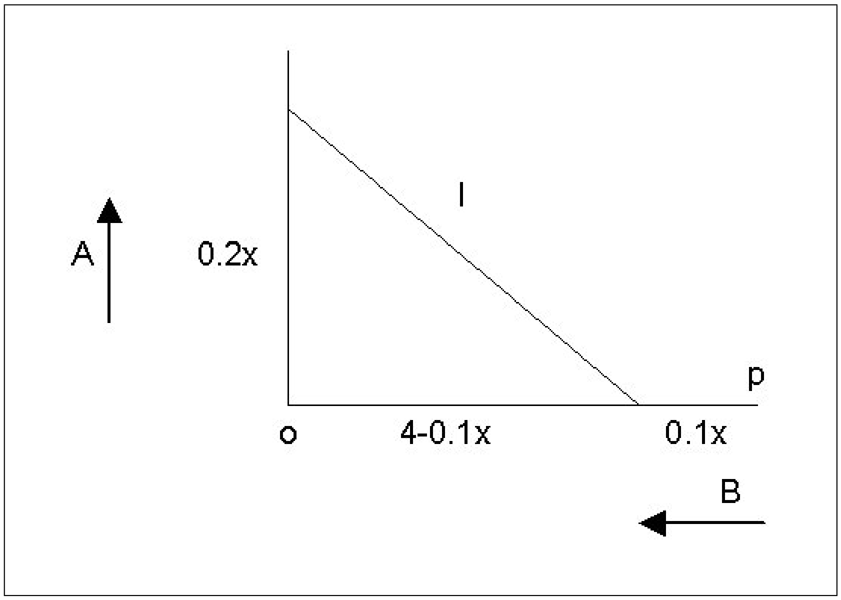


Figure 7: A and B after x minutes

(c)(iii)

B will be closest to A when $\frac{dl}{dx}$ is a minimum.

$$\begin{aligned}
l &= \sqrt{(0.2x)^2 + (4-0.1x)^2} \\
\frac{dl}{dx} &= \frac{1}{2}\left[(0.2x)^2 + (4-0.1x)^2\right]^{-\frac{1}{2}} \times [2(0.2x)(0.2) + 2(4-0.1x)(-0.1)] \\
0 &= 2(0.2x)(0.2) + 2(4-0.1x)(-0.1) \\
0 &= 0.08x - 0.8 + 0.02x \\
0.8 &= 0.1x \\
x &= 8 \text{ minutes}
\end{aligned}$$