Leaving Certificate Honours Level 2002 Paper 2 Solutions

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Introduction

The following are the solutions to Paper 2 of the 2002 Leaving Certificate Honours course.

Question 1

(a)

$$
\begin{aligned}
x &= 4 + 3\cos\theta \\
\therefore \frac{x-4}{3} &= \cos\theta \\
\frac{(x-4)^2}{9} &= \cos^2\theta \\
y &= -2 + 3\sin\theta \\
\therefore \frac{y+2}{3} &= \sin\theta \\
\frac{(y+2)^2}{9} &= \sin^2\theta \\
\frac{(x-4)^2}{9} + \frac{(y+2)^2}{9} &= \cos^2\theta + \sin^2\theta \\
\frac{(x-4)^2}{9} + \frac{(y+2)^2}{9} &= 1 \\
(x-4)^2 + (y+2)^2 &= 9 \\
x^2 - 8x + 16 + y^2 + 4y + 4 &= 9 \\
x^2 + y^2 - 8x + 4y + 11 &= 0
\end{aligned}
$$

(b)(i)
Figure 1 shows a sketch of triangle abc. If there is a right-angle at c then the angle must be $90°$. If this is true then ac is perpendicular to bc and so the

product of their slope is -1.Let's find out if it is.

$$\begin{aligned} m_{ac} &= \frac{-2-6}{4-(-2)} \\ &= \frac{-8}{6} \\ &= -\frac{4}{3} \end{aligned}$$

$$\begin{aligned} m_{bc} &= \frac{0-6}{-10-(-2)} \\ &= \frac{-6}{-8} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} m_{ac} \times m_{bc} &= -\frac{4}{3} \times \frac{3}{4} \\ &= -1 \end{aligned}$$

So the lines are perpendicular and the angle $90°$

(b)(ii)
As there is a right-angle at c, ab must be the diameter of the circle. So the midpoint of ab is the center of the circle. This is $((-2+0)/2, (4-10)/2)$ or $(-1,-3)$ and is shown in figure 1.

The radius r of the circle is half the diameter[1] so we have

$$\begin{aligned} r &= \frac{1}{2}\sqrt{(-2-0)^2+(4+10)^2} \\ &= \frac{1}{2}\sqrt{200} \\ &= \sqrt{50} \end{aligned}$$

Then the equation becomes

$$\begin{aligned} (x-(-1))^2+(y-(-3))^2 &= (\sqrt{50})^2 \\ (x+1)^2+(y+3)^2 &= 50 \\ x^2+y^2+2x+6y-40 &= 0 \end{aligned}$$

[1]It is better to use the diameter than to calculate the radius using the center in case a mistake was made getting the center

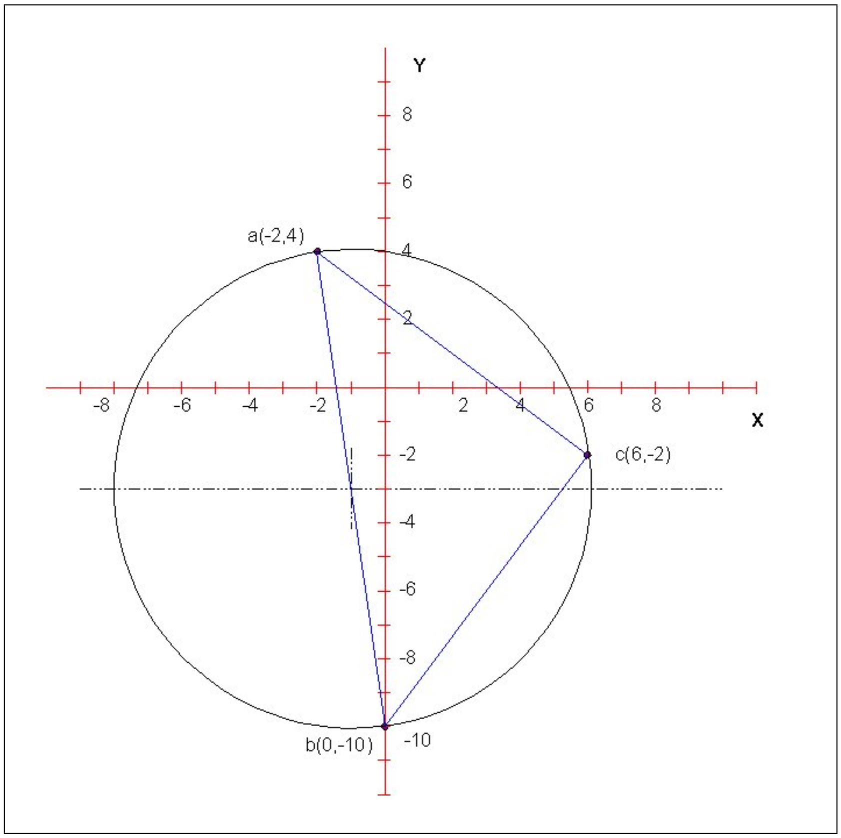

Figure 1: Triangle abc and circle

(c)(i)

Figure 2 shows the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ and the lines L and M. The center of the circle is $(2, -3)$ and the radius is $\sqrt{2^2 + (-3)^2 + 12} = 5$. Let a be the center of $|pq|$, b be the center of $|ts|$ and o the center of the circle. So

$$\begin{aligned} |op| &= 5 \\ |ap| &= 4 \end{aligned}$$

Using Pythagoras's theorem

$$\begin{aligned} |oa| &= \sqrt{5^2 - 4^2} \\ |oa| &= 3 \end{aligned}$$

The same method can be used to show that $|ob| = 3$.

(c)(ii)

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= m(x + 4) \\ mx - y + 4m &= 0 \end{aligned}$$

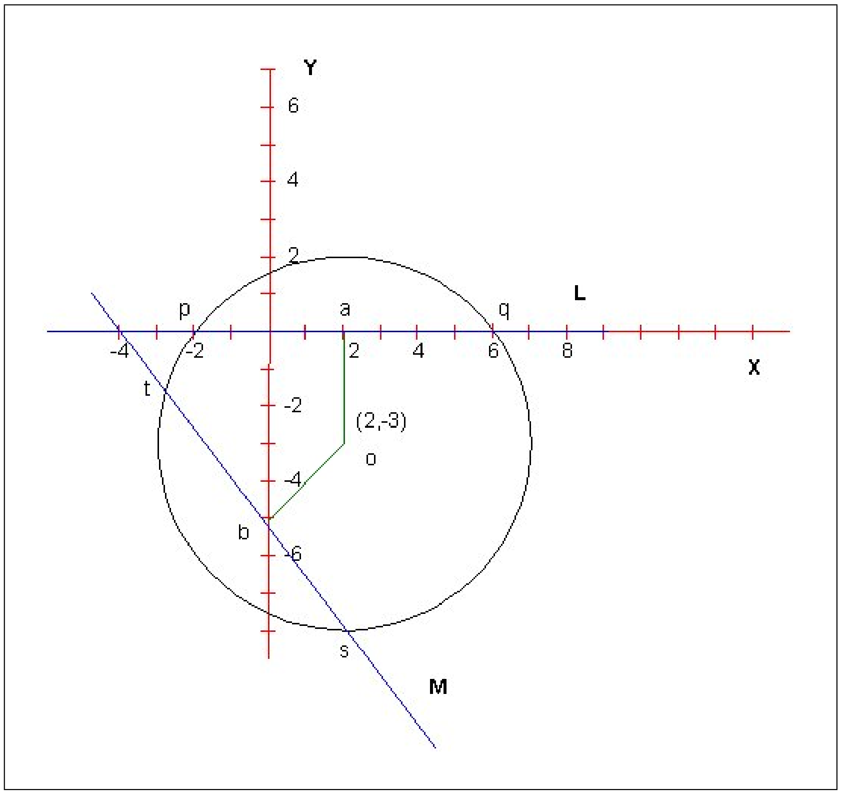

Figure 2: Tangent and Circle

This is the equation of lines L and M with co-efficient's $a = m, b = -1$ and $c = 4m$. Of course m is different for both lines. Also, the perpendicular distance from $(2, -3)$ to L, M is 3. So

$$\begin{aligned}
\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right| &= 3 \\
\left|\frac{m(2) + (-1)(-3) + 4m}{\sqrt{m^2 + 1}}\right| &= 3 \\
6m + 3 &= 3\sqrt{m^2 + 1} \\
(2m + 1)^2 &= m^2 + 1 \\
4m^2 + 4m + 1 &= m^2 + 1 \\
m(3m + 4) &= 0 \\
m &= 0 \\
m &= \frac{-4}{3}
\end{aligned}$$

So the equations are

$$\begin{aligned} y &= 0(x+4) \\ y &= 0 \end{aligned}$$

$$\begin{aligned} y &= \frac{-4}{3}(x+4) \\ 3y &= -4x-16 \\ 4x+3y+16 &= 0 \end{aligned}$$

Question 2

(a)

$\overrightarrow{s} = 4\overrightarrow{i} - 3\overrightarrow{j}$ and $\overrightarrow{t} = 2\overrightarrow{i} - 5\overrightarrow{j}$. Now

$$\begin{aligned} \overrightarrow{st} &= \overrightarrow{t} - \overrightarrow{s} \\ &= 2\overrightarrow{i} - 5\overrightarrow{j} - (4\overrightarrow{i} - 3\overrightarrow{j}) \\ &= -2\overrightarrow{i} - 2\overrightarrow{j} \\ |\overrightarrow{st}| &= \sqrt{-2^2+2^2} \\ &= \sqrt{8} \end{aligned}$$

(b)(i)

$$\begin{aligned} |oq| &= \frac{1}{2}|oc| \\ |ap| &= \frac{3}{4}|ab| \\ &= \frac{3}{4}|oc| \\ \frac{|oq|}{|ap|} &= \frac{\frac{1}{2}|oc|}{\frac{3}{4}|ab|} \\ &= \frac{2}{3} \end{aligned}$$

Now, $\angle orq = \angle arp$ (opposite), $\angle oqr = \angle par$ (alternate) and $\angle roq = \angle rpa$ (alternate). Thus $\triangle orq = \triangle pra$. But since $|oq| = \frac{2}{3}|ap|$ then the ratio of $|or|$ to $|rp|$ must also be $\frac{2}{3}$.

(b)(ii)

$$\begin{aligned}
\overrightarrow{op} &= \overrightarrow{oa} + \frac{3}{4}\overrightarrow{ab} \\
\overrightarrow{p} &= \overrightarrow{a} + \frac{3}{4}\left(\overrightarrow{b} - \overrightarrow{a}\right) \\
&= \frac{\overrightarrow{3b} + \overrightarrow{a}}{4}
\end{aligned}$$

Since $|or| : |rp| = \frac{2}{3}$, $|or| : |op| = \frac{2}{5}$. Thus

$$\begin{aligned}
\overrightarrow{or} &= \frac{2}{5}\overrightarrow{op} \\
\overrightarrow{r} &= \frac{2}{5}\left(\frac{\overrightarrow{3b} + \overrightarrow{a}}{4}\right) \\
&= \left(\frac{\overrightarrow{3b} + \overrightarrow{a}}{10}\right)
\end{aligned}$$

(c)(i)

$$\begin{aligned}
\overrightarrow{kn} &= \overrightarrow{n} - \overrightarrow{k} \\
&= 4\overrightarrow{i} - 2\overrightarrow{j} - \overrightarrow{i} - 3\overrightarrow{j} \\
&= 3\overrightarrow{i} - 5\overrightarrow{j}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{kv} &= \overrightarrow{v} - \overrightarrow{k} \\
&= x\overrightarrow{i} + y\overrightarrow{j} - \overrightarrow{i} - 3\overrightarrow{j} \\
&= (x-1)\overrightarrow{i} + (y-3)\overrightarrow{j}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{kn}.\overrightarrow{kv} &= (3\overrightarrow{i} - 5\overrightarrow{j}).((x-1)\overrightarrow{i} + (y-3)\overrightarrow{j}) \\
&= 3(x-1) - 5(y-3) \\
&= 3x - 5y + 12
\end{aligned}$$

(c)(ii)

$$\begin{aligned}
\overrightarrow{kn}.\overrightarrow{kv} &= \overrightarrow{kn}.\overrightarrow{ku}\\
\overrightarrow{kn}.\overrightarrow{kv} - \overrightarrow{kn}.\overrightarrow{ku} &= 0\\
\overrightarrow{kn}.(\overrightarrow{kv} - \overrightarrow{ku}) &= 0\\
\overrightarrow{kn}.(\overrightarrow{v} - \overrightarrow{k} - \overrightarrow{u} + \overrightarrow{k}) &= 0\\
\overrightarrow{kn}.(\overrightarrow{v} - \overrightarrow{u}) &= 0\\
\overrightarrow{kn}.\overrightarrow{uv} &= 0
\end{aligned}$$

So $\overrightarrow{kn}$ is $\perp$ to $\overrightarrow{uv}$ as their dot product is equal to 0.

Question 3

(a)
The midpoint of (a, b) is $\left(\frac{-1+5}{2}, \frac{4-4}{2}\right) = (2, 0)$.

The slope of (a, b) is $\frac{-4-4}{5+1} = \frac{-4}{3}$.

Since the product of the slopes of two $\perp$ lines is always -1, the slope of the $\perp$ bisector is $\frac{3}{4}$. Hence the equation of the line containing $(2, 0)$ with slope of $\frac{3}{4}$ is

$$\begin{aligned}
y - y_1 &= m(x - x_1)\\
y - 0 &= \frac{3}{4}(x - 2)\\
4y &= 3x - 6\\
3x - 4y - 6 &= 0
\end{aligned}$$

(b)(i)

$$\begin{aligned} f(0,0) &= (3(0)+0, 0-2(0)) \\ &= (0,0) \end{aligned}$$

$$\begin{aligned} f(1,0) &= (3(1)+0, 1-2(0)) \\ &= (3,1) \end{aligned}$$

$$\begin{aligned} f(1,1) &= (3(1)+1, 1-2(1)) \\ &= (4,-1) \end{aligned}$$

$$\begin{aligned} f(0,1) &= (3(0)+1, 0-2(1)) \\ &= (1,-2) \end{aligned}$$

(b)(ii)

$$\begin{aligned} x' &= 3x+y \\ 2x' &= 6x+2y \end{aligned}$$

$$\begin{aligned} y' &= x-2y \\ 2x'+y' &= 7x \\ x &= \frac{2x'+y'}{7} \end{aligned}$$

$$\begin{aligned} x' &= 3\left(\frac{2x'+y'}{7}\right)+y \\ &= \left(\frac{6x'+3y'}{7}\right)+y \\ 7x' &= 6x'+3y'+7y \\ x'-3y' &= 7y \\ y &= \frac{x'-3y'}{7} \end{aligned}$$

(b)(iii)
Lines $ax + by + c$ and $ax + by + d$ are parallel as their slopes are both $-a/b$.

$$\begin{aligned} f(ax + by + c) &= a\left(\frac{2x' + y'}{7}\right) + b\left(\frac{x' - 3y'}{7}\right) + c \\ &= \left(\frac{2a + b}{7}\right)x' + \left(\frac{a - 3b}{7}\right)y' + c \end{aligned}$$

The slope of this line is $-\frac{2a+b}{a-3b}$.

$$\begin{aligned} f(ax + by + d) &= a\left(\frac{2x' + y'}{7}\right) + b\left(\frac{x' - 3y'}{7}\right) + d \\ &= \left(\frac{2a + b}{7}\right)x' + \left(\frac{a - 3b}{7}\right)y' + d \end{aligned}$$

The slope of this line is $-\frac{2a+b}{a-3b}$ which is the same as the slope of $f(ax+by+c)$ and hence the transformed lines are parallel. Hence every pair of parallel lines is mapped to a pair of parallel lines.

(b)(iv)

See figure 3.

(b)(v)

$f(S)$ is a parallelogram so if a diagonal line is drawn from $(0, 0)$ to $(4, -1)$ it bisects $f(S)$ into 2 equal area triangles. So getting the area of one triangle and doubling it will give the area of $f(S)$. The area of a triangle with one point at $(0, 0)$ is

$$\begin{aligned} A_\triangle &= \frac{1}{2}|x_1y_2 - x_2y_1| \\ &= \frac{1}{2}|1(-1) - 4(-2)| \\ &= \frac{7}{2} \end{aligned}$$

So the area of $f(S)$ is twice this, i.e. 7.

Question 4

(a)
First find the angle in the first quadrant whose cosine is $\sqrt{3}/2$. From page 9 of the tables this angle is $\pi/6 = 30°$. Since the cosine is negative this places the angle in the second or fourth quadrants. However, the specified limits are from $0°$ to $180°$ so it must be the second quadrant. In this quadrant the actual angle is $180 - 30 = 150°$.

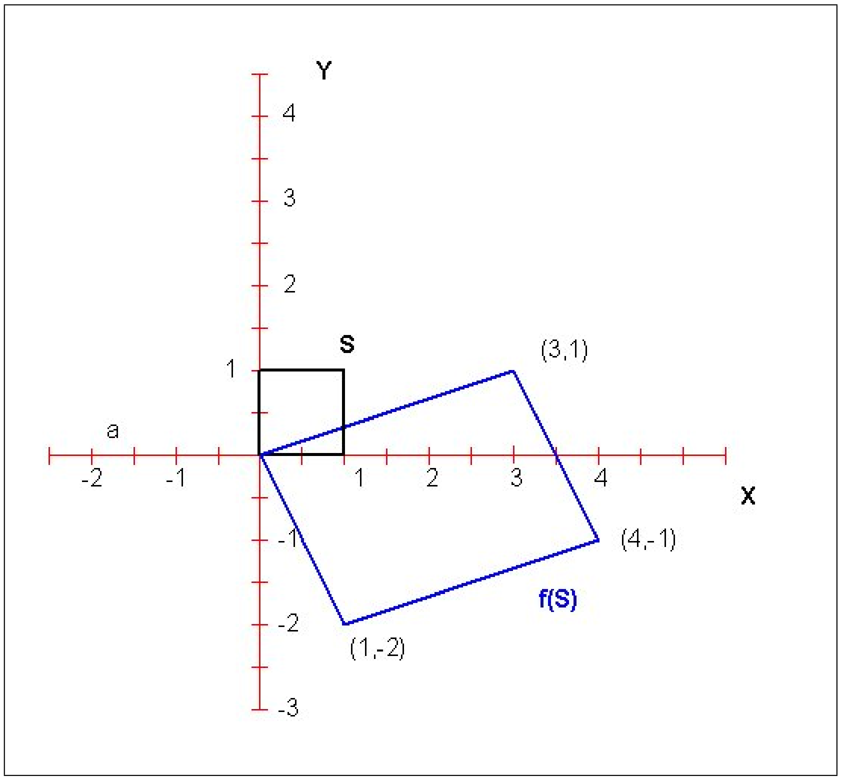

Figure 3: Squares S and $f(S)$

(b)(i)

$$\begin{aligned}
\sin^2 A &= \frac{1}{2}\left(1-\cos 2A\right) \\
\sin^2\left(\frac{x}{2}\right) &= \frac{1}{2}\left(1-\cos 2\left[\frac{x}{2}\right]\right) \\
&= \frac{1}{2}\left(1-\cos x\right)
\end{aligned}$$

(b)(ii)

$$\begin{aligned}
\sin^2\left(\frac{x}{2}\right) - \cos^2 x &= 0\\
\frac{1}{2}(1-\cos x) - \cos^2 x &= 0\\
1 - \cos x - 2\cos^2 x &= 0\\
2\cos^2 x + \cos x - 1 &= 0\\
(2\cos x - 1)(\cos x + 1) &= 0\\
2\cos x - 1 &= 0\\
2\cos x &= 1\\
\cos x &= \frac{1}{2}\\
x &= 60^\circ \text{ and } 300^\circ
\end{aligned}$$

$$\begin{aligned}
\cos x + 1 &= 0\\
\cos x &= -1\\
x &= 180^\circ
\end{aligned}$$

So the answers are 60°, 180° and 300°.

(c)(i)

Let h be the point shown in figure 4. Then, since ae is parallel to bf, $|ah| = 15$ and so $|he| = 60$. Thus, if we let $E = \angle aef$

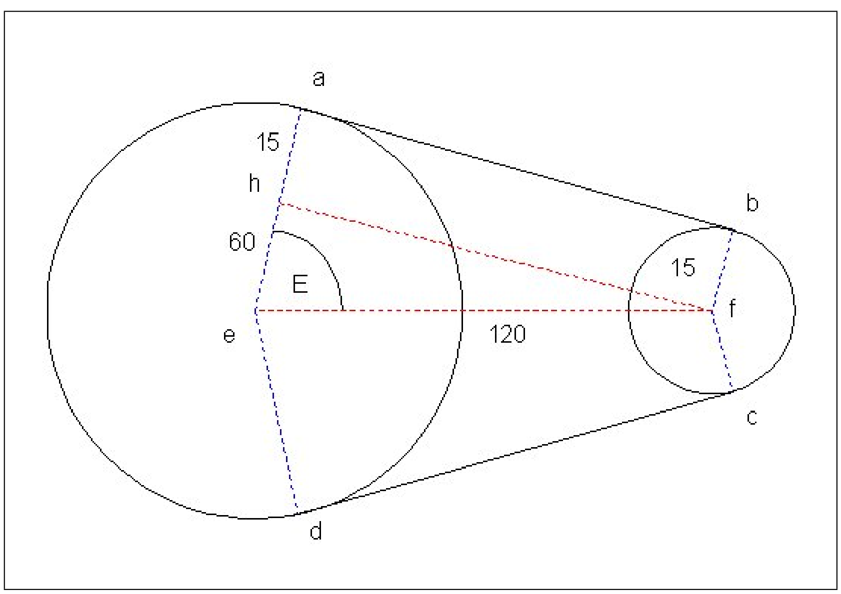

Figure 4: Chain

$$
\begin{aligned}
\cos E &= \frac{|eh|}{|ef|} \\
&= \frac{60}{120} \\
&= \frac{1}{2} \\
E &= 60^\circ
\end{aligned}
$$

(c)(ii)
Using Pythagoras's theorem

$$
\begin{aligned}
|hf|^2 + |eh|^2 &= 120^2 \\
|hf| &= \sqrt{120^2 - 60^2} \\
&= \sqrt{10800} \\
&= 10\sqrt{108} \\
&= 20\sqrt{27} \\
&= 60\sqrt{3}
\end{aligned}
$$

Since $ahfb$ forms a square, $|ab| = |hf| = 60\sqrt{3}$.

(c)(iii)
The length of the chain is equal to $|ad| + |ab| + |bc| + |cd|$. Also, $|ab| = |cd| = 60\sqrt{3}$ from above. Now, $\angle aed = 2\angle aef = 120^\circ = 2\pi/3$ radians. Thus, $|ad| = r\theta = 75(2\pi/3) = 50\pi$ cm.

Now, $\angle hfe = 30^\circ$ so $\angle bfc = 90 + 30 + 30 + 90 = 240^\circ$ (angle towards the big chain). Hence the alternate angle is $360 - 240 = 120^\circ$ and thus $|bc| = r\theta = 15(2\pi/3) = 10\pi$ cm. Hence the total chain length is $120\sqrt{3} + 60\pi$ cm.

Question 5

(a)

$$
\begin{aligned}
A_\triangle &= \frac{1}{2}|ab||bc|\sin B \\
12 &= \frac{1}{2}(8)|bc|\sin 30 \\
&= 2|bc| \\
|bc| &= 6\,\text{cm}
\end{aligned}
$$

(b)(i)
Firstly $\tan A = \sin A / \cos A$. So

$$\begin{aligned}
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\
&= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B} \\
&= \frac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B} - \frac{\sin A\sin B}{\cos A\cos B}} \\
&= \frac{\tan A + \tan B}{1 - \tan A\tan B}
\end{aligned}$$

(b)(ii)

$$\begin{aligned}
\tan(22.5+22.5) &= \frac{\tan 22.5 + \tan 22.5}{1 - \tan 22.5\tan 22.5} \\
\tan(45) &= \frac{\tan 22.5 + \tan 22.5}{1 - \tan^2 22.5} \\
1 &= \frac{2\tan 22.5}{1 - \tan^2 22.5}
\end{aligned}$$

Let $x = \tan 22.5$. Then

$$\begin{aligned}
1 &= \frac{2x}{1-x^2} \\
1 - x^2 &= 2x \\
x^2 + 2x - 1 &= 0 \\
x = \tan 22.5 &= \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2} \\
&= \frac{-2 \pm 2\sqrt{2}}{2} \\
\tan 22.5 &= -1 \pm \sqrt{2}
\end{aligned}$$

However, as $22.5^\circ < 90^\circ$, the tan is positive and so $\tan 22.5 = \sqrt{2} - 1$.

(c)(i)
The solution to this problem is to re-draw it. We know that q is directly above p (vertical mast) and that a, b and c are on horizontal ground. Thus, this can be re-drawn as shown in figure 5(a). Now the problem is quite simple and, using Pythagoras's theorem

$$\begin{aligned}
|ap| = |bp| = |cp| &= \sqrt{52^2 - 48^2} \\
&= 20 \text{ m}
\end{aligned}$$

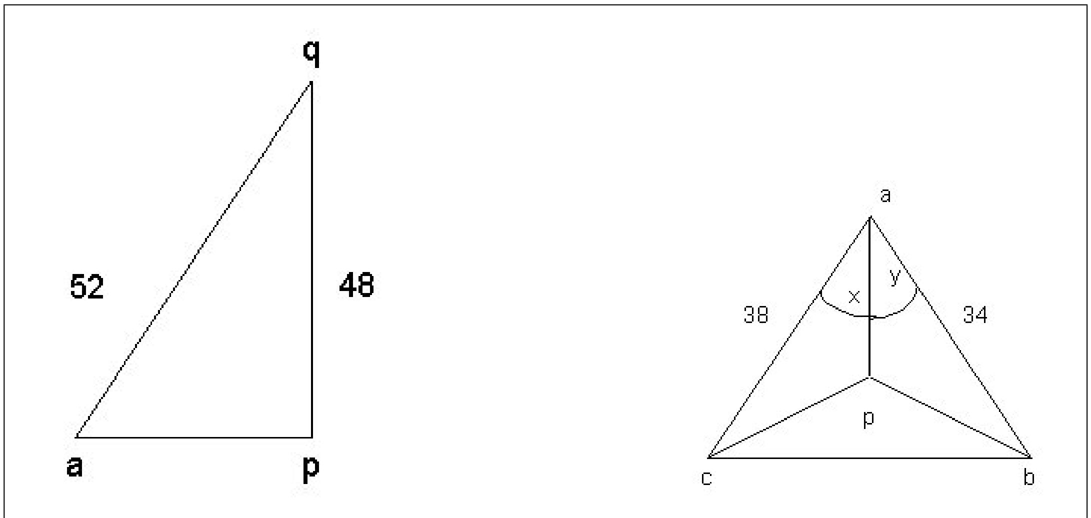

Figure 5: Two views of the problem

(c)(ii)
We first find the angle $\angle pac = x$ using the cosine rule.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ \text{So} \quad \cos A &= \frac{b^2+c^2-a^2}{2bc} \\ \cos x &= \frac{38^2+20^2-20^2}{2(38)(20)} \\ &= 0.95 \\ x &= \cos^{-1}(0.95) \\ &= 18.19\ ^\circ \end{aligned}$$

Similarly

$$\begin{aligned} \cos y &= \frac{20^2+34^2-20^2}{2(34)(20)} \\ y &= 31.79\ ^\circ \end{aligned}$$

Then let $z = x + y$ and so $z = 18.19 + 31.79 = 49.98°$. Thus

$$\begin{aligned} |bc|^2 &= 38^2 + 34^2 - 2(38)(34)\cos 49.98 \\ &= 30.6 \quad \text{m} \end{aligned}$$

Question 6

(a)(i)
If John and Mary are both chosen then this leaves 7 other people with 3 to

be chosen. Thus

$$\binom{7}{3} = 35$$

(a)(ii)

If either John or Mary are chosen but not both then this leaves 7 other people with 4 to be chosen for both cases. Thus

$$\begin{aligned}\binom{7}{4} + \binom{7}{4} &= 35 + 35 \\ &= 70\end{aligned}$$

(a)(iii)

If at least one driver is chosen then this is the opposite of no drivers chosen (i.e. total no. of possible selections minus those where no driver is chosen). For no driver to be chosen 5 people must be selected from the 6 who do not drive. So

$$\begin{aligned}\binom{9}{5} - \binom{6}{5} &= 126 - 6 \\ &= 120\end{aligned}$$

(b)(i)

$$
\begin{aligned}
6u_{n+2} - 5u_{n+1} + u_n &= 0 \quad n \geq 0 \\
6x^2 - 5x + 1 &= 0 \quad \text{(characteristic equation)} \\
(3x-1)(2x-1) &= 0 \\
x &= \frac{1}{3} \quad \text{and} \\
x &= \frac{1}{2} \\
\text{So} \quad u_n &= l\left(\frac{1}{3}\right)^n + m\left(\frac{1}{2}\right)^n \\
u_0 &= l\left(\frac{1}{3}\right)^0 + m\left(\frac{1}{2}\right)^0 \\
5 &= l + m \\
10 &= 2l + 2m \\
u_1 &= l\left(\frac{1}{3}\right)^1 + m\left(\frac{1}{2}\right)^1 \\
2 &= \frac{l}{3} + \frac{m}{2} \\
12 &= 2l + 3m \\
\text{So} \quad 12 - 10 &= m \\
m &= 2 \\
l &= 5 - m \\
l &= 3 \\
u_n &= 3\left(\frac{1}{3}\right)^n + 2\left(\frac{1}{2}\right)^n \\
&= 3^{1-n} + 2^{1-n}
\end{aligned}
$$

(b)(ii)

$$
u_n = 3\left(\frac{1}{3}\right)^n + 2\left(\frac{1}{2}\right)^n
$$

The S_n of a geometric with a common ratio less than one is given by

$$
S_n = \frac{a(1-r)^n}{1-r}
$$

So S_n is given by

$$\begin{aligned} S_n &= 3\left(\frac{1}{3}\right)^n + 2\left(\frac{1}{2}\right)^n \\ &= 3\left(\frac{1-(\frac{1}{3})^n}{1-\frac{1}{3}}\right) + 2\left(\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}\right) \\ &= \frac{9}{2}\left[1-\left(\frac{1}{3}\right)^n\right] + 4\left[1-\left(\frac{1}{2}\right)^n\right] \end{aligned}$$

(b)(iii)

$$\begin{aligned} S_\infty &= \lim_{n\to\infty} S_n \\ &= \lim_{n\to\infty}\left(\frac{9}{2}\left[1-\left(\frac{1}{3}\right)^n\right] + 4\left[1-\left(\frac{1}{2}\right)^n\right]\right) \\ &= \frac{9}{2}[1-0] + 4[1-0] \\ &= \frac{17}{2} \end{aligned}$$

Question 7

(a)(i)
The easiest way to answer dice questions is by drawing a square. The X's in figure 6 shows the possible outcomes where the sum of the 2 dice equals 8. As there are 5 squares out of a total of 36 possible outcomes the probability is 5/36.

(a)(ii)
The O's in figure 6 shows the possible outcomes where the sum of the 2 dice is less than 8. As there are 21 squares out of a total of 36 possible outcomes the probability is $21/36 = 7/12$.

(b)(i)
The weighted mean x_w is defined as

$$\begin{aligned} x_w &\triangleq \frac{\sum wx}{\sum w} \\ &= \frac{8(110)+19(108)+5(116)+16(105)+10(97)+42(105)}{8+19+5+16+10+42} \\ &= \frac{10572}{100} \\ &= 105.72 \end{aligned}$$

6	O	X				
5	O	O	X			
4	O	O	O	X		
3	O	O	O	O	X	
2	O	O	O	O	O	X
1	O	O	O	O	O	O
	1	2	3	4	5	6

Figure 6: Square for Dice

(b)(ii)
If the tobacco is removed then

$$\begin{aligned} x_w &= \frac{8(110) + 19(108) + 16(105) + 10(97) + 42(105)}{8 + 19 + 5 + 16 + 10 + 42} \\ &= \frac{9992}{95} \\ &= 105.18 \end{aligned}$$

So the change is $105.72 - 105.18 = 0.54$

(c)(i)
The next palindromic year is 2112.

(c)(ii)
There are 4 digits to consider. These must take the form $xyyx$ for the year to be palindromic. For the first (and last) number there are 9 possible numbers. For the second and third digits there are 10 possible numbers. Thus there are $9x10 = 90$ palindromic years between 1000 and 9999.

(c)(iii)
The probability of a number being palindromic is the number of palindromic numbers divided by the total number of numbers. The number of numbers

between 9 and 10000 is 9990 as it does not include either 9 or 10000.

The number of two digit palindromic numbers is 9 (e.g. 11,22 etc.). Each 3 digit palindromic number must take the form xyx and so the number of these is $9x10 = 90$ (e.g. 101,111,121..202 etc.). The number of four digit palindromic numbers is also 90 from part (ii) above.

So the total number of palindromic numbers is $9 + 90 + 90 = 189$ and the probability is $189/9990 = 0.0189$.

Question 8

(a)

$$\boxed{\int u dv = uv - \int v du}$$

Let $u = \ln x$ and $dv = xdx$. Then $du = dx/x$ and $v = x^2/2$. So

$$\begin{aligned}
\int x \ln x dx &= \frac{x^2}{2}\ln x - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right) dx \\
&= \frac{x^2}{2}\ln x - \int \left(\frac{x}{2}\right) dx \\
&= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C \\
&= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + C
\end{aligned}$$

(b)(i)
The length of a sector of a circle is $r\theta$. Thus the perimeter of the sector is $r + r + r\theta = 8$. Hence

$$\theta = \frac{8 - 2r}{r}$$

(b)(ii)
The area A of a sector of a circle is $1/2r^2\theta$. Thus

$$\begin{aligned}
A &= \frac{1}{2}r^2\theta \\
&= \frac{1}{2}r^2\frac{8-2r}{r} \\
&= 4r - r^2
\end{aligned}$$

(b)(iii)

The maximum area A of the sector occurs when the rate of change is 0. So

$$\begin{aligned} A &= 4r - r^2 \\ \frac{dA}{dr} &= 4 - 2r \\ 0 &= 4 - 2r \\ r &= 2m \end{aligned}$$

And so the maximum area is $4(2) - 2^2 = 4m^2$.

(c)(i)

$$\begin{aligned} \tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \\ \tan^{-1} \frac{1}{2} &= \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} - \frac{\left(\frac{1}{2}\right)^7}{7} + \dots \end{aligned}$$

(c)(ii)

$$\begin{aligned} \frac{\pi}{4} &= \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} \\ \pi &= 4\left[\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} - \frac{\left(\frac{1}{2}\right)^7}{7} + \dots\right] \\ &+ 4\left[\frac{1}{3} - \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} - \frac{\left(\frac{1}{3}\right)^7}{7} + \dots\right] \end{aligned}$$

(c)(iii)

$$\begin{aligned} \pi &= 4\left[\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} - \frac{\left(\frac{1}{2}\right)^7}{7} + \dots\right] \\ &+ 4\left[\frac{1}{3} - \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5} - \frac{\left(\frac{1}{3}\right)^7}{7} + \dots\right] \\ &= 4(0.5 - 0.04166 + 0.00625 - 0.001116 + 0.33333 - 0.0123456 + 0.000823 - 0.0000653) \\ &= 3.1409 \end{aligned}$$