A MIXTURE APPROACH TO THE MECHANICS OF SKIN

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Abstract—Skin can be considered to be a mixture of a solid and a fluid. A general theory for the description of the behaviour of mixtures is presented and applied to a mixture of a solid and a fluid. A numerical procedure is presented to solve the non-linear field equations describing such a mixture. The abilities of the procedure are demonstrated by means of a confined compression test.

NOMENCLATURE

a	scalar
$\underline{a}$	column
$A = A_{ij}$	matrix
$\mathbf{a}$	vector
A	second order tensor
A^C	the conjugate of a second order tensor
^nA	tensor of order n $(n > 2)$
$\mathbf{ab}$	dyadic product of the vectors $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{a} \cdot \mathbf{b}$	dot product of the vectors a and b
abs $(\mathbf{a})$	length of vector $\mathbf{a}$
$A : B$	dubble dot product of two second order tensors
f^x	property of constituent x
T^x	partial stress tensor
$\mathbf{q}^x$	body force working on constituent x
A^x	specific free energy of constituent x
t	time

INTRODUCTION

Skin is a complex structure of a number of different constituents (Kenedi *et al.*, 1975). It consists of two main layers: the thin top-layer, called epidermis (thickness: 0.07–0.12 mm), and the dermis (thickness: 1–4 mm). At some places the epidermis is relatively thick, for example at the palm of the hand and the sole of the foot. The epidermis is composed of a layer of cells. New cells are continuously being formed at the junction with the dermis. They subsequently migrate towards the external surface. The cells are keratinized as they progress.

The dermis is divided into two structurally distinctive layers. Adjacent to the epidermis the papillary layer is found. It has a rather loose structure and living cells are most abundant in this part of the skin (Brown, 1971). The deeper reticular layer is the important one with regard to the mechanical behaviour of the tissue. It consists of a dense fibrous structure embedded in a gel-like amorphous ground substance. One distin-

guishes between collagen, elastin and reticulin fibres. In tube-like structures extending from the epidermal surface deep into the dermis hairs are attached to the skin. Skin contains different types of glands. Blood vessels are abundant in skin. Further, lymph vessels and nerves are found in skin. Except for the skin of the palms of the hands and the soles of the feet, the role of the epidermis with regard to the mechanical behaviour is usually neglected. Many authors neglect the influence of the blood vessels, lymphatics, nerves, glands and hairs (Kenedi *et al.*, 1975). In the present paper it is also assumed that with regard to the mechanical behaviour of skin a major role is played by the fibre network and the groundsubstance. The fibre network is probably responsible for the highly nonlinear behaviour when the tissue is being stretched (Lanir, 1983). The groundsubstance probably plays a major role when the tissue is under compression. It is a gel-like substance containing a class of chemicals called glycosaminoglycans. A major component is hyaluronate. This is very hydrophilic and binds most of the tissue fluid. These large molecules with the bound tissue fluids behave like a gel, which can be regarded as a solid. Several authors claim that there must be a small amount of free movable fluid in the groundsubstance which is not bound in the gel or the cells (Tregear and Dinhuber, 1965; Guyton *et al.*, 1971; Wiederhielm, 1972, Harrison and Massaro, 1976).

It is rather strange that in most mechanical models used in the past little attention has been paid to the role of the groundsubstance. The role of the fibre structure was emphasized. In continuum approaches the time dependent behaviour was studied with viscoelastic models. These models are phenomenological and do not give any insight in the physical nature of the time-dependent properties of the tissue. In most structural models viscoelasticity was considered to be a property of the fibres. For a review on this type of model the reader is referred to Barbenel (1979) and Lanir (1983). Numerical models of skin are usually limited to elastic behaviour (Chow and Odell, 1978; Shock, 1981). It is

our opinion that at least under compression the groundsubstance is very important and that a lot of the time dependent properties can be explained when the free fluid movement is incorporated in the model. We regard the fibre network embedded in the colloid-rich part of the groundsubstance as some fibre reinforced solid. The colloid-poor fluid component is considered to be a Newtonian fluid. The bulk material can thus be regarded as some kind of sponge filled with fluid. When the sponge is compressed the fluid will be pressed away. Because of the viscosity of the fluid there will be a resistance against this flow. This results in time dependent behaviour of the bulk material. When the sponge is compressed, the size of the pores of the sponge will decrease. This causes an increase of the resistance to flow which results in non-linear time dependent behaviour. Non-linear elastic behaviour can originate from the fibre reinforced solid. It should be noticed that the above model is a very rude approximation of the real mixture. The exact nature of the interstitium is still a point of discussion in the literature (a good overview can be found in Hargens, 1981). Questions concerning the exact role of interstital pressure, the amount of free fluid and the fine structure of soft tissues (the amount of tissue channels for example) are not yet completely solved. So the authors consider the presented model as a first step in using mixture theory for soft tissues.

A mixture approach is not new. In civil engineering this type of modelling is used for the mechanical behaviour of soil. The use of mixtures is common in chemistry, as well as in studies on oil and gas fields and on multi-phase flows. In biomechanics the use of a mixture approach is rare. Mow *et al.* (1980) used it with much success on articular cartilage. Simon *et al.* described a finite element model of a porous intervertebral disk (1985). Although mixture theory is widely used, solutions of non-linear mixture problems are rare. The purpose of the present paper is to present a non-linear general mixture theory, which in the special case of an incompressible elastic solid and an incompressible fluid reduces to the field equations used for soil. Also a numerical procedure is described to solve these non-linear equations and some test-problems with a plain strain element are presented. In a subsequent paper (Oomens *et al.*, in press) the theory is applied to a tissue layer on a rigid foundation and compared with *in vitro* experiments.

THEORY

The review articles from Bowen (1976) and Atkin and Craine (1976) are used as a basis for the presented theory. Also see Mow *et al.* (1980). The basic assumption of the theory is that any mixture may be viewed upon as a superposition of v single continua each following its own motion, and that at any time t each position x in the mixture is occupied by several different particles X^x, each particle corresponding to one constituent α. The position x of a particle X^x is a

function of time as well as of its original or reference position vector, denoted by x_0^x

$$x = x^x := \phi^x(x_o^x, t) \qquad (1)$$

The subscript o refers to the original or reference configuration. This configuration does not necessarily have to coincide with the undeformed configuration. It is important to notice, that although by definition the position vectors x^x coincide for all different phases, this is generally not true for the vectors x_0^x. We can associate with the particles a velocity field v^x defined by

$$v^x := \frac{\partial \phi^x}{\partial t} \qquad (2)$$

In general v^x will be different for each value of α.

In order to define the material time derivative of a physical property, we have to take into account which constituent is our reference. If the observer wishes to move along with constituent α, the material time derivative of a property ψ is given by

$$\dot{\psi}^{(x)} = \frac{D^{(x)}\psi}{Dt} = \frac{\partial \psi}{\partial t} + v^x \cdot \nabla\psi \qquad (3)$$

∇ represents the gradient operator with regard to the coordinates in the deformed configuration. Because of the above, ∇ is the same for each phase. For each constituent we can define a deformation tensor

$$F^x := (\nabla_o^x x^x)^C \qquad (4)$$

where ∇_o^x is the gradient operator with respect to x_o^x. The deformation tensor F is invertible according to

$$(F^x)^{-1} = (\nabla x_o^x)^C. \qquad (5)$$

Because each constituent is modelled as a continuum, properties which are defined per unit area or volume have a 'true' value and an 'apparent' value. The latter is referred to the mixture volume. The true density ρ_*^x is defined as the mass of constituent α divided by the true volume V^x of constituent α. The apparent density ρ^x is the mass of phase α divided by the mixture volume V. Of course

$$\rho^x = n^x \rho_*^x \text{ with } n^x = \frac{V^x}{V}. \qquad (6)$$

It should be noticed that when material components intermingle on a molecular scale, a volume fraction has no true physical meaning. In that case the apparent densities are more suitable as independent variables. Because each constituent is regarded as a continuum following its own motion, it is possible to derive balance laws for each constituent. These balance laws are the same as those for a single phase material except for so-called interaction terms. These terms arise from the presence of the other constituents. The balance laws for the mixture as a whole have to be the same as those for a single phase material, because the mixture does not 'know' that it is a mixture. This leads to restrictions on the interaction terms (Bowen, 1976).

For the local balance of mass for the constituent α

we can write

$$\frac{\partial \rho^{\alpha}}{\partial t} + \nabla \cdot (\mathbf{v}^{\alpha} \rho^{\alpha}) = c^{\alpha}. \tag{7}$$

The quantity c^{α} is an interaction term that represents the mass supply from the other constituents. This term is important with chemically reacting media. In skin, for example, the amorphous groundsubstance binds most of the water in the tissue, but it is possible that under certain external or internal influences the groundsubstance releases water (oedema). The balance of the mixture as a whole leads to

$$\sum_{\alpha=1}^{v} c^{\alpha} = 0. \tag{8}$$

The balance of momentum for the constituent α is given by

$$\rho^{\alpha} \frac{D^{(\alpha)} \mathbf{v}^{\alpha}}{Dt} = \nabla \cdot \mathbf{T}^{\alpha} + \rho^{\alpha} \mathbf{q}^{\alpha} + \pi^{\alpha} \tag{9}$$

where: π^{α} = momentum supplied by other constituents; $\mathbf{T}^{\alpha}$ = Cauchy stress tensor; $\mathbf{q}^{\alpha}$ = body force.

The momentum balance for the mixture leads to

$$\sum_{\alpha=1}^{v} (\pi^{\alpha} + c^{\alpha} \mathbf{v}^{\alpha}) = 0. \tag{10}$$

The balance of moment of momentum may lead to non-symmetrical partial stress tensors $\mathbf{T}^{\alpha}$ (Kelly, 1964), because of the assumption of a moment of momentum supply, due to the presence of the other constituents. We use fairly simple constitutive models for solid and fluid, so we have no reason to believe that such a moment of momentum supply exists. In that case the partial stress tensors will be symmetric. Further we assume that there is no mass exchange between solid and fluid, so $c^{\alpha} = 0$. The first axiom of thermodynamics is assumed to be

$$\rho^{\alpha} \frac{D^{(\alpha)} U^{\alpha}}{Dt} = \mathbf{T}^{\alpha} : \mathbf{D}^{\alpha} - \nabla \cdot \mathbf{h}^{\alpha} + \rho^{\alpha} r^{\alpha} + \varepsilon^{\alpha} \tag{11}$$

where: U^{α} = specific internal energy; $\mathbf{D}^{\alpha}$ = deformation velocity; $\mathbf{h}^{\alpha}$ = heat flux vector per unit area; r^{α} = external specific heat supply; ε^{α} = heat supply by the other constituents.

The energy balance for the mixture as a whole leads to the restriction:

$$\sum_{\alpha=1}^{v} [\varepsilon^{\alpha} + \pi^{\alpha} \cdot \mathbf{v}^{\alpha} + c^{\alpha} (U^{\alpha} + \tfrac{1}{2} \mathbf{v}^{\alpha} \cdot \mathbf{v}^{\alpha})] = 0. \tag{12}$$

For the second law of thermodynamics (Atkin and Craine, 1976) we hypothesize

$$\sum_{\alpha=1}^{v} \left\{ \rho^{\alpha} \frac{D^{(\alpha)} S^{\alpha}}{Dt} + c^{\alpha} S^{\alpha} + \nabla \cdot \frac{\mathbf{h}^{\alpha}}{T^{\alpha}} - \frac{\rho^{\alpha} r^{\alpha}}{T^{\alpha}} \right\} \geqslant 0 \tag{13}$$

where S^{α} denotes the specific entropy for phase α and T^{α} is the local temperature of this phase. It is rather common to introduce at this stage the Helmholtz free energy A $(A = U - TS)$.

By means of equations (11) and (12) and the assumption that $c^{\alpha} = 0$, we are able to eliminate r^{α} in (13). This leads to the inequality

$$-\sum_{\alpha=1}^{v} \left\{ \rho^{\alpha} \frac{D^{(\alpha)} A^{\alpha}}{Dt} + \rho^{\alpha} S^{\alpha} \frac{D^{(\alpha)} T^{\alpha}}{Dt} + \pi^{\alpha} \cdot \mathbf{v}^{\alpha} \right.$$
$$\left. - \mathbf{T}^{\alpha} : \mathbf{L}^{\alpha} + \frac{1}{T^{\alpha}} \mathbf{h}^{\alpha} \cdot \nabla T^{\alpha} \right\} \geqslant 0. \tag{14}$$

As a model for the behaviour of skin we consider a homogeneous, intrinsically incompressible solid with an intrinsically incompressible fluid in it. This means that we assume that the true densities of the solid ρ_{*}^{s} and the fluid ρ_{*}^{f} are constant. In this case the porosity or fluid fraction n^{f} is a suitable independent variable (notice that for the solid fraction $n^{s} = 1 - n^{f}$). Utilizing equation (6) and the assumption of incompressibility we find, after adding the local balance laws of mass for the constituents $(c^{\alpha} = 0)$, the mass balance for the mixture:

$$\nabla \cdot \mathbf{v}^{s} - \nabla \cdot n^{f} (\mathbf{v}^{f} - \mathbf{v}^{s}) = 0. \tag{15}$$

An admissible thermodynamic process has to fulfil the balance laws of mass, momentum, moment of momentum and energy as well as the second law of thermodynamics. Only the balance of mass and the second law of thermodynamics are real restrictions, because the external heat supply and body forces can be chosen arbitrarily. As dependent variables we have

$$A^{f}, A^{s}, \mathbf{T}^{s}, \mathbf{T}^{f}, \pi^{s}. \tag{16}$$

For the independent variables we choose

$$\mathbf{F}^{s}, \rho^{f}, \mathbf{g} = \nabla \rho^{f}, \mathbf{a} = \mathbf{v}^{f} - \mathbf{v}^{s} \tag{17}$$

The property $\mathbf{g}$ is included because of thermochemical reasons (Müller, 1968). The influence of the viscosity of the fluid can be divided into two parts. Firstly we have an influence because the fluid moves through the solid. The second part is due to the deformation rate of the fluid itself. The first part is incorporated in the model by means of the property $\mathbf{a}$, the second part is neglected.

Following the procedure as described by Atkin and Craine (1976) we state that an admissible thermodynamic process has to satisfy the inequality (14) and the balance of mass (15). For this the balance of mass is multiplied by a Lagrange multiplier p and added to the left hand side of equation (14). Using the principle of equipresence (every dependent variable depends on every independent variable, unless otherwise is proved) and substituting (16) and (17) in the second law of thermodynamics leads to a number of restrictions on the constitutive equations. In Appendix A a derivation of the following restrictions is given

$$\mathbf{T}^{s} = \rho^{s} \mathbf{F}^{s} \cdot \left(\frac{\partial A^{sC}}{\partial \mathbf{F}} \right) - n^{s} p \mathbf{I} + \rho^{s} \mathbf{a} \frac{\partial A^{s}}{\partial \mathbf{a}} \tag{18}$$

$$\mathbf{T}^{f} = -p n^{f} \mathbf{I} - (\rho^{f})^{2} \frac{\partial A^{f}}{\partial \rho^{f}} \mathbf{I} - \rho^{f} \rho^{s} \frac{\partial A^{s}}{\partial \rho^{f}} \mathbf{I} - \rho^{s} \mathbf{a} \frac{\partial A^{s}}{\partial \mathbf{a}} \tag{19}$$

$$A^s = A^s(\mathbf{F}^s, \rho^f, \mathbf{a}); A^f = A^f(\rho^f, \mathbf{a}) \qquad (20)$$

$$\rho^f \frac{\partial A^f}{\partial \mathbf{a}} = -\rho^s \frac{\partial A^s}{\partial \mathbf{a}} \qquad (21)$$

$$\boldsymbol{\pi}^s \cdot \mathbf{a} + \left(\rho^s \frac{\partial A^s}{\partial \rho^f} + \frac{p}{\rho^f_*} \right) \mathbf{g} \cdot \mathbf{a} \geqslant 0. \qquad (22)$$

The first term on the right hand side of equation (18) is known as the effective stress $\mathbf{T}_{eff}$ in soil mechanics. The Lagrange multiplier p can be interpreted as a hydrodynamic pressure arising from the condition of incompressibility. The presence of n^s in (18) and n^f in (19) is caused by the assumption that solid and fluid are continua occupying the whole mixture volume. The second and third terms on the right hand side of equation (19) can be interpreted as changes in the free energy of the solid and the fluid as a result of a change in the ratio of fluid and solid. We assume that these can be neglected with regard to the term $-pn^f\mathbf{I}$. For the same reason we neglect the term $\rho^s \partial A^s/\partial \rho^f$ in equation (22).

At this point it is worthwhile to mention some differences with the theoretical derivation by Mow *et al.* (1980). Mow *et al.* use ρ^s and $\alpha = n^s/n^f$ as independent variables. Further they use the free energy A for the mixture as a whole $(A = \rho^s A^s + \rho^f A^f/\rho$ with $\rho = \rho^s + \rho^f)$ instead of A^s and A^f. Mow *et al.* do not use the principle of equipresence and do not neglect the deformation rate of solid and fluid.

As a first approximation let us assume that $\boldsymbol{\pi}^s$ depends linearly on $\mathbf{a}$ and $\mathbf{g}$ according to

$$\boldsymbol{\pi}^s = \mathbf{A} \cdot \mathbf{a} + \mathbf{B} \cdot \mathbf{g}. \qquad (23)$$

Then equation (22) leads to

$$(\mathbf{A} \cdot \mathbf{a}) \cdot \mathbf{a} + \left[(\mathbf{B} + \frac{p}{\rho^f_*} \mathbf{I}) \cdot \mathbf{g} \right] \cdot \mathbf{a} \geqslant 0. \qquad (24)$$

Because $\mathbf{a}$ can be chosen arbitrarily, it is found that $(\mathbf{B} + \frac{p}{\rho^f_*} \mathbf{I}) \cdot \mathbf{g} = 0$. Finally this leads to

$$\boldsymbol{\pi}^s = \mathbf{A} \cdot (\mathbf{v}^f - \mathbf{v}^s) - p\nabla n^f \qquad (25)$$

The first term on the right hand side of equation (25) is called the Stokes drag, the second term the buoyancy force.

If we substitute equations (19) and (25) in the balance of momentum (9) for the fluid and neglect inertial terms (because we consider only slow deformation rates) and body forces, we find the well-known law of Darcy for the motion of a fluid through a porous solid

$$n^f(\mathbf{v}^f - \mathbf{v}^s) = -\mathbf{K} \cdot \nabla p \qquad (26)$$

with: $\qquad \qquad \mathbf{K} = (n^f)^2 \mathbf{A}^{-1}$

$\mathbf{K}$ is called the permeability tensor.

If we add the balance laws of momentum for the solid and the fluid, we find

$$\nabla \cdot (\mathbf{T}_{eff} - p\mathbf{I}) = 0. \qquad (27)$$

After substituting the law of Darcy in the balance of mass (15), we find

$$\nabla \cdot \mathbf{v}^s - \nabla \cdot (\mathbf{K} \cdot \nabla p) = 0. \qquad (28)$$

To be able to apply the above theory to practical problems, relationships for the Helmholtz free energy A^s and the permeability $\mathbf{K}$ have to be provided. They can be derived from experiments on skin. Supplied with the function $A^s(\mathbf{F}^s)$ and $\mathbf{K}(\mathbf{F}^s)$, the set of equations (27) and (28) together with initial and boundary conditions constitute a set of non-linear equations describing the mechanical behaviour of a solid/fluid mixture. This set of equations can be solved numerically by means of the finite element method.

SOLUTION METHOD

In order to solve the coupled set of equations (27) and (28), an integral formulation is obtained by multiplying equation (27) with an arbitrary vector function $\mathbf{h}$ and equation (28) with an arbitrary scalar function g and integrating the result over a volume V of the mixture. After abbreviating from now on $\mathbf{T}_{eff}$ by $\mathbf{T}$ we obtain

$$\int_V (\nabla \cdot \mathbf{T} - \nabla p) \cdot \mathbf{h} \, dV = 0 \qquad (29)$$

$$\int_V (\nabla \cdot \mathbf{v}^s - \nabla \cdot (\mathbf{K} \cdot \nabla p)) g \, dV = 0 \qquad (30)$$

This set of equations is equivalent to (27) and (28). After some tensor manipulations and application of Gauss' theorem, (29) and (30) are found to be replaced by

$$\int_V [\mathbf{T} : (\nabla \mathbf{h})^C - p(\nabla \cdot \mathbf{h})] \, dV = \int_A \mathbf{n}_A \cdot (\mathbf{T} \cdot \mathbf{h} - p\mathbf{h}) \, da \qquad (31)$$

$$\int_V [g(\nabla \cdot v) + \mathbf{K} \cdot \nabla p \cdot \nabla g] \, dV = \int_A \mathbf{n}_A \cdot (\mathbf{K} \cdot \nabla p) g \, da \qquad (32)$$

where $\mathbf{n}_A$ denotes the outward unit normal on A. The term $\mathbf{n}_A \cdot \mathbf{T}$ can be recognized as the boundary stress vector acting on da. The term $\mathbf{n}_A \cdot p$ can be interpreted as a stress vector resulting from the hydrodynamic pressure p. The term $\mathbf{n}_A \cdot (\mathbf{K} \cdot \nabla p)$ can be considered to be the outward flow through the surface per unit area. In order to obtain time-independent integral boundaries, equations (31) and (32) are referred to the reference configuration of the solid matrix

$$\int_{V_0} [\mathbf{F}^{-1} \cdot \mathbf{T} J : (\nabla_0 \mathbf{h})^C - pJ\mathbf{F}^{-C} : (\nabla_0 \mathbf{h})] \, dV_0$$

$$= \int_{A_0} \mathbf{n}_{AO} \cdot [\mathbf{T} \cdot \mathbf{h} - p\mathbf{h}] J^a \, da_0 \qquad (33)$$

$$\int_{V_0} [g(\mathbf{F}^{-C} \cdot \nabla_0 \cdot \mathbf{v}) + \mathbf{K} \cdot \mathbf{F}^{-C} \cdot \nabla_0 p \cdot \mathbf{F}^{-C} \cdot \nabla_0 g] \, dV_0$$

$$= \int_{A_0} \mathbf{n}_{AO} \cdot (\mathbf{K} \cdot \mathbf{F}^{-C} \cdot \nabla_0 p) g J^a \, da_0 \qquad (34)$$

where: $J = \det(\mathbf{F})$; $J^a = J\mathbf{F}^{-1} \cdot \mathbf{n}_{AO}$; $\mathbf{n}_{AO}$ = the outward normal on the undeformed surface.

Substituting the second Piola–Kirchhoff stress $S = F^{-1} \cdot JT \cdot F^{-C}$ in equation (33) and taking the time derivative of (33) leads to

$$\int_{V_0} \dot{S} \cdot F^C : (\nabla_0 h)^C + S \cdot \dot{F}^C : (\nabla_0 h)^C dV_0$$

$$- \int_{V_0} [\dot{p}J(F^{-C}:\nabla_0 h) + p\dot{J}(F^{-C}:\nabla_0 h)$$

$$+ pJ(\dot{F}^{-C}:\nabla_0 h)] dV_0$$

$$= \int_{A_0} (n_A \cdot [\overline{(T - pI) \cdot h}] J^a da_0 \qquad (35)$$

If the strain energy is only a function of E, then we are able to write the time derivative of S in a form: $\dot{S} = {}^4M : \dot{E}$. This transforms (35) into:

$$\int_{V_0} [{}^4M : \dot{E} \cdot F^C : (\nabla_0 h)^C + S \cdot \dot{F}^C : (\nabla_0 h)^C] dV_0$$

$$- \int_{V_0} [\dot{p}J(F^{-C}:\nabla_0 h) + p\dot{J}(F^{-C}:\nabla_0 h)$$

$$+ pJ(\dot{F}^{-C}:\nabla_0 h)] dV_0$$

$$= \int_{A_0} (n_A \cdot [\overline{(T - pI) \cdot h}] J^a da_0 \qquad (36)$$

The set of equations (36) and (34) are solved by means of the finite element method. The observed mechanical system is divided into a number of elements of finite dimensions. The displacements of the solid, as well as the fluid pressures within one element will be approximated by means of an expansion

$$u = H^z \cdot u_z \quad (\alpha = 1, \ldots, n) \qquad (37)$$

$$p = G^z p_z \qquad (38)$$

n = the total number of nodes in one element; u_z = the displacement of node α; p_z = the fluid pressure in node α.

Summation convention is used, i.e. when an index occurs twice in a product term, this implies summation with respect to all its possible values, provided we deal with an upper and a lower index. H^z and G^z are called the interpolation or shape functions. The order and shape of these functions are determined by the shape of the element and the number of nodes used for the discretisation. Generally an exact solution is no longer possible after the above approximation is made. An approximate solution can be found by choosing well determined functions for h and g. We use the Galerkin weighted residual method (Zienkiewicz, 1977) which implies that H^z is used for h and G^z for g. Application of these functions leads in a very straightforward way

to a set of equations of the following form

$$\begin{pmatrix} \Gamma & R \\ R^T & O \end{pmatrix} \begin{pmatrix} \dot{u} \\ \dot{p} \end{pmatrix} + \begin{pmatrix} O & O \\ O & A \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} \dot{L} \\ \dot{Q} \end{pmatrix} \qquad (39)$$

The submatrices Γ, R and A are functions of the deformation and of time. The expressions for the submatrices Γ, R and A and for the columns L and Q are given in Appendix B. Assuming that Γ, R and A are constant in one time interval enables the integration of (38) by means of a one-step integration scheme. This leads to

$$\begin{pmatrix} \Gamma_n & R_n \\ R_n^T & A_n\theta\Delta t \end{pmatrix} \begin{pmatrix} u_{n+1} - u_n \\ p_{n+1} - p_n \end{pmatrix} = \begin{pmatrix} P_{n+1} - L_n \\ \Theta Q_{n+1}\Delta t + (1 - \theta)Q_n - A p_n\Delta t \end{pmatrix} \qquad (40)$$

where a lower index n means the value of the property at time t_n and $\Delta t = t_{n+1} - t_n$. In (40) P_{n+1} is the externally applied load at time t_{n+1}, whereas L_n is the volume integral of the internal stresses at time t_n. The value of θ depends on the type of integration. With $\theta = 0$ the integration is called explicit. Stable and accurate solutions can be found when $\theta > 0.5$. Equation (40) is a linearized equation with regard to the balance of momentum. By means of equation (33) it is possible to calculate the volume integral of the internal stresses L_{n+1} at time t_{n+1}, which can be used for comparison with the applied load P_{n+1} on the boundaries.

Let us assume that we want to calculate the values u_{n+1} at time t_{n+1}, while u_n and p_n are known. We use u_n and p_n to calculate Γ_n, R_n and A_n. After applying a new load increment, we are able to calculate a first estimate of the displacements $u_{n+1}^{(1)}$ and the pressure $p_{n+1}^{(1)}$ at time t_{n+1} by means of equation (40). We substitute $u_{n+1}^{(1)}$ and $p_{n+1}^{(1)}$ into (33). This leads to the actual integrated stresses corresponding to $u_{n+1}^{(1)}$ and $p_{n+1}^{(1)}$ given by $L_{n+1}^{(1)}$. In general $L_{n+1}^{(1)}$ will not be the same as P_{n+1}. The difference is substituted in equation (40) and is used to calculate a new estimate $u_{n+1}^{(2)}$ for u_{n+1} by means of

$$\begin{pmatrix} \Gamma_{n+1}^{(1)} & R_{n+1}^{(1)} \\ R_{n+1}^{T(1)} & A_{n+1}^{(1)}\theta\nabla t \end{pmatrix} \begin{pmatrix} u_{n+1}^{(2)} - u_{n+1}^{(1)} \\ p_{n+1}^{(2)} - p_{n+1}^{(1)} \end{pmatrix} = \begin{pmatrix} P_{n+1} - L_{n+1}^{(1)} \\ Q \end{pmatrix}.$$

$$(41)$$

We proceed with this iteration until $P_{n+1} - L_{n+1}^{(1)}$ is small enough. This iteration within one time interval does not improve the time integration (this can only be done by choosing smaller time steps), but more accurately satisfies equilibrium at t_{n+1}. In this sense the method can be viewed upon as an ordinary Newton–Raphson iteration procedure.

The above theory has been used to develop a four-noded, isoparametric, plain strain element. In the next section some of the results of calculations with this element will be presented.

ONE-DIMENSIONAL CONFINED COMPRESSION

To study the two-phase behaviour of the previously described element several tests can be performed. One of the most simple tests available is a confined compression test. The objective of the test is to study the influence of the integration constant θ, the time step and the element mesh on the solution. We do this by comparing the numerical solution for the linear problem (infinitesimal strain, isotropic, linear elastic material behaviour for the solid, constant permeability) with the analytical solution. An analytical solution is also available for the non-linear problem with a strain dependent permeability. This will be treated at the end of this section.

In a confined compression test a sample of two-phase material is placed in a tightly fitting ring on a porous filter. This filter has a very high permeability compared to that of the sample. During the test a stepwise load F is applied to the tissue sample, which can deform in one direction only by expelling fluid (Fig. 1).

In this one-dimensional case equations (27) and (28) can be combined and the solid displacements can be eliminated. This leads to the following equation for the fluid pressure

$$\frac{\partial p}{\partial t} = HK \frac{\partial^2 p}{\partial y^2} \tag{42}$$

with: H = confined compression modulus or aggregate elastic modulus; K = one-dimensional permeability.

At this stage it is convenient to introduce nondimensional variables

$$Y = \frac{y}{L} \tag{43}$$

$$T = \frac{HK}{L^2} t \tag{44}$$

$$P = \frac{A}{F} p \tag{45}$$

where A is the cross-section of the sample.

The initial condition is given by

$$P = 1 \text{ for } 0 \leqslant Y \leqslant 1 \text{ at } T = 0. \tag{46}$$

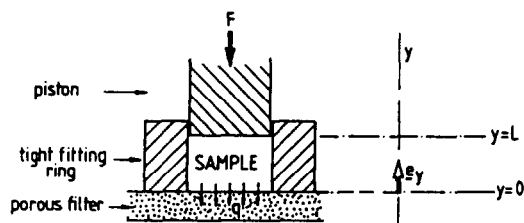The above condition arises from the internal constraint of incompressibility, so the external load is balanced by the hydrodynamic pressure. At $T > 0$ the boundary conditions are given by

$$P = 0 \quad \text{at } Y = 0 \text{ (free drainage)} \tag{47}$$

$$dP/dY = 0 \quad \text{at } Y = 1 \text{ (no flow through } Y = 1). \tag{48}$$

The analytical solution of this problem can be found in the literature on diffusing media (Carlslaw and Jaeger, 1947):

$$P = \sum_{i=0}^{\infty} \frac{2}{N} (\sin NY) \exp(-N^2 T) \tag{49}$$

with: $N = \frac{\pi}{2} (2i + 1)$.

A numerical test has been carried out by means of the five-element mesh shown in Fig. 2. All displacements in the x-direction have been suppressed. The mesh is supported in nodes 11 and 12. At time $T = 0$ a load is applied by means of two prescribed forces $F/2$ on the nodes 1 and 2. These forces are held constant during all time increments.

Three tests have been carried out, each with an exponentially growing time interval ΔT, which was chosen such that T increased from 0.025 to 1 in 10 time increments. These tests have been carried out with a rather large time step and a mesh that is not fine enough for an accurate calculation. The reason for this is to be able to see clearly the effect of possible refinements. We have used three different Crank–Nicholson constants $\theta = 0.5, 0.6$ and 1. In Fig. 3 the hydrodynamic pressure is given as a function of the position Y for three different values of the time T. It becomes immediately clear that the solution with $\theta = 0.5$ oscillates around the analytical solution, but $\theta = 0.6$ and 1 lead to stable solutions. This is in agreement with results of Booker and Small (1975) for the linear case.

In Fig. 4 a solution with $\theta = 1$ is given for small values of the loading time T. It can be seen that immediately after load application we find a spatial instability that disappears when time increases. This is a well-known problem with this kind of elements (Verruyt, 1977). The spatial instability can, if necessary,

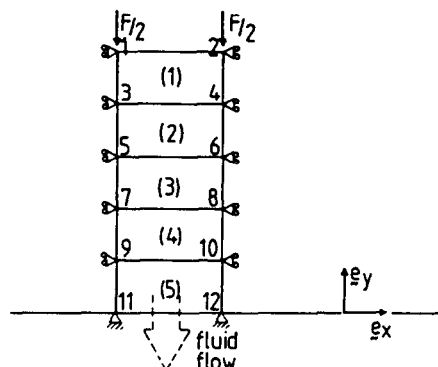Fig. 1. Schematic drawing of the confined compression test.

Fig. 2. Finite element mesh for a confined compression test.
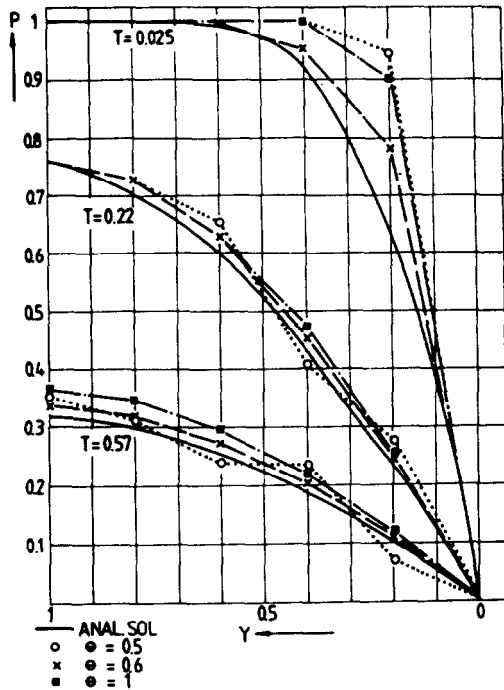
Fig. 3. Non-dimensional pressure against position in a one-dimensional compression test.
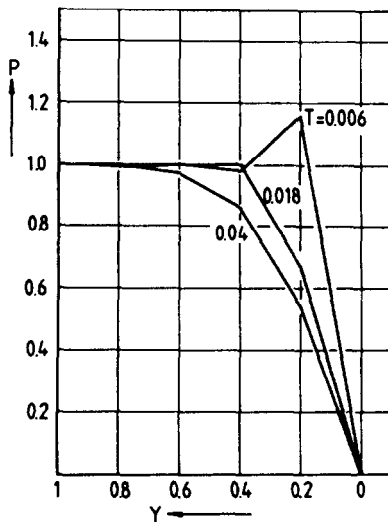

Fig. 4. Non-dimensional pressure against position just after load application.

Fig. 5. Non-dimensional pressure against position using a 15-element mesh.

case with a strain dependent permeability K according to

$$K = K_0 \exp[M\varepsilon] \qquad (50)$$

with: $K_0 =$ permeability before deformation; M = non-dimensional constant; ε = volumetric infinitesimal strain.

We introduce the following non-dimensional parameter for the displacement

$$U = \frac{u(L, T)}{u(L, \infty)} = \frac{uAH}{FL}. \qquad (51)$$

Mow et al. (1984) found in the first (small) period after load application that

$$U = \frac{2}{\sqrt{\pi}} \left[\exp\left(-\frac{FM}{\pi HA} \right) \right] \sqrt{T}. \qquad (52)$$

This result was used as one of the tests for the mixture element. The result is shown in Fig. 6. For small values of T the analytical and numerical solutions are the same. When T increases the asymptotic analytical solution is no longer valid and the two solutions diverge, as is to be expected.

DISCUSSION

As stated in the introduction, we assume that a mixture approach is a promising method for the

be shifted to lower times by mesh refinement, so there is a reasonable progression of the consolidation in each time step. An example of this improvement is illustrated in Fig. 5 where a 15-element mesh has been used.

There are very few non-linear solutions available for mixture problems. Mow et al. (1984) were able to derive an analytical solution valid for a small time period just after load application for the non-linear
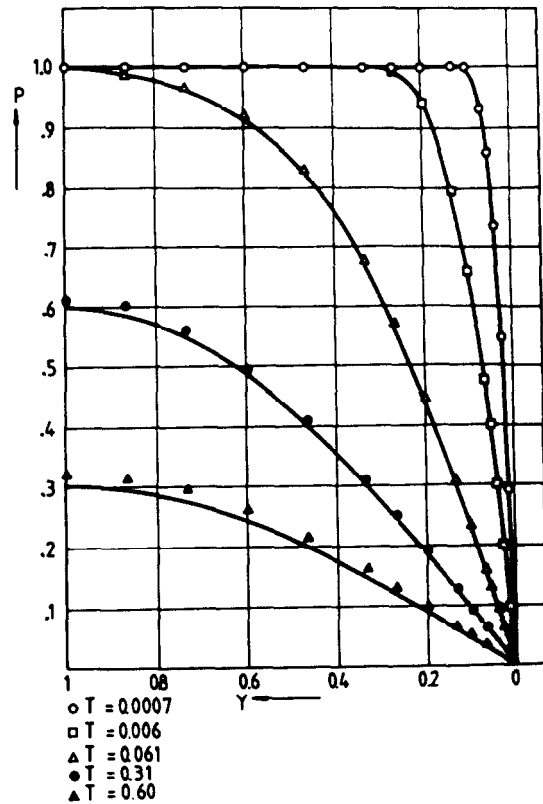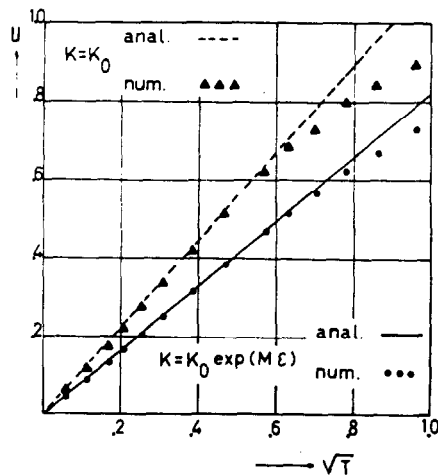
Fig. 6. Reduced displacement of the boundary at $Y = L$ against the square root of time T in a confined compression test. The dotted line represents the analytical solution in the case of a constant permeability (tangent is $2/\sqrt{\pi}$). The solid line represents the solution for a strain dependent permeability. $K_0 = 3.5 * 10^{-14}$ m^4 N^{-1} s^{-1}, $H = 10^5$ N m^{-2}, $M = 10$, $F/A = 10^4$ N m^{-2}.

mechanical description of skin. Oomens et al. (1985) have shown that indeed results of confined compression experiments on porcine skin point in this direction. We also think that a mixture approach can be used for other types of soft tissues, for example subcutaneous fat and muscle (Huyghe et al., 1983).

The numerical method used for the solution of the non-linear field equations, describing a solid/fluid mixture, is very promising. It is possible to incorporate all kinds of non-linear constitutive laws for the solid, the fluid and the interaction terms. With some adjustments it can also be used for the case that mass exchange is not excluded (Snijders, 1986).

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APPENDIX A

In this appendix the equations (18)–(22) will be derived from the second law of thermodynamics. When the mass balance (15), multiplied by a Lagrange multiplier p is added to the second law of thermodynamics (14), and if we assume isothermal conditions to exist within the mixture, a mixture of a solid and a fluid will lead to the following inequality

$$-\rho^f \frac{D^{(f)}A^f}{Dt} - \rho^s \frac{D^{(s)}A^s}{Dt} + \pi^s \cdot \mathbf{a} + \mathbf{T}^f : \mathbf{L}^f + \mathbf{T}^s : \mathbf{L}^s$$

$$+ p(\nabla \cdot \mathbf{v}^s - \nabla \cdot n^f \mathbf{a}^f) \geq 0. \tag{A1}$$

By means of

$$\nabla \cdot \mathbf{v}^s = \mathbf{I} : \nabla \mathbf{v}^s = \mathbf{I} : \mathbf{L}^s \tag{A2}$$

and formula (6), the last term on the left-hand side of (A1) can be transformed to

$$p(\nabla \cdot \mathbf{v}^s - \nabla \cdot n^f \mathbf{a}^f) = pn^s \mathbf{I} : \mathbf{L}^s + pn^f \mathbf{I} : \mathbf{L}^f + \frac{p}{\rho_*^f} \mathbf{a} \cdot \mathbf{g}. \tag{A3}$$

The time differentiation in the first two terms of (A1) needs some further elaboration. When (17) is used and the principle of equipresence, we can write for the time derivative of A^f

$$\frac{D^{(f)} A^f}{Dt} = \frac{\partial A^f}{\partial \mathbf{F}^s} : \frac{D^{(f)} \mathbf{F}^{sC}}{Dt} + \frac{\partial A^f}{\partial \rho^f} \frac{D^{(f)} \rho^f}{Dt}$$

$$+ \frac{\partial A^f}{\partial \mathbf{g}} \cdot \frac{D^{(f)} \mathbf{g}}{Dt} + \frac{\partial A^f}{\partial \mathbf{a}} \cdot \frac{D^{(f)} \mathbf{a}}{Dt}. \tag{A4}$$

Using equality (3), the operator for time derivation can be written as

$$\frac{D^{(f)}}{Dt} = \frac{D^{(s)}}{Dt} + \mathbf{a} \cdot \nabla. \tag{A5}$$

By means of (A5) the first term of (A4) becomes

$$\frac{\partial A^f}{\partial \mathbf{F}^s} : \frac{D^{(f)} \mathbf{F}^{sC}}{Dt} = \frac{\partial A^f}{\partial \mathbf{F}^s} : \frac{D^{(s)} \mathbf{F}^{sC}}{Dt} + \frac{\partial A^f}{\partial \mathbf{F}^s} : (\mathbf{a} \cdot \nabla \mathbf{F}^s)^C \tag{A6}$$

By means of equations (7) and (A2) we are able to write for the second term in the right-hand side of (A4)

$$\frac{\partial A^f}{\partial \rho^f} \frac{D^{(f)} \rho^f}{Dt} = -\rho^f \frac{\partial A^f}{\partial \rho^f} \mathbf{I} : \mathbf{L}^f. \tag{A7}$$

The time derivative of A^s is treated in the same way as above. If we substitute the above results in (A1) and rearrange some terms, equation (A1) becomes

$$\left[-\rho^f \mathbf{F}^s \cdot \left(\frac{\partial A^f}{\partial \mathbf{F}^s} \right)^C - \rho^s \mathbf{F}^s \cdot \left(\frac{\partial A^s}{\partial \mathbf{F}^s} \right)^C + \mathbf{T}^s + n^s p \mathbf{I} \right.$$

$$\left. - \rho^s \mathbf{a} \frac{\partial A^s}{\partial \mathbf{a}} \right] : \mathbf{L}^s$$

$$+ \left[-(\rho^f)^2 \frac{\partial A^f}{\partial \rho^f} \mathbf{I} - \rho^s \rho^f \frac{\partial A^s}{\partial \rho^f} \mathbf{I} + \mathbf{T}^f + n^f p \mathbf{I} \right.$$

$$\left. + \rho^s \mathbf{a} \frac{\partial A^s}{\partial \mathbf{a}} \right] : \mathbf{L}^f$$

$$+ \left(-\rho^f \frac{\partial A^f}{\partial \mathbf{g}} - \rho^s \frac{\partial A^s}{\partial \mathbf{g}} \right) \cdot \frac{D^{(f)} \mathbf{g}}{Dt}$$

$$- \rho^f \frac{\partial A^f}{\partial \mathbf{F}^s} : (\mathbf{a} \cdot \nabla \mathbf{F}^s)^C$$

$$+ \left(-\rho^f \frac{\partial A^f}{\partial \mathbf{a}} - \rho^s \frac{\partial A^s}{\partial \mathbf{a}} \right) \cdot \frac{\partial^{(f)} \mathbf{a}}{\partial t}$$

$$+ \rho^s \frac{\partial A^s}{\partial \mathbf{g}} \cdot (\mathbf{a} \cdot \nabla \mathbf{g})$$

$$+ \left(\rho^s \frac{\partial A^s}{\partial \rho^f} + \frac{p}{\rho_*^f} \right) \mathbf{g} \cdot \mathbf{a} + \pi^f \cdot \mathbf{a} \geqslant 0. \tag{A8}$$

In an admissible thermodynamic process at some point in space the independent variables may take arbitrary values. This also holds for the path along which some value in a point is reached. This means that the following parameters may also take arbitrary values

$$\mathbf{L}^s, \ \mathbf{L}^f, \ \nabla \mathbf{F}^s, \ \frac{D^f \mathbf{g}}{Dt}, \ \frac{D^f \mathbf{a}}{Dt}, \ \nabla \mathbf{g}.$$

Because the above mentioned parameters all appear linearly in (A8), the factors before these parameters have to be zero, to fulfil (A8) for arbitrary values of the independent variables. From this it will be immediately clear that A^f cannot depend on $\mathbf{F}^s$ and neither A^f nor A^s depends on $\mathbf{g}$. The other zero terms lead to the restrictions mentioned in formulas (18)-(22).

APPENDIX B

In this appendix the submatrices of equation (39) are given. The following weighting functions have been used

$$\mathbf{h} = \mathbf{H}^\alpha \cdot \mathbf{g}_{0k} \qquad \begin{aligned} \alpha &= (1, \ldots, n) \\ k &= (1, 2, 3) \end{aligned}$$

$$g = G^\alpha. \tag{B1}$$

$\mathbf{g}_{0k}$ are covariant base vectors referring to the undeformed configuration. A greek index means a nodal number, whereas a roman index means a geometrical degree of freedom. For the sake of convenience we also introduce the following third order tensor, depending on the shape functions $\mathbf{H}^\alpha$

$$^3\mathbf{D}^\alpha = \nabla_0 \mathbf{H}^\alpha. \tag{B2}$$

Substituting (B1) and (B2) in (36) leads after some tensor calculations to the following expression for $\underline{\Gamma}$

$$\underline{\Gamma} = G_{rs}^{\beta\alpha} + E_{rs}^{\beta\alpha} + I_{rs}^{\beta\alpha} + M_{rs}^{\beta\alpha} + N_{rs}^{\beta\alpha} \tag{B3}$$

with

$$G_{rs}^{\beta\alpha} = \int_{V_0} (^3\mathbf{D}^\beta)^C \cdot \mathbf{g}_{0r} : [^4\mathbf{M} : (^3\mathbf{D}^\gamma)^C \cdot \mathbf{u}_\gamma + \mathbf{u}_\delta \cdot {}^3\mathbf{D}^\delta : {}^4\mathbf{M}$$

$$+ \mathbf{u}_\epsilon \cdot {}^3\mathbf{D}^\epsilon \cdot {}^4\mathbf{M} \cdot (^3\mathbf{D}^\eta)^C \cdot \mathbf{u}_\eta] : \mathbf{g}_{0s} \cdot {}^3\mathbf{D}^\alpha \mathrm{d} V_0 \tag{B4}$$

$$E_{rs}^{\beta\alpha} = \int_{V_0} (^3\mathbf{D}^\beta)^C \cdot \mathbf{g}_{0r} : {}^4\mathbf{M} : \mathbf{g}_{0s} \cdot {}^3\mathbf{D}^\alpha \mathrm{d} V_0 \tag{B5}$$

$$I_{rs}^{\beta\alpha} = \int_{V_0} \mathbf{S} : (^3\mathbf{D}^\beta)^C \cdot \mathbf{g}_{0r} \cdot \mathbf{g}_{0s} \cdot {}^3\mathbf{D}^\alpha \mathrm{d} V_0 \tag{B6}$$

$$N_{rs}^{\beta\alpha} = -\int_{V_0} G^\gamma p_\gamma J (\mathbf{F}^{-C} : (^3\mathbf{D}^\beta)^C \cdot \mathbf{g}_{0r})$$

$$\times (\mathbf{F}^{-C} : (^3\mathbf{D}^\alpha)^C \cdot \mathbf{g}_{0s}) \mathrm{d} V_0 \tag{B7}$$

$$M_{rs}^{\beta\alpha} = \int_{V_0} G^\gamma p_\gamma J (\mathbf{F}^{-1} \cdot \mathbf{g}_{0\theta} \cdot {}^3\mathbf{D}^\beta) : (\mathbf{F}^{-1} \cdot \mathbf{g}_{0s} \cdot {}^3\mathbf{D}^\alpha) \mathrm{d} V_0. \tag{B8}$$

The matrix $\underline{R} = R_r^{\beta\alpha}$ is equal to

$$R_r^{\beta\alpha} = \int_{V_0} -G^\alpha J \mathbf{F}^{-1} : (^3\mathbf{D}^\beta)^C \cdot \mathbf{g}_{0r} \mathrm{d} V_0. \tag{B9}$$

For $\underline{A} = A^{\beta\alpha}$ we find

$$A^{\beta\alpha} = \int_{V_0} \mathbf{K} \cdot \mathbf{F}^{-C} \cdot \nabla_0 G^\alpha \cdot \mathbf{F}^{-C} \cdot \nabla_0 G^\beta \mathrm{d} V_0 \tag{B10}$$

For $\underline{L}$ we find

$$\underline{L} = L_r^\beta = \int_{A_0} [\mathbf{k}_0 \cdot \mathbf{H}^\beta \cdot \mathbf{g}_{0r} + \mathbf{d}_0 \cdot \mathbf{H}^\beta \cdot \mathbf{g}_{0r}] \mathrm{d} A_0 \tag{B11}$$

with

$$\mathbf{k}_o = \mathbf{n}_A \cdot \mathbf{T} J^a \quad \text{and} \quad \mathbf{d}_o = \mathbf{n}_A p J^a.$$

For Q we write

$$Q^\beta = \int_{A_0} q_A G^\beta J_0^a \mathrm{d} A_0 \tag{B12}$$

with $q_A = \mathbf{n}_{A0} \cdot (\mathbf{K} \cdot \mathbf{F}^{-C} \cdot \nabla_0 p)$.