

# EE660 Lecture Notes: Mathematical Foundations and Methods of Machine Learning

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## Abstract

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**Note:** This document is a continual work in progress and will be constantly updated throughout the semester. Be sure to frequently check for updates.

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# 1 Supervised learning

We start our study by considering an unknown distribution  $\mathcal{D}$  over an event space  $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ . Here, the space  $\mathcal{X}$  denotes the *covariates*, and we will typically assume for simplicity that  $\mathcal{X}$  is a subset of Euclidean space. On the other hand, the space  $\mathcal{Y}$  denotes the *labels*. We will let  $p(x, y) = p(x)p(y | x)$  denote the joint density function over  $\mathcal{Z}$ .<sup>1</sup>

We first consider *binary classification*, where  $\mathcal{Y} = \{-1, +1\}$ . The binary classification problem is: given i.i.d. *training examples*  $S_n := \{(x_1, y_1), \dots, (x_n, y_n)\}$  from  $\mathcal{D}$ , learn a predictor  $\hat{f}_n : \mathcal{X} \mapsto \mathcal{Y}$ , so that ideally on unseen *test examples*  $(x, y) \sim \mathcal{D}$ , we have that  $\hat{f}_n(x) = y$ .

## 1.1 What form should the predictor take?

In order to propose an algorithm to learn the predictor  $\hat{f}_n$ , we need to specify two things:

- (a) What form should the predictor function  $\hat{f}_n$  take?
- (b) What algorithm should we use to learn its representation?

To shed some insight into question (a), let us set forth a framework to study what an *optimal* predictor looks like. Let  $\ell : \mathcal{Y} \times \mathcal{Y} \mapsto \mathbb{R}$  be a loss function, where we think of  $\ell(\hat{y}, y)$  as the cost of predicting  $\hat{y}$  when the true label is  $y$ . For a concrete example, think of  $\ell(\hat{y}, y) = \mathbf{1}\{\hat{y} \neq y\}$ , which is the *zero-one loss* or the *mistake loss*.

With this setup, we can now pose an optimization problem to compute the best predictor:

$$\min_{f: \mathcal{X} \rightarrow \mathcal{Y}} \mathbb{E}_{(x, y) \sim \mathcal{D}} [\ell(f(x), y)]. \quad (1.1)$$

(Note for what follows, we will usually drop the subscript under the expectation  $\mathbb{E}$ . However, please do not be afraid to ask if it is ever not clear what random variables we are taking expectations and/or probabilities with respect to.) Above, the optimization is taken over all (measurable) functions mapping covariates  $\mathcal{X}$  to labels  $\mathcal{Y}$ . It may seem at first that this is an impossible problem to solve, but it actually has a very simple solution.

**Proposition 1.1.** *Suppose the loss  $\ell$  is proper, i.e.,*

$$\ell(1, -1) > \ell(-1, -1), \quad \text{and} \quad \ell(-1, 1) > \ell(1, 1).$$

*A solution  $\hat{f} : \mathcal{X} \rightarrow \mathcal{Y}$  of Equation (1.2) takes on the form:*<sup>2</sup>

$$\hat{f}(x) = \text{sgn} \left\{ p(y = +1 | x) - \frac{\ell(1, -1) - \ell(-1, -1)}{\ell(-1, 1) - \ell(1, 1)} p(y = -1 | x) \right\}. \quad (1.2)$$

*Proof.* By iterated expectations,

$$\mathbb{E}[\ell(f(x), y)] = \mathbb{E}[\mathbb{E}[\ell(f(x), y) | x]].$$

Let us now study the inner quantity, conditioned on  $x$ . The key is that  $f(x)$ , once conditioned on  $x$ , is no longer random (the jargon I will often use is that  $f(x)$  is “ $x$ -measurable”). Hence, the inner conditional expectation decomposes very cleanly:

$$\mathbb{E}[\ell(f(x), y) | x] = \ell(f(x), 1)p(y = +1 | x) + \ell(f(x), -1)p(y = -1 | x).$$

<sup>1</sup>We will not be overly concerned with measure-theoretic concerns in this course.

<sup>2</sup>The  $\text{sgn}$  function is  $\text{sgn}(a) := \mathbf{1}\{a \geq 0\} - \mathbf{1}\{a < 0\}$ . Note the tie at  $a = 0$ , such that  $\text{sgn}(0) = 1$ , is broken arbitrarily.

Hence, an optimal  $f(x)$ , for this value of  $x$  that we are conditioning on, should pick either  $+1$  or  $-1$  to minimize the RHS of the above equation. That is, abbreviating  $p_1 = p(y = 1 | x)$  and  $p_{-1} = p(y = -1 | x)$ :

$$f(x) = \begin{cases} +1 & \text{if } \ell(-1, 1)p_1 + \ell(-1, -1)p_{-1} \geq \ell(1, 1)p_1 + \ell(1, -1)p_{-1} \\ -1 & \text{otherwise.} \end{cases}$$

The result now follows by re-arranging the expression above, relying on the properness of the loss  $\ell$  to divide by  $\ell(-1, 1) - \ell(1, 1)$ .  $\square$

Let us now consider a special case of Equation (1.2) where  $\ell(\hat{y}, y) = \mathbf{1}\{\hat{y} \neq y\}$  is the zero-one loss. In this case, the predictor  $\hat{f}$  reduces to the *maximum a-posterior* (MAP) predictor:

$$\hat{f}(x) = \arg \max_{y \in \mathcal{Y}} p(y | x). \quad (1.3)$$

Note that by Bayes' rule, we have that:

$$p(y | x) = \frac{p(y)p(x | y)}{p(x)}.$$

Hence, under a uniform prior on the labels, i.e.,  $p(y = 1) = p(y = -1) = 1/2$ , we have that  $\hat{f}$  is also equal to the *maximum-likelihood estimator* (MLE):

$$\hat{f}(x) = \arg \max_{y \in \mathcal{Y}} p(x | y).$$

**Exercise 1.2.** Suppose we are in a multi-class classification setting, so that  $\mathcal{Y} = \{1, \dots, K\}$ . Show that the optimal predictor  $\hat{f} : \mathcal{X} \mapsto \mathcal{Y}$  minimizing the zero-one loss is still the MAP predictor:

$$\hat{f}(x) = \arg \max_{y \in \mathcal{Y}} p(y | x).$$

## 1.2 Perceptron

We now turn to answer the second question (b), what algorithm do we use to learn the representation of a decision function  $f : \mathcal{X} \mapsto \mathcal{Y}$ . We will start from a very simple prototypical algorithm, called the Perceptron, and understand its various properties. It will turn out that while the Perceptron is very simple, the same underlying principles are used to train essentially every modern machine learning model.

The Perceptron starts by making a modeling assumption regarding the form of the predictor  $\hat{f}$ . Conceptually, every predictor  $\hat{f}$  (whether optimal or not) induces a *partition* of  $\mathcal{X}$  (which we now assume to be a subset of  $\mathbb{R}^d$ ):

$$T_1 := \{x \in \mathcal{X} \mid \hat{f}(x) = 1\}, \quad \text{and} \quad T_{-1} := \{x \in \mathcal{X} \mid \hat{f}(x) = -1\}.$$

(In fact, there is a one-to-one correspondence between partitions of  $\mathcal{X}$  and predictors  $\hat{f}$ .) In the Perceptron, it is posited that the optimal predictor induces a *linear* partition of  $\mathcal{X}$ : that is, we cut  $\mathcal{X}$  in half by a hyperplane (that we will assume crosses through the origin for convenience), and assign the label  $+1$  to one side of the hyperplane, and  $-1$  to the other. Mathematically:

$$f_w(x) = \text{sgn}\{\langle w, x \rangle\}, \quad w \in \mathbb{R}^d.$$

Learning here then refers to learning the weights  $w$  which parameterize the separating hyperplane. The Perceptron prescribes the following procedure to learn these weights.

We will now study the behavior of the Perceptron on *linearly separable* data.

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**Algorithm 1** The Perceptron

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**Input:** Dataset  $S_n = \{(x_1, y_1), \dots, (x_n, y_n)\}$ .

```
1: Set  $w = 0$ .
2: while true do
3:   Let  $\mathcal{I}$  be a (possibly random) permutation of  $\{1, \dots, n\}$ .
4:   Set mistake = false.
5:   for  $i \in \mathcal{I}$  do
6:     if  $y_i \langle w, x_i \rangle < 1$  then
7:       Set  $w \leftarrow w + y_i x_i$ .
8:       Set mistake = true.
9:     end if
10:  end for
11:  if not mistake then
12:    return  $w$ .
13:  end if
14: end while
```

---

**Definition 1.3.** A dataset  $S_n = \{(x_1, y_1), \dots, (x_n, y_n)\} \subset \mathcal{Z}$  is linearly separable if there exists a non-zero  $w \in \mathbb{R}^d$  such that:

$$\forall i \in \{1, \dots, n\}, \quad y_i = \text{sgn}\{\langle w, x_i \rangle\}.$$

Such a non-zero  $w \in \mathbb{R}^d$  satisfying the above condition is called a linear separator.

**Exercise 1.4.** Which of the following  $n = 2$  datasets with  $\mathcal{X} = \mathbb{R}^2$  are linearly separable?

(a)  $S_2 = \{((0, 1), +1), ((0, -1), +1)\}$ .

(b)  $S_2 = \{((0, 1), -1), ((0, -1), -1)\}$ .

(c)  $S_2 = \{((0, 0), +1), ((0, 1), -1)\}$ .

Next, we will define the notion of a *margin*, which quantifies how linearly separable a dataset is. Before we do this, let  $S$  be a subset of Euclidean space. The distance function  $\text{dist}(x, S)$  is defined as:

$$\text{dist}(x, S) = \inf\{\|x - z\| \mid z \in S\}.$$

Also, for a vector  $w \in \mathbb{R}^d$ , we define its hyperplane  $H_w$  as:

$$H_w := \{x \in \mathbb{R}^d \mid \langle x, w \rangle = 0\}.$$

We now have enough notation to define the notion of margin.

**Definition 1.5.** Let  $S_n$  be a linearly separable dataset (cf. Definition 1.3), and let  $W \subset \mathbb{R}^d$  denote the set of linear separators. The margin  $\gamma(S_n)$  is defined as:

$$\gamma(S_n) := \sup_{w \in W} \min_{i \in \{1, \dots, n\}} \text{dist}(x_i, H_w). \quad (1.4)$$

The idea behind margin is to assign a quantity to a linear separator which measures how “robust” it is. (Draw some pictures in class). The following exercises check your understanding of margin.

**Exercise 1.6.** Show that the margin defined in Equation (1.4) satisfies the following equalities:

$$\gamma(S_n) = \sup_{w \in W} \min_{i \in \{1, \dots, n\}} \frac{|\langle x_i, w \rangle|}{\|w\|} = \sup_{\|w\|=1} \min_{i \in \{1, \dots, n\}} y_i \langle x_i, w \rangle.$$

**Exercise 1.7.** Suppose  $S_n$  is linearly separable. Consider the alternative definition of margin, which does not take into account whether or not a hyperplane separates the dataset:

$$\gamma'(S_n) = \sup_{w \neq 0} \min_{i \in \{1, \dots, n\}} \text{dist}(x_i, H_w).$$

Is it possible to construct an  $S_n$  such that  $\gamma'(S_n) > \gamma(S_n)$ ? If so, provide one construction. If not, prove it is not possible.

### 1.2.1 Mistake bound

With this notion of margin, we are able to prove our first non-trivial result regarding Perceptron, which is its well-known *mistake bound*. While this is a classic result, we will follow the derivations based on the presentation in [Hardt and Recht \[2022, Ch. 3\]](#).

To start, we will define a sequences of weights  $w_0, w_1, \dots$  recursively, which captures the main Perceptron update. Let  $i_0, i_1, \dots$  denote an *arbitrary* sequence of indices in  $\{1, \dots, n\}$  (meaning, these indices could be repeating, etc.). Our sequence  $w_0, w_1, \dots$  is defined as:

$$w_{t+1} = w_t + y_{i_t} x_{i_t} \mathbf{1}\{y_{i_t} \langle w_t, x_{i_t} \rangle < 1\}, \quad w_0 = 0. \quad (1.5)$$

Note that our notation makes implicit the dependence of the iterates  $\{w_t\}$  on the indices  $\{i_t\}$ . In the sequel, it will be clear from context how the indices  $\{i_t\}$  are specified.

**Lemma 1.8** (Perceptron mistake bound). *Let  $S_n$  be linearly separable. Consider the Perceptron sequence  $\{w_t\}_{t \geq 0}$  defined in Equation (1.5). Let  $m_t := \mathbf{1}\{y_{i_t} \langle w_t, x_{i_t} \rangle < 1\}$  denote the indicator of whether a mistake occurred at time  $t$ , and let  $M_t := \sum_{s=0}^{t-1} m_s$  denote the cumulative number of mistakes made up to time  $t$ . We have that for all  $t \in \mathbb{N}$ :*

$$M_t \leq \frac{2 + \max_{i \in \{1, \dots, n\}} \|x_i\|^2}{\gamma^2(S_n)}. \quad (1.6)$$

Before proceeding to the proof, it is worth reflecting on the nature of this result. It is quite remarkable in that it is *dimension independent*. Furthermore, since the RHS is independent of  $t$ , this result implies there exists a  $t$  such that for all  $t' \geq t$ , we have  $m_{t'} = 0$  (why?). In other words, the Perceptron eventually stops making mistakes. Furthermore, since this result holds for *any* sequence  $i_0, i_1, \dots$ , by setting the sequence  $\{i_t\}$  to be repeated copies of  $\{1, \dots, n\}$ , this ensures that eventually the Perceptron indeed learns a linear separator (why?).

*Proof of Lemma 1.8.* Put  $B := \max_{i \in \{1, \dots, n\}} \|x_i\|^2$ . Fix a  $t$  and suppose that  $m_t = 1$ . Then, by expanding the square:

$$\begin{aligned} \|w_{t+1}\|^2 &= \|w_t + y_{i_t} x_{i_t}\|^2 = \|w_t\|^2 + 2y_{i_t} \langle w_t, x_{i_t} \rangle + \|x_{i_t}\|^2 \\ &\leq \|w_t\|^2 + 2y_{i_t} \langle w_t, x_{i_t} \rangle + B \leq \|w_t\|^2 + 2 + B. \end{aligned}$$

The key here is the last inequality, which is a consequence of  $m_t = 1$ , or equivalently  $y_{i_t} \langle w_t, x_{i_t} \rangle < 1$ . On the other hand, if  $m_t = 0$ , then  $\|w_{t+1}\|^2 = \|w_t\|^2$  by definition. Hence, for every  $t$ :

$$\|w_{t+1}\|^2 \leq \|w_t\|^2 + (2 + B)m_t.$$

Unrolling this recursion to  $t = 0$  and using the definition of  $M_t$  yields for every  $t$ :

$$\|w_t\|^2 \leq (2 + B)M_t.$$

Next, let  $w_\star$  denote a unit norm max-margin linear separator, i.e., a vector such that (cf. Equation (1.4)):

$$\gamma(S_n) = \min_{1 \leq i \leq n} |\langle x_i, w_\star \rangle|.$$

Now, suppose that  $m_t = 1$ , and hence, since  $w_\star$  linearly separates  $S_n$ ,

$$\langle w_\star, w_{t+1} - w_t \rangle = y_{i_t} \langle w_\star, x_{i_t} \rangle = |\langle w_\star, x_{i_t} \rangle| \geq \gamma(S_n).$$

Next, we use a *telescoping sum*, which is a trick often used in optimization proofs. The telescoping sum simply states (recall that  $w_0 = 0$ ):

$$w_t = \sum_{k=1}^t (w_k - w_{k-1}) = \sum_{k=1}^t (w_k - w_{k-1}) m_{k-1}.$$

While this may appear tautological, it is actually quite useful, since:

$$\|w_t\| \geq \langle w_\star, w_t \rangle = \sum_{k=1}^t \langle w_\star, w_k - w_{k-1} \rangle m_{k-1} \geq \gamma(S_n) \sum_{k=1}^t m_{k-1} = \gamma(S_n) M_t.$$

Above, the first inequality holds by Cauchy-Schwarz (recall that  $w_\star$  is unit norm). We now have upper and lower bounds on  $\|w_t\|$ , from which we conclude:

$$\gamma^2(S_n) M_t^2 \leq \|w_t\|^2 \leq (2 + B) M_t \implies M_t \leq \frac{2 + B}{\gamma^2(S_n)}.$$

□

**Exercise 1.9.** Suppose Algorithm 1 is called with a linearly separable dataset  $S_n$ . Let  $B$  bound the covariate norm, i.e.,  $B = \max_{i \in \{1, \dots, n\}} \|x_i\|^2$ . Show that Algorithm 1 terminates with a solution  $w \in \mathbb{R}^d$ , which linearly separates  $S_n$ , in at most  $(2 + B)/\gamma^2(S_n)$  passes through  $S_n$ .

### 1.2.2 Generalization bound

The mistake bound from Section 1.2.1 gives us a way to prove a generalization bound about Perceptron, on *unseen* instances. We will utilize a clever technique known as leave-one-out analysis. We need one more piece of notation before we proceed. Given a linearly separable dataset  $S_n$ , we let  $w(S_n)$  denote the limiting value of the Perceptron sequence  $\{w_t\}$  defined in Equation (1.5), where we fix the indices  $\{i_t\}$  to repeatedly cycle through  $\{1, \dots, n\}$ , i.e.,  $i_t = (t \bmod n) + 1$ . Note that  $w(S_n)$  is well-defined by Lemma 1.8. For what follows, we will assume that the distribution  $\mathcal{D}$  is linearly separable, meaning that almost surely, for any  $n \in \mathbb{N}_+$ ,  $S_n$  sampled i.i.d. from  $\mathcal{D}$  is linearly separable.

**Lemma 1.10** (Perceptron generalization bound). *Suppose that  $\mathcal{D}$  is linearly separable. Also, suppose that  $\|x\| \leq B$  almost surely when  $x \sim p(x)$ . Then, letting  $(x, y) \in \mathcal{Z}$  be drawn independently from  $S_n$ ,*

$$\mathbb{P}\{y \langle w(S_n), x \rangle < 1\} \leq \frac{2 + B}{n + 1} \mathbb{E} \left[ \frac{1}{\gamma^2(S_{n+1})} \right]. \quad (1.7)$$

*Proof.* We will construct an equivalent probability space as follows. Let  $S_{n+1} = \{z_1, \dots, z_{n+1}\}$  be  $n + 1$  i.i.d. draws from  $\mathcal{D}$ , and let  $S_{n+1}^{-k} := \{z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_{n+1}\}$  be the dataset where the  $k$ -th point is left out. Note that by exchangeability, for any  $k$ ,

$$\mathbb{P}\{y \langle w(S_n), x \rangle < 1\} = \mathbb{P}\{y_k \langle w(S_{n+1}^{-k}), x_k \rangle < 1\}.$$

Hence we can average over all indices to conclude:

$$\mathbb{P}\{y \langle w(S_n), x \rangle < 1\} = \frac{1}{n + 1} \sum_{k=1}^{n+1} \mathbb{P}\{y_k \langle w(S_{n+1}^{-k}), x_k \rangle < 1\}.$$

(This type of relabeling argument is common to leave-one-out analyses, and we will see it come up again later when we study algorithmic stability.)

Now, let  $I_m \subset \{1, \dots, n+1\}$  denote the indices for which the Perceptron on  $S_{n+1}$  made a mistake at any point during execution, i.e.,

$$I_m := \{(t \bmod (n+1)) + 1 \mid m_t = 1, t \in \mathbb{N}\}.$$

Let  $k \in \{1, \dots, n+1\}$ , and suppose  $k \notin I_m$ . Then, we have the key equality:

$$w(S_{n+1}^{-k}) = w(S_{n+1}).$$

It is worth taking a moment to think why this equality is true. If  $k \notin I_m$ , then this means that each time Perceptron on  $S_{n+1}$  cycles through the index  $k$ , it makes no update. Hence, it is equivalent to running Perceptron on  $S_{n+1}^{-k}$ , where the  $k$ -th datapoint is altogether removed.

Now here is the final trick. Again let  $k \notin I_m$ . Then we have:

$$1 \leq y_k \langle w(S_{n+1}), x_k \rangle = y_k \langle w(S_{n+1}^{-k}), x_k \rangle.$$

(Check your understanding: why is the first inequality true?) Therefore, chaining these arguments together and applying the mistake bound from Lemma 1.8,

$$\begin{aligned} \sum_{k=1}^{n+1} \mathbf{1}\{y_k \langle w(S_{n+1}), x_k \rangle < 1\} &= \sum_{k \in I_m} \mathbf{1}\{y_k \langle w(S_{n+1}), x_k \rangle < 1\} + \sum_{k \notin I_m} \mathbf{1}\{y_k \langle w(S_{n+1}), x_k \rangle < 1\} \\ &= \sum_{k \in I_m} \mathbf{1}\{y_k \langle w(S_{n+1}^{-k}), x_k \rangle < 1\} \leq |I_m| \leq \frac{2+B}{\gamma^2(S_{n+1})}. \end{aligned}$$

The result now follows by taking expectations on both sides of the above inequality.  $\square$

**Remark 1.11.** Recall in the definition of  $w(S_n)$ , we specifically fixed the indices  $\{i_t\}$  to repeatedly cycle through  $\{1, \dots, n\}$ . Where in the proof of Lemma 1.10 did we use this assumption? What modifications to how the sequence  $\{i_t\}$  is generated can you think of, which allow the proof to still go through?

**Remark 1.12.** Lemma 1.10 states that the margin mistake error  $\mathbb{P}\{y \langle w(S_n), w \rangle < 1\}$  decreases at a rate of  $O(1/n)$ . In the literature, this is referred to as a *fast rate*, since it is faster than the usual  $O(1/\sqrt{n})$  rate which arises out of typical concentration of measure arguments (which we will see in a few lectures). This fast-rate is not to be taken for granted: obtaining a fast-rate in general settings typically involves a much more non-trivial argument, which is out of scope for this course.

**Interpreting the margin mistake bound.** How does the margin mistake bound from Lemma 1.10 relate to the zero-one loss  $\mathbb{P}\{y \neq \text{sgn}(\langle w(S_n), x \rangle)\}$ ? Let us abbreviate  $w = w(S_n)$ . We can upper bound the zero-one loss by the margin mistake bound via the following argument:

$$\begin{aligned} \{y \neq \text{sgn}(\langle w, x \rangle)\} &= \{y = 1, \langle w, x \rangle < 0\} \cup \{y = -1, \langle w, x \rangle \geq 0\} \\ &= \{y = 1, y \langle w, x \rangle < 0\} \cup \{y = -1, y \langle w, x \rangle \leq 0\} \\ &\subset \{y \langle w, x \rangle \leq 0\} \subset \{y \langle w, x \rangle < 1\}. \end{aligned}$$

(Note the asymmetry in the cases is due to the tie-breaking choice made in our definition of the  $\text{sgn}$  function.) Hence by monotonicity of probability,

$$\mathbb{P}\{y \neq \text{sgn}(\langle w(S_n), x \rangle)\} \leq \mathbb{P}\{y \langle w(S_n), x \rangle < 1\}.$$

Therefore, Lemma 1.10 implies a bound on the classification error of the Perceptron.



### 1.3 Empirical Risk Minimization (ERM)

The Perceptron algorithm we studied in Section 1.2, while having nice theoretical properties, is quite limiting. Indeed, its theoretical analysis requires that the underlying distribution  $\mathcal{D}$  is linearly separable, a fairly restrictive assumption for many real world datasets. Therefore, we desire a more general framework which does not require such limiting assumptions.

#### 1.3.1 Framework setup

The main framework we will study in this course, which covers essentially most of modern machine learning, is the empirical risk minimization (ERM) framework. We will spend quite a bit of time studying ERM, as it is the de-facto standard workhorse for modern machine learning.

At its core, ERM is a very simple concept. There are only three core ingredients:

1. A loss function  $\ell : \Lambda \times \mathcal{Y} \mapsto \mathbb{R}$ , where  $\Lambda$  is an auxiliary space.
2. A known, deterministic selection function  $\psi : \Lambda \mapsto \mathcal{Y}$ .
3. A function (hypothesis) class  $\mathcal{F} = \{f : \mathcal{X} \mapsto \Lambda\}$  over which to restrict our search for predictors over. Note, this class induces a set of predictors by composition with the selection function  $\psi$ .
4. A training dataset  $S_n = \{z_1, \dots, z_n\}$ , sampled i.i.d. from an underlying distribution  $\mathcal{D}$  over  $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ .

**Remark 1.13.** While the ERM framework above is stated somewhat abstractly, for binary classification, we will always consider models with auxiliary space  $\Lambda = \mathbb{R}$  and selection rule  $\psi = \text{sgn}$ . This corresponds to models  $f(x)$  which classify examples by thresholding  $\text{sgn}(f(x))$ . This structure will be implicitly assumed moving forward. Note that the extra generality allows for e.g., multi-class classification, regression, etc. to fit under the same framework.

The goal of a learner is to produce a hypothesis  $f \in \mathcal{F}$  to minimize the *population risk*:

$$L[f] := \mathbb{E}[\ell(f(x), y)].$$

Towards this goal, the ERM prescribes to us the following predictor:

$$\hat{f}_{\text{erm}} \in \arg \min_{f \in \mathcal{F}} \hat{L}_n[f] := \left\{ \frac{1}{n} \sum_{i=1}^n \ell(f(x_i), y_i) \right\}. \quad (1.8)$$

The function  $\hat{L}_n[f]$  is referred to as the *empirical risk*. The main object of study in this section of the course is the behavior of the population risk of the ERM, i.e.,  $L[\hat{f}_{\text{erm}}]$ . (Note: our notation for both the population risk  $L[f]$  and the empirical risk  $\hat{L}_n[f]$  makes the dependency on the loss  $\ell$  implicit. In the sequel, it should be clear from context what  $\ell$  is, but please do not hesitate to ask if it is not.)

#### 1.3.2 Generalization error

Given a *fixed* predictor  $f(x)$  and a dataset  $S_n$ , the empirical loss is an unbiased estimator of the true population level error, i.e.,

$$\mathbb{E}[\hat{L}_n[f]] = L[f].$$

However, when the predictor  $f$  is no longer independent of the dataset  $S_n$  (e.g.,  $f$  is constructed as a function of  $S_n$ , as in the ERM (1.8)), then the empirical risk is no longer an unbiased estimator (convince yourself this is true by constructing an example). Thus, what we will need to do is understand how much the empirical risk can differ from the population risk. We start with a tautological decomposition:<sup>3</sup>

$$L[f] = \underbrace{\hat{L}_n[f]}_{\text{empirical risk}} + \underbrace{L[f] - \hat{L}_n[f]}_{\text{generalization error}} =: \hat{L}_n[f] + \text{gen}[f]. \quad (1.9)$$

---

<sup>3</sup>My graduate school advisor Ben would often call this decomposition the “fundamental theorem of machine learning”.

In ERM, we purposefully make the first quantity, the empirical risk  $\hat{L}_n[f]$ , as small as possible. However, the question that remains is how does this affect  $\text{gen}[f]$ ? Again, the difficulty is that  $\hat{f}_{\text{erm}}$  is a function of the dataset  $S_n$ , and therefore  $\mathbb{E}[L_n[\hat{f}_{\text{erm}}]] \neq L[\hat{f}_{\text{erm}}]$ .

One of the key ideas behind generalization theory in machine learning is that by *restricting the function class  $\mathcal{F}$  which we optimize over*, we can indeed control the generalization error by quantities which scale according to the complexity of the function class. The way this works is by doing something admittedly crude. Since characterizing the random variable  $\hat{f}_{\text{erm}}$  is non-trivial in general, we instead consider the *worst-case* generalization error over the class  $\mathcal{F}$ :

$$\text{gen}[\hat{f}_{\text{erm}}] \leq \sup_{f \in \mathcal{F}} \text{gen}[f].$$

Analyzing the RHS of the above expression will be the focus of our next few lectures.

### 1.3.3 Finite hypothesis classes

As a warmup which contains the key ideas, let us suppose that  $\mathcal{F}$  is finite, i.e.,  $|\mathcal{F}| < \infty$ , and that  $\ell$  is the zero-one loss, i.e.,  $\ell(\lambda, y) = \mathbf{1}\{\text{sgn}(\lambda) \neq y\}$ . We will use Hoeffding's inequality combined with a union bound to control the generalization error of every single predictor in  $\mathcal{F}$ .

**Proposition 1.14.** *Let  $\ell$  be the zero-one loss and  $\mathcal{F}$  be a finite function class. Fix any  $\delta \in (0, 1)$ . With probability at least  $1 - \delta$ ,*

$$\max_{f \in \mathcal{F}} \text{gen}[f] \leq \sqrt{\frac{2 \log(|\mathcal{F}|/\delta)}{n}}.$$

*Proof.* As stated above, the main tool is Hoeffding's inequality (cf. Proposition B.12). Fix any  $f \in \mathcal{F}$ . We apply to the sum  $M_n = \sum_{i=1}^n X_i$ , with  $X_i = \mathbf{1}\{y_i \neq \text{sgn}(f(x_i))\}$ . Note that by construction  $X_i \in [-1, 1]$  and hence for any  $t > 0$ , by Hoeffding's inequality,

$$\mathbb{P}\{\text{gen}[f] \geq t\} = \mathbb{P}\{n^{-1}(\mathbb{E}[M_n] - M_n) \geq t\} \leq \exp(-nt^2/2).$$

Hence by a union bound,

$$\mathbb{P}\left\{\max_{f \in \mathcal{F}} \text{gen}[f] \geq t\right\} = \mathbb{P}\left\{\bigcup_{f \in \mathcal{F}} \{\text{gen}[f] \geq t\}\right\} \leq \sum_{f \in \mathcal{F}} \mathbb{P}\{\text{gen}[f] \geq t\} \leq |\mathcal{F}| \exp(-nt^2/2).$$

Now, set the RHS above equal to the given  $\delta$  and solve for  $t$  to conclude.  $\square$

The above bound in Proposition 1.14 is a prototypical statistical learning bound. The  $1/\sqrt{n}$  rate in the denominator is generally unimprovable, as a consequence of the Central Limit Theorem. The  $\sqrt{\log |\mathcal{F}|}$  scaling in the numerator states that we are allowed to test exponentially many in  $n$  hypothesis before our generalization gap dominates the error.

The generalization bound in Proposition 1.14 is quite powerful already. It tells us, via (1.9), that:

$$\forall f \in \mathcal{F}, \quad L[f] \leq \hat{L}_n[f] + \sqrt{\frac{2 \log(|\mathcal{F}|/\delta)}{n}}.$$

Again, it is important to emphasize that this is quite remarkable. It implies that we can test exponentially many hypothesis in the number of data points  $n$ , until the training error  $\hat{L}_n[f]$  fails to be an accurate proxy of the population error  $L[f]$ . Let us conclude this section with an example. Suppose that  $\mathcal{F} = \{f_\theta : \mathbf{X} \mapsto \mathbb{R} \mid \theta \in \mathbb{R}^d\}$ . That is, we optimize over a function class parameterized by  $d$  parameters. Let  $N_{\text{f32}}$  denote the number of valid `float32` numbers (so  $N_{\text{f32}} \leq 2^{32}$ ). Then, on a computer,  $\mathcal{F}$  is a finite hypothesis class of cardinality  $|\mathcal{F}| = N_{\text{f32}}^d$ . So, if we perform optimization in our favorite ML framework,<sup>4</sup> we immediately have:

$$\forall f \in \mathcal{F}, \quad L[f] \leq \hat{L}_n[f] + \sqrt{\frac{2(d \log N_{\text{f32}} + \log(1/\delta))}{n}}. \quad (1.10)$$

<sup>4</sup>You are using `jax` (<https://github.com/google/jax>) right?

From this inequality, we have already uncovered a core tenant of conventional machine learning wisdom: *generalization happens when one has more datapoints  $n$  than parameters  $d$  (i.e.,  $n \gg d$ )*. Now, this wisdom has come into question in the past decade or so, with the success of deep overparameterized neural networks (which seem to be generalizing in ways that defy this logic). We will revisit this later on, but for now  $n \gg d$  implies generalization is a good mental framework to have.

**Exercise 1.15.** Suppose that  $\ell$  is the zero-one loss and  $\mathcal{F}$  is a countably infinite function class of the form:

$$\mathcal{F} = \{f_\kappa \mid \kappa \in \mathbb{N}_+\}.$$

Let  $\delta \in (0, \delta_0]$  where  $0 < \delta_0 < 1$ . Show that with probability at least  $1 - \delta$ , we have:

$$\forall f_\kappa \in \mathcal{F}, \text{ gen}[f_\kappa] \leq c \sqrt{\frac{\log(\kappa/\delta)}{n}},$$

where  $c > 0$  is a constant that depends only on  $\delta_0$ , but not on  $\kappa$ ,  $\delta$ , or  $n$ .

Hint: you may use without proof the well-known solution to the Basel problem:  $\sum_{i=1}^{\infty} i^{-2} = \pi^2/6$ .

### 1.3.4 Fast rates with realizable finite hypothesis classes

In addition to the assumptions of the last section, let us now impose an additional assumption, that there exists an  $f \in \mathcal{F}$  such that  $L[f] = 0$ , that is, no mistakes are made at the population level. This is obviously a strong assumption, but we will see that it provides fast rates (as we saw with the Perceptron when data was linearly separable).

**Proposition 1.16.** Suppose that  $\ell$  is the zero-one loss,  $\mathcal{F}$  is finite, and there exists  $f \in \mathcal{F}$  satisfying  $L[f] = 0$ . Then, the empirical risk minimizer  $\hat{f}_n$  satisfies, with probability at least  $1 - \delta$ ,

$$L[\hat{f}_n] \leq \frac{\log(|\mathcal{F}|/\delta)}{n}.$$

*Proof.* For any  $t > 0$ , we will study the event  $\{L(\hat{f}_n) > t\}$ . Define the set of sub-optimal hypothesis as  $B(t) = \{f \in \mathcal{F} \mid L(f) > t\}$ . Note that  $\{L(\hat{f}_n) > t\} = \{\hat{f}_n \in B(t)\}$ . Now observe that since there exists an  $f \in \mathcal{F}$  with  $L[f] = 0$ , then the ERM  $\hat{f}_n$  will always achieve zero training error, i.e.,  $\hat{L}_n[\hat{f}_n] = 0$ .<sup>5</sup> Hence if  $\hat{f}_n \in B(t)$ , that means there exists some  $f \in B(t)$  with  $\hat{L}_n[f] = 0$ . So now we compute, for a fixed  $f \in B(t)$ ,

$$\begin{aligned} \mathbb{P}\{\hat{L}_n[f] = 0\} &= \mathbb{P}\left\{\bigcap_{i=1}^n \{\text{sgn}(f(x_i)) = y_i\}\right\} = \prod_{i=1}^n (1 - \mathbb{P}\{\text{sgn}(f(x_i)) \neq y_i\}) \\ &= (1 - L[f])^n \leq (1 - t)^n \leq \exp(-tn), \end{aligned}$$

where we used the elementary inequality  $1 - x \leq \exp(x)$  for any  $x \in \mathbb{R}$ . Therefore by a union bound,

$$\begin{aligned} \mathbb{P}\{\hat{f}_n \in B(t)\} &\leq \mathbb{P}\{\exists f \in B(t), \hat{L}_n(f) = 0\} \leq \sum_{f \in B(t)} \mathbb{P}\{\hat{L}_n(f) = 0\} \\ &\leq |B(t)| \exp(-tn) \leq |\mathcal{F}| \exp(-tn). \end{aligned}$$

Now set the RHS equal to  $\delta$  and solve for  $t$ . □

### 1.3.5 Moving towards non-finite hypothesis classes: hinge loss and logistic regression

The analyses of Section 1.3.3 and Section 1.3.4 both suffer from one key issue: they only apply to finite function classes, i.e., when  $|\mathcal{F}| < \infty$ . This is an unnecessary restriction. Indeed we saw that the Perceptron algorithm, which operates over a continuum of linear hypothesis, also enjoys a strong generalization bound (cf. Lemma 1.10). Towards removing this restriction, we first observe that Perceptron is an instance of ERM.

<sup>5</sup>Technically, there could be a measure-zero event where this is not true, but we ignore this technicality.

**Hinge loss and the Perceptron.** We claim that the Perceptron algorithm is an instance of ERM over hypothesis class of linear function  $\mathcal{F} = \{x \mapsto \langle w, x \rangle \mid w \in \mathbb{R}^d\}$ . What is the loss function? Here, we slightly overload notation and consider  $\ell$  as a univariate function  $\ell(\lambda, y) = \ell(\lambda \cdot y)$ . With this univariate notation, we define the *hinge loss*

$$\ell_{\text{hinge}}(s) := \max\{1 - s, 0\}.$$

Now observe that for a datapoint  $(x, y) \in \mathcal{Z}$  and parameters  $w$ , we have that:

$$\nabla_w \ell_{\text{hinge}}(y \langle w, x \rangle) = -yx \cdot \mathbf{1}\{y \langle w, x \rangle < 1\}.$$

This update rule is exactly that of the Perceptron algorithm (cf. Equation (1.5)). Hence, we can see that Perceptron is precisely running stochastic gradient descent (SGD) on the following empirical risk:

$$L_n[w] = \frac{1}{n} \sum_{i=1}^n \ell_{\text{hinge}}(y_i \langle w, x_i \rangle) = \frac{1}{n} \sum_{i=1}^n \max\{1 - y_i \langle w, x_i \rangle, 0\}.$$

We will revisit SGD in more depth later on in the course. Next, we will see that even for linear models, the hinge loss is not the only well-motivated loss function.

**Logistic regression.** Motivated by the MAP estimator (1.3), we now consider a different type of loss function based on directly learning the posterior probability  $p(y \mid x)$ . Sticking with a hypothesis class of linear functions  $\mathcal{F} = \{x \mapsto \langle w, x \rangle \mid w \in \mathbb{R}^d\}$ , consider the following model:

$$p_w(y = 1 \mid x) \propto \exp(\langle w, x \rangle), \quad p_w(y = -1 \mid x) \propto 1.$$

Normalizing across  $\mathcal{Y} = \{\pm 1\}$ ,

$$p_w(y = 1 \mid x) = \frac{\exp(\langle w, x \rangle)}{1 + \exp(\langle w, x \rangle)}, \quad p_w(y = -1 \mid x) = \frac{1}{1 + \exp(\langle w, x \rangle)}.$$

To construct a loss function, let us use the KL-divergence to match  $p_w(y \mid x)$  with  $p(y \mid x)$ . Specifically, conditioned on  $x \in \mathcal{X}$ ,

$$\begin{aligned} & \text{KL}(p(y \mid x) \parallel p_w(y \mid x)) - \mathbb{E}[\log p(y \mid x) \mid x] \\ &= -\mathbb{E}[\log p_w(y \mid x) \mid x] \\ &= -p(y = 1 \mid x) \log p_w(y = 1 \mid x) - p(y = -1 \mid x) \log p_w(y = -1 \mid x) \\ &= p(y = 1 \mid x) \log(1 + \exp(-\langle w, x \rangle)) + p(y = -1 \mid x) \log(1 + \exp(\langle w, x \rangle)) \\ &= \mathbb{E}[\log(1 + \exp(-y \langle w, x \rangle)) \mid x]. \end{aligned}$$

Hence, by the tower property,

$$\mathbb{E}[\text{KL}(p(y \mid x) \parallel p_w(y \mid x))] = \mathbb{E}[\log p(y \mid x)] + \mathbb{E}[\log(1 + \exp(-y \langle w, x \rangle))].$$

Observe that the first (negative) entropy term does not depend on the parameters  $w$ . Hence, this motivates the population level estimator:

$$w_* \in \arg \min_{w \in \mathbb{R}^d} \mathbb{E}[\ell_{\log}(y \langle w, x \rangle)], \quad \ell_{\log}(s) := \log(1 + \exp(-s)),$$

and corresponding empirical risk:

$$\hat{L}_n[w] = \frac{1}{n} \sum_{i=1}^n \ell_{\log}(y_i \langle w, x_i \rangle) = \frac{1}{n} \sum_{i=1}^n \log(1 + \exp(-y_i \langle w, x_i \rangle)).$$

(Indeed, we have  $\mathbb{E}[\hat{L}_n[w]] = L[w]$  for every fixed  $w \in \mathbb{R}^d$ .) The loss  $\ell_{\log}(s)$  is the *logistic loss*, and this setup is referred to as *logistic regression*.

**Exercise 1.17.** Suppose that we are in the multi-class classification setting, with  $\mathcal{Y} = \{1, \dots, K\}$ . Suppose that we posit the following probabilistic model:

$$p_{w_{1:K}}(y = k \mid x) \propto \exp(\langle w_k, x \rangle), \quad k \in \{1, \dots, K\}.$$

Show that the corresponding empirical risk for the population risk

$$L[w] = \mathbb{E}[\text{KL}(p(y \mid x) \parallel p_{w_{1:K}}(y \mid x))]$$

is given by the following *cross-entropy loss*:

$$\hat{L}_n[w] = -\frac{1}{n} \sum_{i=1}^n \left[ \langle w_{y_i}, x_i \rangle - \log \left[ \sum_{j=1}^K \exp(\langle w_j, x_i \rangle) \right] \right].$$

That is, show that  $\mathbb{E}[\hat{L}_n[w]] = L[w] + C$  for every fixed  $w = \{w_k\}_{k=1}^K$ , where  $C$  is a constant that does not depend on  $w$ .

### 1.3.6 Margin theory

We have seen two examples of loss functions: the hinge loss and the logistic loss. However, at this point two key questions remain to be answered:

- (a) Both our derivations of the hinge loss and logistic loss were based on linear predictors. Can we apply these specific losses beyond linear models?
- (b) Ultimately, our goal is to access model quality via the zero-one loss. What is the relationship between the hinge/logistic loss and the zero-one loss? Does low population risk under the former imply low classification error?

Resolving (a) is simple. Observe that both the hinge and logistic loss involve the corresponding linear model  $x \mapsto \langle w, x \rangle$  only through quantities of the form  $y \langle w, x \rangle$ . Hence, for a possibly non-linear hypothesis  $f : \mathcal{X} \mapsto \mathbb{R}$ , we construct the following population and empirical loss:

$$L[f] = \mathbb{E}[\ell(yf(x))], \quad \hat{L}_n[f] = \frac{1}{n} \sum_{i=1}^n \ell(y_i f(x_i)), \quad \ell \in \{\ell_{\text{hinge}}, \ell_{\text{log}}\}.$$

We now turn to resolving (b). For both hinge and logistic loss, there is actually a simple answer. Indeed, a key property of both losses is that they both dominate the zero-one loss in the following way. Specifically, consider a model  $f : \mathcal{X} \mapsto \mathbb{R}$ , and following similar arguments as we did to conclude Section 1.3.3:

$$\mathbf{1}\{y \neq \text{sgn}(f(x))\} \leq \mathbf{1}\{yf(x) \leq 0\}.$$

This characterization of the zero-one loss can then be used to show that both losses dominate the zero-one loss, as formalized by the following exercise.

**Exercise 1.18.** Show that there exists a universal constant  $c > 0$  such that:

$$\forall s \in \mathbb{R}, \mathbf{1}\{s \leq 0\} \leq \ell_{\text{hinge}}(s) \quad \text{and} \quad \mathbf{1}\{s \leq 0\} \leq c \cdot \ell_{\text{log}}(s).$$

What is the sharpest constant  $c$ ?

Hence, having low population risk under both the hinge or logistic loss implies low classification error, i.e.,

$$\mathbb{P}\{\text{sgn}(f(x)) \neq y\} \leq \mathbb{E}[\ell_{\text{hinge}}(yf(x))] \quad \text{and} \quad \mathbb{P}\{\text{sgn}(f(x)) \neq y\} \leq c \cdot \mathbb{E}[\ell_{\text{log}}(yf(x))].$$

We can go beyond these two losses by considering a *family* of *ramp losses* parameterized by  $\gamma > 0$ :

$$\ell_\gamma(s) = \begin{cases} 1 & \text{if } s < 0 \\ 1 - s/\gamma & \text{if } s \in [0, \gamma], \\ 0 & \text{if } s > \gamma \end{cases}.$$

By construction, this loss is sandwiched by the following inequalities:

$$\mathbf{1}\{s \leq 0\} \leq \ell_\gamma(s) \leq \mathbf{1}\{s < \gamma\}. \quad (1.11)$$

Consequently, for every  $f \in \mathcal{F}$ ,

$$\begin{aligned} \mathbb{P}\{y \neq \text{sgn}(f(x))\} &\leq \mathbb{E}[\ell_\gamma(yf(x))] && \text{using (1.11)} \\ &\leq \frac{1}{n} \sum_{i=1}^n \ell_\gamma(y_i f(x_i)) + \sup_{f \in \mathcal{F}} \text{gen}[f; \ell_\gamma] && \text{recall } \text{gen}[f] = L[f] - \hat{L}_n[f] \\ &\leq \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{y_i f(x_i) < \gamma\} + \sup_{f \in \mathcal{F}} \text{gen}[f; \ell_\gamma] && \text{using (1.11)}. \end{aligned} \quad (1.12)$$

(Here, we use the notation  $\text{gen}[f; \ell_\gamma]$  to emphasize that the ramp loss is involved in the generalization bound). This derivation states that we can control the zero-one loss in general by:

- (a) Controlling the generalization error  $\sup_{f \in \mathcal{F}} \text{gen}[f; \ell_\gamma]$  under the ramp loss  $\ell_\gamma$ , and
- (b) Counting the number of mistakes  $y_i f(x_i) < \gamma$  on the training dataset  $S_n$ . Recall that if  $f$  is linear, then we defined (Definition 1.5) the margin of a correctly classified point  $(x, y)$  as  $y\langle w, x \rangle / \|w\|$ . In the general case, the quantity  $yf(x)$  serves as the *un-normalized* margin.

Note that,  $\hat{f}_{\text{erm}}$  can actually optimize any loss function not necessarily related to  $\ell_\gamma$  and still apply the bound from Equation (1.12). We shall see shortly studying generalization of  $\ell_\gamma$  in the non-finite hypothesis class case, is much simpler than directly trying to study the generalization over the zero-one loss, thanks to the Lipschitz continuity of  $\ell_\gamma$  (observe that  $\ell_\gamma$  is  $\gamma^{-1}$ -Lipschitz).

### 1.3.7 Rademacher complexity

Towards a theory for general function classes, we first observe that simply counting the number of parameters of a function class is insufficient. Instead we must consider the geometric structure. For instance, consider the following linear hypothesis classes:

$$\mathcal{F}_p = \{x \mapsto \langle w, x \rangle \mid w \in \mathbb{R}^d, \|w\|_p \leq 1\}, \quad p \in [1, \infty].$$

Note that for all  $p$ ,  $\mathcal{F}_p$  is parameterized by  $d$  parameters. However, in high dimension, the parameter spaces of these sets are drastically different. By a volume calculation,

$$\text{Vol}(\{\|w\|_1 \leq 1\}) = 2^d/d!, \quad \text{Vol}(\{\|w\|_2 \leq 1\}) = \pi^{d/2}/\Gamma(1 + d/2), \quad \text{Vol}(\{\|w\|_\infty \leq 1\}) = 2^d.$$

Note that as  $d \rightarrow \infty$ , the first two volumes tend to 0, whereas the last tends to  $\infty$ . Hence, these parameter sets are geometrically very different, despite all sets being described by  $d$  parameters.

To address this issue, we now introduce a powerful tool for analyzing generalization error, which captures the geometry of the underlying hypothesis class.

**Definition 1.19** (Rademacher complexity). *Let  $\mathcal{F}$  be a set of functions mapping  $\mathbf{X} \mapsto \mathbb{R}$ . The Rademacher complexity of  $\mathcal{F}$  is defined as:*

$$\mathcal{R}_n(\mathcal{F}) := \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i f(x_i),$$

where  $\{\varepsilon_i\}_{i=1}^n$  are independent Rademacher random variables (and also independent of the  $\{x_i\}_{i=1}^n$ ). Note that the expectation is taken jointly over both the data  $\{x_i\}$  and the Rademacher random variables  $\{\varepsilon_i\}$ .

**Notation overload in Definition 1.19.** Note that there is a slight overload of notation in the definition of the Rademacher complexity in Definition 1.19. There, the domain  $\mathsf{X}$  is an abstract domain. In particular, it is *not* necessarily equal to the domain that the covariates in supervised learning reside in. In the sequel, we will often consider the Rademacher complexity of function classes taking as input covariates (i.e., functions mapping  $\mathsf{X} \mapsto \mathbb{R}$ ), but we will also function classes taking as input labelled pairs (i.e., functions mapping  $\mathsf{X} \times \mathsf{Y} \mapsto \mathbb{R}$ ).

The power of the Rademacher complexity comes from the following classical symmetrization lemma.

**Proposition 1.20** (Symmetrization). *Let  $\{x_i\}_{i=1}^n$  be independent random variables in  $\mathsf{X}$ , and let  $\mathcal{F}$  be a set of functions mapping  $\mathsf{X} \mapsto \mathbb{R}$ . We have that:*

$$\mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (\mathbb{E}[f(x_i)] - f(x_i)) \leq 2\mathcal{R}_n(\mathcal{F}).$$

*Proof.* Let  $\{x'_i\}$  be an independent copy of  $\{x_i\}$ , and let  $\{\varepsilon_i\}$  be independent Rademacher random variables. We have:

$$\begin{aligned} & \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (\mathbb{E}[f(x_i)] - f(x_i)) \\ &= \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (\mathbb{E}[f(x'_i)] - f(x_i)) && \text{since } x_i \stackrel{d}{=} x'_i \\ &= \mathbb{E}_{x_{1:n}} \sup_{f \in \mathcal{F}} \mathbb{E}_{x'_{1:n}} \left[ n^{-1} \sum_{i=1}^n (f(x'_i) - f(x_i)) \right] && \text{linearity of expectation} \\ &\leq \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (f(x'_i) - f(x_i)) && \text{since } \sup_f \mathbb{E}[Z_f] \leq \mathbb{E} \sup_f Z_f \\ &= \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i (f(x'_i) - f(x_i)) && \text{since } f(x'_i) - f(x_i) \stackrel{d}{=} \varepsilon_i (f(x'_i) - f(x_i)) \\ &\leq \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i f(x'_i) + \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i f(x_i) && \text{since } \sup_x [f(x) + g(x)] \leq \sup_x f(x) + \sup_x g(x) \\ &= 2\mathcal{R}_n(\mathcal{F}) && \text{since } x_i \stackrel{d}{=} x'_i. \end{aligned}$$

(Note: it is worth reflecting where in the proof we used the fact that the  $x_i$ 's are independent!).  $\square$

Note that given  $\mathcal{F}$ , applying Proposition 1.20 to the function class  $\{-f \mid f \in \mathcal{F}\}$  we also conclude that,

$$\mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (f(x_i) - \mathbb{E}[f(x_i)]) \leq 2\mathcal{R}_n(\{-f \mid f \in \mathcal{F}\}) = 2\mathcal{R}_n(\mathcal{F}).$$

The last equality is a special case the positive homogeneity of Rademacher complexities (convince yourself it is true!). Therefore, we can upgrade the conclusion of Proposition 1.20 to consider both directions:

$$\max \left\{ \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (\mathbb{E}[f(x_i)] - f(x_i)), \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (f(x_i) - \mathbb{E}[f(x_i)]) \right\} \leq 2\mathcal{R}_n(\mathcal{F}). \quad (1.13)$$

Next, we prove a high probability deviation bound on the quantity  $\sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (\mathbb{E}[f(x_i)] - f(x_i))$  using the bounded-differences inequality (Proposition B.16).

**Proposition 1.21.** Let  $\{x_i\}_{i=1}^n$  be independent random variables, and let  $\mathcal{F}$  be a set of uniformly bounded functions mapping  $\mathbf{X} \mapsto [-B, B]$ . Then, with probability at least  $1 - \delta$  (over the randomness of  $\{x_i\}_{i=1}^n$ ),

$$\sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (\mathbb{E}[f(x_i)] - f(x_i)) \leq 2\mathcal{R}_n(\mathcal{F}) + 2B\sqrt{\frac{2\log(1/\delta)}{n}}.$$

*Proof.* Let  $h(x_{1:n}) := \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (\mathbb{E}[f(x_i)] - f(x_i))$ . Pick any index  $i \in \{1, \dots, n\}$ , and let  $x_1, \dots, x_n, x'_i$  be arbitrary. Observe that, since  $\mathcal{F}$  is uniformly bounded,

$$\begin{aligned} & |h(x_{1:i-1}, x_i, x_{i+1:n}) - h(x_{1:i-1}, x'_i, x_{i+1:n})| \\ &= \left| \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (\mathbb{E}[f(x_i)] - f(x_i)) - \sup_{f \in \mathcal{F}} n^{-1} \left[ (\mathbb{E}[f(x_i)] - f(x'_i)) + \sum_{i \neq i} (\mathbb{E}[f(x_i)] - f(x_i)) \right] \right| \\ &\leq \frac{1}{n} \sup_{f \in \mathcal{F}} |f(x_i) - f(x'_i)| \leq \frac{2B}{n}. \end{aligned}$$

Therefore the function  $h$  satisfies bounded-differences, and we can apply the bounded-differences inequality (Proposition B.16) to conclude:

$$\mathbb{P}\{h(x_{1:n}) - \mathbb{E}[h(x_{1:n})] \geq t\} \leq \exp(-nt^2/(8B^2)).$$

The claim now follows by using Proposition 1.20 to bound  $\mathbb{E}[h(x_{1:n})] \leq 2\mathcal{R}_n(\mathcal{F})$ .  $\square$

Again, as we saw following Proposition 1.20, applying Proposition 1.21 to the negated function class  $\{-f \mid f \in \mathcal{F}\}$ , we conclude that with probability at least  $1 - \delta$ ,

$$\sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (f(x_i) - \mathbb{E}[f(x_i)]) \leq 2\mathcal{R}_n(\mathcal{F}) + 2B\sqrt{\frac{2\log(1/\delta)}{n}}. \quad (1.14)$$

At this point, we have seen that the Rademacher complexity is a natural notion of complexity which arises from controlling the expected value of the following supremum of the *empirical process*:

$$\sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (\mathbb{E}[f(x_i)] - f(x_i)).$$

We now discuss some properties of this complexity measure. Immediately from the definition, we see that the Rademacher complexity has the following properties (here,  $\mathcal{F}$  and  $\mathcal{G}$  are arbitrary function class mapping  $\mathbf{X} \mapsto \mathbb{R}$ ):

- (a) (Monotonicity) For any  $\mathcal{G}$  such that  $\mathcal{F} \subseteq \mathcal{G}$ ,  $\mathcal{R}_n(\mathcal{F}) \leq \mathcal{R}_n(\mathcal{G})$ .
- (b) (Positive homogeneous) For any  $c \in \mathbb{R}$ ,  $\mathcal{R}_n(\{x \mapsto cf(x) \mid f \in \mathcal{F}\}) = |c| \cdot \mathcal{R}_n(\mathcal{F})$ .
- (c) (Offset invariance) Let  $f_0 : \mathbf{X} \mapsto \mathbb{R}$  be an arbitrary function. We have

$$\mathcal{R}_n(\{x \mapsto f(x) + f_0(x) \mid f \in \mathcal{F}\}) = \mathcal{R}_n(\mathcal{F}).$$

- (d) (Sub-additivity) For any  $\mathcal{G}$ ,

$$\mathcal{R}_n(\{x \mapsto f(x) + g(x) \mid f \in \mathcal{F}, g \in \mathcal{G}\}) \leq \mathcal{R}_n(\mathcal{F}) + \mathcal{R}_n(\mathcal{G}).$$

(Check that these properties hold!). Furthermore, Rademacher complexities have a key “peeling” property which will end up being important for what follows.



**Proposition 1.22.** Let  $\ell : \mathbb{R} \mapsto \mathbb{R}$  be  $L$ -Lipschitz, i.e.,:

$$\forall x, y \in \mathbb{R}, |\ell(x) - \ell(y)| \leq L|x - y|.$$

Let  $\mathcal{F}$  be a function class mapping  $X \mapsto \mathbb{R}$ . We have that:

$$\mathcal{R}_n(\{x \mapsto \ell(f(x)) \mid f \in \mathcal{F}\}) \leq L \cdot \mathcal{R}_n(\mathcal{F}).$$

Unlike the previously stated properties, the proof of Proposition 1.22 is not as immediate. It is, however, a consequence of the following Ledoux and Talagrand's contraction lemma.

**Lemma 1.23** (Ledoux and Talagrand Contraction). Let  $\phi : \mathbb{R} \mapsto \mathbb{R}$  be a contraction, i.e.,

$$\forall x, y \in \mathbb{R}, |\phi(x) - \phi(y)| \leq |x - y|.$$

Let  $T \subset \mathbb{R}^n$  be any set. We have that:

$$\mathbb{E} \sup_{t \in T} \sum_{i=1}^n \varepsilon_i \phi(t_i) \leq \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \varepsilon_i t_i.$$

*Proof.* We will prove that, for every index  $i \in \{1, \dots, n\}$  and any arbitrary  $A : T \mapsto \mathbb{R}$ ,

$$\mathbb{E} \sup_{t \in T} [A(t) + \varepsilon \phi(t_i)] \leq \mathbb{E} \sup_{t \in T} [A(t) + \varepsilon t_i]. \quad (1.15)$$

We start by expanding the LHS of (1.15):

$$\mathbb{E} \sup_{t \in T} [A(t) + \varepsilon \phi(t_i)] = \frac{1}{2} \sup_{t \in T} [A(t) + \phi(t_i)] + \frac{1}{2} \sup_{t \in T} [A(t) - \phi(t_i)].$$

Suppose that  $u \in T$  achieves the supremum for the first term on the RHS, and  $v \in T$  achieves the supremum for the second term on the RHS (if both supremums are not achieved, consider sequences  $\{u^{(i)}\}, \{v^{(i)}\}$  in  $T$  which approach the supremums instead). First, let us suppose that  $u_i \leq v_i$ . Then, the contraction of  $\phi$  yields  $\phi(u_i) - \phi(v_i) \leq v_i - u_i$ , and therefore

$$\begin{aligned} \frac{1}{2}(A(u) + \phi(u_i)) + \frac{1}{2}(A(v) - \phi(v_i)) &\leq \frac{1}{2}(A(u) - u_i) + \frac{1}{2}(A(v) + v_i) \\ &\leq \frac{1}{2} \sup_{t \in T} (A(t) - t_i) + \frac{1}{2} \sup_{t \in T} (A(t) + t_i) \quad \text{since } u, v \in T \\ &= \mathbb{E} \sup_{t \in T} [A(t) + \varepsilon t_i]. \end{aligned}$$

On the other hand, if  $v_i \leq u_i$ , then  $\phi(u_i) - \phi(v_i) \leq u_i - v_i$ , and a nearly identical argument yields:

$$\frac{1}{2}(A(u) + \phi(u_i)) + \frac{1}{2}(A(v) - \phi(v_i)) \leq \frac{1}{2}(A(u) + u_i) + \frac{1}{2}(A(v) - v_i) \leq \mathbb{E} \sup_{t \in T} [A(t) + \varepsilon t_i].$$

Combining these two cases, we have shown that (1.15) holds. To conclude the proof, we use the independence of the Rademacher random variables  $\{\varepsilon_i\}$  along with repeated applications of (1.15) to “peel” off the  $\phi$  in a step by step manner:

$$\begin{aligned} \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \varepsilon_i \phi(t_i) &= \mathbb{E}_{\varepsilon_{-n}} \mathbb{E}_{\varepsilon_n} \sup_{t \in T} \left[ \sum_{i=1}^{n-1} \varepsilon_i \phi(t_i) + \varepsilon_n \phi(t_n) \right] \\ &\leq \mathbb{E}_{\varepsilon_{-n}} \mathbb{E}_{\varepsilon_n} \sup_{t \in T} \left[ \sum_{i=1}^{n-1} \varepsilon_i \phi(t_i) + \varepsilon_n t_n \right] \quad \text{using (1.15)} \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}_{\varepsilon_{-n-1}} \mathbb{E}_{\varepsilon_{n-1}} \sup_{t \in T} \left[ \left\{ \sum_{i=1}^{n-2} \varepsilon_i \phi(t_i) + \varepsilon_n t_n \right\} + \varepsilon_{n-1} \phi(t_{n-1}) \right] \\
&\leq \mathbb{E}_{\varepsilon_{-n-1}} \mathbb{E}_{\varepsilon_{n-1}} \sup_{t \in T} \left[ \left\{ \sum_{i=1}^{n-2} \varepsilon_i \phi(t_i) + \varepsilon_n t_n \right\} + \varepsilon_{n-1} t_{n-1} \right] \quad \text{using (1.15) again} \\
&\vdots \\
&\leq \mathbb{E} \sup_{t \in T} \sum_{i=1}^n \varepsilon_i t_i.
\end{aligned}$$

Above, the notation  $\varepsilon_{-k}$  for an index  $k \in \{1, \dots, n\}$  denotes all the Rademacher random variables excluding the  $k$ -th index, i.e.,  $\varepsilon_{-k} = (\varepsilon_1, \dots, \varepsilon_{k-1}, \varepsilon_{k+1}, \dots, \varepsilon_n)$ .  $\square$

**Exercise 1.24.** Prove Proposition 1.22 using Lemma 1.23.

### 1.3.8 Generalization bounds via Rademacher complexity

We now have the tools in place to state our first generalization bound via Rademacher complexities.

**Theorem 1.25** (Generalization bound via Rademacher complexity). *Let  $\ell : \mathbb{R} \mapsto \mathbb{R}$  be any  $L$ -Lipschitz loss function. Let  $\mathcal{F}$  be a function class mapping  $\mathbf{X} \mapsto \mathbb{R}$ . Suppose that:*

$$\sup_{f \in \mathcal{F}} \sup_{(x, y) \in \mathbf{Z}} |\ell(yf(x))| \leq B_\ell.$$

With probability at least  $1 - \delta$  (over the randomness of  $\{(x_i, y_i)\}_{i=1}^n$ ),

$$\forall f \in \mathcal{F}, \quad L[f] \leq \frac{1}{n} \sum_{i=1}^n \ell(y_i f(x_i)) + 2L \cdot \mathcal{R}_n(\mathcal{F}) + 2B_\ell \sqrt{\frac{2 \log(1/\delta)}{n}}.$$

*Proof.* Much as we proceeded in (1.9), we have for any  $f \in \mathcal{F}$ ,

$$\begin{aligned}
L[f] &= \frac{1}{n} \sum_{i=1}^n \ell(y_i f(x_i)) + \mathbb{E}[\ell(yf(x))] - \frac{1}{n} \sum_{i=1}^n \ell(y_i f(x_i)) \\
&\leq \frac{1}{n} \sum_{i=1}^n \ell(y_i f(x_i)) + \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (\mathbb{E}[\ell(y_i f(x_i))] - \ell(y_i f(x_i))).
\end{aligned}$$

We use Proposition 1.21 to analyze the second term (noting that the empirical process we want to control is over functions mapping  $\mathbf{X} \times \mathbf{Y} \mapsto \mathbb{R}$ , cf. the discussion after Definition 1.19), which tells us with probability at least  $1 - \delta$ ,

$$\sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n (\mathbb{E}[\ell(y_i f(x_i))] - \ell(y_i f(x_i))) \leq 2\mathcal{R}_n(\{(x, y) \mapsto \ell(yf(x)) \mid f \in \mathcal{F}\}) + 2B_\ell \sqrt{\frac{2 \log(1/\delta)}{n}}. \quad (1.16)$$

The Rademacher complexity term  $\mathcal{R}_n(\{(x, y) \mapsto \ell(yf(x)) \mid f \in \mathcal{F}\})$  in (1.7) is challenging to analyze directly, since it involves the composition of two functions. However, we can use Rademacher contraction to “peel” off the loss  $\ell$  directly exposing the Rademacher complexity of  $\mathcal{F}$ . Specifically, by Proposition 1.22:

$$\mathcal{R}_n(\{(x, y) \mapsto \ell(yf(x)) \mid f \in \mathcal{F}\}) \leq L \cdot \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i y_i f(x_i)$$

$$\begin{aligned}
&= L \cdot \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i f(x_i) && \varepsilon_i y_i \stackrel{d}{=} \varepsilon_i \text{ by symmetry} \\
&= L \cdot \mathcal{R}_n(\mathcal{F}).
\end{aligned}$$

□

Recalling our discussion on margin theory (Section 1.3.6), an immediate corollary of Theorem 1.25 is the following margin bound. The following bound allows one to decouple the question of what loss function to use for training from the desired zero-one loss metric. It states that no matter what loss function one uses for empirical risk minimization (more broadly, no matter what algorithm is used to select a hypothesis from the training data), one can bound the estimator's population zero-one loss by counting the number of margin mistakes the estimator makes on the training data, and then adding some extra terms that account for both the complexity of the hypothesis space  $\mathcal{F}$  and the number of training data points observed.

**Corollary 1.26** (Margin generalization bound). *Fix any  $\gamma > 0$ . Let  $\mathcal{F}$  be a function class mapping  $\mathbf{X} \mapsto \mathbb{R}$ . With probability at least  $1 - \delta$  (over the randomness of  $\{(x_i, y_i)\}_{i=1}^n$ ),*

$$\forall f \in \mathcal{F}, \mathbb{P}\{\text{sgn}(f(x)) \neq y\} \leq \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{y_i f(x_i) < \gamma\} + \frac{2}{\gamma} \cdot \mathcal{R}_n(\mathcal{F}) + 2\sqrt{\frac{2 \log(1/\delta)}{n}}.$$

*Proof.* Apply Theorem 1.25 to the ramp loss  $\ell_\gamma$  (which satisfies  $|\ell_\gamma| \leq 1$  and is  $\gamma^{-1}$ -Lipschitz) from Section 1.3.6, and then apply the ramp loss inequality  $\ell_\gamma(s) \leq \mathbf{1}\{s < \gamma\}$  from (1.11) to conclude. □

One drawback of Corollary 1.26 is that it only applies for a *fixed* margin  $\gamma$ . However, this can be remedied at a small cost to the bound, as shown in the following exercise.

**Exercise 1.27.** Show that one can extend Corollary 1.26 to apply for all margins as follows. Suppose that  $\mathcal{F}$  is uniformly bounded, so that  $|f| \leq B$  for  $f \in \mathcal{F}$ . With probability at least  $1 - \delta$ , for all  $f \in \mathcal{F}$ ,

$$\mathbb{P}\{\text{sgn}(f(x)) \neq y\} \leq \inf_{\gamma \in [0, B]} \left[ \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{y_i f(x_i) < \gamma\} + \frac{4}{\gamma} \cdot \mathcal{R}_n(\mathcal{F}) + c\sqrt{\frac{\log \log(B/\gamma) + \log(1/\delta)}{n}} \right],$$

where  $c > 0$  is a universal constant. Note that the range of  $\gamma \in [0, B]$  is not restrictive, since the assumption that  $|f| \leq B$  implies that for any  $\gamma > B$ , the margin mistake  $yf(x) < \gamma$  always occurs and hence the bound on the RHS becomes vacuous for  $\gamma > B$ .

**Why the ramp loss?** At this point, a natural question to ask is why Theorem 1.25 is applied to the ramp loss  $\ell_\gamma$  and not directly to the zero-one loss. While the simple answer is that the zero-one loss is not Lipschitz thanks to its discontinuity at the origin, inspecting the proof of Theorem 1.25 shows that the Lipschitzness of the loss is only used to “peel” the loss out of the Rademacher complexity and directly expose the Rademacher complexity  $\mathcal{R}_n(\mathcal{F})$  of the hypothesis class  $\mathcal{F}$ . Indeed, the proof up to and including (1.16) can be replayed with the zero-one loss substituted in. In other words, with probability at least  $1 - \delta$ ,

$$\begin{aligned}
\forall f \in \mathcal{F}, \mathbb{P}\{\text{sgn}(f(x)) \neq y\} &\leq \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{\text{sgn}(f(x_i)) \neq y_i\} \\
&\quad + 2\mathcal{R}_n(\{(x, y) \mapsto \mathbf{1}\{\text{sgn}(f(x)) \neq y\} \mid f \in \mathcal{F}\}) + 2\sqrt{\frac{2 \log(1/\delta)}{n}}.
\end{aligned}$$

The Rademacher complexity term to be controlled then becomes:

$$\mathcal{R}_n(\{(x, y) \mapsto \mathbf{1}\{\text{sgn}(f(x)) \neq y\} \mid f \in \mathcal{F}\}) = \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i \mathbf{1}\{y_i \neq \text{sgn}(f(x_i))\}$$

$$\begin{aligned}
&= \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i \left( \frac{1 - \text{sgn}(f(x_i)) y_i}{2} \right) \quad \text{using Exercise 1.28} \\
&= \frac{1}{2} \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i (-y_i) \text{sgn}(f(x_i)) \\
&= \frac{1}{2} \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i \text{sgn}(f(x_i)) \quad \text{since } \varepsilon_i \stackrel{d}{=} -\varepsilon_i y_i \\
&= \frac{1}{2} \mathcal{R}_n(\{x \mapsto \text{sgn}(f(x)) \mid f \in \mathcal{F}\}). \tag{1.17}
\end{aligned}$$

We will see later (Section 1.3.10) that the Rademacher complexity in (1.17) can be controlled by the VC-dimension of the predictor class  $\mathcal{H} = \{x \mapsto \text{sgn}(f(x)) \mid f \in \mathcal{F}\}$ . The VC-dimension, a classical object in the machine learning literature which we will study shortly, is a combinatorial quantity which indicates the cardinality of the largest dataset which can be *shattered* by a predictor. However, this approach of going through the VC-dimension has its drawbacks, one of which is the geometric structure of the underlying hypothesis class  $\mathcal{F}$  is abstracted into a combinatorial quantity, rather than directly exposed via the Rademacher complexity. The latter is much intuitive in my opinion, and is why we study the latter approach first.

We conclude our discussion with the following exercise.

**Exercise 1.28.** Show that for all  $y, y' \in \{\pm 1\}$ ,

$$\mathbf{1}\{y \neq y'\} = \frac{1 - y \cdot y'}{2}.$$

### 1.3.9 Explicit computations of Rademacher complexity

In this section, we will explicitly compute upper and lower bounds on the Rademacher complexity, in order to get a better understanding of the properties of this complexity measure. Recall that from Theorem 1.25, for any  $L$ -Lipschitz and  $B$ -bounded loss, the Rademacher complexity controls the maximum generalization error over the function class in the following way. With probability at least  $1 - \delta$ ,

$$\sup_{f \in \mathcal{F}} \text{gen}[f] \leq 2L \cdot \mathcal{R}_n(\mathcal{F}) + 2B \sqrt{\frac{2 \log(1/\delta)}{n}}. \tag{1.18}$$

As we will see through a series of examples, the first term above is the dominant term.

**Linear function classes.** We will first consider Rademacher complexities of linear function classes,

$$\mathcal{F}_p = \{x \mapsto \langle x, w \rangle \mid \|w\|_p \leq 1\}.$$

We will also assume that  $x_i \sim N(0, I)$  to simplify expressions as much as possible. Our first computation is to simplify  $\mathcal{R}_n(\mathcal{F}_p)$ .

Recall from linear algebra, for any norm  $\|\cdot\|$ , its dual norm  $\|\cdot\|_*$  is defined as:

$$\|x\|_* := \sup\{\langle z, x \rangle \mid \|z\| \leq 1\}.$$

For  $\ell_p$  norms, the dual norms have a simple characterization. For a given  $p \in [1, \infty]$ , the number  $q$  is *dual* to  $p$  if  $1/p + 1/q = 1$ . With this duality pairing, we have that the  $\ell_q$  norm  $\|\cdot\|_q$  is dual to the  $\ell_p$  norm  $\|\cdot\|_p$ , as a consequence of Hölder's inequality. With this context in mind, we have:

$$\mathcal{R}_n(\mathcal{F}_p) = \mathbb{E} \sup_{\|w\|_p \leq 1} n^{-1} \sum_{i=1}^n \varepsilon_i \langle x_i, w \rangle = n^{-1} \mathbb{E} \sup_{\|w\|_p \leq 1} \left\langle w, \sum_{i=1}^n \varepsilon_i x_i \right\rangle = n^{-1} \mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|_q. \tag{1.19}$$

That is, the Rademacher complexity of  $\mathcal{F}_p$  involves computing the dual  $\ell_q$ -norm of a random vector sum. For  $p = q = 2$ , Jensen's inequality yields a simple upper bound:

$$\mathcal{R}_n(\mathcal{F}_2) = n^{-1} \mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\| \leq n^{-1} \sqrt{\mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^2} = n^{-1} \sqrt{\sum_{i=1}^n \mathbb{E} \|x_i\|^2} = \sqrt{\frac{d}{n}}. \quad (1.20)$$

This argument is actually fairly sharp, as shown in the following exercise.

**Exercise 1.29.** Show that  $\mathcal{R}_n(\mathcal{F}_2) \geq c\sqrt{d/n}$  for a universal constant  $c$ . You may assume that  $d \geq c_0$  for any fixed universal constant  $c_0$  to simplify the argument.

Hint: You may use without proof the following fact, known as the *Gaussian Poincaré inequality*. Let  $g \in \mathbb{R}^k$  be an isotropic Gaussian random vector. For any  $f : \mathbb{R}^k \mapsto \mathbb{R}$ , we have that

$$\text{Var}(f(g)) \leq \mathbb{E} \|\nabla f(g)\|^2.$$

Note that this fact is dimension free.

Thus, we see that for  $\mathcal{F}_2$ , the Rademacher complexity scales as  $\sqrt{d/n}$ . Plugging into (1.18), assuming  $B = L = 1$ ,

$$\sup_{f \in \mathcal{F}_2} \text{gen}[f] \leq c \sqrt{\frac{d + \log(1/\delta)}{n}},$$

for a universal constant  $c$ . Note that so far, this matches up to constants the generalization bound we derived by discretizing  $\mathcal{F}_2$  using `float32` numbers in Equation (1.10).

We now turn to study several other linear hypothesis spaces. First, we will see that for  $\mathcal{F}_1$ , the Rademacher machinery yields a significant reduction in model complexity.

**Proposition 1.30.** *We have that:*

$$\mathcal{R}_n(\mathcal{F}_1) \leq \sqrt{\frac{2 \log(2d)}{n}}.$$

*Proof.* Fix an index  $j \in \{1, \dots, d\}$ . It is not hard to see that  $Z_j := \sum_{i=1}^n \varepsilon_i \langle x_i, e_j \rangle \stackrel{d}{=} N(0, n)$  (why?). Hence by the sub-Gaussian maximal inequality (Exercise B.18),

$$\mathbb{E} \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|_\infty = \mathbb{E} \max_{j=1, \dots, d} |Z_j| \leq \sqrt{2n \log(2d)}.$$

The claim now follows from the dual  $\ell_\infty$ -norm characterization of  $\mathcal{R}_n(\mathcal{F}_1)$  (cf. Equation (1.19)). □

Thus we see that from a Rademacher complexity standpoint, the geometry of the  $\ell_1$ -ball yields a significant reduction in model complexity compared to the  $\ell_2$ -ball. Note that this can go the other direction, as we see for the  $\ell_\infty$ -ball.

**Exercise 1.31.** Show that  $\mathcal{R}_n(\mathcal{F}_\infty) = \sqrt{2/\pi} \cdot d/\sqrt{n}$ .

We conclude our last example of linear hypothesis by looking at sparse predictors. Here, we consider a strict subset of  $\mathcal{F}_2$  where the support (non-zero entries) of the weight vector  $w$  is restricted to at most  $s$  (with  $s < d$ ) entries. We do not, however, which coordinates the support can entail. Note that if we restricted  $\mathcal{F}_2$  to one fixed subset of  $s$  non-zero coordinates, then we would have Rademacher complexity scaling as  $\sqrt{s/n}$ ; only the coordinates with non-zero support affect the model complexity. However, it turns out that we only have to pay a modest price to allow for any support of at most  $s$  coordinates.

**Proposition 1.32.** Let  $\mathcal{F}_{2,s}$  denote the following sparse hypothesis class (where  $1 \leq s < d$ ):

$$\mathcal{F}_{2,s} := \{x \mapsto \langle w, x \rangle \mid w \in \mathbb{R}^d, \|w\|_0 \leq s, \|w\| \leq 1\}.$$

Here,  $\|w\|_0 = \sum_{i=1}^d \mathbf{1}\{w_i \neq 0\}$  counts the number of non-zero entries of  $w \in \mathbb{R}^d$ . There exists a universal positive constant  $c_0$  such that,

$$\mathcal{R}_n(\mathcal{F}_{2,s}) \leq c_0 \sqrt{\frac{s \log(d/s)}{n}}.$$

Before we turn to the proof of this result, let us remark about its implication. Using Equation (1.18) with  $B = L = 1$ , for a universal  $c$ ,

$$\sup_{f \in \mathcal{F}_{2,s}} \text{gen}[f] \leq c \sqrt{\frac{s \log(d/s) + \log(1/\delta)}{n}}.$$

Thus, this states that statistically, we can learn  $s$ -sparse predictors with ambient dimension  $d$ , even when we do not know the support, by only requiring  $n \gtrsim s \log(d/s)$  samples. Indeed,  $n \ll d$  can hold and this condition on  $n$  can still be satisfied. This is related to the compressed sensing literature, where one studies how to recover a sparse model when the number of measurements is less than the ambient dimension of the parameter vector. However, note that the result we prove is purely statistical: it does not provide prescribe an efficient algorithm for recovering such a sparse model. Indeed, solving the ERM problem over  $\mathcal{F}_{2,s}$  requires enumerating all possible subsets of  $\{1, \dots, d\}$  of size at most  $s$ , and solving an ERM problem restricted to each subset.

*Proof of Proposition 1.32.* The proof of this result uses the following fact. Let  $g \sim N(0, I)$  be an isotropic random Gaussian vector. The random variable  $\|g\|$  is  $c_1$ -sub-Gaussian, for a universal positive constant. Note that this fact is also dimension free (meaning the constant  $c_1$  does not depend on the dimension of  $g$ ). This fact is a consequence of Gaussian concentration of Lipschitz functions, which is a quite remarkable result [Vershynin, 2018, cf. Section 5.2.1].

Let  $I \subseteq \{1, \dots, d\}$  denote an index set, and let  $2^{[d]}$  denote the powerset (subset of all subsets) of  $[d] := \{1, \dots, d\}$ . Let  $P_I : \mathbb{R}^{|I|} \mapsto \mathbb{R}^d$  denote the linear operator which maps a vector of size  $|I|$  to a vector of size  $d$ , placing the entries of the input into the coordinates listed in  $I$ . For example if  $d = 3$  and  $I = \{1, 3\}$ ,

$$P_{\{1,3\}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ x_2 \end{bmatrix}.$$

Observe that by construction,  $\|P_I x\| = \|x\|$  for all  $x \in \mathbb{R}^{|I|}$ . Also observe that the transpose (adjoint operator) of  $P_I$ , denoted  $P_I^* : \mathbb{R}^d \mapsto \mathbb{R}^{|I|}$ , picks out the coordinates of the  $d$ -dimensional input vector listed in  $I$  to form an  $|I|$ -dimension output vector. For example, again if  $d = 3$  and  $I = \{1, 3\}$ ,

$$P_{\{1,3\}}^* \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}.$$

Furthermore, let

$$\mathcal{I}_s := \{I \in 2^{[d]} \mid |I| \in \{1, \dots, s\}\}$$

denote the set of all non-empty subsets of  $[d]$  which contain at most  $s$  elements. A simple counting argument yields that:

$$|\mathcal{I}_s| = \sum_{k=1}^s \binom{d}{k}.$$

With this notation in place, and letting  $g$  be a  $d$ -dimensional isotropic Gaussian random vector,

$$\begin{aligned}
n^{1/2} \cdot \mathcal{R}_n(\mathcal{F}_{2,s}) &= n^{-1/2} \cdot \mathbb{E} \sup_{w \in \mathcal{F}_{2,s}} \sum_{i=1}^n \varepsilon_i \langle w, x_i \rangle \\
&= \mathbb{E} \sup_{w \in \mathcal{F}_{2,s}} \left\langle w, n^{-1/2} \sum_{i=1}^n \varepsilon_i x_i \right\rangle \\
&= \mathbb{E}_g \sup_{w \in \mathcal{F}_{2,s}} \langle w, g \rangle \\
&= \mathbb{E}_g \sup_{I \in \mathcal{I}_s} \sup_{w \in \mathbb{R}^{|I|}, \|w\| \leq 1} \langle P_I w, g \rangle \\
&= \mathbb{E}_g \sup_{I \in \mathcal{I}_s} \sup_{w \in \mathbb{R}^{|I|}, \|w\| \leq 1} \langle w, P_I^* g \rangle \\
&= \mathbb{E}_g \sup_{I \in \mathcal{I}_s} \|P_I^* g\|.
\end{aligned}$$

Now, we observe that  $P_I^* g$  has the same distribution as an  $|I|$ -dimensional Gaussian random vector, and therefore by Jensen's inequality  $\mathbb{E}_g \|P_I^* g\| \leq \sqrt{|I|} \leq \sqrt{s}$  for any  $I \in \mathcal{I}_s$ . Hence,

$$\begin{aligned}
\mathbb{E}_g \sup_{I \in \mathcal{I}_s} \|P_I^* g\| &= \mathbb{E}_g \sup_{I \in \mathcal{I}_s} \{ \|P_I^* g\| - \mathbb{E}_g \|P_I^* g\| + \mathbb{E}_g \|P_I^* g\| \} \\
&\leq \mathbb{E}_g \sup_{I \in \mathcal{I}_s} \{ \|P_I^* g\| - \mathbb{E}_g \|P_I^* g\| \} + \sup_{I \in \mathcal{I}_s} \mathbb{E}_g \|P_I^* g\| \\
&\leq \sqrt{s} + \mathbb{E}_g \sup_{I \in \mathcal{I}_s} \{ \|P_I^* g\| - \mathbb{E}_g \|P_I^* g\| \}.
\end{aligned}$$

Now we use the fact that the random variable  $(\|P_I^* g\| - \mathbb{E}_g \|P_I^* g\|)$  is zero-mean  $c_1$ -sub-Gaussian for all  $I$ . Hence, by the sub-Gaussian maximal inequality (Proposition B.17),

$$\mathbb{E}_g \sup_{I \in \mathcal{I}_s} \{ \|P_I^* g\| - \mathbb{E}_g \|P_I^* g\| \} \leq c_1 \sqrt{2 \log |\mathcal{I}_s|} = c_1 \sqrt{2 \log \left( \sum_{k=1}^s \binom{d}{k} \right)}.$$

We now use Proposition D.1 to bound:

$$\sum_{k=1}^s \binom{d}{k} \leq \sum_{k=0}^s \binom{d}{k} \leq \left( \frac{ed}{s} \right)^s.$$

This bound implies:

$$n^{1/2} \cdot \mathcal{R}_n(\mathcal{F}_{2,s}) \leq \sqrt{s} + c_1 \sqrt{2s \log(ed/s)} \implies \mathcal{R}_n(\mathcal{F}_{2,s}) \leq c_2 \sqrt{\frac{s \log(d/s)}{n}},$$

for some universal constant  $c_2$ . □

**Beyond linear function classes.** We now move beyond linear hypothesis classes. First, we show a negative result, that without making further assumptions on the hypothesis class, the Rademacher complexity does not necessarily need to decay to zero as  $n \rightarrow \infty$ .

**Exercise 1.33.** Construct a hypothesis class  $\mathcal{F} \subseteq [-1, 1]^{\mathcal{X}}$  such that  $\mathcal{R}_n(\mathcal{F}) \geq 1$  for all  $n \in \mathbb{N}_+$ .

Next, we will study the hypothesis space of a single hidden-layer neural network with ReLU activations. The following example comes from Ma [2022, Section 5.3].

**Proposition 1.34.** Let  $\phi(x) = \max\{x, 0\}$  be the ReLU activation function. Consider the hypothesis space of single hidden-layer neural networks with width  $m \in \mathbb{N}_+$ :

$$\mathcal{F} = \left\{ x \mapsto \sum_{h=1}^m w_h \phi(\langle u_h, x \rangle) \mid \{w_h\}_{h=1}^m \subset \mathbb{R}, \{u_h\}_{h=1}^m \subset \mathbb{R}^d, \sum_{h=1}^m |w_h| \|u_h\| \leq 1 \right\}.$$

We have that:

$$\mathcal{R}_n(\mathcal{F}) \leq 2\sqrt{\frac{d}{n}}.$$

Before we proceed with the proof of this result, let us first remark why it is interesting. First, the number of parameters parameterizing functions in  $\mathcal{F}$  is  $m + m \cdot d = m(d+1)$ . Thus, from a parameter counting perspective, we expect the Rademacher complexity of this class to scale as  $\sqrt{m(d+1)/n}$ . However, this computation shows that by imposing the extra norm constraint  $\sum_{h=1}^m |w_h| \|u_h\| \leq 1$ , the Rademacher scaling improves to  $\sqrt{d/n}$ , meaning it only *implicitly* depends on the hidden width dimension  $m$  via the norm constraint. In fact, if for all  $h \in \{1, \dots, m\}$  and any  $s > 1$ , we have  $\|u_h\| \leq 1$  and  $|w_h| \leq \frac{1}{\zeta(s) \cdot h^s}$  (here,  $\zeta(s) := \sum_{i=1}^{\infty} i^{-s}$ ), then we can let  $m \rightarrow \infty$  while preserving the  $\sqrt{d/n}$  scaling. Indeed, this allows us to have an *infinitely wide* neural network with bounded capacity.

*Proof.* Let  $\theta = (\{w_h\}, \{u_h\})$  denote the parameters of  $\mathcal{F}$ , and let  $\Theta$  denote the parameter set. We have:

$$\begin{aligned} n \cdot \mathcal{R}_n(\mathcal{F}) &= \mathbb{E} \sup_{\theta \in \Theta} \sum_{i=1}^n \varepsilon_i \sum_{h=1}^m w_h \phi(\langle u_h, x_i \rangle) \\ &= \mathbb{E} \sup_{\theta \in \Theta} \sum_{h=1}^m w_h \sum_{i=1}^n \varepsilon_i \phi(\langle u_h, x_i \rangle) && \text{by re-arranging sums} \\ &= \mathbb{E} \sup_{\theta \in \Theta} \sum_{h=1}^m w_h \|u_h\| \sum_{i=1}^n \varepsilon_i \phi(\langle u_h, x_i \rangle / \|u_h\|) && \text{since ReLU is positive homogeneous} \\ &\leq \mathbb{E} \sup_{\theta \in \Theta} \max_{h=1, \dots, m} \left| \sum_{i=1}^n \varepsilon_i \phi(\langle u_h, x_i \rangle / \|u_h\|) \right| && \text{since } \sum_{h=1}^m |w_h| \|u_h\| \leq 1 \\ &\leq \mathbb{E} \sup_{\|u\| \leq 1} \left| \sum_{i=1}^n \varepsilon_i \phi(\langle u, x_i \rangle) \right| && \text{since } u_h / \|u_h\| \text{ is unit-norm.} \end{aligned}$$

We now prove an quick fact that will allow us to remove the absolute value in the last inequality above.

**Proposition 1.35.** Let  $T \subset \mathbb{R}^n$  be an arbitrary set, and suppose that:

$$\forall \varepsilon \in \{\pm 1\}^n, \exists t \in T \text{ s.t. } \langle \varepsilon, t \rangle \geq 0.$$

Then,

$$\mathbb{E} \sup_{t \in T} |\langle \varepsilon, t \rangle| \leq 2 \mathbb{E} \sup_{t \in T} \langle \varepsilon, t \rangle, \quad (1.21)$$

where  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$  and the  $\varepsilon_i$ 's are i.i.d. Rademacher random variables.

*Proof.* Observe that our assumption implies that:

$$\forall \varepsilon \in \{\pm 1\}^n, \sup_{t \in T} \langle \varepsilon, t \rangle \geq 0. \quad (1.22)$$

Hence, for any  $\varepsilon \in \{\pm 1\}^n$ ,

$$\sup_{t \in T} |\langle \varepsilon, t \rangle| = \sup_{t \in T} \max_{\sigma \in \{\pm 1\}} \sigma \langle \varepsilon, t \rangle \quad |x| = \sup_{\sigma \in \{\pm 1\}} \sigma x \text{ for all } x \in \mathbb{R}$$



$$\begin{aligned}
&= \max \left\{ \sup_{t \in T} \langle \varepsilon, t \rangle, \sup_{t \in T} \langle -\varepsilon, t \rangle \right\} && \text{switching order of maximization} \\
&\leq \sup_{t \in T} \langle \varepsilon, t \rangle + \sup_{t \in T} \langle -\varepsilon, t \rangle && \text{using (1.22).}
\end{aligned}$$

To clarify, the last inequality holds since for non-negative  $a, b$ ,  $\max\{a, b\} \leq a + b$ , and by (1.22) both  $\sup_{t \in T} \langle \varepsilon, t \rangle$  and  $\sup_{t \in T} \langle -\varepsilon, t \rangle$  are non-negative. The claim now follows by taking expectation and noting that for  $\varepsilon$  a random Rademacher vector,  $\varepsilon$  and  $-\varepsilon$  have the same distribution.  $\square$

We now define the projection set  $Q(x_{1:n}) \subset \mathbb{R}^n$  as:

$$Q(x_{1:n}) := \{z = (\phi(\langle u, x_i \rangle))_{i=1}^n \mid \|u\| \leq 1\}.$$

Notice that  $0 \in Q(x_{1:n})$  always (by setting  $u = 0$ ), so the hypothesis of the previous proposition hold, and therefore we can apply (1.21) to conclude that:

$$\begin{aligned}
n \cdot \mathcal{R}_n(\mathcal{F}) &\leq \mathbb{E} \sup_{\|u\| \leq 1} \left| \sum_{i=1}^n \varepsilon_i \phi(\langle u, x_i \rangle) \right| \\
&= \mathbb{E}_{x_{1:n}} \mathbb{E}_{\varepsilon} \sup_{q \in Q(x_{1:n})} |\langle \varepsilon, q \rangle| \\
&\leq 2 \mathbb{E}_{x_{1:n}} \mathbb{E}_{\varepsilon} \sup_{q \in Q(x_{1:n})} \langle \varepsilon, q \rangle && \text{using (1.21)} \\
&= 2 \mathbb{E} \sup_{\|u\| \leq 1} \sum_{i=1}^n \varepsilon_i \phi(\langle u, x_i \rangle) \\
&\leq 2 \mathbb{E} \sup_{\|u\| \leq 1} \sum_{i=1}^n \varepsilon_i \langle u, x_i \rangle && \text{using Proposition 1.22} \\
&\leq 2\sqrt{nd} && \text{using (1.20).}
\end{aligned}$$

The claim now follows by dividing both sides of the above inequality by  $n$ .  $\square$

Next, we consider another possibly infinite-dimensional function class, inspired by the theory of reproducing kernel Hilbert spaces (RKHS).

**Exercise 1.36.** Let  $\phi : \mathbf{X} \mapsto \ell_2(\mathbb{N})$  be an infinite-dimensional feature map satisfying  $\|\phi(x)\|_{\infty} \leq 1$  for all  $x \in \mathbf{X}$ . Consider the following hypothesis class:

$$\mathcal{F} = \left\{ x \mapsto \sum_{i=1}^{\infty} w_i \phi(x)_i \mid \sum_{i=1}^{\infty} \frac{w_i^2}{\mu_i} \leq 1 \right\},$$

where  $(\mu_i)_{i \geq 1}$  is a positive element in  $\ell_1(\mathbb{N})$  (i.e.,  $\mu_i \geq 0$  and  $\sum_{i=1}^{\infty} \mu_i < \infty$ ). Show that:

$$\mathcal{R}_n(\mathcal{F}) \leq \sqrt{\frac{\sum_{i=1}^{\infty} \mu_i}{n}}.$$

For those familiar with the theory of RKHS, given a reproducing kernel, Mercer's theorem gives us an ellipsoidal representation (of the form written in the previous exercise) of the induced function class, where  $\phi$  corresponds to the eigenfunctions which diagonalize the kernel operator, and  $(\mu_i)_{i \geq 1}$  corresponds to the resulting eigenvalues.

**Finite function classes.** We conclude our discussion of Rademacher complexity by showing that the Rademacher complexity is well-controlled for any finite function class. The next result is often referred to as Massart's lemma.

**Proposition 1.37** (Massart's lemma). *Let  $Q \subset \mathbb{R}^n$  be a finite set of points, and put  $M = \max_{q \in Q} n^{-1/2} \|q\|$ . Let  $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$  denote a Rademacher random vector. Then,*

$$\mathbb{E}_\varepsilon \max_{q \in Q} n^{-1} \langle \varepsilon, q \rangle \leq \sqrt{\frac{2M^2 \log |Q|}{n}}.$$

*Proof.* For a fixed  $q \in Q$ , we have that  $n^{-1/2} \langle \varepsilon, q \rangle$  is a  $n^{-1/2} \|q\|$ -sub-Gaussian random vector. Hence by the sub-Gaussian maximal inequality (Proposition B.17),

$$\mathbb{E}_\varepsilon \max_{q \in Q} n^{-1} \langle \varepsilon, q \rangle = n^{-1/2} \mathbb{E}_\varepsilon \max_{q \in Q} n^{-1/2} \langle \varepsilon, q \rangle \leq \sqrt{\frac{2M^2 \log |Q|}{n}}.$$

□

The previous result gives us direct control over the Rademacher complexity of a finite class.

**Corollary 1.38.** *Let  $\mathcal{F}$  be a finite function class, and suppose that there exists a finite  $M$  satisfying*

$$\sup_{x \in X, f \in \mathcal{F}} n^{-1} \sum_{i=1}^n f(x_i)^2 \leq M^2.$$

*Then, we have that:*

$$\mathcal{R}_n(\mathcal{F}) \leq \sqrt{\frac{2M^2 \log |\mathcal{F}|}{n}}.$$

*Proof.* Let  $Q(x_{1:n}) := \{(f(x_1), \dots, f(x_n)) \mid f \in \mathcal{F}\} \subset \mathbb{R}^n$  denote the empirical projection of  $\mathcal{F}$  onto  $\mathbb{R}^n$ . By assumption, we have that for any  $q \in Q(x_{1:n})$ ,  $n^{-1/2} \|q\| \leq M$ . Hence,

$$\begin{aligned} \mathcal{R}_n(\mathcal{F}) &= \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i f(x_i) \\ &= \mathbb{E}_{x_{1:n}} \mathbb{E}_\varepsilon \sup_{q \in Q(x_{1:n})} n^{-1} \langle \varepsilon, q \rangle \\ &\leq \mathbb{E}_{x_{1:n}} \sqrt{\frac{2M^2 \log |Q(x_{1:n})|}{n}} && \text{using Proposition 1.37} \\ &\leq \sqrt{\frac{2M^2 \log |\mathcal{F}|}{n}} && \text{since } |Q(x_{1:n})| \leq |\mathcal{F}|. \end{aligned}$$

□

### 1.3.10 Vapnik-Chervonenkis dimension

We now return to our discussion of the Vapnik-Chervonenkis dimension (abbreviated VC-dimension) that we started in Section 1.3.8. Recall from that discussion that with probability at least  $1 - \delta$ , for all  $f \in \mathcal{F}$ ,

$$\mathbb{P}\{\text{sgn}(f(x)) \neq y\} \leq \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{\text{sgn}(f(x_i)) \neq y_i\} + \mathcal{R}_n(\{x \mapsto \text{sgn}(f(x)) \mid f \in \mathcal{F}\}) + 2\sqrt{\frac{2 \log(1/\delta)}{n}}. \quad (1.23)$$

Let  $\mathcal{H}$  denote the predictor class  $\mathcal{H} = \{x \mapsto \text{sgn}(f(x)) \mid f \in \mathcal{F}\}$ . At this point, it is not immediately obvious how to analyze the Rademacher complexity  $\mathcal{R}_n(\mathcal{H})$  which appears in the expression above. It turns out, as we will see, that this quantity is precisely controlled by the VC-dimension of  $\mathcal{H}$ .

To define the VC-dimension, we start with the notion of a growth function. For a fixed set of datapoints  $x_{1:n}$ , let  $\mathcal{H}(x_{1:n}) \subseteq \{\pm 1\}^n$  denote the projection set:

$$\mathcal{H}(x_{1:n}) = \{(h(x_1), \dots, h(x_n)) \mid h \in \mathcal{H}\}.$$

Specficially, the *Vapnik-Chervonenkis dimension*  $\text{VCdim}(\mathcal{H})$  is the following quantity:

$$\text{VCdim}(\mathcal{H}) := \sup\{n \in \mathbb{N}_+ \mid \exists x_{1:n} \text{ s.t. } |\mathcal{H}(x_{1:n})| = 2^n\}. \quad (1.24)$$

Let us take a moment to unpack this abstract definition. To do this, we first discuss the notion of shattering.

**Shattering.** First, we make the simple observation that  $|\mathcal{H}(x_{1:n})| \leq 2^n$  for any  $\mathcal{H}$  and  $x_{1:n}$ , since there are at most  $2^n$  possible values for an  $n$ -length binary string. Thus, the condition  $|\mathcal{H}(x_{1:n})| = 2^n$  states that every  $n$ -length binary string can be realized by the projection set  $\mathcal{H}(x_{1:n})$ ; this condition is referred to as  $\mathcal{H}$  *shatters*  $x_{1:n}$ . Put in the language of classification, this condition states that for any set of labels  $(y_1, \dots, y_n) \in \{\pm 1\}^n$ , there exists a hypothesis  $h \in \mathcal{H}$  such that its predictions on  $x_{1:n}$  precisely match the given labels:

$$(h(x_1), \dots, h(x_n)) = (y_1, \dots, y_n).$$

Intuitively, we expect function classes  $\mathcal{H}$  which are richer to be able to shatter  $x_{1:n}$  for larger values of  $n$ .

With the notion of shattering in place, we see that  $\text{VCdim}(\mathcal{H})$  is defined as the largest cardinality  $n \in \mathbb{N}_+$  such that there *exists* a set of data points  $x_{1:n}$  that is shattered by  $\mathcal{H}$ . Note that if for every  $n \in \mathbb{N}_+$  there exists  $x_{1:n}$  that is shattered by  $\mathcal{H}$ , then  $\text{VCdim}(\mathcal{H}) = \infty$ .

The choice of the existential qualifier here is important. For example, convince yourself that if we had replaced the  $\exists$  quantifier with the  $\forall$  quantifier in the definition of VC-dimension (Equation (1.24)), that the resulting definition would not be very useful. Also note that the definition of shattering satisfies two monotonicity properties:

- (i) If  $\mathcal{H}$  shatters  $x_{1:n}$  for some  $n \in \mathbb{N}_+$ , then  $\mathcal{H}$  also shatters  $x_{1:k}$  for every  $k \in \{1, \dots, n-1\}$ .
- (ii) Suppose that  $\mathcal{F}' \supseteq \mathcal{F}$ . Then its corresponding predictor class  $\mathcal{H}' = \{x \mapsto \text{sgn}(f(x)) \mid f \in \mathcal{F}'\}$  satisfies  $\mathcal{H}' \supseteq \mathcal{H}$ . Consequently, if  $\mathcal{H}$  shatters  $x_{1:n}$ , then so does  $\mathcal{H}'$ , since  $\mathcal{H}(x_{1:n}) \subseteq \mathcal{H}'(x_{1:n})$ . Hence, VC-dimension is monotonic:  $\text{VCdim}(\mathcal{H}) \leq \text{VCdim}(\mathcal{H}')$ .

**Computing the VC-dimension.** In light of our discussion, in order to compute  $\text{VCdim}(\mathcal{H})$  for a given hypothesis class  $\mathcal{H}$ , one needs to check two conditions:

- (a) Find an  $n \in \mathbb{N}_+$  and a specific dataset  $x_{1:n}$  such that  $\mathcal{H}$  shatters  $x_{1:n}$ , i.e.,  $|\mathcal{H}(x_{1:n})| = 2^n$ .
- (b) Show that for *every* dataset  $x_{1:n+1}$ ,  $\mathcal{H}$  *cannot* shatter  $x_{1:n+1}$ , i.e.,  $|\mathcal{H}(x_{1:n+1})| < 2^{n+1}$ .

If both conditions (a) and (b) are met, then that proves that  $\text{VCdim}(\mathcal{H}) = n$ . This is because condition (a) shows that  $\text{VCdim}(\mathcal{H}) \geq n$ , and condition (b) shows that  $\text{VCdim}(\mathcal{H}) < n+1$ .

Shortly, we will compute specific examples of VC-dimension to get more intuition. But first, we will relate the VC-dimension to the Rademacher complexity of  $\mathcal{R}_n(\mathcal{H})$ . To do this, we introduce one last concept. The *growth function*  $\tau_{\mathcal{H}}(n)$  is the following quantity:

$$\tau_{\mathcal{H}}(n) := \sup_{x_{1:n}} |\mathcal{H}(x_{1:n})|.$$

By the definition of VC-dimension, for any  $n \leq \text{VCdim}(\mathcal{H})$ , we immediately have  $\tau_{\mathcal{H}}(n) = 2^n$ . That is, the growth function is exponential in  $n$  in this regime. However, it turns out that when  $n > \text{VCdim}(\mathcal{H})$ , this growth function undergoes a phase transition and the growth turns to *polynomial* in  $n$ . This is the contents of the well-known Sauer-Shelah lemma. Specifically, letting  $d = \text{VCdim}(\mathcal{H})$ ,

$$\tau_{\mathcal{H}}(n) \leq \sum_{i=0}^d \binom{n}{i} \quad \text{and} \quad \sum_{i=0}^d \binom{n}{i} \leq \left(\frac{en}{d}\right)^d \text{ when } n \geq d. \quad (1.25)$$

The proof of this result is out of scope for this course, but can be found in [Vershynin, 2018, Section 8.3.3]. The estimate on the RHS of (1.25) follows from Proposition D.1. With this result in place, we have all the tools needed to connect the Rademacher complexity  $\mathcal{R}_n(\mathcal{H})$  with the growth function  $\tau_{\mathcal{H}}(n)$

**Proposition 1.39.** *Suppose that  $\mathcal{H} = \{x \mapsto \text{sgn}(f(x)) \mid f \in \mathcal{F}\}$  has finite  $\text{VCdim}(\mathcal{H}) = d$ . Then,*

$$\mathcal{R}_n(\mathcal{H}) \leq \sqrt{\frac{2 \log \tau_{\mathcal{H}}(n)}{n}}.$$

*Proof.* The idea is to use the definition of the growth function  $\tau_{\mathcal{H}}(n)$  to reduce the Rademacher complexity to a maximum over a finite set of hypothesis, and appeal to Massart's lemma (Proposition 1.37). Specifically,

$$\begin{aligned} \mathcal{R}_n(\mathcal{H}) &= \mathbb{E} \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i \text{sgn}(f(x_i)) \\ &= \mathbb{E} \sup_{h \in \mathcal{H}(x_{1:n})} n^{-1} \langle \varepsilon, h \rangle && \text{definition of } \mathcal{H}(x_{1:n}) \\ &= \mathbb{E}_{x_{1:n}} \left[ \mathbb{E}_{\varepsilon} \sup_{h \in \mathcal{H}(x_{1:n})} n^{-1} \langle \varepsilon, h \rangle \right] \\ &\leq \mathbb{E}_{x_{1:n}} \sqrt{\frac{2 \log |\mathcal{H}(x_{1:n})|}{n}} && \text{Massart's lemma (Proposition 1.37)} \\ &\leq \sqrt{\frac{2 \log \tau_{\mathcal{H}}(n)}{n}}. && \text{definition of growth function} \end{aligned}$$

□

Now, combining the generalization bound (1.23), the Sauer-Shelah growth function bound (1.25), with the result from Proposition 1.39 yields the following key result.

**Theorem 1.40.** *Let  $\mathcal{F}$  denote a function class and let  $\mathcal{H} = \{x \mapsto \text{sgn}(f(x)) \mid f \in \mathcal{F}\}$  denote its corresponding predictor class. Suppose that the predictor class  $\mathcal{H}$  has finite  $\text{VCdim}(\mathcal{H}) = d$ . There exists a universal constant  $c > 0$  such that for any  $n \geq d + 1$ , with probability at least  $1 - \delta$ ,*

$$\forall f \in \mathcal{F}, \mathbb{P}\{\text{sgn}(f(x)) \neq y\} \leq \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{\text{sgn}(f(x_i)) \neq y_i\} + c \sqrt{\frac{d \log(n/d)}{n}} + 2 \sqrt{\frac{2 \log(1/\delta)}{n}}.$$

We now take a moment to compare the generalization bound from Theorem 1.40 with the previously derived bounds in Theorem 1.25 and Corollary 1.26, respectively.

**Theorem 1.25 vs. Theorem 1.40.** Theorem 1.25 bounds the generalization error of Lipschitz losses, and uses the Rademacher complexity  $\mathcal{R}_n(\mathcal{F})$  to measure complexity. On the other hand, Theorem 1.40 is specific to the zero-one loss, and uses  $\text{VCdim}(\mathcal{H})$  of the induced predictor class  $\mathcal{H}$  to measure complexity.

**Corollary 1.26 vs. Theorem 1.40.** Both results bound the population zero-one loss  $\mathbb{P}\{\text{sgn}(f(x)) \neq y\}$ . However, the main difference is on the right hand sides. For Corollary 1.26, the Rademacher complexity  $\mathcal{R}_n(\mathcal{F})$  is still used to measure complexity (similar to Theorem 1.40). However, the price to pay for this is that training error measures the *margin mistakes*  $\frac{1}{n} \sum_{i=1}^n \mathbf{1}\{y_i f(x_i) < \gamma\}$  for  $\gamma > 0$ ; having no margin ( $\gamma = 0$ ) is not allowed in Corollary 1.26. On the other hand, Theorem 1.40 does *not* require any margin, and directly counts the number of classification mistakes  $\frac{1}{n} \sum_{i=1}^n \mathbf{1}\{\text{sgn}(f(x_i)) \neq y_i\}$  made on the training dataset. The price to pay for this, however, is that the complexity measure is no longer the more intuitive Rademacher complexity  $\mathcal{R}_n(\mathcal{F})$ , but instead it becomes the VC-dimension  $\text{VCdim}(\mathcal{H})$  of the induced predictor class  $\mathcal{H}$ .

In light of this discussion, we see that none of the bounds (Theorem 1.25, Corollary 1.26, and Theorem 1.40) are strictly better than the others, and involve different sets of assumptions and complexity measures. Depending on the specific problem at hand, one may be better than the other.

**Exercise 1.41.** Let  $\{\mathcal{H}_i\}_{i=1}^k$  be  $k$  hypothesis classes. Put  $d := \max_{i=1,\dots,k} \text{VCdim}(\mathcal{H}_i)$ , and suppose that  $d$  is finite. Show that there exists universal constants  $c_0, c_1$  such that

$$\text{VCdim}(\cup_{i=1}^k \mathcal{H}_i) \leq c_0 d + c_1 \log k.$$

Hint: use the Sauer-Shelah lemma and Proposition D.2.

### 1.3.11 Explicit computations of VC-dimension

We now proceed to compute the VC-dimension for several specific predictor classes.

**Threshold functions.** Let us consider first consider the predictor class

$$\mathcal{H} = \{x \mapsto h_t(x) := \text{sgn}(t - x) \mid t \in \mathbb{R}\}.$$

This is the set of all threshold functions in one-dimension; any  $h_t \in \mathcal{H}$  is equal to:

$$h_t(x) = \begin{cases} 1 & \text{if } x \leq t \\ -1 & \text{o.w.} \end{cases}$$

We will show that  $\text{VCdim}(\mathcal{H}) = 1$ . Following the recipe from Section 1.3.10, we first check there exists a dataset of one point can be shattered by  $\mathcal{H}$ . We choose this dataset as  $x_1 = 0$ . Note that if we choose  $t = 1$ , then  $h_1(x_1) = h_1(0) = \text{sgn}(1) = 1$ . On the other hand, if we choose  $t = -1$ , then  $h_{-1}(x_1) = h_{-1}(0) = \text{sgn}(-1) = -1$ , so  $|\mathcal{H}(x_1)| = 2$ . Now we show that no  $n = 2$  points can be shattered. Without loss of generality assume that  $x_1 \leq x_2$ . Notice that the labels  $(y_1, y_2) = (-1, +1)$  are not contained in  $\mathcal{H}(x_{1:2})$ , since for any  $t \in \mathbb{R}$  such that  $h_t(x_2) = 1$ , then  $x_2 \leq t$  which implies that  $x_1 \leq t$ , but which then implies  $h_t(x_1) = 1$ . Hence  $|\mathcal{H}(x_{1:2})| < 4$ .

**Interval functions.** Now let us consider the prediction class

$$\mathcal{H} = \{x \mapsto h_{a,b}(x) := \text{sgn}(\mathbf{1}\{x \in [a, b]\} - 1/2) \mid a, b \in \mathbb{R}, a \leq b\}.$$

Note that  $h_{a,b} \in \mathcal{H}$  is equal to:

$$h_{a,b}(x) = \begin{cases} 1 & \text{if } x \in [a, b] \\ -1 & \text{o.w.} \end{cases}$$

We now show that  $\text{VCdim}(\mathcal{H}) = 2$ . Again, following the recipe from Section 1.3.10, we first construct a dataset of  $n = 2$  points that can be shattered by  $\mathcal{H}$ . Let this dataset be  $x_1 = 0$  and  $x_2 = 1$ . Now we need to check that all four labelings  $(y_1, y_2) \in \{\pm 1\}^2$  are contained in  $\mathcal{H}(x_{1:2})$ . We do this by enumerating all four cases and choosing values of  $(a, b) \in \mathbb{R}^2$  such  $(h_{a,b}(x_1), h_{a,b}(x_2)) = (y_1, y_2)$ :

- (i) Case  $(y_1, y_2) = (-1, -1)$ : select  $a = b = 1/2$ .
- (ii) Case  $(y_1, y_2) = (-1, +1)$ : select  $a = b = 1$ .
- (iii) Case  $(y_1, y_2) = (+1, -1)$ : select  $a = b = 0$ .
- (iv) Case  $(y_1, y_2) = (+1, +1)$ : select  $a = 0, b = 1$ .

Next, we need to show that any set of  $n = 3$  points cannot be shattered by  $\mathcal{H}$ . Note that we can assume wlog that  $x_1 < x_2 < x_3$ , since if there are any equalities, then clearly  $|\mathcal{H}(x_{1:3})| < 8$ . Now we will argue that for any  $x_1 < x_2 < x_3$ , the label  $(y_1, y_2, y_3) = (+1, -1, +1)$  cannot be achieved. To see this, the first requirement that  $h_{a,b}(x_1) = 1$  implies that  $a \leq x_1$ . On the other hand, the second requirement that  $h_{a,b}(x_2) = -1$  then requires that  $b \leq x_2$ . But then this necessitates that  $h_{a,b}(x_3) = -1$  since  $x_3 > x_2 \geq b$ . Hence,  $|\mathcal{H}(x_{1:3})| < 8$ .

**Linear functions.** We now consider the set of  $d$ -dimensional linear functions:

$$\mathcal{H} = \{x \mapsto h_w(x) := \text{sgn}(\langle x, w \rangle) \mid w \in \mathbb{R}^d\}.$$

We will show that  $\text{VCdim}(\mathcal{H}) = d$ . First, let  $e_i \in \mathbb{R}^d$  denote the  $i$ -th standard basis vector. Set  $x_i = e_i$  for  $i \in \{1, \dots, d\}$ . Now for a given labeling  $(y_1, \dots, y_n) \in \{\pm 1\}^d$ , set  $w = (y_1, \dots, y_n)$ , and observe that:

$$h_w(x_i) = \text{sgn}(\langle x_i, w \rangle) = \text{sgn}(\langle e_i, w \rangle) = \text{sgn}(w_i) = \text{sgn}(y_i) = y_i.$$

This shows that  $\mathcal{H}$  shatters  $x_{1:d}$ .

Now we show that no set of  $d+1$  points can be shattered by  $\mathcal{H}$ . Let  $x_{1:d+1}$  be arbitrary. Since the  $x_i$ 's live in a  $d$ -dimensional vector space, the points  $x_{1:d+1}$  must be linearly dependent. This means there exists a non-zero  $\alpha \in \mathbb{R}^{d+1}$  such that

$$\sum_{i=1}^{d+1} \alpha_i x_i = 0.$$

Let  $I_+ = \{i \in \{1, \dots, d+1\} \mid \alpha_i \geq 0\}$  and  $I_- = \{i \in \{1, \dots, d+1\} \mid \alpha_i < 0\}$ . Wlog, let us assume that  $\alpha$  has at least one negative element, that is  $|I_-| \geq 1$  (otherwise, negate  $\alpha$ ). Re-arranging the above equation,

$$\sum_{i \in I_+} \alpha_i x_i = - \sum_{i \in I_-} \alpha_i x_i = \sum_{i \in I_-} |\alpha_i| x_i. \quad (1.26)$$

Now let us suppose  $x_{1:d+1}$  is shattered by  $\mathcal{H}$ . Choose the labeling  $(y_1, \dots, y_{d+1}) \in \{\pm 1\}^{d+1}$

$$y_i = \begin{cases} 1 & \text{if } i \in I_+, \\ -1 & \text{if } i \in I_-, \end{cases}$$

and let  $w \in \mathbb{R}^{d+1}$  be such that  $h_w(x_i) = y_i$  for  $i \in \{1, \dots, d+1\}$ ; such a  $w$  must exist by the definition of shattering. Notice this means that for all  $i \in I_+$ ,  $\langle w, x_i \rangle \geq 0$ , and for all  $i \in I_-$ ,  $\langle w, x_i \rangle < 0$ . However, using (1.26),

$$0 > \sum_{i \in I_-} |\alpha_i| \langle w, x_i \rangle = \left\langle w, \sum_{i \in I_-} |\alpha_i| x_i \right\rangle = \left\langle w, \sum_{i \in I_+} \alpha_i x_i \right\rangle = \sum_{i \in I_+} \alpha_i \langle w, x_i \rangle \geq 0,$$

which is a contradiction. Note that the strict inequality above follows since  $|I_-| \geq 1$ . Thus,  $\mathcal{H}$  cannot shatter  $x_{1:d+1}$ .

**Finite hypothesis class.** Suppose now we consider the prediction class:

$$\mathcal{H} = \{x \mapsto \text{sgn}(f(x)) \mid f \in \mathcal{F}\}, \quad |\mathcal{F}| < \infty.$$

Then immediately we have that for any  $x_{1:n}$ ,  $|\mathcal{H}(x_{1:n})| \leq |\mathcal{F}|$ . Hence for any  $n > \log_2 |\mathcal{F}|$ , we must have  $|\mathcal{H}(x_{1:n})| \leq |\mathcal{F}| < 2^n$ . Hence, this yields an upper bound of  $\text{VCdim}(\mathcal{H}) \leq \log_2 |\mathcal{F}|$ . Note that this upper bound can be very loose though. For example, consider the subset of threshold functions:

$$\mathcal{H}' = \{x \mapsto \text{sgn}(t - x) \mid t \in I\},$$

where  $I$  is a finite subset of  $\mathbb{R}$ . Thus, this finite class bound yields that  $\text{VCdim}(\mathcal{H}') \leq \log_2 |I|$ , but since  $\mathcal{H}' \subset \mathcal{H}'' = \{x \mapsto \text{sgn}(x - t) \mid t \in \mathbb{R}\}$  and we know that  $\text{VCdim}(\mathcal{H}'') = 1$  from our previous analysis, we actually by monotonicity of VC-dimension that  $\text{VCdim}(\mathcal{H}') \leq \text{VCdim}(\mathcal{H}'') = 1$ . Therefore, the finite class bound can be arbitrarily loose.

**An example of infinite VC-dimension.** We now consider a one parameter prediction class, and show that it has infinite VC-dimension. The prediction class we consider is:

$$\mathcal{H} = \{x \mapsto h_\theta(x) := \text{sgn}(\sin(\theta x)) \mid \theta \in \mathbb{R}\}.$$

Fix any  $n \in \mathbb{N}_+$ , and consider the dataset  $x_{1:n}$  where  $x_i = 10^{-i}$ . We will show that  $\mathcal{H}$  shatters  $x_{1:n}$ . Let  $(y_1, \dots, y_n) \in \{\pm 1\}^n$  be any labeling, and let  $I_- = \{i \in \{1, \dots, n\} \mid y_i = -1\}$ . Choose the frequency parameter  $\theta$  as:

$$\theta = \pi \left( 1 + \sum_{i \in I_-} 10^i \right).$$

Now for any  $j \in \{1, \dots, n\}$ , we compute:

$$\begin{aligned} \theta x_j &= \pi 10^{-j} + \pi \sum_{i \in I_-} 10^{i-j} \\ &= \pi 10^{-j} + \pi \mathbf{1}\{j \in I_-\} + \pi \sum_{i \in I_-, i < j} 10^{i-j} + \pi \sum_{i \in I_-, i > j} 10^{i-j}. \end{aligned}$$

We next observe that, whenever  $i > j$ , then  $10^{i-j}$  is an even integer, and hence  $\pi \sum_{i \in I_-, i > j} 10^{i-j} = 2\pi k(j)$  for a non-negative integer  $k(j)$ . On the other hand, we can bound,

$$0 \leq \varepsilon(j) := \sum_{i \in I_-, i < j} 10^{i-j} \leq \sum_{i=1}^{\infty} 10^{-i} = \frac{1}{1 - 1/10} - 1 = 1/9.$$

Hence,

$$\begin{aligned} \sin(\theta x_j) &= \sin(\pi 10^{-j} + \pi \mathbf{1}\{j \in I_-\} + \pi \varepsilon(j) + 2\pi k(j)) \\ &= \sin(\pi 10^{-j} + \pi \mathbf{1}\{j \in I_-\} + \pi \varepsilon(j)) && \text{since } \sin(x + 2\pi k) = \sin(x) \text{ for } k \in \mathbb{N} \\ &= \sin(\pi(\mathbf{1}\{j \in I_-\} + 10^{-j} + \varepsilon(j))). \end{aligned}$$

Next defining  $\bar{\varepsilon}(j) := 10^{-j} + \varepsilon(j)$ , we observe that  $0 < \bar{\varepsilon}(j) \leq 1/10 + 1/9 < 1$ . Therefore, let us first suppose that  $j \notin I_-$ , so that  $y_j = +1$ . Then, since  $\sin(x) > 0$  for  $x \in (0, \pi)$ ,

$$\sin(\theta x_j) = \sin(\pi \bar{\varepsilon}(j)) > 0 \implies h_\theta(x_j) = \text{sgn}(\sin(\theta x_j)) = y_j.$$

On the other hand, suppose that  $j \in I_-$ , so that  $y_j = -1$ . Then, since  $\sin(x) < 0$  for  $x \in (\pi, 2\pi)$ ,

$$\sin(\theta x_j) = \sin(\pi(1 + \bar{\varepsilon}(j))) < 0 \implies h_\theta(x_j) = \text{sgn}(\sin(\theta x_j)) = y_j,$$

This shows that  $\mathcal{H}$  shatters  $x_{1:n}$ . Since the choices of  $n$  and  $x_{1:n}$  are arbitrary, we have that  $\text{VCdim}(\mathcal{H}) = \infty$ .

### 1.3.12 Agnostic PAC learnability and the No Free Lunch theorem

We now have all the tools in place to formally define what it means for a hypothesis to be *learnable*. Furthermore, we will consider the question of whether or not learnability is even possible in general. In this section, we will focus on the zero-one loss.

We start with a general definition of learnability.

**Definition 1.42** (Agnostically PAC learnable). *A hypothesis class  $\mathcal{H}$  of functions mapping  $\mathbf{X} \mapsto \mathbf{Y}$  is called agnostically probably approximately correct learnable (agnostically PAC learnable) if there exists an algorithm  $\mathcal{A} : \mathbf{Z}^* \mapsto \mathcal{H}$  with the following property. For every  $\varepsilon, \delta \in (0, 1)$ , there exists a finite  $n = n(\varepsilon, \delta) \in \mathbb{N}_+$  such that running  $\mathcal{A}$  on  $n$  iid examples  $S_n = \{(x_i, y_i)\}_{i=1}^n$  drawn from any distribution  $\mathcal{D}$  on  $\mathbf{Z} = \mathbf{X} \times \mathbf{Y}$*

satisfies the following property. With probability at least  $1 - \delta$  over the randomness of  $S_n$ ,  $\mathcal{A}$  returns a hypothesis  $\hat{h}_n = \mathcal{A}(S_n)$  such that

$$\mathbb{P}_{\mathcal{D}}\{y \neq \hat{h}_n(x)\} \leq \inf_{h \in \mathcal{H}} \mathbb{P}_{\mathcal{D}}\{y \neq h(x)\} + \varepsilon. \quad (1.27)$$

The function  $n(\varepsilon, \delta)$  is referred to as the sample complexity of learning.

We now highlight a few key characteristics of Definition 1.42.

- (a) Agnostic PAC learnability does *not* prescribe a rate of learning; the sample complexity  $n(\varepsilon, \delta)$  can scale arbitrarily poorly on  $\varepsilon, \delta$ , it just needs to be finite for every fixed  $\varepsilon, \delta$ .
- (b) Agnostic PAC learnability also does *not* require computational efficiency. The algorithm  $\mathcal{A}$  does not have to be efficiently computable, it just has to be a well-defined mathematical function.
- (c) Agnostic PAC learnability does not require realizability (i.e.,  $\exists h \in \mathcal{F}$  such that  $\mathbb{P}_{\mathcal{D}}\{y \neq h(x)\} = 0$ ). It simply requires that the learning algorithm  $\mathcal{A}$  emits a hypothesis  $h \in \mathcal{H}$  that can compete with the best possible predictor in  $\mathcal{H}$ . In other words, agnostic PAC learnability is still meaningful for  $\inf_{h \in \mathcal{H}} \mathbb{P}_{\mathcal{D}}\{y \neq h(x)\}$  arbitrarily close to 1. On the other hand, standard PAC learnability (without the agnostic) requires realizability.
- (d) Note that the order of quantifiers in the definition is extremely important. For example, if the definition were to require instead that for every distribution  $\mathcal{D}$  over  $Z$ , there exists an algorithm  $\mathcal{A}$  satisfying (1.27), then the definition would be vacuous (why?).

With the definition of agnostically PAC learnable in place, we are ready to prove our first result providing a sufficient condition for agnostic PAC learnability.

**Theorem 1.43** (Finite VC-dimension implies learnability). *Let  $\mathcal{H}$  have finite VC-dimension  $d$ . Then,  $\mathcal{H}$  is agnostically PAC learnable via the empirical risk minimizer. Furthermore, it suffices to take  $n$  satisfying:*

$$n \geq \frac{c_0}{\varepsilon^2} \left[ d \log \left( \frac{2}{\varepsilon} \right) + \log \left( \frac{2}{\delta} \right) \right],$$

where  $c_0$  is a universal positive constant.

*Proof.* Fix  $\varepsilon, \delta \in (0, 1)$ . We already saw by Theorem 1.40, that for any distribution over  $Z$ , for any  $n \geq d+1$ , with probability at least  $1 - \delta/2$  over the samples  $\{(x_i, y_i)\}_{i=1}^n$ ,

$$\forall h \in \mathcal{H}, \mathbb{P}\{y \neq h(x)\} \leq \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{y_i \neq h(x_i)\} + c \sqrt{\frac{d \log(n/d)}{n}} + 2 \sqrt{\frac{2 \log(2/\delta)}{n}}. \quad (1.28)$$

Here,  $c$  is a universal constant. We will assume  $c \geq 1$  without loss of generality. Note that by Equation (1.14), Theorem 1.40 also tells us that with probability at least  $1 - \delta/2$  over the samples,

$$\forall h \in \mathcal{H}, \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{y_i \neq h(x_i)\} \leq \mathbb{P}\{y \neq h(x)\} + c \sqrt{\frac{d \log(n/d)}{n}} + 2 \sqrt{\frac{2 \log(2/\delta)}{n}}. \quad (1.29)$$

By a union bound, with probability at least  $1 - \delta$ , both (1.28) and (1.29) hold. We will work assuming this event holds for the remainder of the proof.

Setting  $\hat{h}_n \in \arg \min_{h \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{y_i \neq h(x_i)\}$  to be the ERM and letting  $h' \in \mathcal{H}$  be arbitrary,

$$\mathbb{P}\{y \neq \hat{h}_n(x)\} \leq \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{y_i \neq \hat{h}_n(x_i)\} + c \sqrt{\frac{d \log(n/d)}{n}} + 2 \sqrt{\frac{2 \log(2/\delta)}{n}} \quad \text{using (1.28)}$$



$$\begin{aligned}
&\leq \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{y_i \neq h'(x_i)\} + c\sqrt{\frac{d \log(n/d)}{n}} + 2\sqrt{\frac{2 \log(2/\delta)}{n}} && \text{since } \hat{h}_n \text{ is the ERM} \\
&\leq \mathbb{P}\{y \neq h'(x)\} + 2c\sqrt{\frac{d \log(n/d)}{n}} + 4\sqrt{\frac{2 \log(2/\delta)}{n}}. && \text{using (1.29)}
\end{aligned} \tag{1.30}$$

Since (1.30) holds for any  $h' \in \mathcal{H}$ , we can take the infimum over the RHS to conclude:

$$\mathbb{P}\{y \neq \hat{h}_n(x)\} \leq \inf_{h \in \mathcal{H}} \mathbb{P}\{y \neq h(x)\} + 2c\sqrt{\frac{d \log(n/d)}{n}} + 4\sqrt{\frac{2 \log(2/\delta)}{n}}.$$

Now we simply need to set  $n$  large enough so the RHS is  $\leq \varepsilon$ . We will choose a sufficient (but not necessary) condition instead. We will ask that  $n$  be large enough so that:

$$2c\sqrt{\frac{d \log(n/d)}{n}} \leq \frac{\varepsilon}{2} \quad \text{and} \quad 4\sqrt{\frac{2 \log(2/\delta)}{n}} \leq \frac{\varepsilon}{2}. \tag{1.31}$$

Clearly if both these conditions hold, then we have achieved the agnostic PAC learnability condition (1.27):

$$\mathbb{P}\{y \neq \hat{h}_n(x)\} \leq \inf_{h \in \mathcal{H}} \mathbb{P}\{y \neq h(x)\} + \varepsilon.$$

We now turn to ensuring (1.31). The RHS of (1.31) is simple to ensure by requiring that:

$$n \geq \frac{128 \log(2/\delta)}{\varepsilon^2}.$$

Turning to the LHS of Equation (1.31), we first re-arrange the condition as:

$$n \geq \left(\frac{4c}{\varepsilon}\right)^2 d \log(n/d). \tag{1.32}$$

Given the appearance of the  $n$  inside the logarithm on the RHS of (1.32), it is not straightforward to precisely solve for the smallest  $n$  satisfying (1.32). We can, however, derive a sufficient condition. In particular, Proposition D.2 states that if  $\alpha, \beta$  are positive reals such that  $\alpha\beta \geq 1$ , then for any  $x \in \mathbb{R}$ :

$$x \geq 2\alpha \log(2\alpha\beta) \implies x \geq \alpha \log(\beta x). \tag{1.33}$$

We now set:

$$\alpha = \left(\frac{4c}{\varepsilon}\right)^2 d, \quad \beta = \frac{1}{d}.$$

To verify that  $\alpha\beta \geq 1$ , since  $c \geq 1$  and  $\varepsilon \in (0, 1)$ , we have:

$$\alpha\beta = \left(\frac{4c}{\varepsilon}\right)^2 d \cdot \frac{1}{d} = \left(\frac{4c}{\varepsilon}\right)^2 \geq 1.$$

Thus, by (1.33), a sufficient condition for (1.32) to hold is:

$$n \geq \frac{32c^2 d}{\varepsilon^2} \log\left(\frac{32c^2}{\varepsilon^2}\right) \implies n \geq \left(\frac{4c}{\varepsilon}\right)^2 d \log(n/d).$$

Combining our two requirements on  $n$ , we see that for the final sample complexity bound, it suffices to take

$$n \geq \max \left\{ \frac{32c^2 d}{\varepsilon^2} \log\left(\frac{32c^2}{\varepsilon^2}\right), \frac{128 \log(2/\delta)}{\varepsilon^2} \right\}.$$

□

We note that the sample complexity bound given in Theorem 1.43 is sub-optimal order-wise. It can actually be improved to:

$$n \geq \frac{c_0}{\varepsilon^2} \left[ d + \log \left( \frac{2}{\delta} \right) \right],$$

using a technique called *chaining*. This more advanced technique is developed in Section 1.3.13, although we will not have time to discuss it in class.

At this point we have seen that finite VC-dimension implies agnostic PAC learnability. What about the converse? Is it possible that hypothesis classes with infinite VC-dimension are still agnostically PAC learnable? We will show that this is not possible. The main step towards showing this is the following classic “no free lunch” theorem.

**Theorem 1.44** (No Free Lunch). *Let  $\mathbf{X}$  be a domain of infinite cardinality, and let  $\mathcal{H}$  denote hypothesis class of functions mapping  $\mathbf{X} \mapsto \mathbf{Y}$  with infinite VC-dimension (i.e.,  $\text{VCdim}(\mathcal{H}) = \infty$ ). Fix  $n \in \mathbb{N}_+$ . Let  $\mathcal{A} : \mathbf{Z}^n \mapsto \mathcal{H}$  be any algorithm which takes in as input  $n$  pairs  $S_n = \{(x_i, y_i)\}_{i=1}^n$  and outputs a hypothesis in  $\mathcal{H}$ . Then, there exists a distribution  $\mathcal{D}$  over  $\mathbf{Z}$  such that the following hold:*

- (i) *There exists an  $h \in \mathcal{H}$  satisfying  $\mathbb{P}_{\mathcal{D}}\{h(x) \neq y\} = 0$ .*
- (ii) *Let  $S_n$  consist of  $n$  iid draws from  $\mathcal{D}$ . With probability at least  $1/7$  over the randomness of  $S_n$ ,*

$$\mathbb{P}_{\mathcal{D}}\{\mathcal{A}(S_n)(x) \neq y\} \geq \frac{1}{8}.$$

*Proof.* The proof of this result relies on the probabilistic method. To do this, we set up distributions  $\mu$  and  $\nu$  over  $\mathbf{X}$  and  $\mathcal{H}$ , respectively. Since  $\text{VCdim}(\mathcal{H}) = \infty$ , there exists  $2n$  distinct points  $x'_{1:2n} = (x'_1, \dots, x'_{2n}) \subset \mathbf{X}$  such that the projection set  $\mathcal{H}(x'_{1:2n})$  has cardinality  $2^{2n}$ . This implies there exists a subset of  $\mathcal{H}'_{2n}$  of  $\mathcal{H}$  containing  $2^{2n}$  distinct functions such that  $\mathcal{H}'_{2n}(x'_{1:2n}) = \mathcal{H}(x'_{1:2n})$ . Importantly, note there is a bijection between  $\mathcal{H}'_{2n}(x'_{1:2n})$  and  $\{\pm 1\}^{2n}$ . We set  $\mu$  to be the uniform distribution over  $x'_{1:2n}$ , and we set  $\nu$  to be the uniform distribution over  $\mathcal{H}'_{2n}$ .

Next, for any arbitrary sequence of points  $X_n = (x_1, \dots, x_n) \subset \mathbf{X}$  and a hypothesis  $h \in \mathcal{H}$ , we set  $S(X_n, h) := \{(x_i, h(x_i))\}_{i=1}^n$ . We will now lower bound the following expectation:

$$Q := \mathbb{E}_{X_n \sim \mu^n} \mathbb{E}_{h \sim \nu} \mathbb{E}_{x' \sim \mu} \mathbf{1}\{\mathcal{A}(S(X_n, h))(x') \neq h(x')\}.$$

For notational brevity, for what follows we will write:

$$\mathbb{E}_{X_n \sim \mu^n} \mathbb{E}_{h \sim \nu} \mathbb{E}_{x' \sim \mu} = \mathbb{E}_{\mu^n} \mathbb{E}_{\nu} \mathbb{E}_{\mu}.$$

Now for a given  $X_n$  and  $h$ , let  $h(X_n) = (h(x_1), \dots, h(x_n))$ . Fix  $X_n, x'$  such that  $x' \notin X_n$ , and observe that:

$$\begin{aligned} \mathbb{E}_{\nu} \mathbf{1}\{\mathcal{A}(S(X_n, h))(x') \neq h(x')\} &= \sum_{y \in \{\pm 1\}^n} \mathbb{E}_{\nu} [\mathbf{1}\{\mathcal{A}(S(X_n, h))(x') \neq h(x') \mid h(X_n) = y\}] \mathbb{P}_{\nu}\{h(X_n) = y\} \\ &= \sum_{y \in \{\pm 1\}^n} \frac{1}{2} \cdot \mathbb{P}_{\nu}\{h(X_n) = y\} = \frac{1}{2}. \end{aligned}$$

This is the key idea here so let us be clear about why this equality is true. First, observe that if  $h(X_n) = y$ , then the quantity  $\mathcal{A}(S(X_n, h))(x')$  is constant over the remaining randomness of  $h$ . Hence, since  $\nu$  is uniform over  $\mathcal{H}'_{2n}$ , conditioned on  $h(X_n) = y$ , since  $x' \notin X_n$ ,  $h(x')$  takes on  $-1$  and  $+1$  both with equal probability. To make this more clear, consider the bijection between  $\mathcal{H}'_{2n}$  and  $\{\pm 1\}^{2n}$ . If  $\zeta$  is a random Rademacher random variable, the distribution of  $h(x')$  conditioned on  $h(X_n) = y$  for  $x \notin X_n$  is equivalent to conditioning  $\zeta$  on  $n$  coordinates and then considering the distribution on one of the remaining  $n$  free coordinates (i.e., it is uniform on  $\{\pm 1\}$ ). This allows us to conclude, by swapping the order of expectations,

$$Q = \mathbb{E}_{\mu^n} \mathbb{E}_{\mu} \mathbb{E}_{\nu} \mathbf{1}\{\mathcal{A}(S(X_n, h))(x') \neq h(x')\}$$

$$\begin{aligned}
&\geq \mathbb{E}_{\mu^n} \mathbb{E}_{\mu} [\mathbf{1}\{x' \notin X_n\} \cdot \mathbb{E}_{\nu} \mathbf{1}\{\mathcal{A}(S(X_n, h))(x') \neq h(x')\}] \\
&= \mathbb{E}_{\mu^n} \mathbb{E}_{\mu} \left[ \mathbf{1}\{x' \notin X_n\} \cdot \frac{1}{2} \right] \\
&= \frac{1}{2} \cdot \mathbb{P}_{\mu^n, \mu} \{x' \notin X_n\} \\
&\geq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.
\end{aligned}$$

Above, the last inequality holds since  $\mu$  is uniform over  $x'_{1:2n}$ . Hence, since there are at most  $n$  distinct elements in  $X_n$ ,  $\mathbb{P}_{\mu} \{x' \notin X_n\} \geq 1/2$ . By swapping order of expectations again,

$$\mathbb{E}_{\nu} \mathbb{E}_{\mu^n} \mathbb{E}_{\mu} \mathbf{1}\{\mathcal{A}(S(X_n, h))(x') \neq h(x')\} \geq \frac{1}{4}.$$

By the probabilistic method (applied to the outermost expectation over  $\nu$  only), there exists an  $h \in \mathcal{H}'_{2n}$  such that

$$\mathbb{E}_{\mu^n} \mathbb{E}_{\mu} \mathbf{1}\{\mathcal{A}(S(X_n, h))(x') \neq h(x')\} \geq \frac{1}{4}.$$

To conclude the claim, we need a technical fact.

**Proposition 1.45.** *Suppose that  $X \in [0, 1]$  is a random variable with  $\mathbb{E}[X] \geq \mu$ . Then,*

$$\forall t \in (0, 1), \mathbb{P}\{X > t\} \geq \frac{\mu - t}{1 - t}.$$

*Proof.* Observe that  $1 - X$  is non-negative, and hence we can apply Markov's inequality as follows:

$$1 - \mathbb{P}\{1 - t < X\} = \mathbb{P}\{1 - X \geq t\} \leq \frac{\mathbb{E}[1 - X]}{t} \leq \frac{1 - \mu}{t}.$$

Rearranging, this yields,

$$\mathbb{P}\{X > 1 - t\} \geq 1 - \frac{1 - \mu}{t}.$$

Now set  $t' = 1 - t$  to conclude:

$$\mathbb{P}\{X > t'\} \geq \frac{t - 1 + \mu}{t} = \frac{\mu - t'}{1 - t'}.$$

□

The claim now follows by applying the above proposition to the random variable

$$\psi(X_n) := \mathbb{E}_{\mu} \mathbf{1}\{\mathcal{A}(S(X_n, h))(x') \neq h(x')\}.$$

Since  $\mathbb{E}_{\mu^n} [\psi(X_n)] \geq 1/4$ , we conclude:

$$\mathbb{P}_{\mu^n} \{\psi(X_n) > 1/8\} \geq \frac{1/4 - 1/8}{1 - 1/8} = \frac{1}{7}.$$

Note that the distribution  $\mathcal{D}$  is given by sampling  $x \sim \mu$  and setting  $y = h(x)$ . □

The no free lunch theorem immediate gives us the following converse result for agnostic PAC learnability.

**Theorem 1.46** (Infinite VC-dimension classes are not agnostically PAC learnable). *Let  $X$  be a domain of infinite cardinality, and let  $\mathcal{H}$  denote hypothesis class of functions mapping  $X \mapsto Y$  with infinite VC-dimension (i.e.,  $\text{VCdim}(\mathcal{H}) = \infty$ ). Then,  $\mathcal{H}$  is not agnostically PAC learnable (cf. Definition 1.42).*

*Proof.* Suppose that  $\mathcal{H}$  is agnostically PAC learnable via algorithm  $\mathcal{A}$ . Set  $\varepsilon = 1/16$ ,  $\delta = 1/14$ , and  $n = n(\varepsilon, \delta)$ . Let  $\mathcal{D}$  denote the distribution from Theorem 1.44 such that:

- (a) there exists an  $h \in \mathcal{H}$  satisfying  $\mathbb{P}_{\mathcal{D}}\{h(x) \neq y\} = 0$ ,
- (b) there exists an event  $\mathcal{E}_1$  on  $\mathcal{D}^n$  such that for  $S_n \sim \mathcal{D}^n$ ,  $\mathbb{P}_{\mathcal{D}^n}\{S_n \in \mathcal{E}_1\} \geq 1/7$ , and
- (c) on  $\mathcal{E}_1$ ,  $\mathbb{P}_{\mathcal{D}}\{\mathcal{A}(S_n)(x) \neq y\} \geq 1/8$ .

However, since  $\mathcal{H}$  is agnostically PAC learnable, there also exists another event  $\mathcal{E}_2$  on  $\mathcal{D}^n$  such that:

- (a) for  $S_n \sim \mathcal{D}^n$ ,  $\mathbb{P}_{\mathcal{D}^n}\{S_n \in \mathcal{E}_2\} \geq 1 - \delta$ ,
- (b) on  $\mathcal{E}_2$ ,  $\mathbb{P}_{\mathcal{D}}\{\mathcal{A}(S_n)(x) \neq y\} \leq \inf_{h \in \mathcal{H}} \mathbb{P}_{\mathcal{D}}\{h(x) \neq y\} + \varepsilon = \varepsilon$ .

Observe that for  $S_n \in \mathcal{E}_1 \cap \mathcal{E}_2$ , we have that:

$$\frac{1}{8} \leq \mathbb{P}_{\mathcal{D}}\{\mathcal{A}(S_n)(x) \neq y\} \leq \varepsilon = \frac{1}{16},$$

which is impossible. So we must have that  $\mathbb{P}_{\mathcal{D}^n}\{\mathcal{E}_1 \cap \mathcal{E}_2\} = 0$ . But, we have:

$$\begin{aligned} \mathbb{P}_{\mathcal{D}^n}\{\mathcal{E}_1 \cap \mathcal{E}_2\} &= 1 - \mathbb{P}_{\mathcal{D}^n}\{\mathcal{E}_1^c \cup \mathcal{E}_2^c\} \\ &\geq 1 - \mathbb{P}_{\mathcal{D}^n}\{\mathcal{E}_1^c\} - \mathbb{P}_{\mathcal{D}^n}\{\mathcal{E}_2\} && \text{by a union bound} \\ &\geq 1 - (1 - 1/7) - \delta = 1/7 - 1/14 = 1/14 > 0. \end{aligned}$$

□

Theorem 1.43 combined with Theorem 1.46 immediately yields the following characterization of when agnostic PAC learning is possible. The following corollary is a key foundational result in machine learning.

**Corollary 1.47.** *A hypothesis class  $\mathcal{H}$  is agnostically PAC learnable if and only if  $\text{VCdim}(\mathcal{H}) < \infty$ .*

### 1.3.13 Chaining and Dudley's inequality

**Note:** we will not have time to cover this section in this course.

**Definition 1.48** (Covering number). *Let  $(T, \rho)$  be a metric space. A collection of points  $S = \{s_i\} \subset T$  is an  $\varepsilon$ -covering if for every  $t \in T$  there exists an  $s_i \in S$  such that  $\rho(s_i, t) \leq \varepsilon$ . The covering number of  $T$  at resolution  $\varepsilon$ , denoted  $N(\varepsilon; T, \rho)$ , is the minimum number of points needed to form an  $\varepsilon$ -cover of  $T$ .*

**Definition 1.49** (Packing number). *Let  $(T, \rho)$  be a metric space. A collection of points  $S = \{s_i\} \subset T$  is an  $\varepsilon$ -packing if for any  $s_i, s_j \in S$  with  $s_i \neq s_j$ , we have  $\rho(s_i, s_j) > \varepsilon$ . The packing number of  $T$  at resolution  $\varepsilon$ , denoted  $M(\varepsilon; T, \rho)$ , is the maximum achievable size amongst all  $\varepsilon$ -packings of  $T$ .*

Covering and packing numbers are closely related.

**Proposition 1.50.** *Let  $(T, \rho)$  be a metric space. We have:*

$$M(2\varepsilon; T, \rho) \leq N(\varepsilon; T, \rho) \leq M(\varepsilon; T, \rho).$$

*Proof.* We will abbreviate the covering and packing numbers as  $N(\varepsilon)$  and  $M(\varepsilon)$ , respectively. We first prove that  $N(\varepsilon) \leq M(\varepsilon)$ . Let  $\{t_1, \dots, t_M\}$  be a maximal  $\varepsilon$ -packing. We claim that  $\{t_1, \dots, t_M\}$  is also an  $\varepsilon$ -covering. To see this, suppose by contradiction there exists a  $t \in T$  satisfying  $\min_{i=1, \dots, M} \rho(t, t_i) > \varepsilon$ . Then,  $\{t_1, \dots, t_M, t\}$  is an  $\varepsilon$ -packing by definition with cardinality  $M + 1$ , a contradiction.

Now we show that  $M(2\varepsilon) \leq N(\varepsilon)$ . Let  $\{t_1, \dots, t_M\}$  be a maximal  $2\varepsilon$ -packing and let  $\{u_1, \dots, u_N\}$  be a minimal  $\varepsilon$ -covering. Suppose that  $M \geq N + 1$ . Since  $\{u_1, \dots, u_N\}$  covers  $T$  and  $M > N$ , there must exist a  $t_i \neq t_j$  and  $u_k$  such that  $t_i, t_j \in B_\rho(u_k, \varepsilon)$  (here,  $B_\rho(u, \varepsilon) = \{v \in T \mid \rho(u, v) \leq \varepsilon\}$ .) By triangle inequality,  $\rho(t_i, t_j) \leq \rho(t_i, u_k) + \rho(u_k, t_j) \leq 2\varepsilon$ . But, since  $\{t_i\}$  is a  $2\varepsilon$ -packing, by definition we must have  $\rho(t_i, t_j) > 2\varepsilon$ , a contradiction, and hence  $M < N + 1$  or equivalently  $M \leq N$ .  $\square$

**Remark 1.51.** The quantities  $\log N(\varepsilon; T, \rho)$  and  $\log M(\varepsilon; T, \rho)$  are often referred to as the *metric entropy* of  $T$  (at resolution  $\varepsilon$ ).

**Proposition 1.52** (Volume comparison inequalities). *Let  $T \subset \mathbb{R}^d$ , and let  $\|\cdot\|$  be a norm on  $\mathbb{R}^d$  (not necessarily Euclidean). Let  $B(x, r)$  denote the closed ball of radius  $r$  around  $x$  (measured using the  $\|\cdot\|$  norm). Finally, let  $\text{Vol}(\cdot)$  denote the Lebesgue measure on  $\mathbb{R}^d$ . For any  $\varepsilon > 0$ ,*

$$\left(\frac{1}{\varepsilon}\right)^d \frac{\text{Vol}(T)}{\text{Vol}(B(0, 1))} \leq N(\varepsilon; T, \|\cdot\|) \leq \frac{\text{Vol}(T + B(0, \varepsilon/2))}{\text{Vol}(B(0, \varepsilon/2))}.$$

*Proof.* We first prove the LHS inequality. Let  $\{t_1, \dots, t_N\}$  be any  $\varepsilon$ -covering of  $T$ . By the definition of an  $\varepsilon$ -covering,  $T \subseteq \bigcup_{i=1}^N B(t_i, \varepsilon)$ , and hence,

$$\text{Vol}(T) \stackrel{(a)}{\leq} \text{Vol}\left(\bigcup_{i=1}^N B(t_i, \varepsilon)\right) \stackrel{(b)}{\leq} \sum_{i=1}^N \text{Vol}(B(t_i, \varepsilon)) \stackrel{(c)}{=} \sum_{i=1}^N \varepsilon^d \text{Vol}(B(0, 1)) = \varepsilon^d N \cdot \text{Vol}(B(0, 1)).$$

Above, (a) follows from monotonicity of measure, (b) follows from sub-additivity of measure, and (c) follows from the translation invariance and scaling properties of the Lebesgue measure. Rearranging the above inequality yields  $N \geq \varepsilon^{-d} \text{Vol}(T) / \text{Vol}(B(0, 1))$ , and taking the infimum over  $\varepsilon$ -covers yields the LHS inequality.

We now proceed to the RHS inequality. Let  $\{t_1, \dots, t_M\}$  be a maximal  $\varepsilon$ -packing of  $T$ . By construction,

$$\text{Vol}\left(\bigcup_{i=1}^M B(t_i, \varepsilon/2)\right) = \sum_{i=1}^M \text{Vol}(B(t_i, \varepsilon/2)) = M \cdot \text{Vol}(B(0, \varepsilon/2)),$$

since the balls  $B(t_i, \varepsilon/2)$  are disjoint. On the other hand, we clearly have  $\bigcup_{i=1}^M B(t_i, \varepsilon/2) \subseteq T + B(0, \varepsilon/2)$ , and hence by monotonicity of measure,

$$\text{Vol}\left(\bigcup_{i=1}^M B(t_i, \varepsilon/2)\right) \leq \text{Vol}(T + B(0, \varepsilon/2)).$$

Combining these two inequalities yields  $M \leq \text{Vol}(T + B(0, \varepsilon/2)) / \text{Vol}(B(0, \varepsilon/2))$ . The claim follows by using the inequality  $N(\varepsilon; T, \|\cdot\|) \leq M(\varepsilon; T, \|\cdot\|)$  from Proposition 1.50.  $\square$

**Corollary 1.53.** *Let  $\|\cdot\|$  be any norm on  $\mathbb{R}^d$ . For any  $\varepsilon > 0$ ,*

$$\left(\frac{1}{\varepsilon}\right)^d \leq N(\varepsilon; B(0, 1), \|\cdot\|) \leq \left(1 + \frac{2}{\varepsilon}\right)^d.$$

*Proof.* The LHS inequality follows immediately from Proposition 1.52. For the RHS inequality, we simply observe that  $B(0, 1) + B(0, \varepsilon/2) \subseteq B(0, 1 + \varepsilon/2)$ , in which case:

$$\text{Vol}(B(0, 1) + B(0, \varepsilon/2)) \leq (1 + \varepsilon/2)^d \text{Vol}(B(0, 1)).$$

The RHS inequality now follows again from Proposition 1.52.  $\square$

For a function class  $\mathcal{F}$  mapping  $\mathbf{X} \mapsto \mathbb{R}$  and a realization of data points  $\{x_i\}_{i=1}^n$ , we define the empirical  $L_2(\mathbf{P}_n)$  (semi-)norm as:

$$\|f\|_{L_2(\mathbf{P}_n)}^2 := \frac{1}{n} \sum_{i=1}^n f(x_i)^2.$$

Let us make a first attempt to bound the Rademacher complexity by covering numbers. Recall Massart's lemma (Proposition 1.37), which gives a simple bound for finite function classes. Thus, we can use the idea of a covering to discretize a continuous function class, and pay a little extra for the discretization error (which is controlled by the scale of the covering).

**Proposition 1.54** (One step discretization). *The following holds for every realization  $x_1, \dots, x_n \in \mathbf{X}$ :*

$$\mathbb{E}_\varepsilon \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i f(x_i) \leq \inf_{\alpha > 0} \left\{ \alpha + B_n \sqrt{\frac{2 \log N(\alpha; \mathcal{F}, L_2(\mathbf{P}_n))}{n}} \right\}, \quad B_n = \sup_{f \in \mathcal{F}} \|f\|_{L_2(\mathbf{P}_n)}.$$

*Proof.* We start with the projection set  $Q = \{(f(x_1), \dots, f(x_n)) \in \mathbb{R}^n \mid f \in \mathcal{F}\}$ . Fix any  $\alpha > 0$  and let  $C$  denote a minimal  $\alpha$ -covering of  $Q$  in the weighted  $x \mapsto n^{-1/2}\|x\|$  norm. For any  $q \in Q$  let  $\varphi(q)$  denote the element in  $C$  of minimal distance, such that  $n^{-1/2}\|q - \varphi(q)\| \leq \alpha$ . We can now proceed by:

$$\begin{aligned} \mathbb{E} \sup_{q \in Q} n^{-1} \langle \varepsilon, q \rangle &= \mathbb{E} \sup_{q \in Q} n^{-1} \langle \varepsilon, \varphi(q) + q - \varphi(q) \rangle \\ &\leq \mathbb{E} \sup_{q \in Q} n^{-1} \langle \varepsilon, \varphi(q) \rangle + \mathbb{E} \sup_{q \in Q} n^{-1} \langle \varepsilon, q - \varphi(q) \rangle. \end{aligned}$$

Now, we first observe that:

$$n^{-1} \langle \varepsilon, q - \varphi(q) \rangle \leq n^{-1} \|\varepsilon\| \|q - \varphi(q)\| = n^{-1/2} \|q - \varphi(q)\| \leq \alpha,$$

and hence the second term is bounded by  $\alpha$ . As for the first term, we apply Massart's lemma (Proposition 1.37), noting that  $n^{-1/2}\|q\| \leq B_n$ , and therefore,

$$\mathbb{E} \sup_{q \in Q} n^{-1} \langle \varepsilon, \varphi(q) \rangle \leq B_n \sqrt{\frac{2 \log |C|}{n}} \leq B_n \sqrt{\frac{2 \log N(\alpha; \mathcal{F}, L_2(\mathbf{P}_n))}{n}}.$$

Since  $\alpha > 0$  was arbitrary, we can optimize the bound over  $\alpha$ , from which the claim follows.  $\square$

While the one step discretization argument is certainly a step in the right direction in terms of using covering numbers to bound Rademacher complexity, as we will see later it can be un-necessarily loose. The issue is that the argument is quite conservative by only choosing to discretize at one resolution. By repeatedly discretizing at finer and finer resolutions, we get a sharper result known as Dudley's inequality.

**Lemma 1.55** (Dudley's entropy inequality). *The following holds for every realization  $x_1, \dots, x_n \in \mathbf{X}$ :*

$$\mathbb{E}_\varepsilon \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i f(x_i) \leq 12 \int_0^{B_n} \sqrt{\frac{\log N(\varepsilon; \mathcal{F}, L_2(\mathbf{P}_n))}{n}} d\varepsilon, \quad B_n = \sup_{f \in \mathcal{F}} \|f\|_{L_2(\mathbf{P}_n)}.$$

*Proof.* Put  $\varepsilon_0 := B_n$  and for  $i = 1, 2, \dots$ , define a sequence of decreasing resolutions  $\varepsilon_i = \varepsilon_0 2^{-i}$ . As before, let the projection set  $Q$  be

$$Q = \{(f(x_1), \dots, f(x_n)) \in \mathbb{R}^n \mid f \in \mathcal{F}\}.$$

With the definition of  $\varepsilon_0$ , observe that for any  $q \in Q$ ,

$$n^{-1/2} \|q\| = \sqrt{\frac{1}{n} \sum_{i=1}^n f(x_i)^2} \leq \varepsilon_0.$$

For any  $i = 1, 2, \dots$ , let  $C_i$  denote a minimal  $\varepsilon_i$ -cover of  $Q$  in the weighted  $\ell_2$  norm  $n^{-1/2}\|\cdot\|$ . Furthermore, for any  $q \in Q$ , let  $\varphi_i(q)$  denote an element in  $C_i$  satisfying  $n^{-1/2}\|q - \varphi_i(q)\| \leq \varepsilon_i$ . Fix any  $q \in Q$ , and note that for any finite  $N$ ,

$$\varphi_1(q) + \sum_{i=1}^N (\varphi_{i+1}(q) - \varphi_i(q)) = \varphi_{N+1}(q).$$

Hence, since  $\lim_{i \rightarrow \infty} \varphi_i(q) = q$ ,

$$q = \varphi_1(q) + \sum_{i=1}^{\infty} (\varphi_{i+1}(q) - \varphi_i(q)).$$

Therefore,

$$\begin{aligned} \mathbb{E} \sup_{q \in Q} n^{-1} \langle \varepsilon, q \rangle &= \mathbb{E} \sup_{q \in Q} n^{-1} \left\langle \varepsilon, \varphi_1(q) + \sum_{i=1}^{\infty} (\varphi_{i+1}(q) - \varphi_i(q)) \right\rangle \\ &\leq \mathbb{E} \sup_{q \in Q} n^{-1} \langle \varepsilon, \varphi_1(q) \rangle + \sum_{i=1}^{\infty} \mathbb{E} \sup_{q \in Q} n^{-1} \langle \varepsilon, \varphi_{i+1}(q) - \varphi_i(q) \rangle. \end{aligned}$$

Let us control the second term first. First, observe that for any  $q \in Q$ ,

$$n^{-1/2} \|\varphi_{i+1}(q) - \varphi_i(q)\| \leq n^{-1/2} \|q - \varphi_{i+1}(q)\| + n^{-1/2} \|q - \varphi_i(q)\| \leq \varepsilon_{i+1} + \varepsilon_i = 3\varepsilon_{i+1}.$$

On the other hand, the number of unique vectors  $\{\varphi_{i+1}(q) - \varphi_i(q) \mid q \in Q\}$  is clearly upper bounded by  $|C_{i+1}| \times |C_i|$ . Hence by Massart's lemma (Proposition 1.37),

$$\mathbb{E} \sup_{q \in Q} n^{-1} \langle \varepsilon, \varphi_{i+1}(q) - \varphi_i(q) \rangle \leq 3\varepsilon_{i+1} \sqrt{\frac{2 \log |C_{i+1}| |C_i|}{n}} \stackrel{(a)}{\leq} 6\varepsilon_{i+1} \sqrt{\frac{\log |C_{i+1}|}{n}},$$

where (a) follows since  $|C_i| \leq |C_{i+1}|$ .

Now turning to the first term, we observe that  $n^{-1/2} \|\varphi_1(q)\| \leq \varepsilon_0$ , and hence again by Massart's lemma,

$$\mathbb{E} \sup_{q \in Q} n^{-1} \langle \varepsilon, \varphi_1(q) \rangle \leq \varepsilon_0 \sqrt{\frac{2 \log |C_1|}{n}} = 2\varepsilon_1 \sqrt{\frac{2 \log |C_1|}{n}}.$$

Hence combining both inequalities,

$$\mathbb{E} \sup_{q \in Q} n^{-1} \langle \varepsilon, q \rangle \leq 6 \sum_{i=1}^{\infty} \varepsilon_i \sqrt{\frac{\log |C_i|}{n}} = 6 \sum_{i=1}^{\infty} \varepsilon_i \sqrt{\frac{\log N(\varepsilon_i; \mathcal{F}, L_2(\mathbf{P}_n))}{n}}.$$

To finish the proof, we observe that since  $\varepsilon \mapsto N(\varepsilon; \mathcal{F}, L_2(\mathbf{P}_n))$  is a monotonically decreasing function of  $\varepsilon$ , we have the following inequality:

$$(\varepsilon_i - \varepsilon_{i+1}) \sqrt{\log N(\varepsilon_i; \mathcal{F}, L_2(\mathbf{P}_n))} \leq \int_{\varepsilon_{i+1}}^{\varepsilon_i} \sqrt{\log N(\varepsilon; \mathcal{F}, L_2(\mathbf{P}_n))} d\varepsilon.$$

Since  $\varepsilon_i = 2(\varepsilon_i - \varepsilon_{i+1})$ , we conclude that

$$\begin{aligned} 6 \sum_{i=1}^{\infty} \varepsilon_i \sqrt{\log N(\varepsilon_i; \mathcal{F}, L_2(\mathbf{P}_n))} &\leq 12 \sum_{i=1}^{\infty} \int_{\varepsilon_{i+1}}^{\varepsilon_i} \sqrt{\log N(\varepsilon; \mathcal{F}, L_2(\mathbf{P}_n))} d\varepsilon \\ &= 12 \int_0^{\varepsilon_1} \sqrt{\log N(\varepsilon; \mathcal{F}, L_2(\mathbf{P}_n))} d\varepsilon \\ &\leq 12 \int_0^{\varepsilon_0} \sqrt{\log N(\varepsilon; \mathcal{F}, L_2(\mathbf{P}_n))} d\varepsilon. \end{aligned}$$

□

**Remark 1.56.** Dudley's inequality (Lemma 1.55) is stated conditionally on the data  $\{x_i\}_{i=1}^n$ . Taking another expectation over the data, we obtain:

$$\mathcal{R}_n(\mathcal{F}) \leq 12\mathbb{E} \int_0^{B_n} \sqrt{\frac{\log N(\varepsilon; \mathcal{F}, L_2(\mathbf{P}_n))}{n}} d\varepsilon, \quad B_n = \sup_{f \in \mathcal{F}} \|f\|_{L_2(\mathbf{P}_n)}.$$

While Dudley's inequality is an improvement over the one step discretization argument in that it considers covering numbers at multiple resolutions, the version stated in Lemma 1.55 suffers from one key drawback, which is that in situations where  $\log N(\varepsilon; \mathcal{F}, L_2(\mathbf{P}_n))$  is not integrable near the origin, the bound is vacuous. On the other hand the one step argument Proposition 1.54 does not suffer from this issue. We can, however, combine both bounds and obtain the best of both—multiscale covering numbers and truncating the lower limit of the integral before  $\varepsilon \rightarrow 0$ .

**Exercise 1.57.** Modify the proof of Lemma 1.55 to prove the following refined Dudley's inequality:

$$\mathbb{E}_\varepsilon \sup_{f \in \mathcal{F}} n^{-1} \sum_{i=1}^n \varepsilon_i f(x_i) \leq \inf_{\alpha > 0} \left\{ 4\alpha + 12 \int_\alpha^{B_n} \sqrt{\frac{\log N(\varepsilon; \mathcal{F}, L_2(\mathbf{P}_n))}{n}} d\varepsilon \right\}.$$

Let us now apply Dudley's inequality to study linear hypothesis classes, as we did in Section 1.3.9. Specifically, let us consider  $\mathcal{F} = \{x \mapsto \langle x, w \rangle \mid w \in \mathbb{R}^d, \|w\| \leq 1\}$ . We need to estimate the  $L_2(\mathbf{P}_n)$  covering number of  $\mathcal{F}$ . Let  $X \in \mathbb{R}^{n \times d}$  denote the data matrix with row  $i$  containing example  $x_i$ . Now observe that for any  $f_u, f_v \in \mathcal{F}$ ,

$$\|f_u - f_v\|_{L_2(\mathbf{P}_n)}^2 = \frac{1}{n} \sum_{i=1}^n \langle x_i, u - v \rangle^2 = \frac{1}{n} (u - v)^\top X^\top X (u - v) \leq \frac{\lambda_{\max}(X^\top X)}{n} \|u - v\|^2.$$

Thus, we immediately see that, with  $B_n := \|X/\sqrt{n}\|_{\text{op}}$ ,

$$N(\varepsilon; \mathcal{F}, L_2(\mathbf{P}_n)) \leq N(\varepsilon/B_n, B(0, 1), \|\cdot\|) \leq \left(1 + \frac{2B_n}{\varepsilon}\right)^d,$$

where the last estimate follows from Corollary 1.53. Now, letting  $B$  be an almost sure upper bound on  $B_n$ , we compute:

$$\begin{aligned} \int_0^B \sqrt{\log N(\varepsilon; \mathcal{F}, L_2(\mathbf{P}_n))} d\varepsilon &\leq \sqrt{d} \int_0^B \sqrt{\log(1 + 2B/\varepsilon)} d\varepsilon \\ &= B\sqrt{d} \int_0^1 \sqrt{\log(1 + 2/\varepsilon)} d\varepsilon && \text{change of variables } \varepsilon \leftarrow \varepsilon/B \\ &\leq 2B\sqrt{d} && \text{using Mathematica.} \end{aligned}$$

Hence, from Dudley's inequality,

$$\mathcal{R}_n(\mathcal{F}) \leq 24B\sqrt{\frac{d}{n}}.$$

Now, as a point of comparison, observe that the one-step discretization bound Proposition 1.54 yields the looser bound  $\mathcal{R}_n(\mathcal{F}) \lesssim B\sqrt{\frac{d \log(n/d)}{n}}$  by choosing  $\alpha \asymp \sqrt{d/n}$ . Hence, the Dudley argument allows us to shave off a log factor in the bound for this example.

Let us now compare the bound from Dudley's inequality to the direct calculations made in Section 1.3.9. Recall that we computed that if  $x_i$  are i.i.d. from  $N(0, I)$ , then  $\mathcal{R}_n(\mathcal{F}) \leq \sqrt{d/n}$ . Immediately we see that the Dudley bound above does not apply to case, since there is no finite almost sure bound on  $B$ . However, let us suppose that instead the  $x_i$ 's are i.i.d. where each entry is an independent Rademacher random variable. Then, the first and second moments  $\mathbb{E}\|x\|$  and  $\mathbb{E}\|x\|^2$  match the Gaussian distribution order wise, but we have that  $B = \sqrt{d}$  (check!). Thus, we actually get a looser result  $\mathcal{R}_n(\mathcal{F}) \lesssim d/\sqrt{n}$ , which is not sharp. However, this can be fixed, as the contents of the following exercise:



**Exercise 1.58.** Suppose that the random vector  $x_i$ 's are i.i.d. where each entry is an independent Rademacher random variable. Let  $\mathcal{F} = \{x \mapsto \langle x, w \rangle \mid w \in \mathbb{R}^d, \|w\| \leq 1\}$ . Show using Remark 1.56 that when  $n \geq d$ , there exists a universal constant  $c$  such that:

$$\mathcal{R}_n(\mathcal{F}) \leq c \sqrt{\frac{d}{n}}.$$

Hint: use the following random matrix deviation bound (which you may use without proof, although at this point you actually have all the tools to prove this inequality!). Let  $X \in \mathbb{R}^{n \times d}$  be the data matrix with row  $i$  corresponding to  $x_i$ . There exists universal constants  $c_0, c_1$  such that:

$$\forall t > 0, \mathbb{P} \left\{ \|X\|_{\text{op}} \geq \sqrt{n} + c_0 \sqrt{d} + t \right\} \leq \exp(-c_1 t^2).$$

The real advantage of Dudley's inequality, however, is to utilize it beyond linear hypothesis classes.

**Exercise 1.59.** Let  $\mathcal{F} = \{x \mapsto f_\theta(x) \mid \theta \in \Theta \subset \mathbb{R}^d, \|\theta\| \leq B\}$ . Suppose furthermore that the following conditions hold:

$$\forall x \in \mathbf{X}, \theta, \theta_1, \theta_2 \in \Theta, |f_\theta(x)| \leq 1, |f_{\theta_1}(x) - f_{\theta_2}(x)| \leq L \|\theta_1 - \theta_2\|.$$

Show that there exists a universal constant  $c$  such that:

$$\mathcal{R}_n(\mathcal{F}) \leq c \sqrt{\frac{d \log(BL)}{n}}.$$

**Exercise 1.60.** Let  $\mathcal{F}$  denote the following 1-Lipschitz function class:

$$\mathcal{F} = \{f : [-1, 1]^d \mapsto [-1, 1] \mid \forall x, y \in [-1, 1]^d, |f(x) - f(y)| \leq \|x - y\|_\infty\}.$$

Suppose that  $d > 2$ . There exists an constant  $c$  such that:

$$\mathcal{R}_n(\mathcal{F}) \leq \frac{c}{n^{1/d}}.$$

Hint: Use the fact that  $\log N(\varepsilon; \mathcal{F}, \|\cdot\|_\infty) \leq (c_0/\varepsilon)^d$  for some universal constant  $c_0$ . Here,  $\|f\|_\infty = \sup_{x \in [-1, 1]^d} |f(x)|$  is the supremum norm over  $[-1, 1]^d$ .

## 1.4 Algorithmic stability

We now study alternative techniques for ensuring generalization. One particular method is based on algorithmic stability. The main idea is that algorithms which are more robust to small changes in the data (i.e., small perturbations to the data do not drastically alter the result) generalize better.

We will slightly redefine notation in this section to aide the analysis. As before, we fix a hypothesis class  $\mathcal{F}$  and a loss function  $\ell$ . However, in this second, the loss function maps  $\ell : \mathcal{F} \times \mathbf{Z} \rightarrow \mathbb{R}$ . That is, the loss function takes the pair of hypothesis and labelled data point, and produces a score (think  $\ell(f, (x, y)) = \ell(f(x), y)$ ). Finally, we will introduce the notion of an algorithm, which is a mapping  $A : \mathbf{Z}^n \times \Xi \rightarrow \mathcal{F}$ . The set  $\Xi$  encapsulates any randomness used by the algorithm.

**Definition 1.61** (Uniform stability). *An algorithm  $A : \mathbf{Z}^n \times \Xi \rightarrow \mathcal{F}$  is  $\varepsilon$ -uniformly-stable if for every pair of datasets  $S, S' \in \mathbf{Z}^n$  which differ in at most one example,*

$$\sup_{z \in \mathbf{Z}} \mathbb{E}_\xi [\ell(A(S, \xi), z) - \ell(A(S', \xi), z)] \leq \varepsilon.$$

It is worth noting that the trivial algorithm  $A(S, \xi) = f_0$  for some fixed  $f_0$  (e.g., an algorithm which is *not* a function of the data) is trivially  $\varepsilon$  stable for any  $\varepsilon \geq 0$ . Of course this is not actually useful for learning, but it demonstrates that algorithmic stability only deals with generalization error and not whether the algorithm  $A$  has capacity to fit data.

For what follows we need some extra notation. Let  $\{z_i\}_{i=1}^n$  and  $\{z'_i\}_{i=1}^n$  be i.i.d. draws. Put  $S = (z_1, \dots, z_n)$ ,  $S' = (z'_1, \dots, z'_n)$ , and  $S^{(i)} = (z_1, \dots, z_{i-1}, z'_i, z_{i+1}, \dots, z_n)$  for  $i = 1, \dots, n$ . We slightly overload the  $\hat{L}_n[f]$  notation with:

$$\hat{L}_S[f] := \frac{1}{n} \sum_{i=1}^n \ell(f, z_i).$$

#### 1.4.1 Uniform stability implies generalization

Now, we show that uniform stability directly implies generalization.

**Lemma 1.62** (Uniform stability implies generalization in expectation). *Let  $A : Z^n \times \Xi \rightarrow \mathcal{F}$  be an  $\varepsilon$ -uniformly-stable algorithm (cf. Definition 1.61). Then,*

$$|\mathbb{E}_{S, \xi}[L[A(S, \xi)]] - \hat{L}_S[A(S, \xi)]| \leq \varepsilon.$$

*Proof.* This argument is essentially an exercise in exchangeability. For what follows, we will abbreviate  $A(S, \xi) = A(S)$  to reduce the notational overhead. The first trick is to observe that for any  $i$ , because  $(S, z)$  and  $(S^{(i)}, z_i)$  are exchangeable pairs (since  $(S, z) \stackrel{d}{=} (S^{(i)}, z_i)$ ),

$$\mathbb{E}_{S, \xi}[L[A(S)]] = \mathbb{E}_{S, z, \xi}[\ell(A(S), z)] = \mathbb{E}_{S', z_i, \xi}[\ell(A(S^{(i)}), z_i)] = \mathbb{E}_{S, S', \xi}[\ell(A(S^{(i)}), z_i)].$$

Since this identity holds for any  $i = 1, \dots, n$ , we can average the RHS expression as follows:

$$\mathbb{E}_{S, \xi}[L[A(S)]] = \mathbb{E}_{S, S', \xi} \left[ \frac{1}{n} \sum_{i=1}^n \ell(A(S^{(i)}), z_i) \right].$$

Therefore by  $\varepsilon$ -uniform-stability,

$$\mathbb{E}_{S, \xi}[L[A(S)]] - \hat{L}_S[A(S)] = \mathbb{E}_{S, S', \xi} \left[ \frac{1}{n} \sum_{i=1}^n (\ell(A(S^{(i)}), z_i) - \ell(A(S), z_i)) \right] \leq \varepsilon.$$

Bounding  $\mathbb{E}_{S, \xi}[\hat{L}_S[A(S)] - L[A(S)]]$  proceeds in a nearly identical way, from which the claim follows.  $\square$

With this result in hand, we can utilize the bounded differences inequality (Proposition B.16) to prove a high probability bound on the generalization error.

**Lemma 1.63.** *Suppose that  $|\ell| \leq B$ , and that  $A : Z^n \times \Xi \rightarrow \mathcal{F}$  is an  $\varepsilon$ -uniformly-stable algorithm. With probability at least  $1 - \delta$ ,*

$$\mathbb{E}_\xi[L[A(S, \xi)]] \leq \mathbb{E}_\xi[\hat{L}_S[A(S, \xi)]] + \varepsilon + 2(\varepsilon n + B) \sqrt{\frac{2 \log(1/\delta)}{n}}.$$

*Proof.* Again as before, we will abbreviate  $A(S, \xi) = A(S)$  to reduce notational burden. Define the random variable  $f(S) = \mathbb{E}_\xi[L[A(S)]] - \hat{L}_S[A(S)]$ . We will show that  $f(S)$  satisfies the bounded differences property. Let  $S^{(i)}$  be a dataset which differs in  $S$  in the  $i$ -th location. We first observe that,

$$\mathbb{E}_\xi[L[A(S)]] - \mathbb{E}_\xi[L[A(S^{(i)})]] = \mathbb{E}_z[\mathbb{E}_\xi[\ell(A(S), z) - \ell(A(S^{(i)}), z)]].$$

Hence by the  $\varepsilon$ -uniform-stability assumption,  $\mathbb{E}_\xi[L[A(S)]] - \mathbb{E}_\xi[L[A(S^{(i)})]] \leq \varepsilon$ . Furthermore, bounding  $\mathbb{E}_\xi[L[A(S^{(i)})]] - \mathbb{E}_\xi[L[A(S)]] \leq \varepsilon$  proceeds identically.

Next, we observe that,

$$\begin{aligned} & \mathbb{E}_\xi[\hat{L}_S[A(S)] - \hat{L}_{S^{(i)}}[A(S^{(i)})]] \\ &= \mathbb{E}_\xi \left[ \frac{1}{n} \sum_{j \neq i} [\ell(A(S), z_j) - \ell(A(S^{(i)}), z_j)] + \frac{1}{n} [\ell(A(S), z_i) - \ell(A(S^{(i)}), z'_i)] \right]. \end{aligned}$$

Hence by the  $\varepsilon$ -uniform-stability assumption and the boundedness of  $\ell$ ,

$$\mathbb{E}_\xi[\hat{L}_S[A(S)] - \hat{L}_{S^{(i)}}[A(S^{(i)})]] \leq \varepsilon + \frac{2B}{n}.$$

Again, bounding  $\mathbb{E}_\xi[\hat{L}_{S^{(i)}}[A(S^{(i)})] - \hat{L}_S[A(S)]] \leq \varepsilon + 2B/n$  proceeds identically. Putting these inequalities together, we see that

$$|f(S) - f(S^{(i)})| = |\mathbb{E}_\xi[L(A(S))] - \mathbb{E}_\xi[L(A(S^{(i)}))] + \mathbb{E}_\xi[\hat{L}_S[A(S)] - \hat{L}_{S^{(i)}}[A(S^{(i)})]]| \leq 2\varepsilon + \frac{2B}{n}.$$

This shows that  $f$  satisfies the bounded differences inequality. Hence, by Proposition B.16, with probability at least  $1 - \delta$ ,

$$f(S) \leq \mathbb{E}[f(S)] + \sqrt{8n \left( \varepsilon + \frac{B}{n} \right)^2 \log(1/\delta)} = \mathbb{E}[f(S)] + 2(\varepsilon n + B) \sqrt{\frac{2 \log(1/\delta)}{n}}.$$

Finally, by Lemma 1.62, we have that  $\mathbb{E}[f(S)] \leq \varepsilon$ . The claim now follows.  $\square$

An immediate consequence of Lemma 1.63 is that we require that the uniform stability constant  $\varepsilon$  to scale as  $\varepsilon = O(n^{-1})$  in order for the generalization bound from Lemma 1.63 to scale as the expected  $O(n^{-1/2})$ .

**Remark 1.64.** Note that Lemma 1.63 only prescribes a high probability bound over the randomness of the training data, and *not* over the algorithm's internal randomness. This is because our definition of uniform stability (Definition 1.61) only considers the average stability over the algorithm randomness (for a fixed  $(S, S', z)$ ).

**Exercise 1.65.** Modify the definition of Definition 1.61 and the corresponding results in Lemma 1.62 and Lemma 1.63, so that the high probability bound holds jointly over drawing both the training data and the specific algorithm randomness. Note that there is quite a bit of flexibility in altering the definitions (as in there is no one right answer), so be sure to also discuss the pros/cons of your approach.

### 1.4.2 Stability of regularized least-squares regression

We now consider regularized least-squares regression. Given a dataset  $S = \{(x_i, y_i)\}_{i=1}^n$  (here,  $y_i$  is not necessarily restricted to  $\{\pm 1\}$ ), consider the algorithm, parameterized by  $\lambda > 0$ :

$$A_\lambda(S) = \left( \frac{1}{n} \sum_{i=1}^n x_i x_i^\top + \lambda I \right)^{-1} \frac{1}{n} \sum_{i=1}^n x_i y_i.$$

Note that  $A_\lambda(S)$  is the unique solution to the following optimization problem:

$$\arg \min_{\theta \in \mathbb{R}^d} L_S[\theta] := \frac{1}{2n} \sum_{i=1}^n (\langle x_i, \theta \rangle - y_i)^2 + \frac{\lambda}{2} \|\theta\|^2.$$

(Convince yourself this is true!). Now let us study the stability of  $A_\lambda(S)$ . To do this, we will consider the absolute value loss instead of the square loss:

$$\ell(\theta, (x, y)) = |\langle x, \theta \rangle - y|.$$

This is merely for simplicity, the analysis also goes through for the square loss. Next, let us impose some regularity conditions. We assume that  $\|x_i\| \leq 1$  and  $|y_i| \leq 1$  for all  $i$ . Observe that under these conditions, for any  $\theta_1, \theta_2 \in \mathbb{R}^d$  and  $(x, y)$ ,

$$\begin{aligned} ||\ell(\theta_1, (x, y))| - |\ell(\theta_2, (x, y))|| &= ||\langle x, \theta_1 \rangle - y| - |\langle x, \theta_2 \rangle - y|| \\ &\leq |\langle x, \theta_1 - \theta_2 \rangle| \leq \|x\| \|\theta_1 - \theta_2\| \leq \|\theta_1 - \theta_2\|. \end{aligned}$$

Let  $S, S'$  be two datasets differing in the  $i$ -th location. To show that  $A$  is  $\varepsilon$ -uniformly-stable, it suffices to bound  $\|A(S) - A(S')\|$ . To do this, we start with a simple algebraic inequality. Let  $A_1, A_2$  be two invertible matrices, and  $b_1, b_2$  be two vectors. We have:

$$A_1^{-1}b_1 - A_2^{-1}b_2 = A_1^{-1}(b_1 - b_2) + [A_1^{-1} - A_2^{-1}]b_2.$$

Therefore,

$$\begin{aligned} \|A_1^{-1}b_1 - A_2^{-1}b_2\| &\leq \|A_1^{-1}\|_{\text{op}}\|b_1 - b_2\| + \|A_1^{-1} - A_2^{-1}\|_{\text{op}}\|b_2\| \\ &= \frac{1}{\sigma_{\min}(A_1)}\|b_1 - b_2\| + \|A_1^{-1} - A_2^{-1}\|_{\text{op}}\|b_2\|. \end{aligned}$$

Next, we note another simple algebraic fact:

$$A_1^{-1} - A_2^{-1} = -A_2^{-1}(A_1 - A_2)A_1^{-1}.$$

Plugging this into the inequality above yields,

$$\|A_1^{-1}b_1 - A_2^{-1}b_2\| \leq \frac{1}{\sigma_{\min}(A_1)}\|b_1 - b_2\| + \frac{1}{\sigma_{\min}(A_1)\sigma_{\min}(A_2)}\|A_1 - A_2\|_{\text{op}}\|b_2\|. \quad (1.34)$$

Note this generalizes the scalar identity  $\frac{1}{a_1} - \frac{1}{a_2} = \frac{a_2 - a_1}{a_1 a_2}$  for non-zero  $a_1, a_2 \in \mathbb{R}$ . To proceed, we now set:

$$A(S) = A_1^{-1}b_1, \quad A_1 = \frac{1}{n} \sum_{i=1}^n x_i x_i^\top + \lambda I, \quad b_1 = \frac{1}{n} \sum_{i=1}^n x_i y_i.$$

Similarly,

$$A(S') = A_2^{-1}b_2, \quad A_2 = A_1 + \frac{1}{n}(x'_i x'_i{}^\top - x_i x_i{}^\top), \quad b_2 = b_1 + \frac{1}{n}(x'_i y'_i - x_i y_i).$$

Note that since  $A_1 \succcurlyeq \lambda I$ , we have that  $\sigma_{\min}(A_1) = \lambda_{\min}(A_1) \geq \lambda$ . By the same reasoning, we also have  $\sigma_{\min}(A_2) \geq \lambda$ . Furthermore, by triangle inequality and our assumptions on  $(x_i, y_i)$ , we have  $\|b_2\| \leq 1$ . Plugging these estimates into (1.34), and again using our assumptions on  $(x_i, y_i)$ ,

$$\begin{aligned} \|A(S) - A(S')\| &\leq \frac{1}{\lambda} \left\| \frac{1}{n}(x_i y_i - x'_i y'_i) \right\| + \frac{1}{\lambda^2} \left\| \frac{1}{n}(x_i x_i{}^\top - x'_i x'_i{}^\top) \right\| \\ &\leq \frac{2}{\lambda n} + \frac{2}{\lambda^2 n} = \frac{2}{n} \left[ \frac{1}{\lambda} + \frac{1}{\lambda^2} \right]. \end{aligned}$$

Thus, we see that  $A$  is  $\varepsilon$ -uniformly-stable for  $\varepsilon = 2n^{-1}(1/\lambda + 1/\lambda^2)$ . Here we see the influence of the regularization parameter  $\lambda$  on stability: the larger  $\lambda$  is, the more stable  $A$  is. Indeed, as  $\lambda \rightarrow \infty$ , we have that the stability parameter  $\varepsilon \rightarrow 0$ . However, there is a trade-off: since  $\|A(S)\| \leq 1/\lambda$ , we see that  $\lambda$  reduces the possible hypothesis that the algorithm can select; in the limit as  $\lambda \rightarrow \infty$ , the algorithm is forced to choose  $A(S) = 0$ . Thus, we see again the trade-off between expressivity and generalization error.

### 1.4.3 Stability of stochastic gradient descent

With the previous section in place, we now turn to the studying the stability of specific algorithms. We will first consider the classical stochastic gradient descent (SGD) algorithm. In order to simplify the analysis, we will study a stylized version with no batching and where the data is drawn uniformly at random (instead of random shuffling of the data). Specifically, suppose we have a loss function  $\ell(w, z)$ , where now we let  $w \in \mathbb{R}^d$ . The SGD algorithm uses the following recipe for  $t = 0, 1, \dots$ ,

1. Let  $i_t \sim \text{Unif}(\{1, \dots, n\})$ .
2. Set  $w_{t+1} = w_t - \eta_t \nabla_w \ell(w_t, z_{i_t})$ .

To study this update rule, let us define the update operator  $G_{\ell, \eta}(w, z)$  as:

$$G_{\ell, \eta}(w, z) := w - \eta \nabla_w \ell(w, z),$$

so that the SGD update rule is  $w_{t+1} = G_{\ell, \eta_t}(w_t, z_{i_t})$ .

**Proposition 1.66** (Update operator is non-expansive). *Suppose for every  $z \in \mathbf{Z}$ , we have  $w \mapsto \ell(w, z)$  is convex and  $\beta$ -smooth. Then for any  $u, v \in \mathbb{R}^d$ , as long as  $\eta \in [0, 2/\beta]$ , we have that the update operator  $G_{\ell, \eta}$  is non-expansive, i.e.,*

$$\forall z \in \mathbf{Z}, \|G_{\ell, \eta}(u, z) - G_{\ell, \eta}(v, z)\| \leq \|u - v\|.$$

*Proof.* From Proposition C.6, since for every  $z$  the function  $w \mapsto \ell(w, z)$  is convex and  $\beta$ -smooth, we have the following co-coercivity condition:

$$\langle \nabla_w \ell(u, z) - \nabla_w \ell(v, z), u - v \rangle \geq \frac{1}{\beta} \|\nabla_w \ell(u, z) - \nabla_w \ell(v, z)\|^2.$$

Therefore,

$$\begin{aligned} & \|G_{\ell, \eta}(u, z) - G_{\ell, \eta}(v, z)\|^2 \\ &= \|(u - v) - \eta(\nabla_w \ell(u, z) - \nabla_w \ell(v, z))\|^2 \\ &= \|u - v\|^2 - 2\eta \langle u - v, \nabla_w \ell(u, z) - \nabla_w \ell(v, z) \rangle + \eta^2 \|\nabla_w \ell(u, z) - \nabla_w \ell(v, z)\|^2 \\ &\leq \|u - v\|^2 - (2\eta/\beta - \eta^2) \|\nabla_w \ell(u, z) - \nabla_w \ell(v, z)\|^2 \\ &\leq \|u - v\|^2. \end{aligned} \quad \text{since } \eta \in [0, 2/\beta]$$

□

**Lemma 1.67** (SGD is stable on convex functions). *Suppose that for every  $z \in \mathbf{Z}$ , we have  $w \mapsto \ell(w, z)$  is convex,  $L$ -Lipschitz, and  $\beta$ -smooth. Then, as long as  $\eta_t \in [0, 2/\beta]$  for all  $t$ , running the SGD algorithm for  $T$  iterations is  $\varepsilon$ -uniformly-stable for*

$$\varepsilon \leq \frac{2L^2}{n} \sum_{t=0}^{T-1} \eta_t.$$

*Proof.* Let us consider running SGD on two datasets  $S = \{z_i\}_{i=1}^n$  and  $S' = \{z'_i\}_{i=1}^n$ , where  $S, S'$  differ in exactly one data point. Let  $\kappa \in \{1, \dots, n\}$  denote the index which the datasets differ. We are going to use what is known as a *coupling argument*. We let the iterates  $\{u_t\}$  denote SGD running on  $S$ , and  $\{v_t\}$  denote SGD running on  $S'$ . The coupling argument works by using the *same* randomness  $\xi$  for both executions (in this case, the same indices  $\{i_t\}$  are drawn), and then bounding  $\mathbb{E}_\xi \|u_T - v_T\|$  (the dependence of the  $u_t$ 's and  $v_t$ 's on  $\xi$  is suppressed in our notation). To understand why this is correct, recall the definition of uniform stability (Definition 1.61):

$$\sup_{z \in \mathbf{Z}} \mathbb{E}_\xi [\ell(u_T, z) - \ell(v_T, z)] \leq \varepsilon.$$

To control the LHS, we use the  $L$ -Lipschitz assumption on  $\ell$ , which implies

$$\sup_{z \in \mathcal{Z}} \mathbb{E}_\xi [\ell(u_T, z) - \ell(v_T, z)] \leq L \mathbb{E}_\xi \|u_T - v_T\|.$$

Thus it suffices to focus on the difference of the iterates, averaged over the randomness of  $\xi$ .

To proceed, observe that since SGD initializes its weights independently of the data (but possibly using the internal randomness  $\xi$ ), we start with  $u_0 = v_0$ . Let  $\delta_t := \|u_t - v_t\|$ . We will write a recursion for  $\delta_t$  by decomposing as follows:

$$\delta_{t+1} = \delta_{t+1} \mathbf{1}\{i_t = \kappa\} + \delta_{t+1} \mathbf{1}\{i_t \neq \kappa\}.$$

Let us first consider  $\delta_{t+1} \mathbf{1}\{i_t \neq \kappa\}$ , for which we have:

$$\begin{aligned} \delta_{t+1} \mathbf{1}\{i_t \neq \kappa\} &= \|G_{\ell, \eta_t}(u_t, z_{i_t}) - G_{\ell, \eta_t}(v_t, z'_{i_t})\| \mathbf{1}\{i_t \neq \kappa\} \\ &= \|G_{\ell, \eta_t}(u_t, z_{i_t}) - G_{\ell, \eta_t}(v_t, z_{i_t})\| \mathbf{1}\{i_t \neq \kappa\} && \text{since } z_{i_t} = z'_{i_t} \text{ when } i_t \neq \kappa \\ &\leq \|u_t - v_t\| \mathbf{1}\{i_t \neq \kappa\} && \text{using Proposition 1.66} \\ &= \delta_t \mathbf{1}\{i_t \neq \kappa\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \delta_{t+1} \mathbf{1}\{i_t = \kappa\} &= \|u_t - v_t - \eta_t(\nabla_w \ell(u_t, z_{i_t}) - \nabla_w \ell(v_t, z'_{i_t}))\| \mathbf{1}\{i_t = \kappa\} \\ &\leq (\|u_t - v_t\| + 2\eta_t L) \mathbf{1}\{i_t = \kappa\} && \text{since } \ell \text{ is } L\text{-Lipschitz} \\ &= (\delta_t + 2\eta_t L) \mathbf{1}\{i_t = \kappa\}. \end{aligned}$$

Combining these two inequalities,

$$\delta_{t+1} \leq \delta_t + 2\eta_t L \mathbf{1}\{i_t = \kappa\} \implies \delta_T \leq 2L \sum_{t=0}^{T-1} \eta_t \mathbf{1}\{i_t = \kappa\}.$$

Since this inequality holds for every  $\xi$ , by linearity of expectation,

$$\mathbb{E}_\xi[\delta_T] \leq \frac{2L}{n} \sum_{t=0}^{T-1} \eta_t.$$

The claim now follows since  $\ell$  is  $L$ -Lipschitz:

$$\sup_{z \in \mathcal{Z}} \mathbb{E}_\xi [\ell(u_T, z) - \ell(v_T, z)] \leq L \mathbb{E}_\xi \|u_T - v_T\| = L \mathbb{E}_\xi[\delta_T] \leq \frac{2L^2}{n} \sum_{t=0}^{T-1} \eta_t.$$

□

**Remark 1.68.** Note that the proof of Lemma 1.67 illustrates why the definition of uniform stability in Definition 1.61 requires that stability only hold in expectation over the internal randomness of the algorithm. If, for instance, the definition required that the algorithm was stable for almost every realization of the randomness, then SGD would not be stable under this definition, since the realization  $\xi$  which repeatedly picks the index corresponding to the differing point  $\kappa$  is not stable (despite being very unlikely).

**Sub-optimal rates for typical SGD step sizes.** Note that one drawback of Lemma 1.67 is that for the typical  $\eta_t \asymp t^{-1/2}$  step sizes usually prescribed for running SGD on convex functions, the  $\varepsilon$ -uniform-stability constant scales as  $\varepsilon = O(\sqrt{T}/n)$ . So for any non-trivial amount of steps  $T$ , we have that  $\varepsilon$  is slower than the  $n^{-1}$  scaling required so that Lemma 1.63 yields  $O(n^{-1/2})$  generalization bounds. On the other hand, if we use a more aggressive step size  $\eta_t \asymp t^{-1}$ , then we have that  $\varepsilon = O(\log T/n)$  which improves the overall generalization rate, but is generally too aggressive. If we impose extra conditions on the function  $\ell$ , specifically that it is strongly convex (Definition C.7), then it turns out we can get a stability bound which is independent of the number of iterations run.

**Exercise 1.69.** Suppose that for every  $z \in \mathbf{Z}$ , we have that  $w \mapsto \ell(w, z)$  is  $\mu$ -strongly-convex (cf. Definition C.7),  $L$ -Lipschitz, and  $\beta$ -smooth.

(a) Show that if  $\eta \leq 2/(\mu + \beta)$ , the update operator  $G_{\ell, \eta}$  becomes *contractive*, i.e., for any  $u, v \in \mathbb{R}^d$ :

$$\forall z \in \mathbf{Z}, \|G_{\ell, \eta}(u, z) - G_{\ell, \eta}(v, z)\| \leq \left(1 - \frac{\eta\mu\beta}{\mu + \beta}\right) \|u - v\|.$$

Hint: you may assume the following fact about  $\mu$ -strongly-convex and  $\beta$ -smooth functions without proof. For all  $u, v \in \mathbb{R}^d$ ,

$$\langle \nabla f(u) - \nabla f(v), u - v \rangle \geq \frac{\mu\beta}{\mu + \beta} \|u - v\|^2 + \frac{1}{\mu + \beta} \|\nabla f(u) - \nabla f(v)\|^2.$$

(b) Modify the proof of Lemma 1.67 to show the following recursion, valid whenever  $\eta_t \leq 2/(\mu + \beta)$ :

$$\delta_{t+1} \leq (1 - \eta_t m) \delta_t + 2\eta_t L \mathbf{1}\{i_t = \kappa\}, \quad m := \frac{\mu\beta}{\mu + \beta}.$$

(c) Now suppose we use a *constant* step size  $\eta_t = \eta \leq 2/(\mu + \beta)$ . Conclude that:

$$\mathbb{E}[\delta_T] \leq \frac{2L}{mn},$$

and hence SGD is  $\varepsilon$ -uniformly-stable for  $\varepsilon \leq \frac{2L^2}{mn}$ .

#### 1.4.4 Stability of Gibbs ERM

We now study the stability of an idealized algorithm which returns a sample from a particular weighted Gibbs distribution. In particular, let  $p_\beta(w; S)$  be the density of a distribution on  $\mathbb{R}^d$  defined as:

$$p_\beta(w; S) = \exp(-\beta \hat{L}_S[w]) / Z_\beta[w; S], \quad Z_\beta[w; S] = \int \exp(-\beta \hat{L}_S[w]) dw. \quad (1.35)$$

The parameter  $\beta \in \mathbb{R}$  is referred to as the *temperature* parameter. The idea behind the Gibbs ERM is to select hypothesis  $w \in \mathbb{R}^d$  which correspond to low empirical risk  $\hat{L}_S[w]$ . However, by not just selecting the true ERM, but instead allowing for a randomized selection rule, we actually do not require any convexity assumptions on  $\hat{L}_S$  to ensure uniform stability.

**Proposition 1.70** (Gibbs ERM is uniformly-stable). *Let  $A(S, \xi)$  denote the algorithm which returns a sample from the Gibbs distribution defined by Equation (1.35). Suppose that  $|\ell| \leq M$ . We have that  $A$  is  $\varepsilon$ -uniformly-stable with  $\varepsilon \leq (\exp(4\beta M/n) - 1)M$ .*

*Proof.* Let  $S, S'$  differ in only the  $i$ -th index. Note that:

$$\begin{aligned} \mathbb{E}_\xi[\ell(A(S, \xi), z) - \ell(A(S', \xi), z)] &= \int \ell(w, z) p_\beta(w; S) dw - \int \ell(w, z) p_\beta(w; S') dw \\ &= \int \ell(w, z) \left( \frac{p_\beta(w; S)}{p_\beta(w; S')} - 1 \right) p_\beta(w; S') dw \\ &\leq M \left| \frac{p_\beta(w; S)}{p_\beta(w; S')} - 1 \right|. \end{aligned}$$

To bound the likelihood ratio, we first observe that,

$$\frac{\exp(-\beta \hat{L}_S[w])}{\exp(-\beta \hat{L}_{S'}[w])} = \exp\left(-\frac{\beta}{n}(\ell(w, z_i) - \ell(w, z'_i))\right) \leq \exp(2\beta M/n).$$

On the other hand,

$$\frac{Z_\beta[w; S]}{Z_\beta[w; S']} = \frac{\int \exp(-\beta \hat{L}_S[w]) dw}{\int \exp(-\beta \hat{L}_{S'}[w]) dw} = \frac{\int \exp\left(-\frac{\beta}{n}(\ell(w, z_i) - \ell(w, z'_i))\right) \exp(-\beta \hat{L}_{S'}[w]) dw}{\int \exp(-\beta \hat{L}_{S'}[w]) dw} \leq \exp(2\beta M/n).$$

Thus, we have:

$$\max \left\{ \frac{\exp(-\beta \hat{L}_S[w])}{\exp(-\beta \hat{L}_{S'}[w])}, \frac{Z_\beta[w; S]}{Z_\beta[w; S']} \right\} \leq \exp(2\beta M/n).$$

Similar arguments show that

$$\min \left\{ \frac{\exp(-\beta \hat{L}_S[w])}{\exp(-\beta \hat{L}_{S'}[w])}, \frac{Z_\beta[w; S]}{Z_\beta[w; S']} \right\} \geq \exp(-2\beta M/n).$$

Hence,

$$\begin{aligned} \left| \frac{p_\beta(w; S)}{p_\beta(w; S')} - 1 \right| &= \left| \frac{\exp(-\beta \hat{L}_S[w])}{\exp(-\beta \hat{L}_{S'}[w])} \cdot \frac{Z_\beta[w; S']}{Z_\beta[w; S]} - 1 \right| \\ &= \max \left\{ \frac{\exp(-\beta \hat{L}_S[w])}{\exp(-\beta \hat{L}_{S'}[w])} \cdot \frac{Z_\beta[w; S']}{Z_\beta[w; S]} - 1, 1 - \frac{\exp(-\beta \hat{L}_S[w])}{\exp(-\beta \hat{L}_{S'}[w])} \cdot \frac{Z_\beta[w; S']}{Z_\beta[w; S]} \right\} \\ &\leq \max \left\{ \exp\left(\frac{4\beta M}{n}\right) - 1, 1 - \exp\left(-\frac{4\beta M}{n}\right) \right\} \\ &= \exp(4\beta M/n) - 1. \end{aligned}$$

The last equality holds since for all  $x \in \mathbb{R}$ ,

$$\frac{e^x + e^{-x}}{2} = \cosh(x) \geq 1.$$

□

**Remark 1.71.** Note that for  $\beta, M$  fixed, as  $n$  grows larger, we have that:

$$(\exp(4\beta M/n) - 1)M = \frac{4\beta M^2}{n} + o(1/n).$$

Hence the Gibbs-ERM has  $O(1/n)$  uniform stability scaling.

We conclude this section with a discussion on the computation aspects of sampling from the Gibbs distribution (1.35). Without any particular structure imposed on  $\hat{L}_S$ , the most general algorithm for sampling from the Gibbs distribution is a type of Markov Chain Monte Carlo (MCMC) algorithm known as *stochastic gradient Langevin dynamics (SGLD)*. To derive the SGLD update, the first step is to consider using *Langevin dynamics* to sample from (1.35):

$$w_{t+1} = w_t - \eta \beta \nabla \hat{L}_S[w_t] + \sqrt{2\eta} \xi_t, \quad \xi_t \sim N(0, I).$$

We will return to discussing Langevin dynamics in more detail when we study energy-based models. The SGLD update simply uses an unbiased estimator  $g_t$  such that  $\mathbb{E}[g_t | w_t] = \nabla \hat{L}_S[w_t]$  (as is done in standard SGD):

$$w_{t+1} = w_t - \eta \beta g_t + \sqrt{2\eta} \xi_t, \quad \xi_t \sim N(0, I).$$

The convergence rate of SGLD is quite an advanced topic, and we will only briefly touch on it later in this course.



## 1.5 PAC-Bayes inequalities

We now study an alternative approach to controlling generalization error. When we studied the generalization of ERM, we essentially proved uniform convergence of training and population risk over the entire hypothesis space  $\mathcal{F}$ . One downside of this approach is that it uniformly weights every hypothesis in the function class. An alternative approach is to construct a prior distribution over hypothesis, and take a “weighted” union bound over the entire function class which takes into account the prior. This is what the PAC-Bayes approach does. After we derive the main PAC-Bayes deviation inequality, we will instantiate it for a few special cases, including what are referred to as compression-based bound in the literature.

Perhaps the main conceptual difference of PAC-Bayes analysis compared with ERM analysis is that the learning algorithm is now a *distribution* over hypothesis  $\mathcal{F}$ , which we will refer to as the posterior (hence where the “Bayes” part of the name stems from). In PAC-Bayes, we start with a prior  $\pi$  over  $\mathcal{F}$ , data  $\{z_i\}_{i=1}^n$  is observed, and then a posterior distribution  $\rho_n$  over  $\mathcal{F}$  is formed by incorporating the observed data. Unlike ERM analysis, the main quantity which governs the generalization error is the KL-divergence between the posterior  $\rho_n$  and the prior  $\pi$ .

While there are many variants of PAC-Bayes inequalities, we will study one due to Cantoni, described by [Alquier \[2021\]](#).

**Theorem 1.72** (PAC-Bayes deviation inequality). *Suppose that  $|\ell| \leq B$ . For any  $\lambda > 0$  and  $\delta \in (0, 1)$ , we have with probability at least  $1 - \delta$ ,*

$$\forall \rho \in \mathcal{P}(\mathcal{F}), \mathbb{E}_{f \sim \rho}[L[f]] \leq \mathbb{E}_{f \sim \rho}[\hat{L}_n[f]] + \frac{\lambda B^2}{2n} + \frac{\text{KL}(\rho \parallel \pi) + \log(1/\delta)}{\lambda}.$$

Here,  $\mathcal{P}(\mathcal{F})$  refers to the set of probability measures over the hypothesis class  $\mathcal{F}$ .

*Proof.* The main tool used to derive the PAC-Bayes inequality is the Donsker-Varadhan representation of KL-divergence (Lemma A.3). We first set things up in order to apply Donsker-Varadhan.

We start with a fixed  $f \in \mathcal{F}$ . By Hoeffding’s inequality (Proposition B.12),

$$\mathbb{E}_S \exp(\lambda(L[f] - \hat{L}_n[f])) \leq \exp(\lambda^2 B^2 / (2n)).$$

(Here  $\mathbb{E}_S$  denotes expectation w.r.t. the data  $S = \{z_i\}$ .) Note that since this inequality holds for every fixed  $f \in \mathcal{F}$ , it also holds if we integrate it w.r.t. the prior  $\pi$  (this works because the prior *does not depend on* the data  $S$ ). Hence,

$$\mathbb{E}_{f \sim \pi} \mathbb{E}_S \exp(\lambda(L[f] - \hat{L}_n[f])) \leq \exp(\lambda^2 B^2 / (2n)).$$

Now by Fubini’s theorem, we exchange the order of expectations on the LHS,

$$\mathbb{E}_S \mathbb{E}_{f \sim \pi} \exp(\lambda(L[f] - \hat{L}_n[f])) \leq \exp(\lambda^2 B^2 / (2n)).$$

The next step is to apply the Donsker-Varadhan lemma (Lemma A.3). Specifically, we will use the following identity which holds for any fixed  $g$ ,

$$\log \mathbb{E}_\pi[\exp(g)] = \sup_{\rho \in \mathcal{P}(\mathcal{F})} \{ \mathbb{E}_\rho[g] - \text{KL}(\rho \parallel \pi) \}.$$

Applying this identity to  $g = \lambda(L[f] - \hat{L}_n[f])$  and taking an expectation w.r.t.  $S$  on both sides,

$$\mathbb{E}_S \mathbb{E}_\pi \exp(\lambda(L[f] - \hat{L}_n[f])) = \mathbb{E}_S \exp \left( \sup_{\rho \in \mathcal{P}(\mathcal{F})} \left\{ \lambda \mathbb{E}_\rho[L[f] - \hat{L}_n[f]] - \text{KL}(\rho \parallel \pi) \right\} \right) \leq \exp(\lambda^2 B^2 / (2n)).$$

Hence for any  $t > 0$ , by the Laplace transform method (Proposition B.2),

$$\mathbb{P}_S \left\{ \sup_{\rho \in \mathcal{P}(\mathcal{F})} \left\{ \lambda \mathbb{E}_\rho[L[f] - \hat{L}_n[f]] - \text{KL}(\rho \parallel \pi) \right\} - \frac{\lambda^2 B^2}{2n} \geq t \right\}$$

$$\leq e^{-t} \mathbb{E}_S \exp \left( \sup_{\rho \in \mathcal{P}(\mathcal{F})} \left\{ \lambda \mathbb{E}_\rho[L[f]] - \hat{L}_n[f] - \text{KL}(\rho \parallel \pi) \right\} - \frac{\lambda^2 B^2}{2n} \right) \leq e^{-t}.$$

The claim now follows by setting  $t = \log(1/\delta)$ .  $\square$

**Remark 1.73.** The order of the quantifier for  $\lambda$  in Theorem 1.72 is important. The statement requires that you choose a  $\lambda$  first, and then the resulting deviation inequality holds. Note in particular, that the following is in general **not true** as a direct consequence of Theorem 1.72: for all  $\rho \in \mathcal{P}(\mathcal{F})$ ,

$$\begin{aligned} \mathbb{E}_{f \sim \rho}[L[f]] &\leq \mathbb{E}_{f \sim \rho}[\hat{L}_n[f]] + \inf_{\lambda > 0} \left[ \frac{\lambda B^2}{2n} + \frac{\text{KL}(\rho \parallel \pi) + \log(1/\delta)}{\lambda} \right] \\ &= \mathbb{E}_{f \sim \rho}[\hat{L}_n[f]] + B \sqrt{\frac{2(\text{KL}(\rho \parallel \pi) + \log(1/\delta))}{n}}. \end{aligned} \quad (1.36)$$

This is because the optimizing  $\lambda$  above, which is

$$\lambda = \sqrt{\frac{\text{KL}(\rho \parallel \pi) + \log(1/\delta)}{B^2/(2n)}},$$

is a function of the posterior  $\rho$ . Hence (1.36) does not hold for every posterior  $\rho$  generally. However, there are some special cases where an inequality of the form (1.36) does hold over a subset of posteriors  $\mathcal{P}' \subset \mathcal{P}(\mathcal{F})$ . In particular, if every  $\rho \in \mathcal{P}'$  has the same value of  $\text{KL}(\rho \parallel \pi)$ , then (1.36) does indeed hold for every  $\rho \in \mathcal{P}'$  (convince yourself why). Furthermore, we can optimize  $\lambda$  over any fixed range with only a small penalty, as shown next.

**Exercise 1.74.** Suppose that  $|\ell| \leq B$ . Fix constants  $0 < \alpha \leq \beta < \infty$  and  $\delta \in (0, 1)$ . Show that with probability at least  $1 - \delta$ ,

$$\forall \rho \in \mathcal{P}(\mathcal{F}), \quad \mathbb{E}_{f \sim \rho}[L[f]] \leq \mathbb{E}_{f \sim \rho}[\hat{L}_n[f]] + \inf_{\lambda \in [\alpha, \beta]} \left\{ \frac{\lambda B^2}{2n} + \frac{\text{KL}(\rho \parallel \pi) + \log((\lfloor \log(\beta/\alpha) \rfloor + 1)/\delta)}{\lambda/e} \right\}.$$

**Gibbs posterior.** Since Theorem 1.72 holds for every  $\rho \in \mathcal{P}(\mathcal{F})$ , it is natural to ask the question which posterior minimizes the bound on the RHS. In particular, what is the solution to:

$$\begin{aligned} \arg \min_{\rho \in \mathcal{P}(\mathcal{F})} \left\{ \mathbb{E}_{f \sim \rho}[\hat{L}_n[f]] + \frac{\lambda B^2}{2n} + \frac{\text{KL}(\rho \parallel \pi) + \log(1/\delta)}{\lambda} \right\} &= \arg \min_{\rho \in \mathcal{P}(\mathcal{F})} \left\{ \mathbb{E}_{f \sim \rho}[\hat{L}_n[f]] + \frac{\text{KL}(\rho \parallel \pi)}{\lambda} \right\} \\ &= \arg \max_{\rho \in \mathcal{P}(\mathcal{F})} \left\{ -\mathbb{E}_{f \sim \rho}[\hat{L}_n[f]] - \frac{\text{KL}(\rho \parallel \pi)}{\lambda} \right\} \\ &= \arg \max_{\rho \in \mathcal{P}(\mathcal{F})} \left\{ \mathbb{E}_{f \sim \rho}[-\lambda \hat{L}_n[f]] - \text{KL}(\rho \parallel \pi) \right\}. \end{aligned}$$

But this question is precisely answered by the Donsker-Varadhan lemma (Lemma A.3), which states that the solution to the maximization above has density w.r.t. the prior  $\pi$  defined as:

$$\frac{d\rho_\lambda}{d\pi}(f) = \frac{\exp(-\lambda \hat{L}_n[f])}{\mathbb{E}_{f \sim \pi}[\exp(-\lambda \hat{L}_n[f])]}.$$

Compare this with the Gibbs ERM distribution introduced in Section 1.4.4 (cf. Equation (1.35)), which is essentially a special case of the above display with a uniform prior.

We now consider some specific instantiations of Theorem 1.72.

**Finite hypothesis classes.** Let us first consider to what extent PAC-Bayes generalizes the uniform convergence analysis of ERM. First, we see that the PAC-Bayes inequality implies the uniform convergence result from Proposition 1.14. Indeed, suppose  $\mathcal{F}$  is finite, and put  $M = |\mathcal{F}|$ . Let us consider the uniform prior on  $\mathcal{F}$ , i.e.,

$$\pi(f) = \frac{1}{M}.$$

Now consider a posterior  $\rho_f = \delta_f$ , which is a Dirac mass on a fixed  $f \in \mathcal{F}$ . An immediate computation yields that:

$$\text{KL}(\rho_f \parallel \pi) = \log \left( \frac{1}{\pi(f)} \right) = \log M.$$

Note that since  $\text{KL}(\rho_f \parallel \pi)$  is independent of  $f$ , then we can apply the argument from Remark 1.73 (using the subset of posteriors  $\mathcal{P}' = \{\rho_f \mid f \in \mathcal{F}\}$ ) to conclude that, with probability at least  $1 - \delta$ ,

$$\forall f \in \mathcal{F}, L[f] \leq \hat{L}_n[f] + B \sqrt{\frac{2 \log(M/\delta)}{n}}.$$

This is precisely the statement of Proposition 1.14.

**Compression based generalization.** Let us now explore an idea from the deep learning literature, known as compression-based bounds. There is a vast literature here, so we will only cover the very basics. The following discussion is motivated by Zhou et al. [2019]. The intuition is that many deep learning models, while vastly overparameterized, can actually be compressed (in a lossy manner) but still retain much of its performance. The PAC-Bayes framework suggests a way to utilize this insight in order to derive sharper generalization bounds, by building a prior distribution which upweights compressible models. Let us describe a simplistic model of this. Let  $M$  denote a max bound on the hypothesis we will consider, denoting the number of bits used to represent the model (hence there are  $|\mathcal{F}| = \sum_{i=1}^M 2^i = 2(2^M - 1)$  possible models in total). For each model  $f \in \mathcal{F}$ , we let  $|f| \in \{1, \dots, M\}$  denote the number of bits used to represent  $f$ . With this notation, let us consider a prior  $\pi$  on  $\mathcal{F}$  such that:

$$\pi(f) = \frac{2^{-|f|}}{M}.$$

This prior distribution is described by the following sampling scheme: (a) choose an index  $i \sim \text{Unif}(\{1, \dots, M\})$ , and (b) choose an  $f$  with  $|f| = i$  uniformly at random.

Now if we let  $\rho_f$  denote a point mass on model  $f$ , a quick computation yields:

$$\text{KL}(\rho_f \parallel \pi) = \log \left( \frac{1}{\pi(f)} \right) = |f| \log 2 + \log M.$$

This prior can now be used to construct the following compression-bound.

**Exercise 1.75.** Suppose that  $\mathcal{F}$  is as described above. Show that the PAC-Bayes deviation bound (Theorem 1.72) implies that with probability at least  $1 - \delta$ ,

$$\forall f \in \mathcal{F}, L[f] \leq \hat{L}_n[f] + cB \sqrt{\frac{|f| + \log(M/\delta)}{n}}.$$

Here,  $c > 0$  is a universal constant.

*Proof.* Let us define, for  $k \in \{1, \dots, M\}$ ,

$$\lambda_k := \sqrt{\frac{k \log 2 + \log M + \log(M/\delta)}{B^2/(2n)}}.$$

Now, invoke the PAC-Bayes result  $M$  times, with  $\delta \leftarrow \delta/M$  and  $\lambda \leftarrow \lambda_k$ . By a union bound, we have that with probability at least  $1 - \delta$ ,

$$\begin{aligned} \forall f \in \mathcal{F}, L[f] &\leq \hat{L}_n[f] + \min_{k=1, \dots, M} \left\{ \frac{\lambda_k B^2}{2n} + \frac{|f| \log 2 + \log M + \log(M/\delta)}{\lambda_k} \right\} \\ &\leq \hat{L}_n[f] + \frac{\lambda_{|f|} B^2}{2n} + \frac{|f| \log 2 + \log M + \log(M/\delta)}{\lambda_{|f|}} \\ &= \hat{L}_n[f] + B \sqrt{\frac{2(|f| \log 2 + \log M + \log(M/\delta))}{n}} \\ &\leq \hat{L}_n[f] + 2B \sqrt{\frac{|f| + \log(M/\delta)}{n}}. \end{aligned}$$

□

Applying the standard uniform convergence bound for finite hypothesis classes (Proposition 1.14) to this setting yields a bound of:

$$\max_{f \in \mathcal{F}} \text{gen}[f] \leq B \sqrt{\frac{2 \log(|\mathcal{F}|/\delta)}{n}} \leq cB \sqrt{\frac{M + \log(1/\delta)}{n}},$$

for an absolute constant  $c$ . Compared this with Exercise 1.75 where compressible models (e.g.  $|f| \ll M$ ) enjoy a much tighter generalization bound. Of course, in the worst case when  $|f| \asymp M$ , the Exercise 1.75 yields orderwise the same bound.

**Gaussian priors.** We now consider the case when our function class  $\mathcal{F}$  is parametric of the form:

$$\mathcal{F} = \{f_\theta(x) \mid \theta \in \mathbb{R}^d\},$$

where  $f_\theta$  is some arbitrary parametric function of  $\theta$ . In this case, a very natural prior is to set  $\pi = N(\mu, \sigma^2 I)$ , for some fixed  $\mu, \sigma$ . Then, considering posteriors of the form  $\rho_\theta = N(\theta, \sigma^2 I)$ , we can compute, using the formula for the KL divergence of two Gaussians (cf. Proposition A.4):

$$\text{KL}(\rho_\theta \parallel \pi) = \frac{1}{2\sigma^2} \|\theta - \mu\|^2.$$

Therefore, the PAC-Bayes guarantee tells us that for any  $\lambda > 0$ , with probability at least  $1 - \delta$ ,

$$\forall \theta \in \mathbb{R}^d, \mathbb{E}_{w \sim N(0, I)}[L[f_{\theta+\sigma w}]] \leq \mathbb{E}_{w \sim N(0, I)}[\hat{L}_n[f_{\theta+\sigma w}]] + \frac{\lambda B^2}{2n} + \frac{\frac{1}{2\sigma^2} \|\theta - \mu\|^2 + \log(1/\delta)}{\lambda}. \quad (1.37)$$

Thus, we see that the  $\ell_2$  distance from our prior parameter vector  $\mu$  controls our generalization error. Indeed, there is *no a-prior* bound on  $\theta$  here (which is typically needed for Rademacher type analysis). Instead, we simply pay a larger price for generalization the more our weight vector  $\theta$  deviates from  $\mu$ .

One way to utilize the guarantee in (1.37) is to use the RHS of (1.37) as the training objective. Indeed, since the RHS gives us a bound on the generalization error, and since it holds for every  $\theta \in \mathbb{R}^d$ , then we can minimize it as a way of selecting the final model  $\theta$ . This approach was taken recently in Dziugaite and Roy [2017] which allowed them to compute a non-trivial (i.e., a resulting bound less than one for the zero-one loss) generalization bound for a deep network trained on MNIST. Many more recent follow-up works have employed more PAC-Bayes tricks (beyond the score of this course) to extend the models and datasets for which a non-trivial generalization bound can be computed.

**Data splits and trainable priors.** We now turn to the question of how to choose the prior. One approach is to split the dataset in a way that the prior is actually learnable from data. Let us again consider the parametric family  $\mathcal{F} = \{f_\theta(x) \mid \theta \in \mathbb{R}^d\}$ , and consider Gaussian priors of the form  $\pi = N(\mu, \sigma^2 I)$ . Suppose we are given  $n$  training examples  $S_n = \{(x_i, y_i)\}_{i=1}^n$ . One valid way to choose the  $\mu$  is to split the dataset into two. That is, for any  $m \in \{1, \dots, n-1\}$ , let us split  $S_n$  into two sets:

$$S_{1:m} = \{(x_i, y_i)\}_{i=1}^m, \quad S_{m+1:n} = \{(x_i, y_i)\}_{i=m+1}^n.$$

We use  $S_{1:m}$  to learn  $\mu$ , for example by solving an ERM problem:

$$\hat{\mu}_{1:m} \in \arg \min_{\theta \in \mathbb{R}^d} \hat{L}_{S_{1:m}}[f_\theta] = \frac{1}{m} \sum_{i=1}^m \ell(f_\theta(x_i), y_i).$$

We then apply Theorem 1.72 using the prior  $\pi = N(\hat{\mu}_{1:m}, \sigma^2 I)$ , to conclude that with probability at least  $1 - \delta$  over  $S_{m+1:n}$ ,

$$\forall \theta \in \mathbb{R}^d, \mathbb{E}_{w \sim N(0, I)}[L[f_{\theta+\sigma w}]] \leq \mathbb{E}_{w \sim N(0, I)}[\hat{L}_n[f_{\theta+\sigma w}]] + \frac{\lambda B^2}{2(n-m)} + \frac{\frac{1}{2\sigma^2} \|\theta - \hat{\mu}_{1:m}\|^2 + \log(1/\delta)}{\lambda}.$$

## 2 Unsupervised learning: generative modeling

We now turn our attention towards unsupervised learning. Unsupervised learning is a very broad topic. To focus our discussion, we will consider a subfield of unsupervised learning, specifically generative modeling. Let us first setup the learning problem. As before, we have covariates  $x \in \mathbf{X}$  and an underlying distribution  $p(x)$  over these covariates. (Again, we will typically assume Euclidean structure on  $\mathbf{X}$ , namely  $\mathbf{X} \subseteq \mathbb{R}^d$ , although many of the ideas we will discuss transfer over the case when  $\mathbf{X}$  is countable.) However, we no longer have any labels  $y \in \mathbf{Y}$ , making this an unsupervised learning setup. The goal then is no longer prediction, but rather sampling. Indeed, given a set of samples  $x_1, \dots, x_n$ , the goal is to learn a sampler  $\hat{p}_n$  such that samples drawn from  $\hat{p}_n$  approximate samples drawn from  $p(x)$  as close as possible.

### 2.1 Energy based models

Our first attempt at solving this problem will be using classic energy based models. This is a classic idea dating back to Yann LeCun's earlier work [LeCun et al., 2006]. Suppose that the distribution  $p(x)$  over  $\mathbf{X}$  has a density, and with slight abuse of notation we let  $p(x)$  denote this density. Energy based models (EBMs) model the *un-normalized density*, i.e., for a function class  $\mathcal{F}$  mapping  $\mathbf{X} \mapsto \mathbb{R}$ ,

$$p(x) \propto \exp(f(x)), \quad f \in \mathcal{F}. \quad (2.1)$$

The important property of this model is that there is no constraint placed on  $f$  that enforces normalization, i.e., generically we have that

$$Z[f] := \int \exp(f(x)) dx \neq 1.$$

In reference to the statistical physics motivations behind EBMs, the function  $Z[f]$  is called the *partition function*. Of course for a given  $f \in \mathcal{F}$ , the resulting normalized density is hence:

$$p_f(x) := \exp(f(x))/Z[f].$$

Since computing  $Z[f]$  is typically not tractable, as it involves high dimensional integration, the exact form  $p_f$  is to be thought of as a purely analytical tool. As stated before, the loss function we want to minimize is now the KL-divergence. This motivates the following maximum likelihood (MLE) procedure at the population level:

$$f_\star = \arg \min_{f \in \mathcal{F}} \text{KL}(p \parallel p_f) = \arg \min_{f \in \mathcal{F}} \{-H[p] - \mathbb{E}_p[\log p_f]\} = \arg \min_{f \in \mathcal{F}} \{-\mathbb{E}_p[\log p_f]\} =: \arg \min_{f \in \mathcal{F}} L[f]. \quad (2.2)$$

Here,  $H[p] := -\mathbb{E}_p[\log p]$  is the (differential) entropy of  $p$ . Note the loss  $L[f]$  is referred to as the *cross-entropy loss*. Let us now suppose that  $\mathcal{F} = \{f_\theta \mid \theta \in \Theta\}$  where  $\Theta$  is a subset of Euclidean space, and compute what the gradient of the cross-entropy loss is. Slightly overloading notation with  $L[\theta] = L[f_\theta]$ ,  $p_\theta = p_{f_\theta}$ , and  $Z[\theta] = Z[f_\theta]$ , a quick computation yields:

$$\begin{aligned}\nabla_\theta L[\theta] &= -\nabla_\theta \int \log p_\theta(x) p(x) \, dx \\ &\stackrel{(a)}{=} -\int \nabla_\theta \log p_\theta(x) p(x) \, dx \\ &= -\int \nabla_\theta [f_\theta(x) - \log Z[\theta]] p(x) \, dx \\ &= -\mathbb{E}_p[\nabla_\theta f_\theta] + \nabla_\theta \log Z[\theta].\end{aligned}$$

In (a), we exchanged the order of differentiation and integration.<sup>6</sup> Next we compute:

$$\begin{aligned}\nabla_\theta \log Z[\theta] &= \frac{1}{Z[\theta]} \nabla_\theta \int \exp(f_\theta(x)) \, dx \stackrel{(a)}{=} \frac{1}{Z[\theta]} \int \nabla_\theta \exp(f_\theta(x)) \, dx \\ &= \frac{1}{Z[\theta]} \int \nabla_\theta f_\theta(x) \exp(f_\theta(x)) \, dx = \mathbb{E}_{p_\theta}[\nabla_\theta f_\theta].\end{aligned}$$

Again in (a) we exchanged the order of differentiation and integration. Above, the blue color  $p_\theta$  is used to emphasize that the sampling is done w.r.t. the model  $p_\theta$  and not the true data generating distribution  $p$ . Combining these calculations,

$$\nabla_\theta L[\theta] = -(\mathbb{E}_p[\nabla_\theta f_\theta] - \mathbb{E}_{p_\theta}[\nabla_\theta f_\theta]). \quad (2.3)$$

Thus, we see that to form a stochastic estimate of this gradient, we need to *sample* from the model  $p_\theta$ . We now turn to how to generate these samples. Note that if  $\mathbf{X}$  is finite, then sampling is simple, since computing  $p_f(x)$  exactly for each  $x \in \mathbf{X}$  is feasible. So we focus on the setting when  $\mathbf{X}$  is not finite. In particular, we will focus on the case when  $\mathbf{X}$  has Euclidean structure.

### 2.1.1 Markov Chain Monte Carlo sampling

Here, suppose that  $\mathbf{X} \subseteq \mathbb{R}^d$ . The standard algorithm for sampling from  $p_\theta$  is Langevin sampling, which sets up a Markov chain with a stationary distribution equal to  $p_\theta$ . A classic fact from statistical physics (by way of the Fokker-Planck equation) states that the following Itô stochastic differential equation (SDE), known as *Langevin dynamics*, has its stationary measure equal to  $p_\theta$ :

$$dX_t = \nabla_x \log p_\theta(X_t) dt + \sqrt{2} dB_t. \quad (2.4)$$

Here,  $(B_t)_{t \geq 0}$  is standard Brownian motion in  $\mathbb{R}^d$ . For those unfamiliar with SDEs, we can think of the process  $(X_t)_{t \geq 0}$  described in Equation (2.4) as the continuous time limit as  $\eta \rightarrow 0$  of the following discrete-time stochastic process parameterized by  $\eta > 0$ , which does not require any fancy machinery to describe:<sup>7</sup>

$$X_{t+1} = X_t + \eta \nabla_x \log p_\theta(X_t) + \sqrt{2\eta} W_t. \quad (2.5)$$

Here,  $W_t \sim N(0, I)$  is drawn independently for  $t \in \mathbb{N}$ . Note the appearance of  $\log p_\theta$  above is critical, since it allows the partition function, which is not a function of  $x$ , to disappear in the gradient:

$$\nabla_x \log p_\theta(x) = \nabla_x [f_\theta(x) - \log Z[\theta]] = \nabla_x f_\theta(x).$$

<sup>6</sup>Technically, we should check some integrability conditions to ensure this is valid (using Lebesgue's dominated convergence theorem), but we will ignore this technicality and assume this step holds.

<sup>7</sup>This is known as the Euler–Maruyama discretization of the Langevin SDE.

The object  $\nabla_x \log p_\theta(x)$  is an important enough quantity that we give it a particular name: the *score function*. Let us denote  $X_t$  from the discrete-time process (2.5) as:

$$X_t = \text{MCMC}_\eta(f_\theta, X_0, t),$$

with the dependence on the random  $\{W_t\}$ 's made implicit.

**Stochastic gradient estimate for  $L[\theta]$ .** Recall that the population level gradient for the cross-entropy loss  $L[\theta] = -\mathbb{E}_p[\log p_\theta]$  is given by  $\nabla_\theta L[\theta] = -(\mathbb{E}_p[\nabla_\theta f_\theta] - \mathbb{E}_{p_\theta}[\nabla_\theta f_\theta])$ . We now have the necessary notation to write down a stochastic estimator for this gradient. This estimator has two hyperparameters,  $\eta > 0$  and  $\kappa \in \mathbb{N}_+$ . Consider the following estimator

$$\hat{g}_n := -(\nabla_\theta f_\theta(x_i) - \nabla_\theta f_\theta(\text{MCMC}_\eta(f_\theta, 0, \kappa))), \quad i \sim \text{Unif}(\{1, \dots, n\}).$$

This estimator approximately satisfies  $\mathbb{E}[\hat{g}_n] \approx \nabla_\theta L[\theta]$ , although quantifying the bias of  $\hat{g}_n$  is non-trivial and out of scope for this course (we will briefly discuss the sources of error in a moment). Furthermore, it is not necessary to start  $X_0 = 0$  in the MCMC; more sophisticated variants use warm-starting techniques (e.g. using the last sample generated) to improve MCMC convergence. See [Song and Kingma \[2021\]](#) for more advanced techniques.

**Gaussian example: Langevin sampling.** To make Langevin sampling more concrete, let us consider what happens when we apply (2.5) to a Gaussian score function. Of course, for a Gaussian distribution, we do not require Langevin sampling to generate samples, but nevertheless we can calculate what Langevin sampling applied to a Gaussian score function does. Specifically, suppose that  $p(x) = N(\mu, \sigma^2 I)$ . A quick calculation yields that the score function is:

$$\nabla_x \log p(x) = -\sigma^{-2}(x - \mu).$$

Therefore, the discrete-time Langevin sampling process (2.5) simplifies to:

$$\begin{aligned} x_{t+1} &= x_t + \eta \nabla_x \log p(x) + \sqrt{2\eta} g_t & [g_t \sim N(0, I), x_0 = 0] \\ &= (1 - \eta\sigma^{-2})x_t + \eta\sigma^{-2}\mu + \sqrt{2\eta} g_t. \end{aligned}$$

Unrolling this recursion yields:

$$x_t = \sum_{i=0}^{t-1} (1 - \eta\sigma^{-2})^{t-1-i} [\eta\sigma^{-2}\mu + \sqrt{2\eta} g_i].$$

Immediately, we see that  $x_t$  is the result of an affine transformation of Gaussian random variables, and is hence Gaussian. Thus, to characterize the distribution of  $x_t$ , it suffices to compute its mean and covariance. Let us assume  $\eta < \sigma^2$ . Starting with its mean,

$$\mathbb{E}[x_t] = \sum_{i=0}^{t-1} (1 - \eta\sigma^{-2})^i \eta\sigma^{-2}\mu = [1 - (1 - \eta\sigma^{-2})^t]\mu.$$

On the other hand, the covariance is:

$$\text{Cov}(x_t) = 2\eta \sum_{i=0}^{t-1} (1 - \eta\sigma^{-2})^{2i} = 2\eta \frac{1 - (1 - \eta\sigma^{-2})^{2t}}{1 - (1 - \eta\sigma^{-2})^2} = \frac{2\eta[1 - (1 - \eta\sigma^{-2})^{2t}]}{2\eta\sigma^{-2} - (\eta\sigma^{-2})^2}.$$

As  $t \rightarrow \infty$ , the process  $x_t$  converges to its stationary measure  $N(\mu, \sigma_\eta^2 I)$  with

$$\sigma_\eta^2 := \frac{2\sigma^2}{2 - \eta\sigma^{-2}}.$$

Furthermore, we also have that  $\lim_{\eta \rightarrow 0} \sigma_\eta^2 = \sigma^2$ , and hence we see that  $\text{Law}(x_t)$  converges to the correct distribution  $N(\mu, \sigma^2 I)$  as both the number of Langevin steps  $t \rightarrow \infty$ , and the Langevin step size  $\eta \rightarrow 0$ . Of course, we see that for finite  $t < \infty$  and non-zero  $\eta > 0$ , the distribution  $\text{Law}(x_t)$  is not equal to the target  $N(\mu, \sigma^2 I)$ , and hence this introduces bias in the sampling.

**Gaussian example: EBM learning.** Let us instantiate an idealized version of EBM learning for clarity, where we ignore the bias in MCMC sampling. Suppose the true distribution is an isotropic Gaussian with unknown mean  $\mu \in \mathbb{R}^d$ , i.e.,  $p(x) = N(\mu, I)$  (we set  $\sigma^2 = 1$  here to simplify the following expressions). Suppose furthermore that our energy model class is:

$$\mathcal{F} = \left\{ x \mapsto f_\theta(x) := -\frac{1}{2} \|x - \theta\|^2 \mid \theta \in \mathbb{R}^d \right\}.$$

That is, we model the log density of an isotropic Gaussian with a non-zero mean up to the normalization constant. Now, let us suppose we are in an idealized streaming setting, where  $x_0, x_1, \dots$  are a stream of iid examples from  $p(x)$ . Following Equation (2.3), we set:

$$\theta_{t+1} = \theta_t + \eta_t (\nabla_\theta f_{\theta_t}(x_t) - \nabla_\theta f_{\theta_t}(\tilde{x}_t)), \quad \tilde{x}_t \sim N(\theta_t, I), \quad \theta_0 = 0.$$

Note in this iteration above, we are using  $-(\nabla_\theta f_{\theta_t}(x_t) - \nabla_\theta f_{\theta_t}(\tilde{x}_t))$  as an estimate of the cross-entropy gradient (2.3); this equation assumes the MCMC sampling is exact (i.e.,  $t \rightarrow \infty$  and  $\eta \rightarrow 0$ ). Now, using the fact that  $\nabla_\theta f_\theta(x) = x - \theta$ ,  $x_t = \mu + g_t$ , and  $\tilde{x}_t = \theta_t + \tilde{g}_t$ , we have that:

$$\begin{aligned} \theta_{t+1} - \mu &= \theta_t - \mu + \eta_t (x_t - \theta_t - (\tilde{x}_t - \theta_t)) \\ &= \theta_t - \mu + \eta_t (\mu - \theta_t + g_t - \tilde{g}_t) \\ &= (1 - \eta_t)(\theta_t - \mu) + \eta_t \hat{g}_t. \end{aligned} \quad \text{defining } \hat{g}_t := g_t - \tilde{g}_t.$$

Letting  $\Delta_t := \theta_t - \mu$ , we have the identity

$$\Delta_{t+1} = (1 - \eta_t)\Delta_t + \eta_t \hat{g}_t.$$

Unrolling the recursion,

$$\Delta_t = \sum_{i=0}^{t-1} \left[ \prod_{j=i+1}^{t-1} (1 - \eta_j) \right] \eta_i \hat{g}_i.$$

Therefore, since all the  $\hat{g}_i$ 's are mutually independent,

$$\mathbb{E} \|\Delta_t\|^2 = 2 \sum_{i=0}^{t-1} \left[ \prod_{j=i+1}^{t-1} (1 - \eta_j) \right]^2 \eta_i^2.$$

Now let us choose  $\eta_i = 1/(i+1)$  for all  $i \in \mathbb{N}$ . With this choice,

$$\prod_{i=k}^{t-1} (1 - \eta_i) = \frac{k}{t}$$

Hence we can compute,

$$\sum_{i=0}^{t-1} \left[ \prod_{j=i+1}^{t-1} (1 - \eta_j) \right]^2 \eta_i^2 = \sum_{i=0}^{t-1} \left( \frac{i+1}{t} \right)^2 \frac{1}{(i+1)^2} = \frac{1}{t}.$$

Therefore,

$$\mathbb{E} \|\Delta_t\|^2 = \frac{2}{t}.$$

So we see that the estimator  $\theta_t$  converges to the true mean  $\mu$  at a  $O(1/t)$  rate.



**Divergence, Laplacian, and integration by parts.** For what follows, we will make use of the following notation and tools from multivariable calculus. For a vector field  $v : \mathbb{R}^d \mapsto \mathbb{R}^d$ , the divergence  $\nabla \cdot v$  is:

$$(\nabla \cdot v)(x) = \sum_{i=1}^d \frac{\partial v_i(x)}{\partial x_i}.$$

Next, for a function  $f : \mathbb{R}^d \mapsto \mathbb{R}$ , the Laplacian  $\Delta f$  is:

$$(\Delta f)(x) = \sum_{i=1}^d \frac{\partial^2 f(x)}{\partial x_i^2}.$$

(Check the following identity that  $\nabla \cdot \nabla f = \Delta f$ .)

A key technical tool we will use is the classic integration by parts identity, which states that for a smooth function  $f : \mathbb{R}^d \mapsto \mathbb{R}$  and smooth vector field  $v : \mathbb{R}^d \mapsto \mathbb{R}^d$  whose behavior at infinity vanishes,<sup>8</sup>

$$\int \langle v(x), \nabla f(x) \rangle dx = - \int \nabla \cdot v(x) f(x) dx. \quad (2.6)$$

**Stationarity of Langevin dynamics (2.4).** To check that  $p_\theta$  is the stationary measure of the Langevin dynamics (2.4), we will appeal to the Fokker-Planck (FP) equation, which states the following. Suppose we have a SDE of the form:

$$dX_t = v(X_t) dt + \sqrt{2} dB_t.$$

Then, the probability density function  $\rho_t$  of  $X_t$  satisfies the following partial differential equation (PDE):<sup>9</sup>

$$\frac{\partial \rho_t}{\partial t} = -\nabla \cdot (\rho_t v) + \Delta \rho_t. \quad (2.7)$$

Now, plugging (2.4) into the FP equation (2.7), we have:

$$\begin{aligned} \frac{\partial p_\theta}{\partial t} &= -\nabla \cdot (p_\theta \nabla \log p_\theta) + \Delta p_\theta \\ &= -\nabla \cdot (p_\theta \nabla \log p_\theta) + \nabla \cdot \nabla p_\theta && \text{since } \Delta p_\theta = \nabla \cdot \nabla p_\theta \\ &= -\nabla \cdot (p_\theta \nabla \log p_\theta) + \nabla \cdot (p_\theta \nabla \log p_\theta) && \text{since } p_\theta \nabla \log p_\theta = \nabla p_\theta \\ &= 0. \end{aligned}$$

Therefore, tells us that the density of  $p_\theta$  remains invariant over time under the SDE (2.4). In other words, if we start  $X_0 \sim p_\theta$ , then  $\text{Law}(X_t) = p_\theta$  for all  $t \geq 0$ .

**Sources of MCMC error.** We briefly describe the sources of error which arise in using MCMC to generate samples from  $p_\theta$ . We first consider the continuous-limit SDE (2.4). Via the Fokker-Planck equation, we previous showed that  $p_\theta$  is the stationary measure of the Langevin dynamics (2.4). As a consequence of this, we have that:

$$\lim_{t \rightarrow \infty} \text{KL}(\text{Law}(X_t) \parallel p_\theta) = 0. \quad (2.8)$$

However, for any *finite*  $t > 0$ ,  $\text{KL}(\text{Law}(X_t) \parallel p_\theta)$  is not necessarily zero. Furthermore, the convergence speed of (2.8), without any further assumptions on  $p_\theta$ , is known to be prohibitively slow in the worst case. This happens in particular when the distribution  $p_\theta$  is *multi-modal*. As an extreme case, consider a distribution with two modes that have equal probability, which are well-separated. Since both modes are equally likely,

<sup>8</sup>We will not be overly concerned with the technical conditions here.

<sup>9</sup>The fact that  $p_t$  has a well-defined density respect to the Lebesgue measure on  $\mathbb{R}^d$  is a consequence of the fact that the Brownian motion driving the process  $(X_t)_{t \geq 0}$  is full dimensional.

the Langevin dynamics should spend equal time traversing between the two modes, in order to accurately reflect the underlying distribution. However, since Langevin dynamics noisily follow the gradient of the log density (cf. Equation (2.4)), traversing across modes can take a long time (exponential in the ambient dimension  $d$ ). Thus, in the worst case, convergence of Langevin dynamics to the stationary distribution  $p_\theta$  requires running the SDE for very large  $t$ . This presents one source of error.

The next source of error involves the relationship between the continuous-limit SDE (2.4) and the discrete-time stochastic process (2.5). To avoid confusion, let  $\{Z_t\}_{t \in \mathbb{N}}$  denote the discrete-time process (2.5). Conceptually, we can think of there being a natural correspondence between  $Z_k \leftrightarrow X_{k\eta}$ . However, the discrepancy between  $Z_k$  and  $X_{k\eta}$  grows as  $\eta$  increases; for any finite  $\eta > 0$  there is always discretization error. This presents another source of error.

### 2.1.2 Convergence of Langevin dynamics in Fisher information

Despite our previous discussion on the convergence of Langevin dynamics, it turns out we can prove a rather general positive result for a particular divergence measure. This result will give us some intuition for the type of resulting distributions which we can reasonably expect Langevin dynamics to converge to. This section is motivated by the results in Balasubramanian et al. [2022].

We define the following divergence over smooth measures (that is, measures with differentiable densities). For two smooth measures  $\mu, \nu$ , the *Fisher information* of  $\mu$  with respect to  $\nu$  is defined as:

$$\text{FI}(\mu \parallel \nu) := \mathbb{E}_\mu \|\nabla \log(\mu/\nu)\|^2. \quad (2.9)$$

We consider the following Langevin dynamics (here  $p$  is an arbitrary smooth density),

$$dX_t = \nabla \log p(X_t) dt + \sqrt{2} dB_t.$$

Recall that  $p$  is the stationary measure of the process  $(X_t)_{t \geq 0}$ . Let  $p_t = \text{Law}(X_t)$  denote both the measure and density of the Langevin dynamics at time  $t$ . It turns out that under the Fisher information divergence, we have that  $\text{FI}(p_t \parallel p) \rightarrow 0$  almost immediately.

The key identity is that the negative Fisher information is the time derivative of the KL-divergence  $\text{KL}(p_t \parallel p)$ :

$$\frac{d}{dt} \text{KL}(p_t \parallel p) = -\text{FI}(p_t \parallel p). \quad (2.10)$$

We will prove this fundamental identity in a moment. But taking it for granted, we first see that (2.10) confirms the stationarity of  $p$ , since if we set  $p_t = p$ , then  $\frac{d}{dt} \text{KL}(p_t \parallel p) = 0$ . Furthermore, by the fundamental theorem of calculus, we conclude from (2.10) that for any  $T > 0$ ,

$$\text{KL}(p_T \parallel p) = \text{KL}(p_0 \parallel p) - \int_0^T \text{FI}(p_t \parallel p) dt.$$

Since the Fisher information is always non-negative, we immediately see a nice property of Langevin dynamics, which is that  $\text{KL}(p_T \parallel p) \leq \text{KL}(p_0 \parallel p)$  (the KL-divergence never increases as a result of running Langevin). Furthermore, since  $\text{KL}(p_T \parallel p) \geq 0$ , re-arranging the above display and dividing by  $T$  yields:

$$\frac{1}{T} \int_0^T \text{FI}(p_t \parallel p) dt \leq \frac{\text{KL}(p_0 \parallel p)}{T}.$$

Now consider the mixture measure  $\bar{p}_T := \frac{1}{T} \int_0^T p_t dt$ . By the convexity of  $\mu \mapsto \text{FI}(\mu \parallel \nu)$ ,<sup>10</sup> Jensen's inequality yields that:

$$\text{FI}(\bar{p}_T \parallel p) \leq \frac{1}{T} \int_0^T \text{FI}(p_t \parallel p) dt \leq \frac{\text{KL}(p_0 \parallel p)}{T}.$$

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<sup>10</sup>This follows by first writing  $\text{FI}(\mu \parallel \nu) = \int \frac{\|\nabla \rho\|^2}{\rho} d\nu$  for  $\rho = d\mu/d\nu$ , and then using the convexity of  $(x, y) \mapsto \|x\|^2/y$  on  $\mathbb{R}^d \times \mathbb{R}_{>0}$ .

Note that this mixture measure  $\bar{p}_T$  can be sampled from by sampling  $t \sim \text{Unif}([0, T])$ , and outputting  $X_t$ . Hence, we see that if we set  $T \geq \text{KL}(p_0 \parallel p)/\varepsilon$ , then  $\text{FI}(\bar{p}_T \parallel p) \leq \varepsilon$ .

The question now becomes, what exactly does this guarantee of  $\text{FI}(\bar{p}_T \parallel p) = O(1/T)$  mean. To shed some insight, Balasubramanian et al. [2022] constructed an example of two distributions, parameterized by an scalar parameter  $m$ , such that the TV-distance is bounded away from zero uniformly for all  $m$ , but the Fisher information tends to zero. The example is:

$$\begin{aligned}\pi_m &= \frac{1}{2}N(-m, 1) + \frac{1}{2}N(m, 1), \\ \mu_m &= \frac{3}{4}N(-m, 1) + \frac{1}{4}N(m, 1).\end{aligned}$$

Balasubramanian et al. [2022, Proposition 1] shows that:

$$\inf_{m \geq 1/80} \|\mu_m - \pi_m\|_{\text{tv}} \geq \frac{1}{800}, \quad \text{FI}(\mu_m \parallel \pi_m) \leq 4m^2 \exp(-m^2/2).$$

The intuition here is that, for the Fisher information to be small, it suffices to get the local structure correct (i.e., the  $N(\pm m, 1)$  modes), but the global weights can be incorrect. On the other hand, TV-distance requires that the global weights are correct (in addition to the local structure).

**Proof of Equation (2.10).** We now finish this section with a proof the time derivative identity (2.10). The key is to use the Fokker-Planck equation (2.7) in conjunction with integration by parts:

$$\begin{aligned}\frac{d}{dt} \text{KL}(p_t \parallel p) &= \partial_t \int p_t \log \frac{p_t}{p} dx \\ &= \int \partial_t \left( p_t \log \frac{p_t}{p} \right) dx \\ &= \int (\partial_t p_t) \log \frac{p_t}{p} dx + \int p_t \cdot \partial_t \log p_t dx \\ &\stackrel{(a)}{=} \int (\partial_t p_t) \log \frac{p_t}{p} dx \\ &\stackrel{(b)}{=} \int \nabla \cdot \left( p_t \nabla \log \frac{p_t}{p} \right) \log \frac{p_t}{p} dx && \text{Fokker-Planck equation} \\ &= - \int \left\| \nabla \log \frac{p_t}{p} \right\|^2 p_t dx && \text{integration by parts} \\ &= -\mathbb{E}_{p_t} \left\| \nabla \log \frac{p_t}{p} \right\|^2 \\ &= -\text{FI}(p_t \parallel p).\end{aligned}\tag{2.11}$$

To see (a) above, note that:

$$\int p_t \cdot \partial_t \log p_t dx = \int \partial_t p_t dx = \partial_t \int p_t dx = 0.$$

To see how (b) arises from the Fokker-Planck equation, from (2.7),

$$\begin{aligned}\partial_t p_t &= -\nabla \cdot (p_t \nabla \log p) + \Delta p_t \\ &= -\nabla \cdot [p_t \nabla \log p - \nabla p_t] && \text{since } \Delta p_t = \nabla \cdot \nabla p_t \\ &= -\nabla \cdot (p_t \nabla \log p - p_t \nabla \log p_t) \\ &= \nabla \cdot \left( p_t \nabla \log \frac{p_t}{p} \right).\end{aligned}$$

### 2.1.3 Exponential convergence of Langevin dynamics under log-Sobolev inequalities

**Note:** this is an advanced topic we will not discuss in the course, but it is included for readers who are interested in diving further into this topic.

In the previous section, we saw that with no extra assumptions on  $p$ , we could show that Langevin dynamics drives the Fisher information to zero at a  $O(1/T)$  rate. However, we also saw that having small Fisher information does not necessarily mean the global structure of the distribution is preserved. In this section, we introduce an extra assumption on the distribution  $p$  to ensure that we have convergence of Langevin in KL-divergence, and furthermore at an exponential rate.

Equation (2.10) gives us a hint as to what condition to enforce on  $p_t$  and  $p$  to ensure convergence. In particular, suppose we are assured that for some  $\alpha > 0$  and all  $t \geq 0$ ,

$$\text{FI}(p_t \parallel p) = \mathbb{E}_{p_t} \left\| \nabla \log \frac{p_t}{p} \right\|^2 \geq \alpha \cdot \text{KL}(p_t \parallel p). \quad (2.12)$$

Then, combining inequality (2.12) with (2.10),

$$\forall t \geq 0, \quad \frac{d}{dt} \text{KL}(p_t \parallel p) \leq -\alpha \cdot \text{KL}(p_t \parallel p),$$

and therefore by the comparison lemma for ODEs,

$$\forall t \geq 0, \quad \text{KL}(p_t \parallel p) \leq e^{-\alpha t} \cdot \text{KL}(p_0 \parallel p).$$

Now the question remains, what assumption on  $p_t$  and  $p$  ensures the condition (2.12) holds? It turns out that this condition is ensured by a standard functional inequality in probability theory known as the *log-Sobolev* inequality. Log-Sobolev inequalities have a rich history in probability theory which we will not have time to discuss. We will simply state its definition and see how it implies (2.12). The first definition we need is that of *concentration entropy* (not to be confused with the information theoretic notion of entropy). For a measure  $\mu$  and a non-negative random variable  $X$ , the concentration entropy  $\text{Ent}_\mu(X)$  (or entropy for short) of  $X$  is defined as:

$$\text{Ent}_\mu(X) := \mathbb{E}_\mu[X \log X] - \mathbb{E}_\mu[X] \cdot \log \mathbb{E}_\mu[X].$$

A measure  $\mu$  satisfies the *log-Sobolev inequality* with constant  $C > 0$  if for all smooth functions  $f : X \mapsto \mathbb{R}$ ,

$$\text{Ent}_\mu(f^2) \leq 2C \cdot \mathbb{E}_\mu \|\nabla f\|^2. \quad (2.13)$$

While this definition is seemingly arbitrary, it turns out that it directly implies (2.12). To see this, first let us compute the entropy of the function  $f = \sqrt{p_t/p}$  under the measure  $p$ :

$$\text{Ent}_p \left( \frac{p_t}{p} \right) = \mathbb{E}_p \left[ \frac{p_t}{p} \log \frac{p_t}{p} \right] - \mathbb{E}_p \left[ \frac{p_t}{p} \right] \log \mathbb{E}_p \left[ \frac{p_t}{p} \right] = \mathbb{E}_p \left[ \frac{p_t}{p} \log \frac{p_t}{p} \right] = \text{KL}(p_t \parallel p).$$

Therefore, if we suppose that the stationary measure  $p$  satisfies the log-Sobolev inequality (LSI) with constant  $C$ , then applying the LSI inequality to  $f = \sqrt{p_t/p}$ ,

$$\text{Ent}_p \left( \frac{p_t}{p} \right) = \text{KL}(p_t \parallel p) \leq 2C \cdot \mathbb{E}_p \left\| \nabla \sqrt{\frac{p_t}{p}} \right\|^2 = \frac{C}{2} \cdot \mathbb{E}_{p_t} \left\| \nabla \log \frac{p_t}{p} \right\|^2.$$

Hence, this implies (2.12) with  $\alpha = 2/C$ . That is, we have shown that if  $p$  satisfies the LSI inequality with constant  $C$ , then for all  $t > 0$ ,

$$\text{KL}(p_t \parallel p) \leq e^{-2t/C} \cdot \text{KL}(p_0 \parallel p).$$

That is, if we want to ensure that  $\text{KL}(p_t \parallel p) \leq \varepsilon$ , it suffices to ensure that  $t \geq \frac{C}{2} \log \left( \frac{\text{KL}(p_0 \parallel p)}{\varepsilon} \right)$ .

**Which distributions satisfy log-Sobolev inequalities?** In general, verifying the functional inequality (2.13) for a given distribution  $p$  is not a simple task. There is, however, a very clean sufficient condition, known as the Bakry-Émery criterion. Suppose that  $p = \exp(-U)$ , and suppose that  $U$  is  $\alpha$  strongly-convex. Then  $p$  satisfies the LSI inequality with constant  $C = 1/\alpha$ . The proof of this result is non-trivial and out of the scope of this course: see e.g., Bakry et al. [2014] for a proof. Distributions of this form are known as log-concave distributions, and include e.g., Gaussian distributions.

## 2.2 Score matching

The MCMC sampling dynamics (2.5) motivate an alternative loss which does not rely on the the cross-entropy loss (2.2). Indeed, the idea is to *directly* learn the score function  $\nabla_x \log p(x)$ . It turns out that we can setup a very simple loss function, called the score matching loss, which bypasses the requirement to sample from the current model in order to implement the gradient of the cross-entropy loss (cf. Equation (2.3)).

**Proposition 2.1** (Score matching loss). *Let  $p(x)$  denote a distribution over  $\mathbf{X} \subset \mathbb{R}^d$ , and suppose its density is sufficiently smooth. For any smooth vector field  $f : \mathbb{R}^d \mapsto \mathbb{R}^d$  vanishing at infinity,*

$$\mathbb{E}_p \|\nabla \log p - f\|^2 = \mathbb{E}_p [\|f\|^2 + 2\nabla \cdot f + \|\nabla \log p\|^2] .$$

*Proof.* First, we use the identity integration by parts to derive,

$$\begin{aligned} \mathbb{E}_p [\langle \nabla \log p, f \rangle] &= \int \langle \nabla \log p(x), f(x) \rangle p(x) \, dx \\ &= \int \langle \nabla p(x), f(x) \rangle \, dx = - \int \nabla \cdot f(x) p(x) \, dx = -\mathbb{E}_p [\nabla \cdot f] . \end{aligned}$$

We now complete the proof as follows:

$$\mathbb{E}_p \|\nabla \log p - f\|^2 = \mathbb{E}_p [\|f\|^2 - 2\langle \nabla \log p, f \rangle + \|\nabla \log p\|^2] = \mathbb{E}_p [\|f\|^2 + 2\nabla \cdot f + \|\nabla \log p\|^2] .$$

□

An immediate consequence of this result is that:

$$\arg \min_{f \in \mathcal{F}} \mathbb{E}_p \|\nabla \log p - f\|^2 = \arg \min_{f \in \mathcal{F}} L[f] := \mathbb{E}_p [\|f\|^2 + 2\nabla \cdot f] .$$

The loss  $L[f]$  is called the (population) score matching loss. Given samples  $x_1, \dots, x_n \sim p(x)$  iid, we can form the empirical score matching loss:

$$\hat{L}_n[f] := \frac{1}{n} \sum_{i=1}^n [\|f(x_i)\|^2 + 2\nabla \cdot f(x_i)] . \quad (2.14)$$

Note that unlike the cross-entropy loss, computing gradients of the score matching loss is straightforward given a reasonable auto-differentiation library.

**Curl-free vector fields.** One issue which arises from directly representing the vector field corresponding to the score function, is that in general it is difficult to directly ensure that a vector field is *curl-free*, meaning that the Jacobian of the vector field is a symmetric matrix. However, we do know that the true score function has this property, since

$$\frac{d\nabla \log p(x)}{dx} = \nabla^2 \log p(x),$$

and Hessian matrices are symmetric. One simple way to enforce this symmetry condition is to indirectly parameterize a vector field through the gradient of a scalar potential, e.g. using a function class of the form:

$$\mathcal{F} = \{x \mapsto \nabla \phi(x) \mid \phi : \mathbf{X} \mapsto \mathbb{R}, \phi \in \mathcal{G}\}, \quad (2.15)$$

where  $\mathcal{G}$  is a function class of smooth potential functions (functions mapping  $\mathbf{X} \mapsto \mathbb{R}$ ). Note that this parameterization also comes with the advantage that it directly searches for an energy model (2.1), showing that we can actually think of the score matching loss as an alternative loss for learning an energy model (e.g., score matching and EBMs are not incompatible).

While the parameterization (2.15) does enforce the curl-free condition, it is not often used in practice since it comes with additional computational overhead. Specifically, we see that the empirical loss  $\hat{L}_n[f]$  (cf. Equation (2.14)) now involves the Laplacian of  $\phi$ . Hence, taking the gradient of  $\hat{L}_n[f]$  requires taking *three* derivatives, which can become a computational bottleneck. Therefore in practice the lack of curl-free vector fields is often simply ignored, and it does not seem to result in any performance loss.

### 2.2.1 Score matching with Gaussians

Let us now work through a specific example of score matching for intuition, instantiating it on Gaussian distributions.

**Isotropic Gaussians.** Suppose that  $p(x) = N(\mu, I)$ , where  $\mu \in \mathbb{R}^d$  is unknown (but the covariance  $I$  is known). A quick calculation yields that the score function of  $p(x)$  is the following affine function:

$$\nabla \log p(x) = -(x - \mu).$$

Suppose our family of score functions  $\mathcal{F}$  is the set of Gaussian score functions corresponding to different means, e.g.,

$$\mathcal{F} = \{x \mapsto -(x - \theta) \mid \theta \in \mathbb{R}^d\}.$$

Then we have that:

$$L[\theta] = \mathbb{E}_{x \sim N(\mu, I)} [\|x - \theta\|^2 - 2\Delta \cdot (x - \theta)] = \mathbb{E}_{x \sim N(\mu, I)} \|x - \theta\|^2 - 2d = \|\theta - \mu\|^2 - d.$$

Clearly at the population level,  $\arg \min_{\theta \in \mathbb{R}^d} L[\theta] = \mu$ , which is correct. Now at the empirical level, we have:

$$\hat{L}_n[\theta] = \frac{1}{n} \sum_{i=1}^n \|x_i - \theta\|^2 - 2d.$$

Therefore, the empirical risk minimizer of  $\hat{L}_n[\theta]$  is the sample mean (check this!):

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n x_i.$$

**General multivariate Gaussians.** Now let us suppose that  $p(x) = N(\mu, \Sigma)$ , where  $\Sigma$  is a positive definite matrix. Here, we assume both  $\mu$  and  $\Sigma$  are unknown. The true score function of  $p(x)$  now takes the form:

$$\nabla \log p(x) = -\Sigma^{-1}(x - \mu).$$

Consider now using the family of score functions  $\mathcal{F}$  as:

$$\mathcal{F} = \{x \mapsto -G(x - \theta) \mid \theta \in \mathbb{R}^d, G \in \mathbb{R}^{d \times d}\}.$$

Note that even though  $G$  needs to be symmetric and positive definite to be correct, we will not constrain it to be symmetric positive definite in our formulation. A quick computation yields that the divergence of an  $f \in \mathcal{F}$ :

$$\nabla \cdot (-G(x - \theta)) = -\nabla \cdot (Gx) = -\text{tr}(G).$$

So the empirical score matching loss is:

$$\hat{L}_n[\theta, G] = \frac{1}{n} \sum_{i=1}^n \|G(x_i - \theta)\|^2 - 2 \text{tr}(G).$$

Let us look at the stationary points of this objective  $\hat{L}_n[\theta, G]$ . Computing the gradients,

$$\begin{aligned}\nabla_{\theta} \hat{L}_n[\theta, G] &= \frac{2}{n} \sum_{i=1}^n G^{\top} G(x_i - \theta) = 2G^{\top} G \cdot \left[ \frac{1}{n} \sum_{i=1}^n x_i - \theta \right], \\ \nabla_G \hat{L}_n[\theta, G] &= \frac{2}{n} \sum_{i=1}^n G(x_i - \theta)(x_i - \theta)^{\top} - 2I = 2G \cdot \left[ \frac{1}{n} \sum_{i=1}^n (x_i - \theta)(x_i - \theta)^{\top} \right] - 2I.\end{aligned}$$

Thus, we see that the following pair  $(\hat{\theta}_n, \hat{G}_n)$  is a stationary point, assuming the inverse is well defined,

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^n x_i, \quad \hat{G}_n = \left[ \frac{1}{n} \sum_{i=1}^n (x_i - \hat{\theta}_n)(x_i - \hat{\theta}_n)^{\top} \right]^{-1}.$$

This essentially reduces to a “plug-in estimate” of the mean and covariance of the data points  $x_1, \dots, x_n$ .

### 2.2.2 Score matching with exponential families

We can generalize beyond the Gaussian example we saw previously. Suppose that  $p(x)$  has the form of an exponential family:

$$p(x) = \exp(\langle \phi(x), \theta_{\star} \rangle) / Z[\theta_{\star}], \quad Z[\theta] = \int \exp(\langle \phi(x), \theta \rangle) dx, \quad \theta \in \Theta \subseteq \mathbb{R}^p.$$

Here, we assume the sufficient statistic  $\phi : \mathbf{X} \mapsto \mathbb{R}^p$  is known. Let the component-wise functions of  $\phi$  be denoted as  $\phi = (\phi_1, \dots, \phi_p)$ , and let  $J_{\phi}(x) \in \mathbb{R}^{p \times d}$  denote the Jacobian of  $\phi$ , i.e.,  $J_{\phi} = \frac{d\phi}{dx}$ . A quick calculation yields:

$$\nabla \log p(x) = J_{\phi}^{\top}(x) \theta_{\star}.$$

As we did in the realizable case, let us consider the following function class:

$$\mathcal{F} = \{x \mapsto J_{\phi}^{\top}(x) \theta \mid \theta \in \Theta\}.$$

The score matching loss has the following least-squares form given as follows:

$$L[\theta] = \mathbb{E}_p [\|J_{\phi}^{\top}(x) \theta\|^2 + 2\langle h_{\phi}(x), \theta \rangle], \quad h_{\phi}(x) = (\Delta \phi_1(x), \dots, \Delta \phi_p(x)).$$

To see this, we observe that  $J_{\phi}^{\top}(x) \theta = \sum_{i=1}^p \nabla \phi_i(x) \theta_i$ , and therefore:

$$\nabla \cdot J_{\phi}^{\top}(x) \theta = \sum_{i=1}^p \nabla \cdot \nabla \phi_i(x) \theta_i = \sum_{i=1}^p \Delta \phi_i(x) \theta_i = \langle h_{\phi}(x), \theta \rangle.$$

Hence the empirical loss  $\hat{L}_n[\theta]$  has the following form:

$$\hat{L}_n[\theta] = \frac{1}{n} \sum_{i=1}^n [\|J_{\phi}^{\top}(x_i) \theta\|^2 + 2\langle h_{\phi}(x_i), \theta \rangle].$$

This is a least-squares problem with one solution given by:

$$\hat{\theta}_n = - \left( \sum_{i=1}^n J_{\phi}(x_i) J_{\phi}^{\top}(x_i) \right)^{\dagger} \left( \sum_{i=1}^n h_{\phi}(x_i) \right).$$

Here,  $(\cdot)^{\dagger}$  denotes the pseudo-inverse of a matrix.

### 2.2.3 Sliced score matching

Despite the empirical score matching loss  $\hat{L}_n[f]$  not requiring MCMC sampling to differentiate, computing the divergence of  $f$  does introduce some computational overhead. However, we can utilize randomized estimators to mitigate this issue. These ideas fall under the family of sliced score matching.

Let  $f : \mathbb{R}^d \mapsto \mathbb{R}$  be a smooth function, and let  $\partial f(x) \in \mathbb{R}^{d \times d}$  denote its Jacobian matrix evaluated at  $x$ . The first observation is the following identity which holds for every  $x$ :

$$\mathbb{E}_v \langle v, \partial f(x) \cdot v \rangle = \nabla \cdot f(x), \quad v \sim N(0, I).$$

To see this, observe that:

$$\mathbb{E}_v \langle v, \partial f(x) \cdot v \rangle = \mathbb{E}_v \text{tr}(v^\top \partial f(x) \cdot v) = \mathbb{E}_v \text{tr}(\partial f(x) \cdot v v^\top) = \text{tr}(\partial f(x)) = \sum_{i=1}^d \frac{\partial f_i}{\partial x_i}(x) = \nabla \cdot f(x).$$

Thus, we can replace the empirical score matching loss with a randomized variant, while still preserving the expectation:

$$\hat{L}_{n,1}[f] := \frac{1}{n} \sum_{i=1}^n [\|f(x_i)\|^2 + 2\langle v_i, \partial f(x_i) \cdot v_i \rangle], \quad v_i \sim N(0, I).$$

The computational advantages of this loss over  $\hat{L}_n[f]$  are as follows. In order to compute  $\nabla \cdot f(x)$ , one has to make  $d$  Jacobian vector product (JVP) calls. However, in order to compute  $\langle v, \partial f(x_i) \cdot v \rangle$ , only one JVP call is needed. Hence, this sliced score matching loss offers a factor of  $d$  savings in computation, which is non-trivial when  $d$  is large. Furthermore, we can actually create a family of losses  $\hat{L}_{n,k}$ :

$$\hat{L}_{n,k}[f] := \frac{1}{n} \sum_{i=1}^n \left[ \|f(x_i)\|^2 + 2 \sum_{j=1}^k \langle v_i^{(j)}, \partial f(x_i) \cdot v_i^{(j)} \rangle \right], \quad v_i^{(j)} \sim N(0, I).$$

Choosing  $k \in \{1, \dots, d\}$  allows us to trade off computation for variance reduction.

**Exercise 2.2.** Sliced score matching takes advantage of the fact that for a fixed matrix  $M \in \mathbb{R}^{d \times d}$ , the following identity holds:

$$\mathbb{E}_v \langle v, Mv \rangle = \text{tr}(M), \quad v \sim N(0, I).$$

Therefore,  $\langle v, Mv \rangle$  can be used as an unbiased estimate of the trace of  $M$ . Come with a few other non-Gaussian distributions such that the same identity holds, and run some computer simulations to compare the relative performance of the  $\langle v, Mv \rangle$  estimator using different distributions.

### 2.2.4 Disadvantages of score matching

While the score matching loss is an improvement over the cross-entropy loss, there are a few disadvantages:

- (a) The divergence calculation in the score matching loss (2.14) adds non-negligible computational overhead. Indeed, computing a gradient of  $\hat{L}_n[f]$  requires at least two derivatives (three if using the scalar potential parameterization (2.15)). While the score matching loss is a clear improvement over the cross-entropy loss, there is still some computational burden.
- (b) Score matching requires that the underlying distribution  $p(x)$  to be learned is absolutely continuous w.r.t. the Lebesgue measure. This means that  $p(x)$  is not allowed to be supported on a lower dimensional manifold. However, when the event space  $\mathbf{X}$  corresponds to say, RGB images, this may not be a realistic assumption. Note that this disadvantage is also present with EBMs, which also starts by postulating the existence of a density on  $\mathbf{X}$ .



- (c) Despite the improve score matching loss over the cross-entropy loss, sampling from a learned score function still involves MCMC sampling (cf. Section 2.1.1), and therefore comes with all the disadvantages of MCMC sampling.

Our goal in the subsequent sections will be to address these concerns. Foreshadowing a bit, it turns out denoising diffusion models actually address all these shortcomings (quite cleverly actually). However, in order to fully appreciate the ingenuity behind diffusion models, we will proceed more or less in chronological order and consider the solutions which came before in the literature (which all contain fundamental ideas in generative modeling which are still extremely relevant).

## 2.3 Denoising score matching

One important step towards addressing the issues brought up in Section 2.2.4 regarding score matching is the idea of *denoising score matching*, which we will cover here. The idea is quite straightforward and intuitive. Regardless of what properties the distribution  $p(x)$ , the noised distribution  $p_\sigma = p \star N(0, \sigma^2 I)$ , where  $\star$  denotes convolution, has the following properties:

- (a) For any  $\sigma > 0$ ,  $p_\sigma$  is fully supported on  $\mathbb{R}^d$ , meaning that it has a well-defined density function on  $\mathbb{R}^d$ . This holds even if  $p(x)$  does not.
- (b) As long as  $\sigma$  is not too large, then  $p$  and  $p_\sigma$  should not be too different in some sense.

Given these observations, denoising score matching simply applies score matching to  $p_\sigma$  instead of  $p$ . However, because of the special structure of  $p_\sigma$ , the resulting loss function simplifies even more compared to the vanilla score matching loss. To see this, we let  $q_\sigma(\tilde{x} | x)$  denote the density of the distribution  $N(x, \sigma^2 I)$ , so that we can write the density of  $p_\sigma$  as:

$$p_\sigma(\tilde{x}) = \int q_\sigma(\tilde{x} | x) p(x) dx = \mathbb{E}_{x \sim p}[q_\sigma(\tilde{x} | x)].$$

With this notation, letting  $(x, \tilde{x}) \sim p \times q_\sigma$  denote sampling from the joint distribution with  $x \sim p(x)$  and  $\tilde{x} \sim q_\sigma(\tilde{x} | x)$ , observe that for any  $f : \mathbf{X} \mapsto \mathbb{R}$ ,

$$\begin{aligned} \mathbb{E}_{p_\sigma} \langle f, \nabla \log p_\sigma \rangle &= \int \langle f(\tilde{x}), \nabla \log p_\sigma(\tilde{x}) \rangle p_\sigma(\tilde{x}) d\tilde{x} \\ &= \int \langle f(\tilde{x}), \nabla p_\sigma(\tilde{x}) \rangle d\tilde{x} \\ &= \int \langle f(\tilde{x}), \nabla \mathbb{E}_{x \sim p}[q_\sigma(\tilde{x} | x)] \rangle d\tilde{x} \\ &= \int \langle f(\tilde{x}), \mathbb{E}_{x \sim p}[\nabla q_\sigma(\tilde{x} | x)] \rangle d\tilde{x} \\ &= \mathbb{E}_{x \sim p} \int \langle f(\tilde{x}), \nabla q_\sigma(\tilde{x} | x) \rangle d\tilde{x} \\ &= \mathbb{E}_{x \sim p} \int \langle f(\tilde{x}), \nabla \log q_\sigma(\tilde{x} | x) \rangle q_\sigma(\tilde{x} | x) d\tilde{x} \\ &= \mathbb{E}_{(x, \tilde{x}) \sim p \times q_\sigma} \langle f(\tilde{x}), \nabla \log q_\sigma(\tilde{x} | x) \rangle. \end{aligned} \tag{2.16}$$

With this calculation, letting  $(x, \tilde{x}) \sim p \times q_\sigma$ , the score matching loss on  $p_\sigma$  simplifies to:

$$\begin{aligned} \mathbb{E}_{p_\sigma} \|f - \nabla \log p_\sigma\|^2 &= \mathbb{E}_{p_\sigma} [\|f\|^2 + \|\nabla \log p_\sigma\|^2] - 2\mathbb{E}_{p_\sigma} \langle f, \nabla \log p_\sigma \rangle \\ &= \mathbb{E}_{p_\sigma} [\|f\|^2 + \|\nabla \log p_\sigma\|^2] - 2\mathbb{E}_{(x, \tilde{x})} \langle f(\tilde{x}), \nabla \log q_\sigma(\tilde{x} | x) \rangle \quad \text{using (2.16)} \\ &= \mathbb{E}_{p_\sigma} [\|f\|^2 + \|\nabla \log p_\sigma\|^2] - 2\mathbb{E}_{(x, \tilde{x})} \langle f(\tilde{x}), \nabla \log q_\sigma(\tilde{x} | x) \rangle \\ &\quad + \mathbb{E}_{(x, \tilde{x})} [\|\nabla \log q_\sigma(\tilde{x} | x)\|^2 - \|\nabla \log q_\sigma(\tilde{x} | x)\|^2] \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}_{(x, \tilde{x})} \|f(\tilde{x}) - \nabla \log q_\sigma(\tilde{x} | x)\|^2 \\
&\quad + \mathbb{E}_{(x, \tilde{x})} [\|\nabla \log p_\sigma(\tilde{x})\|^2 - \|\nabla \log q_\sigma(\tilde{x} | x)\|^2].
\end{aligned}
\tag{2.16}$$

Hence for any function class  $\mathcal{F}$ ,

$$\arg \min_{f \in \mathcal{F}} \mathbb{E}_{p_\sigma} \|f - \nabla \log p_\sigma\|^2 = \arg \min_{f \in \mathcal{F}} L[f] := \mathbb{E}_{(x, \tilde{x})} \|f(\tilde{x}) - \nabla \log q_\sigma(\tilde{x} | x)\|^2. \tag{2.17}$$

The loss  $L[f]$  is known as the *denoising loss*. Its empirical counterpart  $\hat{L}_n[f]$  can be constructed by taking  $\{x_i\}_{i=1}^n$  drawn i.i.d. from  $p(x)$ , taking  $\{w_i\}_{i=1}^n$  drawn i.i.d. from  $N(0, I)$  (and independent of the  $x_i$ 's), and then setting:

$$\hat{L}_n[f] = \frac{1}{n} \sum_{i=1}^n \|f(x_i + \sigma w_i) - \nabla \log q_\sigma(x_i + \sigma w_i | x_i)\|^2.$$

Observe that the denoising loss resolves disadvantages (a) and (b) described in Section 2.2.4. Indeed, (a) is resolved since the denoising loss is a standard least-squares loss; no divergence calculations are needed anymore. Furthermore, (b) is resolved since adding Gaussian noise makes  $p_\sigma$  full dimensional even with  $p$  is not.

Actually, we can create a family of losses  $\hat{L}_{n,k}[f]$  by drawing more noisy samples  $\tilde{x}$ . That is, letting  $\{w_i^{(j)}\}_{i=1, j=1}^{n,k}$  be i.i.d. from  $N(0, I)$ , let

$$\hat{L}_{n,k}[f] = \frac{1}{nk} \sum_{i=1}^n \sum_{j=1}^k \|f(x_i + \sigma w_i^{(j)}) - \nabla \log q_\sigma(x_i + \sigma w_i^{(j)} | x_i)\|^2.$$

Let us see where the term “denoising” comes from. A quick calculation yields that (recall from the derivation that the gradient below is with respect to  $\tilde{x}$ ):

$$\nabla \log q_\sigma(\tilde{x} | x) = -\frac{1}{\sigma^2}(\tilde{x} - x),$$

which is proportional to the (negative) noise vector added to make the noised sample  $\tilde{x}$  from the pure sample  $x$ . Thus, we can interpret the denoising loss as trying to, given a noised sample  $\tilde{x}$ , predict the direction  $f(\tilde{x})$  such that  $\tilde{x} + \sigma^2 f(\tilde{x}) \approx x$  is a denoised version of  $\tilde{x}$ .

**Exercise 2.3.** Use the identity established in Equation (2.17) to prove *Tweedie’s formula*:

$$\nabla \log p_\sigma(\tilde{x}) = \frac{\mathbb{E}[x | \tilde{x}] - \tilde{x}}{\sigma^2}.$$

To be clear, here  $\mathbb{E}[x | \tilde{x}]$  is understood to be the function  $\tilde{x} \mapsto \mathbb{E}[X | \tilde{X} = \tilde{x}]$  with  $(X, \tilde{X}) \sim p \times q_\sigma$ .

## 2.4 Latent models and Variational Autoencoders

We now turn to a probabilistic model based on latent representations. As before, suppose our event space is  $\mathbf{X} = \mathbb{R}^d$ . However, we posit the following probabilistic model for generation. Let  $\mathbf{Z} = \mathbb{R}^\ell$  denote a latent space. Then,

$$\begin{aligned}
z &\sim \nu(z), \\
x &\sim p(x | z).
\end{aligned}$$

Here,  $\nu(z)$  is a *prior* distribution over the latent space  $\mathbf{Z}$ , and the model  $p(x | z)$  *encodes* the latent into the event space  $\mathbf{X}$ . The marginal distribution  $p(x)$  is then given through integrating the latent, i.e.,

$$p(x) = \int p(x | z) \nu(z) dz. \tag{2.18}$$

In this model, we assume the prior is known and fix, and as a further simplification we even assume it is Gaussian, i.e.,  $\nu = N(0, I)$ . The encoder distribution is not known and to be learned.

Our first attempt to learn  $p(x)$  is through maximum likelihood. That is, we parameterize a family of encoder models  $\mathcal{F} = \{p_\theta(x | z) | \theta \in \Theta\}$ , and consider

$$p_{\theta_*} = \arg \max_{p_\theta(\cdot | \cdot) \in \mathcal{F}} \mathbb{E}_p[\log p_\theta(x)] := \mathbb{E}_p \left[ \log \int p_\theta(x | z) \nu(z) dz \right].$$

The notation above means that for a encoder model  $p_\theta(x | z)$ , the marginal  $p_\theta(x)$  is derived via the integration in Equation (2.18). In general, solving this optimization problem is very difficult since the marginal distribution  $p_\theta(x)$  most likely does not have a closed form. We will resolve this issue by deriving a family of lower bounds on  $\log p_\theta(x)$ , parameterized by a family of auxiliary distributions, and then *jointly* optimize over both the encoder  $p_\theta(x | z)$  and this family of auxiliary distributions. This general procedure is known as a *variational lower bound*; in the context of latent models it is also referred to as the *evidence lower bound* (ELBO).

To derive the ELBO, let  $p(x | z)$  be an arbitrary encoder. Let us introduce an auxiliary distribution  $q(z | x)$ . Then, we can cleverly insert this auxiliary distribution inside (2.18) as follows:

$$\begin{aligned} \log p(x) &= \log \int p(x | z) \nu(z) dz \\ &= \log \int \frac{p(x | z) \nu(z)}{q(z | x)} q(z | x) dz \\ &= \log \mathbb{E}_{q(z|x)} \left[ \frac{p(x | z) \nu(z)}{q(z | x)} \right] \\ &\geq \mathbb{E}_{q(z|x)} \log \left[ \frac{p(x | z) \nu(z)}{q(z | x)} \right] && \text{Jensen's inequality} \\ &= \mathbb{E}_{q(z|x)} [\log p(x | z)] - \text{KL}(q(\cdot | x) \parallel \nu). && \text{Definition of KL} \end{aligned} \quad (2.19)$$

Equation (2.19) is the ELBO lower bound which is maximized in variational inference. At first, it is not immediately clear that the ELBO is any more tractable than computing  $\log p(x)$ . However, we can make it tractable by limiting the family of decoders and encoders we consider. In particular, let us parameterize both the encoder and decoder as multivariate Gaussians with learned means and variances. Specifically:

$$p_\theta(x | z) = N(d_{\mu, \theta}(z), \text{ddiag}(d_{\sigma^2, \theta}(z))), \quad q_\phi(z | x) = N(e_{\mu, \phi}(x), \text{ddiag}(e_{\sigma^2, \phi}(x))). \quad (2.20)$$

Here,  $d_{\mu, \theta} : \mathbb{R}^\ell \mapsto \mathbb{R}^d$ ,  $d_{\sigma^2, \theta} : \mathbb{R}^\ell \mapsto \mathbb{R}_{>0}^d$ ,  $e_{\mu, \phi} : \mathbb{R}^d \mapsto \mathbb{R}^\ell$ ,  $e_{\sigma^2, \phi} : \mathbb{R}^d \mapsto \mathbb{R}_{>0}^\ell$ , and  $\text{ddiag}$  maps a vector of inputs to a diagonal matrix where the inputs are located on the diagonal. With this parameterization, we observe firstly that, using the closed-form formula for the KL-divergence between two multivariate Gaussians (cf. Proposition A.4):

$$\text{KL}(q_\phi(\cdot | x) \parallel N(0, I)) = \frac{1}{2} \left\{ \langle \mathbf{1}, e_{\sigma^2, \phi}(x) \rangle + \|e_{\mu, \phi}(x)\|^2 - \ell - \langle \mathbf{1}, \log e_{\sigma^2, \phi}(x) \rangle \right\},$$

where  $\log$  above is applied entry-wise. Next, we apply what is known as the *reparameterization trick* to compute the first term. First, we observe that  $\log p_\theta(x | z)$  has a closed form. Second,

$$\mathbb{E}_{q_\phi(z|x)} [\log p_\theta(x | z)] = \mathbb{E}_g [\log p_\theta(x | z = e_{\mu, \phi}(x) + \text{ddiag}(e_{\sigma^2, \phi}^{1/2}(x))g)], \quad g \sim N(0, I).$$

Hence,

$$\mathbb{E}_p[\log p_\theta(x)] \geq \mathbb{E}_{p, g} \left[ \log p_\theta(x | z = e_{\mu, \phi}(x) + \text{ddiag}(e_{\sigma^2, \phi}^{1/2}(x))g) - \text{KL}(q_\phi(\cdot | x) \parallel N(0, I)) \right].$$

Given i.i.d. samples  $x_1, \dots, x_n$ , the RHS loss can be replaced with its empirical counterpart:

$$\hat{L}_n(\theta, \phi) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}_g \left[ \log p_\theta(x_i | z = e_{\mu, \phi}(x_i) + \text{ddiag}(e_{\sigma^2, \phi}(x_i))g) - \text{KL}(q_\phi(\cdot | x_i) \parallel N(0, I)) \right].$$

The empirical objective above can be optimized via stochastic gradient descent, where fresh noise samples  $g \sim N(0, I)$  are drawn at each stochastic gradient step.

### 2.4.1 The ELBO gap

As shown in Equation (2.19), the ELBO is a valid lower bound for the log probability. We now quantify what the gap between the actual log probability and the lower bound is. We start our analysis by computing the KL-divergence between the encoder  $q(z | x)$  and the true encoder  $p(z | x)$  (which is the posterior distribution of  $z$  given the observation  $x$ ):

$$\begin{aligned} \text{KL}(q(\cdot | x) \parallel p(\cdot | x)) &= \mathbb{E}_{q(z|x)} \left[ \log \frac{q(z | x)}{p(z | x)} \right] \\ &= \mathbb{E}_{q(z|x)} \left[ \log \frac{q(z | x)p(x)}{\nu(z)p(x | z)} \right] && \text{Bayes' rule} \\ &= \log p(x) + \text{KL}(q(\cdot | x) \parallel \nu) - \mathbb{E}_{q(z|x)}[\log p(x | z)]. \end{aligned}$$

Rearranging yields:

$$\log p(x) - \text{KL}(q(\cdot | x) \parallel p(\cdot | x)) = \mathbb{E}_{q(z|x)}[\log p(x | z)] - \text{KL}(q(\cdot | x) \parallel \nu).$$

Thus, we see from this derivation that  $\text{KL}(q(\cdot | x) \parallel p(\cdot | x))$  is precisely the gap between  $\log p(x)$  and the ELBO. (Actually, you can think of derivation as an alternative derivation for the ELBO, since KL-divergence is always non-negative.) Thus maximizing the ELBO lower bound is equivalent to maximizing log likelihood if the encoder model  $q(z | x)$  is sufficiently expressive that it can express the true encoder  $p(z | x)$ . Of course, actually computing  $\text{KL}(q(\cdot | x) \parallel p(\cdot | x))$  to quantify the loss on a particular problem is typically intractable, but this identity tells us conceptually what our approximation error is with the ELBO.

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## A Basic properties of probability divergences

We first state the definition of the *Kullback–Leibler Divergence*. We state it in a general measure-theoretic form, but we will instantiate the definition for some special cases as follows.

**Definition A.1** (KL-Divergence). *Let  $\mu, \nu$  be two probability measures on the same measure space. Suppose that  $\mu$  is absolutely continuous w.r.t.  $\nu$ , i.e.,  $\mu \ll \nu$ , and let  $\frac{d\mu}{d\nu}$  denote the Radon-Nikodym derivative of  $\mu$  w.r.t.  $\nu$ . The KL-divergence  $\text{KL}(\mu \parallel \nu)$  is defined as:*

$$\text{KL}(\mu \parallel \nu) = \mathbb{E}_\mu \left[ \log \frac{d\mu}{d\nu} \right].$$

**Proposition A.2.** *Suppose that  $\mu \ll \nu$ . We have that  $\text{KL}(\mu \parallel \nu) \geq 0$ .*

*Proof.* This proof uses the  $f$ -divergence representation of KL-divergence to deal with the case when  $\nu \ll \mu$  does not hold. Let  $f(t) = t \log t$ . One can check that this function is convex on  $\mathbb{R}_+$ . Furthermore, by a change of measure, and Jensen’s inequality

$$\text{KL}(\mu \parallel \nu) = \int \log \frac{d\mu}{d\nu} d\mu = \int \frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu} d\nu = \int f\left(\frac{d\mu}{d\nu}\right) d\nu \geq f\left(\int \frac{d\mu}{d\nu} d\nu\right) = f(1) = 0.$$

□

**Lemma A.3** (Donsker-Varadhan). *Let  $p, q$  be two probability measures on the same measure space, and suppose that  $p$  is absolutely continuous w.r.t.  $q$ . We have that:*

$$\text{KL}(p \parallel q) = \sup_g \{ \mathbb{E}_p[g] - \log \mathbb{E}_q[\exp(g)] \},$$

where the supremum is taken over all measurable  $g : \Theta \rightarrow \mathbb{R}$  satisfying  $\mathbb{E}_q[\exp(g)] < \infty$ .

Equivalently, let  $q$  be a probability measure and let  $g$  be a measurable function satisfying  $\mathbb{E}_q[\exp(g)] < \infty$ . We have that:

$$\log \mathbb{E}_q[\exp(g)] = \sup_p \{ \mathbb{E}_p[g] - \text{KL}(p \parallel q) \},$$

where the supremum is taken over probability measures  $p$  which are absolutely continuous w.r.t.  $q$ .

*Proof.* Let us define the Gibbs measure with density:

$$\frac{d\pi}{dq}(\theta) = \frac{\exp(g(\theta))}{\mathbb{E}_q[\exp(g)]}.$$

Computing the KL-divergence between  $p$  and  $\pi$ ,

$$\begin{aligned}\text{KL}(p \parallel \pi) &= \int \log \left[ \frac{dp}{d\pi} \right] dp = \int \log \left[ \frac{dp}{dq} \frac{dq}{d\pi} \right] dp \\ &= \text{KL}(p \parallel q) + \int \log \left[ \frac{\mathbb{E}_q[\exp(g)]}{\exp(g)} \right] dp \\ &= \text{KL}(p \parallel q) + \log \mathbb{E}_q[\exp(g)] - \mathbb{E}_p[g].\end{aligned}$$

Since  $\text{KL}(p \parallel \pi) \geq 0$ , this shows:

$$\text{KL}(p \parallel q) \geq \mathbb{E}_p[g] - \log \mathbb{E}_q[\exp(g)] \iff \log \mathbb{E}_q[\exp(g)] \geq \mathbb{E}_p[g] - \text{KL}(p \parallel q).$$

Furthermore, equality is achieved when  $p = \pi$ , i.e., when  $g = \log \frac{dp}{dq}$ .  $\square$

Our next result is the well-known formula for the KL-divergence between two non-degenerate multivariate Gaussians.

**Proposition A.4** (KL-divergence for multivariate Gaussians). *Let  $\mu_1, \mu_2 \in \mathbb{R}^d$  and  $\Sigma_1, \Sigma_2$  be  $d \times d$  positive definite matrices. We have:*

$$\text{KL}(N(\mu_1, \Sigma_1) \parallel N(\mu_2, \Sigma_2)) = \frac{1}{2} \left\{ \text{tr}(\Sigma_2^{-1} \Sigma_1) + (\mu_2 - \mu_1)^\top \Sigma_2^{-1} (\mu_2 - \mu_1) - d + \log \frac{\det \Sigma_2}{\det \Sigma_1} \right\}.$$

*Proof.* **TODO: write proof.**  $\square$

Another often used probability divergence is the total-variation distance.

**Definition A.5.** *Let  $\mu, \nu$  be two probability measures on the same measure space  $(\Omega, \mathcal{B})$ . The total-variation distance (TV-distance) is defined as:<sup>11</sup>*

$$\|\mu - \nu\|_{\text{tv}} = \sup_{A \in \mathcal{B}} |\mu(A) - \nu(A)|.$$

## B Concentration inequalities

We give a brief primer on concentration equalities needed for this course. First some motivation. Suppose that  $X_1, \dots, X_n$  are i.i.d. random variables drawn from some underlying distribution. Put  $\mu := \mathbb{E}[X_1]$  and  $\bar{X}_n := n^{-1} \sum_{i=1}^n X_i$ . By the Central Limit Theorem, we know that:

$$\sqrt{n}(\bar{X}_n - \mu) \rightsquigarrow N(0, \text{Var}(X_1)).$$

Here,  $\rightsquigarrow$  denotes convergence in distribution. Hence for large enough  $n$ , we expect that with high probability,

$$\left| \frac{1}{n} \sum_{i=1}^n (X_i - \mathbb{E}[X_i]) \right| \lesssim \sqrt{\frac{\text{Var}(X_1)}{n}}. \quad (\text{B.1})$$

Here, the notation  $\lesssim$  is just to remind us that this inequality is an approximation, since the CLT is an asymptotic result that is not exact for any finite  $n$ . The purpose of this section is to derive analogous non-asymptotic bounds to Equation (B.1), which hold for any  $n \geq 1$ .

Our starting point is Markov's inequality.

**Proposition B.1** (Markov's inequality). *Let  $X$  be a non-negative random variable. Then,*

$$\forall t > 0, \mathbb{P}\{X \geq t\} \leq \frac{\mathbb{E}[X]}{t}.$$

---

<sup>11</sup>For those unfamiliar with measure theory, you can think of the supremum as being taken over all subsets of  $\Omega$ .

*Proof.* Using the fact that  $X$  is non-negative,

$$\mathbb{E}[X] = \mathbb{E}[X\mathbf{1}\{X \geq t\}] + \mathbb{E}[X\mathbf{1}\{X < t\}] \geq \mathbb{E}[t\mathbf{1}\{X \geq t\}] = t\mathbb{P}\{X \geq t\}.$$

□

Markov's inequality is often used in conjunction with a monotonically increasing bijection, i.e.,  $\{X \geq t\} = \{\phi(X) \geq \phi(t)\}$  for any monotonically increasing bijection  $\phi$ . A very useful transform is  $\phi(x) = \exp(\lambda x)$  for any  $\lambda > 0$ :

**Proposition B.2** (Laplace transform). *Let  $X$  be a random variable. Then,*

$$\forall t \in \mathbb{R}, \mathbb{P}\{X \geq t\} \leq \inf_{\lambda > 0} e^{-\lambda t} \mathbb{E} \exp(\lambda X).$$

*Proof.* For any  $\lambda > 0$ ,  $\{X \geq t\} = \{\exp(\lambda X) \geq \exp(\lambda t)\}$ . Now apply Markov's inequality to the RHS event. Since  $\lambda > 0$  is arbitrary, take the infimum over all  $\lambda > 0$  to conclude. □

Note that the Laplace transform is often referred to in the literature as the “Chernoff bound”.

The quantity  $\mathbb{E} \exp(\lambda X)$  is referred to as the *moment generating function* (MGF) of  $X$ . It has a very useful property that it “tensorizes” in the following way for independent sums. If  $S_n = X_1 + \dots + X_n$  where  $X_i \perp X_j$  for  $i \neq j$ , then

$$\mathbb{E} \exp(\lambda S_n) = \prod_{i=1}^n \mathbb{E} \exp(\lambda X_i).$$

Notice that the RHS is the product of the individual MGFs, a property we will soon utilize.

Our next order of business will be to obtain control of MGFs. Without any assumptions, the MGF does not even need to be finite. So, we will need to impose some conditions on our random variables so that the MGFs do exist. The starting point of study will be the Gaussian distribution. Suppose that  $X \sim N(0, \sigma^2)$ . Then, one can show that:

$$\forall \lambda \in \mathbb{R}, \mathbb{E} \exp(\lambda X) = \exp(\lambda^2 \sigma^2 / 2). \quad (\text{B.2})$$

**Exercise B.3.** Show that (B.2) is true.

Hint: write out  $\mathbb{E} \exp(\lambda X)$  in integral form and complete the square.

The Gaussian MGF motivates the following definition of a random variable, which we will utilize quite extensively in this course.

**Definition B.4** (sub-Gaussian random variable). *Let  $X$  be a random variable with finite expectation. The random variable  $X$  is  $\sigma$ -sub-Gaussian if:*

$$\forall \lambda \in \mathbb{R}, \mathbb{E} \exp(\lambda(X - \mathbb{E}[X])) \leq \exp(\lambda^2 \sigma^2 / 2).$$

Clearly, a  $N(\mu, \sigma^2)$  Gaussian random variable is  $\sigma$ -sub-Gaussian by definition. Also, it is not hard to see that constant scalings of sub-Gaussian random variables remain sub-Gaussian. That is, if  $a \in \mathbb{R}$  is fixed and  $X$  is  $\sigma$ -sub-Gaussian, then  $aX$  is  $|a|\sigma$ -sub-Gaussian, since:

$$\mathbb{E} \exp(\lambda(aX - \mathbb{E}[aX])) \leq \exp(\lambda^2 a^2 \sigma^2 / 2) = \exp(\lambda^2 (a\sigma)^2 / 2),$$

Next, we will show that bounded random variables are also sub-Gaussian. To do this, we start with a definition of a *Rademacher random variable*, which will play a critical role in this class.

**Definition B.5** (Rademacher random variable). *A Rademacher random variable  $\varepsilon$  satisfies:*

$$\mathbb{P}\{\varepsilon = +1\} = \mathbb{P}\{\varepsilon = -1\} = 1/2.$$

**Proposition B.6.** *A Rademacher random variable is 1-sub-Gaussian.*

*Proof.* Let  $\varepsilon$  be a Rademacher random variable, and let  $\lambda \in \mathbb{R}$  be fixed. We have:

$$\begin{aligned}
\mathbb{E} \exp(\lambda \varepsilon) &= \frac{1}{2} (\exp(\lambda) + \exp(-\lambda)) \\
&= \frac{1}{2} \left[ \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} + \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \right] && \text{Taylor series of } \exp(x) \\
&= \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!} && \text{odd terms cancel} \\
&\leq \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{2^k k!} && \text{since } (2k)! \geq 2^k k! \\
&= \sum_{k=0}^{\infty} \frac{(\lambda^2/2)^k}{k!} = \exp(\lambda^2/2).
\end{aligned}$$

□

We can now show that bounded random variables are sub-Gaussian.

**Proposition B.7** (Hoeffding's lemma). *Let  $X$  be a random variable satisfying  $X \in [-1, 1]$  a.s. Then,  $X$  is 1-sub-Gaussian.*

*Proof.* Note: we will prove this result with a slightly worse constant. Specifically, we will prove that  $X$  is 2-sub-Gaussian. However, the proof technique we will use illustrates a lot of the key ideas in studying concentration inequalities. The proof for the sharper constant is somewhat less intuitive.

This is our first example of a symmetrization argument, which will be a technique we return to many times. Let  $\varepsilon$  be a Rademacher random variable, and let  $X'$  be a copy of  $X$  (all of  $X, X', \varepsilon$  are mutually independent). The key observation is that  $(X - X')$  has the same distribution as  $\varepsilon(X - X')$ , due to symmetry. We now let  $\lambda \in \mathbb{R}$  be fixed, and proceed as follows:

$$\begin{aligned}
\mathbb{E} \exp(\lambda(X - \mathbb{E}[X])) &= \mathbb{E} \exp(\lambda(X - \mathbb{E}[X'])) && \text{since } X \stackrel{(d)}{=} X' \\
&\leq \mathbb{E} \exp(\lambda(X - X')) && \text{Jensen's inequality} \\
&= \mathbb{E} \exp(\lambda \varepsilon(X - X')) && \text{since } X - X' \stackrel{(d)}{=} \varepsilon(X - X') \\
&= \mathbb{E}[\mathbb{E}[\exp(\lambda \varepsilon(X - X')) \mid X, X']] && \text{tower property} \\
&= \mathbb{E} \exp(\lambda^2(X - X')^2/2) && \text{since } \varepsilon \text{ is 1-sub-Gaussian} \\
&\leq \exp(\lambda^2 2^2/2) && \text{since } X \in [-1, 1] \text{ a.s.}
\end{aligned}$$

□

**Exercise B.8.** Let  $X$  be a random variable satisfying  $X \in [-a, a]$  a.s. for some  $a > 0$ . Show that  $X$  is  $a$ -sub-Gaussian.

**Exercise B.9.** Let  $X_1, \dots, X_n$  be independent random variables, and suppose that  $X_i$  is  $\sigma_i$ -sub-Gaussian for  $i = 1, \dots, n$ . Show that  $S_n = \sum_{i=1}^n X_i$  is  $\sqrt{\sum_{i=1}^n \sigma_i^2}$ -sub-Gaussian.

We are now in a position to prove our first concentration inequality.



**Proposition B.10** (sub-Gaussian tail bound). *Let  $X$  be a  $\sigma$ -sub-Gaussian random variable. We have that:*

$$\forall t > 0, \mathbb{P}\{X - \mathbb{E}[X] \geq t\} \leq \exp(-t^2/(2\sigma^2)).$$

Consequently,

$$\forall t > 0, \mathbb{P}\{|X - \mathbb{E}[X]| \geq t\} \leq 2 \exp(-t^2/(2\sigma^2)).$$

*Proof.* This is a simple consequence of the Laplace transform in addition to the sub-Gaussian MGF bound. For any  $\lambda > 0$ ,

$$\begin{aligned} \mathbb{P}\{X - \mathbb{E}[X] \geq t\} &\leq e^{-\lambda t} \mathbb{E} \exp(\lambda(X - \mathbb{E}[X])) \\ &\leq \exp\{-\lambda t + \lambda^2 \sigma^2 / 2\}. \end{aligned}$$

Now choosing  $\lambda = t/\sigma^2$  to minimize the quadratic form inside the exponential, we obtain  $\mathbb{P}\{X - \mathbb{E}[X] \geq t\} \leq \exp(-t^2/(2\sigma^2))$ . Applying the same argument to the random variable  $-X$  yields  $\mathbb{P}\{X - \mathbb{E}[X] \leq -t\} \leq \exp(-t^2/(2\sigma^2))$  (why is  $-X$  also  $\sigma$ -sub-Gaussian?). Now we conclude by a union bound:

$$\begin{aligned} \mathbb{P}\{|X - \mathbb{E}[X]| \geq t\} &= \mathbb{P}\{\{X - \mathbb{E}[X] \geq t\} \cup \{X - \mathbb{E}[X] \leq -t\}\} \\ &\leq \mathbb{P}\{X - \mathbb{E}[X] \geq t\} + \mathbb{P}\{X - \mathbb{E}[X] \leq -t\} \leq 2 \exp(-t^2/(2\sigma^2)). \end{aligned}$$

□

**Exercise B.11.** Let  $X$  be a zero-mean, 1-sub-Gaussian random variable. Show that there exists a universal constant  $c > 0$  such that for all  $p \geq 1$ ,  $\|X\|_{L_p} := (\mathbb{E}|X|^p)^{1/p} \leq c\sqrt{p}$ .

Hint: You will need to utilize the following facts (which you can use without proof):

- (a) For a non-negative random variable  $X$ ,  $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > t) dt$ .
- (b) For all  $s \geq 1/2$ ,  $\Gamma(s) \leq 3s^s$ , where  $\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt$  is the Gamma function.
- (c) The quantity  $\sup_{p \geq 1} p^{1/p} < \infty$ .

A nearly immediate consequence is Hoeffding's inequality.

**Proposition B.12** (Hoeffding's inequality). *Let  $X_1, \dots, X_n$  be independent random variables with  $X_i \in [-a_i, a_i]$  a.s. Put  $S_n = \sum_{i=1}^n X_i$ . We have that:*

$$\forall t > 0, \mathbb{P}\{S_n - \mathbb{E}[S_n] \geq t\} \leq \exp\left(-\frac{t^2}{2 \sum_{i=1}^n a_i^2}\right).$$

Consequently,

$$\forall t > 0, \mathbb{P}\{|S_n - \mathbb{E}[S_n]| \geq t\} \leq 2 \exp\left(-\frac{t^2}{2 \sum_{i=1}^n a_i^2}\right).$$

*Proof.* By Exercise B.8, we have that each  $X_i$  is  $a_i$ -sub-Gaussian. Furthermore, by Exercise B.9, we have that  $S_n$  is  $\sqrt{\sum_{i=1}^n a_i^2}$ -sub-Gaussian. Hence by the sub-Gaussian tail bound Proposition B.10, we have:

$$\forall t > 0, \mathbb{P}\{S_n - \mathbb{E}[S_n] \geq t\} \leq \exp\left(-\frac{t^2}{2 \sum_{i=1}^n a_i^2}\right).$$

To control  $\mathbb{P}\{|S_n - \mathbb{E}[S_n]| \geq t\}$ , use the same argument in Proposition B.10. □

Our next result will be the well known bounded differences inequality. To do this, we will need the concept of a martingale.

**Definition B.13.** Let  $\{X_t\}_{t \geq 0}$  be a stochastic process adapted to the filtration  $\{\mathcal{F}_t\}_{t \geq 0}$ . The process  $\{X_t\}$  is a martingale if

$$\forall t \geq 0, \mathbb{E}[X_{t+1} \mid \mathcal{F}_t] = X_t.$$

**Definition B.14.** Let  $\{X_t\}_{t \geq 0}$  be a martingale. We say that  $\{X_t\}_{t \geq 0}$  is a  $\{\sigma_t\}_{t \geq 0}$ -sub-Gaussian martingale if for all  $t \geq 0$ , the random variable  $(X_{t+1} - X_t) \mid \mathcal{F}_t$  is  $\sigma_t$ -sub-Gaussian almost surely, i.e.,

$$\forall t \geq 0, \lambda \in \mathbb{R}, \mathbb{E}[\exp(\lambda(X_{t+1} - X_t)) \mid \mathcal{F}_t] \leq \exp(\lambda^2 \sigma_t^2 / 2) \text{ a.s.}$$

**Proposition B.15** (sub-Gaussian martingale tail inequality). Let  $\{X_t\}_{t \geq 0}$  be a  $\{\sigma_t\}_{t \geq 0}$ -sub-Gaussian martingale. We have that:

$$\forall t > 0, \mathbb{P}\{X_n - X_0 \geq t\} \leq \exp\left(-t^2 / \left(2 \sum_{i=0}^{n-1} \sigma_i^2\right)\right).$$

Consequently,

$$\forall t > 0, \mathbb{P}\{|X_n - X_0| \geq t\} \leq 2 \exp\left(-t^2 / \left(2 \sum_{i=0}^{n-1} \sigma_i^2\right)\right).$$

*Proof.* By the Laplace transform method (cf. Proposition B.2) and the tower property, for any  $\lambda > 0$ :

$$\begin{aligned} \mathbb{P}\{X_n - X_0 \geq t\} &\leq e^{\lambda t} \mathbb{E} \exp(\lambda(X_n - X_0)) \\ &= e^{-\lambda t} \mathbb{E} \exp\left(\sum_{i=0}^{n-1} \lambda(X_{i+1} - X_i)\right) \\ &= e^{-\lambda t} \mathbb{E} \left[ \exp\left(\sum_{i=0}^{n-2} \lambda(X_{i+1} - X_i)\right) \mathbb{E}[\exp(\lambda(X_n - X_{n-1})) \mid \mathcal{F}_{n-1}] \right] \\ &\leq \exp\{-\lambda t + \lambda^2 \sigma_{n-1}^2 / 2\} \mathbb{E} \exp\left(\sum_{i=0}^{n-2} \lambda(X_{i+1} - X_i)\right) \\ &\vdots \\ &\leq \exp\left\{-\lambda t + \lambda^2 \left(\sum_{i=0}^{n-1} \sigma_i^2\right) / 2\right\}. \end{aligned}$$

Now choose  $\lambda = t / \sum_{i=0}^{n-1} \sigma_i^2$ , from which the claim follows.  $\square$

## B.1 Bounded differences inequality

**Proposition B.16** (Bounded differences inequality). Let  $f : \mathbb{X}^n \mapsto \mathbb{R}$  satisfy the following bounded differences property:

$$\forall i \in \{1, \dots, n\}, \sup_{x_1, \dots, x_n, x' \in \mathbb{X}} |f(x_{1:i-1}, x_i, x_{i+1:n}) - f(x_{1:i-1}, x'_i, x_{i+1:n})| \leq c_i.$$

Let  $x_1, \dots, x_n \in \mathbb{X}$  be independent random variables. Then,

$$\forall t > 0, \mathbb{P}\{f(x_1, \dots, x_n) - \mathbb{E}[f(x_1, \dots, x_n)] \geq t\} \leq \exp\left(-t^2 / \left(2 \sum_{i=1}^n c_i^2\right)\right).$$

Consequently,

$$\forall t > 0, \mathbb{P}\{|f(x_1, \dots, x_n) - \mathbb{E}[f(x_1, \dots, x_n)]| \geq t\} \leq 2 \exp\left(-t^2 / \left(2 \sum_{i=1}^n c_i^2\right)\right).$$

*Proof.* The idea here is to construct a martingale, known as the Doob martingale. Let  $\mathcal{F}_i = \sigma(x_{1:i})$  denote the minimal  $\sigma$ -algebra for  $x_{1:i}$ ,<sup>12</sup> and set  $Z_i = \mathbb{E}[f(x_{1:n}) \mid \mathcal{F}_i]$ . Let us first check this is a martingale. We have by the tower property of conditional expectations:

$$\mathbb{E}[Z_{i+1} \mid \mathcal{F}_i] = \mathbb{E}[\mathbb{E}[f(x_{1:n}) \mid \mathcal{F}_{i+1}] \mid \mathcal{F}_i] = \mathbb{E}[f(x_{1:n}) \mid \mathcal{F}_i] = Z_i.$$

Second, observe that:

$$Z_n = f(x_{1:n}), \quad Z_0 = \mathbb{E}[f(x_{1:n})] \implies Z_n - Z_0 = f(x_{1:n}) - \mathbb{E}[f(x_{1:n})].$$

Now let us bound, conditioned on  $\mathcal{F}_{i-1}$ ,

$$\begin{aligned} |Z_i - Z_{i-1}| &= \left| \int f(x_{1:n}) dx_{i+1:n} - \int f(x_{1:n}) dx_{i:n} \right| \\ &= \left| \int f(x_{1:i-1}, x_i, x_{i+1:n}) dx_{i+1:n} - \int f(x_{1:i-1}, x'_i, x_{i+1:n}) dx'_i dx_{i+1:n} \right| \\ &= \left| \int [f(x_{1:i-1}, x_i, x_{i+1:n}) - f(x_{1:i-1}, x'_i, x_{i+1:n})] dx'_i dx_{i+1:n} \right| \\ &\leq \int |f(x_{1:i-1}, x_i, x_{i+1:n}) - f(x_{1:i-1}, x'_i, x_{i+1:n})| dx'_i dx_{i+1:n} && \text{Jensen's inequality} \\ &\leq c_i. \end{aligned}$$

(Check your understanding: where did we use independence of the  $x_i$ 's above?) Hence, we have that for all  $i \in \mathbb{N}_+$ , that  $(Z_i - Z_{i-1}) \mid \mathcal{F}_{i-1}$  is a  $c_i$ -sub-Gaussian random variable almost surely (cf. Exercise B.8). Therefore, the Doob martingale  $\{Z_i\}$  is a  $\{c_i\}$ -sub-Gaussian martingale. By Proposition B.15, we conclude that for all  $t > 0$ ,

$$\mathbb{P}\{f(x_{1:n}) - \mathbb{E}[f(x_{1:n})] \geq t\} = \mathbb{P}\{Z_n - Z_0 \geq t\} \leq \exp\left(-t^2 / \left(2 \sum_{i=1}^n c_i^2\right)\right).$$

□

## B.2 Maximal inequalities

**Proposition B.17** (sub-Gaussian maximal inequality). *Let  $X_i$ ,  $i = 1, \dots, n$  be zero-mean  $\sigma$ -sub-Gaussian random variables (not necessarily independent). Then,*

$$\mathbb{E} \max_{i=1, \dots, n} X_i \leq \sigma \sqrt{2 \log n}.$$

*Proof.* Fix any  $\lambda > 0$ . We have:

$$\begin{aligned} \mathbb{E} \max_{i=1, \dots, n} X_i &= \lambda^{-1} \mathbb{E} \max_{i=1, \dots, n} \lambda X_i && \text{since } \lambda > 0 \\ &= \lambda^{-1} \mathbb{E} \log \exp\left(\max_{i=1, \dots, n} \lambda X_i\right) \\ &\leq \lambda^{-1} \log \mathbb{E} \exp\left(\max_{i=1, \dots, n} \lambda X_i\right) && \text{Jensen's inequality} \\ &= \lambda^{-1} \log \mathbb{E} \max_{i=1, \dots, n} \exp(\lambda X_i) && \text{exp is monotonically increasing} \\ &\leq \lambda^{-1} \log \left( \sum_{i=1}^n \mathbb{E} \exp(\lambda X_i) \right) && \max_i a_i \leq \sum_i a_i \text{ if } \forall i, a_i \geq 0 \end{aligned}$$

<sup>12</sup>If you are unfamiliar with this terminology, just think of  $\mathcal{F}_i$  as encoding the information available in the first  $i$  datapoints.

$$\begin{aligned} &\leq \lambda^{-1} \log(n \exp(\lambda^2 \sigma^2 / 2)) && \text{each } X_i \text{ is } \sigma\text{-sub-Gaussian} \\ &= \frac{\log n}{\lambda} + \frac{\sigma^2 \lambda}{2}. \end{aligned}$$

Since the inequality holds for any  $\lambda > 0$ , we can optimize over the bound by choosing  $\lambda = \sqrt{2 \log n / \sigma^2}$ .  $\square$

Note that this result is actually sharp: if  $\{X_i\}_{i=1}^\infty$  is a sequence of i.i.d.  $N(0, 1)$  random variables, then one can show:

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} \max_{i=1, \dots, n} X_i}{\sqrt{2 \log n}} = 1.$$

**Exercise B.18.** Let  $X_i, i = 1, \dots, n$  be zero-mean  $\sigma$ -sub-Gaussian random variables. Show that

$$\mathbb{E} \max_{i=1, \dots, n} |X_i| \leq \sigma \sqrt{2 \log(2n)}.$$

## C Convex functions

We give a very brief primer on basic properties of convex functions.

**Definition C.1.** A set  $D$  is convex if for all  $x, y \in D$ ,  $\lambda x + (1 - \lambda)y \in D$  for all  $\lambda \in [0, 1]$ .

**Definition C.2.** A function  $f : D \mapsto \mathbb{R}$  is convex if the domain  $D$  is convex and the function satisfies for all  $x, y \in D$  and  $\theta \in [0, 1]$ ,

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

**Proposition C.3.** Let  $D$  be a convex domain and  $f : D \mapsto \mathbb{R}$ :

(a) Suppose that  $f \in C^1(D)$ . Then,  $f$  is convex iff for every  $x, y \in D$ ,

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle.$$

(b) Suppose that  $f \in C^2(D)$ . Then,  $f$  is convex iff for every  $x \in D$ ,

$$\nabla^2 f(x) \succcurlyeq 0.$$

**Definition C.4.** A differentiable function  $f : D \mapsto \mathbb{R}$  has  $L$ -Lipschitz gradients (alternatively,  $L$ -smooth), if  $\nabla f$  is  $L$ -Lipschitz,

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \quad \forall x, y \in D.$$

**Proposition C.5.** Let  $f : \mathbb{R}^n \mapsto \mathbb{R}$  be  $L$ -smooth. Then for all  $x \in \mathbb{R}^n$ ,

$$\inf_{x \in \mathbb{R}^n} f(x) := f_\star \leq f(x) - \frac{1}{2L} \|\nabla f(x)\|^2.$$

*Proof.* Taking a second order Taylor series expansion, we have that for all  $x, y$ :

$$f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|x - y\|^2.$$

Choosing  $y = x - \frac{1}{L} \nabla f(x)$  (which minimizes the RHS over  $y \in \mathbb{R}^n$ ), we obtain:

$$f(y) \leq f(x) - \frac{1}{2L} \|\nabla f(x)\|^2.$$

By definition,  $f_\star \leq f(y)$ , which yields the claim.  $\square$

**Proposition C.6** (Co-coercive gradients). *Let  $f : \mathbb{R}^n \mapsto \mathbb{R}$  be convex and  $L$ -smooth. Then,*

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{1}{L} \|\nabla f(x) - \nabla f(y)\|^2 \quad \forall x, y \in \mathbb{R}^n.$$

*Proof.* Fix an  $x \in \mathbb{R}^n$ , and define the function:

$$f_x(z) = f(z) - \langle \nabla f(x), z \rangle.$$

Since  $f$  is convex, so is  $f_x(z)$  (why?). Hence, we can study its stationary points  $\nabla f_x(z) = 0$  to understand its minimizers. Taking  $\nabla f_x(z) = \nabla f(z) - \nabla f(x)$ , we find that  $z = x$  minimizes  $f_x(z)$ . Furthermore, since  $f$  is  $L$ -smooth, so is  $f_x(z)$  (why?). Therefore, applying Proposition C.5, we conclude that for any  $y \in \mathbb{R}^n$ ,

$$f(y) - f(x) - \langle \nabla f(x), y - x \rangle = f_x(y) - f_x(x) \geq \frac{1}{2L} \|\nabla f_x(y)\|^2 = \frac{1}{2L} \|\nabla f(y) - \nabla f(x)\|^2.$$

Next, we can flip the role of  $x$  and  $y$  in the above argument and conclude

$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle \geq \frac{1}{2L} \|\nabla f(y) - \nabla f(x)\|^2.$$

Combining these two inequalities yields the claim.  $\square$

**Definition C.7** (Strong convexity). *A function  $f : D \mapsto \mathbb{R}$  is  $\mu$ -strongly-convex if the domain  $D$  is convex and the following holds for all  $\theta \in [0, 1]$ :*

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y) - \frac{\theta(1 - \theta)\mu}{2} \|x - y\|^2 \quad \forall x, y \in D.$$

## D Miscellaneous results

**Proposition D.1.** *Let  $n, d \in \mathbb{N}_+$  such that  $n \geq d$ . Then,*

$$\sum_{i=0}^d \binom{n}{i} \leq \left(\frac{en}{d}\right)^d.$$

*Proof.* Since  $d/n \leq 1$ , this implies that  $(d/n)^a \leq (d/n)^b$  for any  $a, b \in \mathbb{N}$  with  $a \geq b$ . Therefore,

$$\left(\frac{d}{n}\right)^d \sum_{i=0}^d \binom{n}{i} = \sum_{i=0}^d \left(\frac{d}{n}\right)^d \binom{n}{i} \leq \sum_{i=0}^d \left(\frac{d}{n}\right)^i \binom{n}{i} \leq \sum_{i=0}^n \left(\frac{d}{n}\right)^i \binom{n}{i}.$$

Now recall that the binomial theorem states for every  $x \in \mathbb{R}$ ,

$$(1 + x)^n = \sum_{i=0}^n \binom{n}{i} x^i$$

Thus, applying the binomial theorem with  $x = d/n$ , we have:

$$\sum_{i=0}^n \left(\frac{d}{n}\right)^i \binom{n}{i} = \left(1 + \frac{d}{n}\right)^n \leq e^d.$$

The claim now follows.  $\square$

**Proposition D.2.** *Let  $b, c$  be positive reals satisfying  $bc \geq 1$ . For every  $n > 0$ , we have that*

$$n \geq 2b \log(2bc) \implies n \geq b \log(cn).$$

*Proof.* For  $x > 0$ , define  $f(x) := x - b \log(cx)$ . A quick computation shows that  $f'(x) = 1 - b/x$ , and hence for  $x \geq b$  we have that  $f'(x) \geq 0$ , or equivalently that  $f(x)$  is increasing. Thus, if we can find an  $n_0$  satisfying (a)  $n_0 \geq b$  and (b)  $f(n_0) \geq 0$ , then  $f(n) \geq 0$  for all  $n \geq n_0$ . We select  $n_0 = 2b \log(2bc)$ . We first check that (a) holds:

$$n_0 = 2b \log(2bc) = b \cdot (2 \log(2bc)) \geq b \cdot (2 \log 2) \geq b.$$

We now check that (b) holds:

$$\begin{aligned} b \log(cn_0) &= b \log(2bc \log(2bc)) \\ &= b \log(2 \log(2)bc + 2bc \log(bc)) \\ &\leq b \log(2 \log(2)bc + 2(bc)^2) && \text{since } \log x \leq x \ \forall x > 0 \\ &\leq b \log(2(1 + \log 2)(bc)^2) && \text{since } bc \geq 1 \\ &= 2b \log(\sqrt{2(1 + \log 2)}bc) \\ &\leq 2b \log(2bc) = n_0. \end{aligned}$$

Hence,  $f(n_0) = n_0 - b \log(cn_0) \geq 0$ . We now conclude that for any  $n \geq n_0$ , then  $f(n) \geq 0$ . □