Leaving Certificate Honours Level 2003 Paper 1 Solutions

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Introduction

The following are the solutions to Paper 1 of the 2003 Leaving Certificate Honours course.

Question 1

(a)

$$\frac{6y}{x(x+4y)} - \frac{3}{2x} = \frac{2(6y) - 3(x+4y)}{2x(x+4y)} = \frac{-3x}{2x(x+4y)} = \frac{-3}{2(x+4y)}$$

(b)(i)

$$\begin{aligned}
f(x) &= ax^2 + bx + c, \qquad a, b, c \in R \\
f(x) - f(k) &= ax^2 + bx + c - (ak^2 + bk + c) \\
&= a(x^2 - k^2) + b(x - k) \\
&= a(x-k)(x+k) + b(x-k) \\
f(x) - f(k) &= (x-k)[a(x+k) + b]
\end{aligned}$$

However, $f(k) = 0$ so,

$$f(x) = (x-k)[a(x+k) + b]$$

and hence $(x - k)$ is a factor of $f(x)$

(b)(ii)

Let $f(x) = 4x^2 - 2(1+\sqrt{3})x + \sqrt{3}$. If $2x - \sqrt{3}$ is a factor of $f(x)$ then

$x = \sqrt{3}/2$ is a root of $f(x)$ by the Factor Theorem. Thus,

$$\begin{aligned}
f(x) &= 4x^2 - 2(1+\sqrt{3})x + \sqrt{3} \\
f\left(\frac{\sqrt{3}}{2}\right) &= 4\left(\frac{\sqrt{3}}{2}\right)^2 - 2(1+\sqrt{3})\left(\frac{\sqrt{3}}{2}\right) + \sqrt{3} \\
&= 4\left(\frac{3}{4}\right) - \sqrt{3} - 3 + \sqrt{3} \\
f\left(\frac{\sqrt{3}}{2}\right) &= 0
\end{aligned}$$

Since $f(\sqrt{3}/2) = 0$ then $\sqrt{3}/2$ is a root of $f(x)$ and hence $2x - \sqrt{3}$ is a factor of $f(x)$

Let α be the other root. Then the sum of the roots is

$$\begin{aligned}
\alpha + \frac{\sqrt{3}}{2} &= \frac{-(-2(1+\sqrt{3}))}{4} \\
&= \frac{1+\sqrt{3}}{2} \\
\alpha &= \frac{1}{2}
\end{aligned}$$

(c)(i)

Let the roots be α and $\alpha + 2p$. Then,

$$\alpha + \alpha + 2p = -10$$

and

$$\alpha(\alpha + 2p) = c$$

Solving we get

$$\alpha = -5 - p$$

Thus,

$$\begin{aligned}
(-5-p)(-5-p+2p) &= c \\
25 - 5p + 5p - p^2 &= c \\
p^2 &= 25 - c
\end{aligned}$$

(c)(ii)

The roots of the quadratic are

$$\frac{-10 \pm \sqrt{100-4c}}{2} = -5 \pm \sqrt{25-c} = -5 \pm p$$

As $p > 0$, $-5 - p$ must be negative. Thus the positive root must be $-5 + p$. So $-5 + p > 0$ or $p > 5$.

Question 2

(a)

$$\begin{aligned} 3x - y &= 8 \\ x^2 + y^2 &= 10 \end{aligned}$$

Thus

$$\begin{aligned} y &= 3x - 8 \\ x^2 + (3x-8)^2 &= 10 \\ x^2 + 9x^2 - 48x + 64 &= 10 \\ 10x^2 - 48x + 54 &= 0 \\ 5x^2 - 24x + 27 &= 0 \qquad \text{(dividing across by 2)} \\ (5x-9)(x-3) &= 0 \\ (5x-9) &= 0 \qquad \text{thus} \\ x &= \frac{9}{5} \qquad \text{or} \\ (x-3) &= 0 \qquad \text{so} \\ x &= 3 \end{aligned}$$

(b)(i)

$$| \, 4x + 7 \, | < 1$$

The trick to a modulus inequality is to square both sides. This is necessary in order to guarantee that the inequality stays true. So

$$\begin{aligned} (4x+7)^2 &< 1 \\ 16x^2 + 56x + 49 &< 1 \\ 16x^2 + 56x + 48 &< 0 \\ 2x^2 + 7x + 6 &< 0 \qquad \text{(dividing across by 8)} \end{aligned}$$

Now it is necessary to solve for $2x^2 + 7x + 6 = 0$. This will yield 2 roots. So

$$\begin{aligned} 2x^2 + 7x + 6 &= 0 \\ (2x+3)(x+2) &= 0 \\ x &= -\frac{3}{2} \qquad \text{or} \\ x &= -2 \end{aligned}$$

The next step is normally to test a value between the two roots in the original inequality. If the inequality is true for this value then the solution lies *between* the roots. If it doesn't then the solution lies *outside* outside the roots. However, here the values are $-3/2$ and -2 which are very close together. Instead test a value outside the roots, for example 0. Now, if the inequality is true then the solution lies *outside* outside the roots while if it is false the solution lies *between* the roots. Let's test 0:

$$\text{Is } |\, 4(0) + 7 \,| \quad < \quad 1?$$

The inequality is false and thus the solution lies between the roots. Hence

$$-2 < x < -\frac{3}{2}$$

Note that this is always easy to check. Just pop in a couple of values into the original inequality (in your head) and check that it yields what you expect.

(b)(ii)

If $x^2 - ax - 3$ is a factor of $x^3 - 5x^2 + bx + 9$ then it must divide evenly into, i.e. it has a remainder of 0. Thus,

$$\begin{array}{rl} & x + (a-5) \\ \cline{2-2} x^2 - ax - 3 \quad | & x^3 - 5x^2 + bx + 9 \\ & \underline{x^3 - ax^2 - 3x} \\ & (a-5)x^2 + (b+3)x + 9 \\ & \underline{(a-5)x^2 + (a-5)(-a)x - 3(a-5)} \\ & 0 \end{array}$$

As the remainder is 0 ($x^2 - ax - 3$ is a factor), $(b+3)$ must equal $(a-5)(-a) = 5a - a^2$ and 9 must equal $-3(a-5)$. Hence $(a-5) = -3$ so $a = 2$. Thus $(b+3) = (-3)(-2) = 6$, so $b = 3$.

(c)(i)

$$\begin{aligned} 2^{2y+1} - 5(2^y) + 2 &= 0 \qquad \text{Thus,} \\ 2(2^{2y}) - 5(2^y) + 2 &= 0 \qquad \text{or,} \\ 2([2^y]^2) - 5(2^y) + 2 &= 0 \end{aligned}$$

Let $x = 2^y$. Then

$$\begin{aligned} 2x^2 - 5x + 2 &= 0 \\ (2x-1)(x-2) &= 0 \\ (2x-1) &= 0 \qquad \text{thus} \\ x &= \frac{1}{2} \qquad \text{or,} \\ (x-2) &= 0 \qquad \text{so} \\ x &= 2 \end{aligned}$$

Now if $x = 1/2$ then $2^y = 1/2$ or $y = \log_2(1/2)$.Similarly, if $x = 2$ then $y = \log_2(2) = 1$.

(c)(ii) α and β are the solutions (i.e roots) of the equation.

Now,

$$\begin{aligned}
\alpha^2 + \beta^2 &= (\alpha+\beta)^2 - 2\alpha\beta \\
&= \left(-\frac{2kt}{2k^2}\right)^2 - 2\left(\frac{t^2-3k^2}{2k^2}\right) \\
&= \left(-\frac{t}{k}\right)^2 - \left(\frac{t^2-3k^2}{k^2}\right) \\
&= \frac{t^2}{k^2} - \frac{t^2}{k^2} + 3 \\
\alpha^2 + \beta^2 &= 3
\end{aligned}$$

Thus $\alpha^2 + \beta^2$ is independent of k and t.

Question 3

(a)

$$\begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ -7 \end{pmatrix} = 17$$

(b)(i)

If $z = 2 - \iota$ then $z^2 = (2-\iota)^2$. So

$$\begin{aligned}
(2-\iota)^2 &= (2-\iota)(2-\iota) \\
&= 4 - 2\iota - 2\iota + \iota^2 \\
z^2 &= 3 - 4\iota
\end{aligned}$$

Thus

$$\begin{aligned}
z^2 - z + 3 &= (3-4\iota) - (2-\iota) + 3 \\
&= 4 - 3\iota
\end{aligned}$$

And so,

$$\begin{aligned}
\mid z^2 - z + 3 \mid &= \mid 4 - 3\iota \mid \\
&= \sqrt{4^2+3^2} \\
&= 5
\end{aligned}$$

(b)(ii)

$$
\begin{aligned}
k\iota &= \frac{-1+\sqrt{3}\iota}{-4\sqrt{3}-4\iota} \\
&= \left(\frac{-1+\sqrt{3}\iota}{-4\sqrt{3}-4\iota}\right)\left(\frac{-4\sqrt{3}+4\iota}{-4\sqrt{3}+4\iota}\right) \\
&= \left(\frac{4\sqrt{3}-4\iota-12\iota+4\sqrt{3}\iota^2}{(-4\sqrt{3})^2+4^2}\right) \\
&= \left(\frac{-16\iota}{64}\right) \\
&= -\frac{\iota}{4} \\
k &= -\frac{1}{4}
\end{aligned}
$$

(c)(i)

$$
\begin{aligned}
z^3 &= 1 \\
&= 1+0\iota
\end{aligned}
$$

The polar form of $1+0\iota$ is $r(\cos\theta+\iota\sin\theta)$ where $r=|1+0\iota|=\sqrt{1^2+0^2}=1$ and $\theta=tan^{-1}0=0$. Thus $1+0\iota=\cos 0+\iota\sin 0$. The general polar form is $1+0\iota=\cos(2n\pi+0)+\iota\sin(2n\pi+0)$. Hence

$$
\begin{aligned}
1+0\iota &= \cos(2n\pi+0)+\iota\sin(2n\pi+0) \\
(1+0\iota)^{\frac{1}{3}} &= (\cos(2n\pi+0)+\iota\sin(2n\pi+0))^{\frac{1}{3}} \\
&= (\cos(2n\pi)+\iota\sin(2n\pi))^{\frac{1}{3}} \\
&= \cos\left(\frac{2n\pi}{3}\right)+\iota\sin\left(\frac{2n\pi}{3}\right) \qquad \text{(by De Moivre's Theorem)}
\end{aligned}
$$

Let $n=0$. Then $\cos 0+\iota\sin 0=1+0\iota=1$

Let $n=1$. Then $\cos\left(\frac{2\pi}{3}\right)+\iota\sin\left(\frac{2\pi}{3}\right)=-\frac{1}{2}+\frac{\sqrt{3}}{2}\iota=\omega$

Let $n=2$. Then $\cos\left(\frac{4\pi}{3}\right)+\iota\sin\left(\frac{4\pi}{3}\right)=-\frac{1}{2}-\frac{\sqrt{3}}{2}\iota=\omega^2$

So

$$
\begin{aligned}
1+\omega+\omega^2 &= 1+(-\frac{1}{2}+\frac{\sqrt{3}}{2}\iota)+(-\frac{1}{2}-\frac{\sqrt{3}}{2}\iota) \\
&= 1-\frac{1}{2}-\frac{1}{2}+\frac{\sqrt{3}}{2}\iota-\frac{\sqrt{3}}{2}\iota \\
&= 0
\end{aligned}
$$

(c)(ii)

$$\begin{aligned}(1-\omega-\omega^2)^5 &= \left[1-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\iota\right)-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\iota\right)\right]^5 \\ &= \left[1+\frac{1}{2}-\frac{\sqrt{3}}{2}\iota+\frac{1}{2}+\frac{\sqrt{3}}{2}\iota\right]^5 \\ &= 2^5 \\ &= 32\end{aligned}$$

Question 4

(a) Let

$$\begin{aligned}x &= 0.252525\ldots \\ 100x &= 25.252525\ldots \\ 99x &= 25 \\ \\ x &= \frac{25}{99}\end{aligned}$$

(b)(i)

$$\begin{array}{rrcll}\text{Let} & T_1 & = & a & \\ \text{Then} & T_2 & = & a+d & \quad\text{where d is the common difference} \\ \text{Thus} & T_3 & = & a+2d & \\ & T_4 & = & a+3d & \\ & T_5 & = & a+4d & \\ & T_6 & = & a+5d & \end{array}$$

Then

$$\begin{aligned}(a+d)+(a+4d) &= 18 \\ 2a+5d &= 18 \\ (a+5d)-(a+2d) &= 9 \\ 3d &= 9 \\ d &= 3 \\ 2a+15 &= 18 \\ 2a &= 3 \\ a &= \frac{3}{2}\end{aligned}$$

(b)(ii)

$$
\begin{aligned}
S_n &= \frac{n}{2}\left[2a+(n-1)d\right] \\
S_n &= \frac{n}{2}\left[2\left(\frac{3}{2}\right)+(n-1)3\right] \\
&= \frac{n}{2}\left[3+3n-3\right] \\
S_n &= \frac{3n^2}{2} \\
\frac{3n^2}{2} &> 600 \\
3n^2 &> 1200 \\
n^2 &> 400 \\
n^2-400 &> 0 \\
(n-20)^2 &> 0 \\
n-20 &> 0 \\
n &> 20 \\
n &= 21
\end{aligned}
$$

(c)(i)

$u_1 = 2$ and $u_{n+1} = (-1)^n u_n + 3$. Thus,

$$
\begin{aligned}
u_2 &= (-1)^{2-1}u_1 + 3 \\
&= -2+3 \\
&= 1 \\
u_3 &= (-1)^{3-1}u_2 + 3 \\
&= 1+3 \\
&= 4 \\
u_4 &= (-1)^{4-1}u_3 + 3 \\
&= -4+3 \\
&= -1 \\
u_5 &= (-1)^{5-1}u_4 + 3 \\
&= -1+3 \\
&= 2
\end{aligned}
$$

Thus $u_5 = u_1$. Similarly, $u_6 = u_2, u_7 = u_3$etc. The pattern repeats so that $u_9 = u_5$ and $u_{10} = u_6 = u_2 = 1$.

(c)(ii)
Let r be the common ratio of the sequence. Then $r = b/a$ or $b = ar$. Similarly, $r = c/b$ or $c = br = ar^2$. Also $r = d/c$ so $d = cr = ar^3$. So,

$$\begin{aligned}
a^2 - b^2 - c^2 + d^2 &\geq 0 \\
a^2 - (ar)^2 - (ar^2)^2 + (ar^3)^2 &\geq 0 \\
a^2 - a^2r^2 - a^2r^4 + a^2r^6 &\geq 0 \\
a^2\left[1 - r^2 - r^4 + r^6\right] &\geq 0 \\
a^2\left[(r^4 - 1)(r^2 - 1)\right] &\geq 0 \\
a^2(r^2 - 1)(r^2 + 1)(r^2 - 1) &\geq 0 \\
a^2(r^2 - 1)^2(r^2 + 1) &\geq 0
\end{aligned}$$

This is true because all terms are positive. Thus $a^2 - b^2 - c^2 + d^2 \geq 0$.

Question 5

(a)

$$\begin{aligned}
x &= \sqrt{7x - 6} + 2 \\
x - 2 &= \sqrt{7x - 6} \\
(x - 2)^2 &= 7x - 6 \\
x^2 - 4x + 4 &= 7x - 6 \\
x^2 - 11x + 10 &= 0 \\
(x - 10)(x - 1) &= 0 \\
x &= 10 \\
\text{or} \quad x &= 1
\end{aligned}$$

(b)
Step 1: Check $P(1), n = 1$

$$\begin{aligned}
\frac{7^{2(1)+1} + 1}{8} &= \frac{344}{8} \\
&= 43
\end{aligned}$$

Now, for $P(k), n = k$,

$$\frac{7^{2k+1} + 1}{8} = C_1 \qquad C_1 \in N$$

For $P(k + 1), n = k + 1$,

$$\frac{7^{2k+3} + 1}{8} = C_2 \qquad C_2 \in N$$

if 8 is a factor for both $n = k$ and $n = k+1$ then $P(k+1) - P(k) = C$ where $C \in N$.

$$\begin{aligned}
\frac{7^{2k+3}+1}{8} - \frac{7^{2k+1}+1}{8} &= C \\
\frac{7^2 7^{2k+1}+1}{8} - \frac{7^{2k+1}+1}{8} &= C \\
\frac{7^2 7^{2k+1}+1-7^{2k+1}-1}{8} &= C \\
\frac{48(7^{2k+1})}{8} &= C \\
6(7^{2k+1}) &= C
\end{aligned}$$

Since this is also a positive integer then the proposition must be true for $n = k+1$. If it's true for $n = 1$ then it must be true for $n = 2$, $n = 3$ etc.

(c)(i)

$$\begin{aligned}
U_{r+1} &= \binom{8}{r}(ax)^{8-r}\left(\frac{1}{bx}\right)^r \\
&= \binom{8}{r}a^{8-r}\frac{1}{b^r}x^{8-2r}
\end{aligned}$$

(c)(ii)

For the coefficient of x^2, $8 - 2r = 2$ or $r = 3$. For the coefficient of x^4, $8 - 2r = 4$ or $r = 2$. Thus,

$$\begin{aligned}
\binom{8}{3}a^{8-3}\frac{1}{b^3} &= \binom{8}{2}a^{8-2}\frac{1}{b^2} \\
56\frac{a^5}{b^3} &= 28\frac{a^6}{b^2} \\
2\frac{a^5}{a^6} &= \frac{b^3}{b^2} \\
\frac{2}{a} &= b \\
2 &= ab
\end{aligned}$$

Question 6

(a)

$$\begin{aligned}
\frac{d}{dx}\left(\sqrt{1+4x}\right) &= \frac{1}{2}(1+4x)^{-\frac{1}{2}} \times 4 \\
&= \frac{2}{\sqrt{1+4x}}
\end{aligned}$$

(b)

$$\begin{aligned} 2^3 - (4 \times 2) - 2 &= -2 < 0 \\ 3^3 - (4 \times 3) - 2 &= 13 > 0 \end{aligned}$$

Thus, $x^3 - 4x - 2$ crosses the x-axis somewhere between 2 and 3.

The Newton-Rhapson method is defined as

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x)}}$$

Now $f'(x) = 3x^2 - 4$. Hence

$$\begin{aligned} x_2 &= 2 - \left[\frac{2^3 - 4(2) - 2}{3(2^2) - 4}\right] \\ &= 2 - \left[\frac{-2}{8}\right] \\ &= \frac{9}{4} \end{aligned}$$

And

$$\begin{aligned} x_3 &= \frac{9}{4} - \left[\frac{\left(\frac{9}{4}\right)^3 - 4(\frac{9}{4}) - 2}{3(\frac{9}{4})^2 - 4}\right] \\ &= 2.22 \end{aligned}$$

(c)(i)

$$\begin{aligned} f(x) &= \frac{1}{1-x} \\ f'(x) &= (-1)(1-x)^{-2}(-1) \\ &= \frac{1}{(1-x)^2} \end{aligned}$$

This can never equal 0 and thus the function $f(x)$ has no maximum or minimum points.

$$\begin{aligned} f''(x) &= (-2)(1-x)^{-3}(-1) \\ &= \frac{2}{(1-x)^3} \end{aligned}$$

Again, this can never be 0 and thus the function $f(x)$ has no points of inflection.

(c)(ii)

$f'(x)$ is the slope of the tangent to the curve at any point on the curve. Because $f'(x) = 1/(1-x)^2$ it is always positive and thus the slope is always increasing.

(c)(iii)

The slope of the tangent $y = x + k$ is 1. Thus

$$\begin{aligned} \frac{1}{(1-x)^2} &= 1 \\ (1-x)^2 &= 1 \\ 1-x &= \pm 1 \\ x &= 0, 2 \end{aligned}$$

When $x = 0, f(0) = 1/(1-0) = 1$ and $y = 1(0) + k = 1$ yielding $k = 1$. When $x = 2, f(2) = 1/(1-2) = -1$ and $y = 1(2) + k = -1$ yielding $k = -3$.

Question 7

(a)(i)

$$\begin{aligned} \frac{d}{dx}\left(\cos^4 x\right) &= 4(\cos^3 x)(-\sin x) \\ &= -4\cos^3 x \sin x \end{aligned}$$

(a)(ii)

$$\begin{aligned} \frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{5}\right)\right] &= \frac{1}{\sqrt{1-\left(\frac{x}{5}\right)^2}} \times \frac{1}{5} \\ &= \frac{1}{\sqrt{25-x^2}} \end{aligned}$$

(b)(i)

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ \frac{dy}{dt} &= \cos t - [t(-\sin t) + \cos t] \\ &= t \sin t \\ \frac{dx}{dt} &= -\sin t + [t\cos t + \sin t] \\ &= t\cos t \\ \frac{dt}{dx} &= \frac{1}{t\cos t} \end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{t\sin t}{t\cos t}\\ &= \tan t\end{aligned}$$

(b)(ii)

$$\begin{aligned}\frac{1}{x}+\frac{1}{y} &= \frac{1}{6}\\ (-1)x^{-2}+(-1)y^{-2}\frac{dy}{dx} &= 0\\ -\frac{1}{y^2}\frac{dy}{dx} &= \frac{1}{x^2}\\ \frac{dy}{dx} &= -\frac{y^2}{x^2}\end{aligned}$$

So at the point $(2,-3)$ $\quad dy/dx=-9/4$.

(c)(i)

$$\begin{aligned}y &= \ln\left(\frac{1+x^2}{1-x^2}\right)\\ \frac{dy}{dx} &= \frac{1}{\left(\frac{1+x^2}{1-x^2}\right)}\times\left[\frac{(1-x^2)(2x)-(1+x^2)(-2x)}{(1-x^2)^2}\right]\\ &= \left(\frac{1-x^2}{1+x^2}\right)\left[\frac{4x}{(1-x^2)^2}\right]\\ &= \frac{4x}{(1+x^2)(1-x^2)}\\ &= \frac{4x}{1-x^4}\end{aligned}$$

(c)(ii)

$$\begin{aligned}f(\theta) &= \sin(\theta+\pi)\cos(\theta-\pi)\\ f'(\theta) &= \sin(\theta+\pi)(-\sin(\theta-\pi))+\cos(\theta-\pi)\cos(\theta+\pi)\\ &= \cos(\theta+\pi)\cos(\theta-\pi)-\sin(\theta+\pi)\sin(\theta-\pi)\\ &= [\cos\theta\cos\pi-\sin\theta\sin\pi]\,[\cos\theta\cos\pi+\sin\theta\sin\pi]\\ &\quad -[\sin\theta\cos\pi+\cos\theta\sin\pi]\,[\sin\theta\cos\pi-\cos\theta\sin\pi]\\ &= (-\cos\theta)(-\cos\theta)-[(-\sin\theta)(-\sin\theta)]\\ &= \cos^2\theta-\sin^2\theta\\ &= \cos 2\theta\end{aligned}$$

Question 8

(a)(i)

$$\int (x^3+2)dx = \frac{x^4}{3} + 2x + C$$

(a)(ii)

$$\int e^{7x}dx = \frac{e^{7x}}{7} + C$$

(b)(i)
Let $u = 1+x^2$. Then $du = 2xdx$. When $x = 0, u = 1+(0)^2 = 1$ and similarly when $x = 1, u = 2$. Thus we have

$$\begin{aligned}\int_1^2 \frac{du}{\sqrt{u}} &= 2\sqrt{u}\,\Big|_1^2 \\ &= 2(\sqrt{2}-1)\end{aligned}$$

(b)(ii)
Let $u = \sin x$. Then $du = \cos x dx$. When $x = 0, u = 0$ and when $x = \pi/2, u = \sin(\pi/2) = 1$. Thus we have

$$\begin{aligned}\int_0^1 u^6 du &= \frac{u^7}{7}\,\Big|_0^1 \\ &= \frac{1}{7}\end{aligned}$$

(c)(i)

$$\begin{aligned}\int_a^{2a} \sin 2x dx &= -\frac{1}{2}\cos 2x\,\Big|_a^{2a} \\ &= -\frac{1}{2}[\cos 4a - \cos 2a] \\ &= -\frac{1}{2}\left[-2\sin\left(\frac{4a+2a}{2}\right)\sin\left(\frac{4a-2a}{2}\right)\right] \\ &= \sin\left(\frac{6a}{2}\right)\sin\left(\frac{2a}{2}\right) \\ &= \sin 3a \sin a\end{aligned}$$

(c)(ii) The volume of a solid revolving about the x-axis is given by

$$\boxed{V = \int_a^b \pi y^2 dx}$$

The equation of a circle with center $(0,0)$ and radius r is $x^2+y^2=r^2$. Thus, $y^2=r^2-x^2$. Thus the volume of a hemisphere is

$$\begin{aligned}
V_{hs} &= \int_0^r \pi y^2 dx \\
&= \int_0^r \pi(r^2-x^2)dx \\
&= \pi\left(r^2x-\frac{x^3}{3}\right)\Big|_0^r \\
&= \pi\left(r^3-\frac{r^3}{3}\right) \\
&= \frac{2\pi r^3}{3}
\end{aligned}$$

Thus the volume of a sphere which is twice that of the hemisphere is $4\pi r^3/3$.