COUNTING THE RATIONALS
PROMYS 2021 EXPLORATION LAB
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Abstract. For our exploration lab, we explored the function f and the binary tree T. We compare the relationship between the two functions and use their relationship to aid us in further study of the function f and potential applications to other topics in number theory.

a. Introduction

The function f takes a natural number as an input and returns a positive rational number as an output. f is defined as:

$$f(1) = \frac{1}{1} \text{ and } f(n+1) = \frac{1}{[f(n)] + 1 - \{f(n)\}}.$$

Where $[x]$ is the greatest integer less than x and $\{x\}$ is the fractional part of x.
T is an infinite binary tree with a root of $\frac{1}{1}$. From each node, $\frac{p}{q}$, the left child is $\frac{p+q}{q}$ and the right child is $\frac{p}{p+q}$.
We were first intrigued by this topic because of the seemingly random recursive formula that correlates so well with the binary tree. We began this project by doing some calculations to look for patterns. We then showed that the function is one-to-one and onto, which may show that the tree is one-to-one and onto. We did this with the purpose of showing the function is bijective.

We also explored a potential method of proving the correlation between the tree and the function and a method to describe the function non-recursively. In the search for an inverse, we found a potential connection to number theory. In addition, we labelled the binary tree using binary, and found a connection to continued fractions.

All of these show that further exploration of this function and tree may reveal deep implications of number theory.

b. Numerical Exploration

$$f(2) = \frac{1}{2} \quad f(3) = 2 \quad f(4) = \frac{1}{3} \quad f(5) = \frac{3}{2}$$

$$f(6) = \frac{2}{3} \quad f(7) = 3 \quad f(8) = \frac{1}{4} \quad f(9) = \frac{4}{3}$$

$$f(10) = \frac{3}{5} \quad f(11) = \frac{5}{2} \quad f(12) = \frac{2}{5} \quad f(13) = \frac{5}{3}$$

$$f(14) = \frac{3}{4} \quad f(15) = 4 \quad f(16) = \frac{1}{5}$$

The following are some graphs that was plotted to visualize the data.

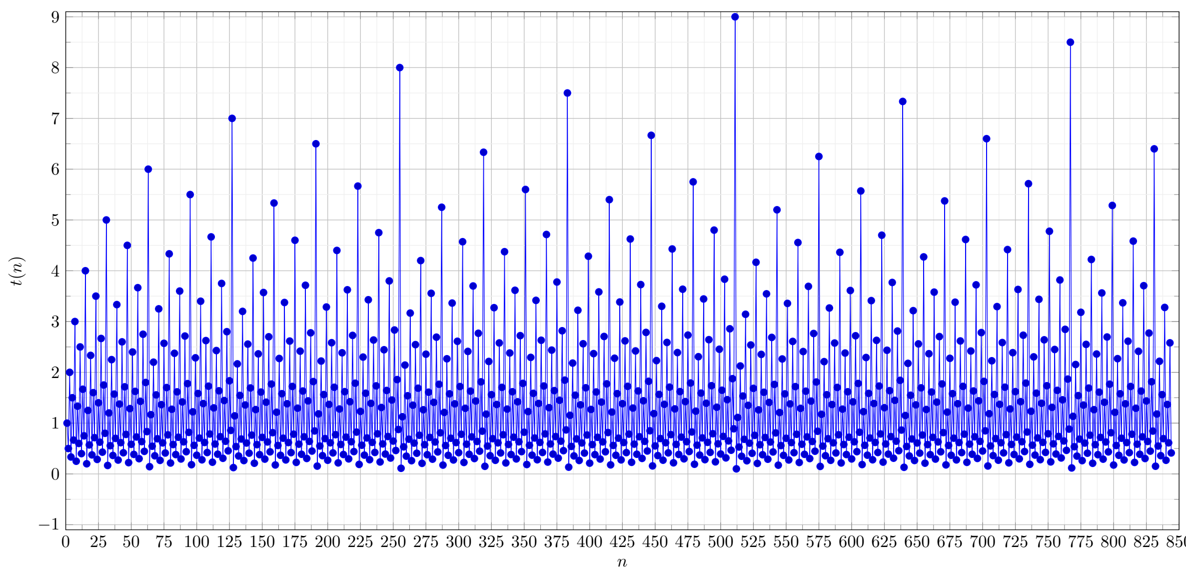

FIGURE 1. A graph of the first ~ 850 values of $f(n)$

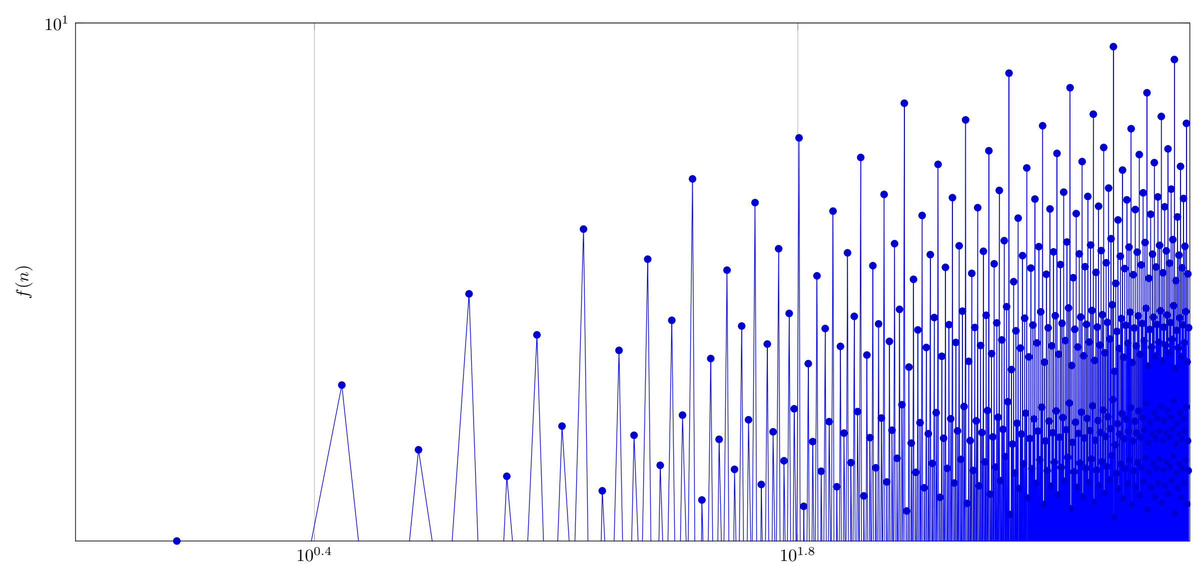

FIGURE 2. A logarithmic graph of the first ~ 850 values of $f(n)$

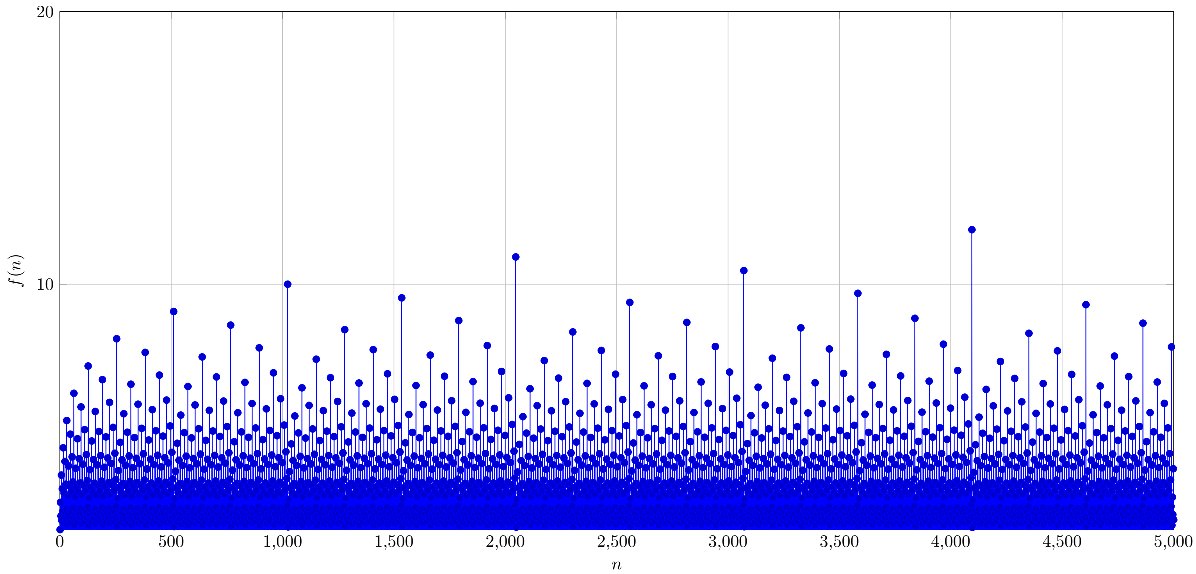

FIGURE 3. A graph of the first 5000 values of $f(n)$

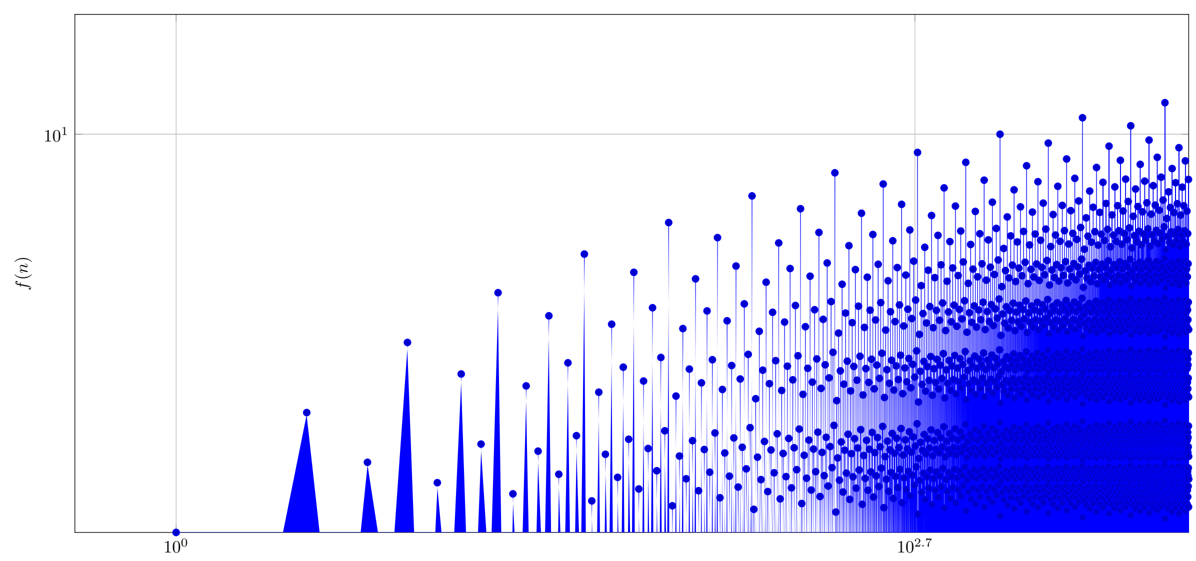

FIGURE 4. A logarithmic graph of the first 5000 values of $f(n)$ (note that it looks like there is quantized levels)

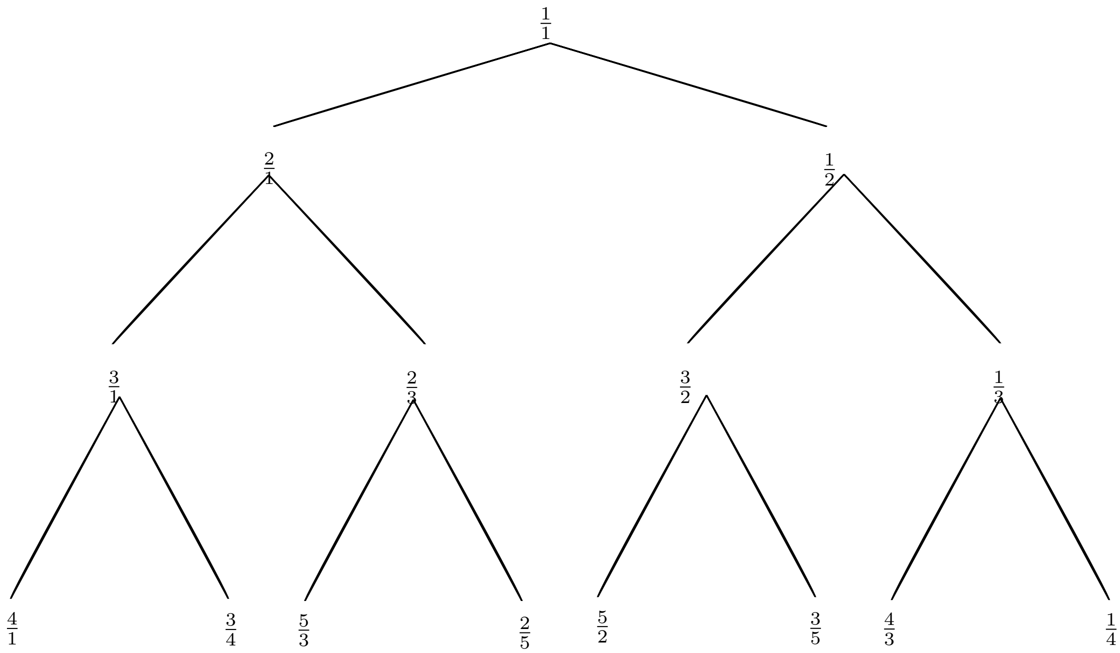

FIGURE 5. Our binary tree diagram for the first four rows

The values of the function f appear to correspond with the values of the tree (see Fig. 5). It appears to be filled from top to bottom, right to left. From the tree we conjecture that

Conjecture 1. $f(2^{n-1})$ always gives output $\frac{1}{n}$ and every $f(2^n-1)$ gives output $\frac{n}{1}$.

Conjecture 2. For every odd n, $f(n)>1$, and for every even n, $f(n)<1$.

Conjecture 3. Given a row m, $f(\frac{3\cdot 2^m}{4}-1-w)=\dfrac{1}{f(\frac{3\cdot 2^m}{4}+w)}$, for an element in row m that is w indices from the right side (describes the symmetry, the rightmost is 0).

c. Basic Facts about f and T

Theorem 1. The tree T always gives fractions in lowest terms.

Proof. A fraction is simplified if their terms are coprime. Being coprime implies that their greatest common factor is 1. We will use induction to show that all nodes have numerator and denominator coprime to each other.
Base case: The root. The first fraction $\frac{1}{1}$ is simplified because $(1,1)=1$.
Let the parent node be $\frac{p}{q}$ and $(p,q)=1$. For its left child, $\frac{p+q}{q}$, we can say that $(p+q,p) \implies (p+q-p,p)=(q,p)$, which we know is 1. Similarly, for $\frac{p}{p+q}$, $(p,p+q)=(p,p+q-p)=(p,q)=1$. Thus, they are coprime. Therefore, if the parent node are in simplest terms, the children nodes are as well. $\square$

Proposition 1. For all even n, $f(n)<1$. For all odd $n\neq 1$, $f(n)>1$.

Proof. Let n be a natural number and let $f(n)=m+\frac{p}{q}$ for $m,p,q\in\mathbb{N}$, $(p,q)=1$ and $p<q$. Suppose $f(n)>1$. By the given definition, $f(n+1)=\frac{1}{m+1-\frac{p}{q}}$. Since $\frac{p}{q}<1$ we know that $1-\frac{p}{q}$ is a positive fraction. Thus,

$$m\geq 1 \implies m+1-\frac{p}{q}>1.$$

We can divide both sides by the LHS to see that

$$1>\frac{1}{m+1-\frac{p}{q}}.$$

Thus $f(n+1)$ is less than one.
Suppose $f(n) < 1$. Then $m = 0$. Then, by the formula,

$$f(n+1) = \frac{1}{0 + 1 - \frac{p}{q}} = \frac{1}{\frac{q-p}{q}} = \frac{q}{q-p}$$

Since we know that $q > q - p$ since $p \in \mathbb{N}$, $\frac{q}{q-p} > 1$. Since $f(2) = \frac{1}{2}$, which satisfies that $f(n) < 1$ and that n is even, the above properties are true. $\square$

Theorem 2. This function is one-to-one.

Proof. Consider the set $S = \{x | \exists y \text{ such that } x \neq y, f(x) = f(y), \}$. Assume that this set is non-empty. By WOP, there exists a least element in this set. Let x_1 be the smallest element of this set (since x, the input of the function, is defined to be a natural number). Then, we can say that there exists a y_1 such that $f(x_1) = f(y_1)$.

$$\begin{aligned}
\frac{1}{[f(x_1-1)] + 1 - \{f(x_1-1)\}} &= \frac{1}{[f(y_1-1)] + 1 - \{f(y_1-1)\}} \\
[f(x_1-1)] + 1 - \{f(x_1-1)\} &= [f(y_1-1)] + 1 - \{f(y_1-1)\} \\
[f(x_1-1)] - \{f(x_1-1)\} &= [f(y_1-1)] - \{f(y_1-1)\} \\
[f(x_1-1)] - [f(y_1-1)] &= \{f(x_1-1)\} - \{f(y_1-1)\}
\end{aligned}$$

Consider the LHS. Since $[f(x_1-1)]$ and $[f(y_1-1)]$ are both integers and the integers are closed under addition, then the LHS is an integer. Thus, the RHS has to be an integer as well. By definition of the fractional part, we know that

$$\begin{aligned}
0 \leq& \{f(x_1-1)\} < 1 \\
-\{f(y_1-1)\} \leq& \{f(x_1-1)\} - \{f(y_1-1)\} < 1 - \{f(y_1-1)\} \\
-1 < -\{f(y_1-1)\} \leq& \{f(x_1-1)\} - \{f(y_1-1)\} < 1 - \{f(y_1-1)\} < 1 \\
-1 <& \{f(x_1-1)\} - \{f(y_1-1)\} < 1
\end{aligned}$$

Thus, the only integer value of $\{f(x_1-1)\} - \{f(y_1-1)\}$ would be 0. Then, $\{f(x_1-1)\} = \{f(y_1-1)\}$. Substituting, we get

$$[f(x_1-1)] - \{f(y_1-1)\} = [f(y_1-1)] - \{f(y_1-1)\} \implies [f(x_1-1)] = [f(y_1-1)]$$

Since the fractional and the integer part of $f(x_1-1)$ and $f(y_1-1)$ are equal, we can conclude that $f(x_1-1) = f(y_1-1)$. However, x_1 is the smallest such x that this is true. Therefore, $x_1 - 1$ must not be a natural number. Then, we assume that $x_1 - 1 = 0$. So, $x_1 = 1$. So, $f(x_1) = f(1) = 1$. We have to find a $f(y)$ such that $f(y) = 1$.

$$\begin{aligned}
f(y) = \frac{1}{[f(y-1)] + 1 - \{f(y-1)\}} &= 1 \\
[f(y-1)] + 1 - \{f(y-1)\} &= 1 \\
[f(y-1)] - \{f(y-1)\} &= 0 \\
[f(y-1)] &= \{f(y-1)\}
\end{aligned}$$

As previously shown, the only integer that $\{f(y-1)\}$ can be is 0. Then $[f(y-1)] = 0$. This implies that $f(y-1) = 0 + 0 = 0$. This is a contradiction because the function outputs positive rationals and 0 is not a positive rational. Therefore, there does not exist a $f(y)$ where $y \neq 1$ such that $f(y) = 1$. Therefore, the set S is empty, which implies that no two inputs have the same output, so each input has at most one output. $\square$

Theorem 3. This function is onto. Suppose that there is a rational number $\frac{p}{q}$ in simplest terms where $p, q \in \mathbb{N}$ such that $\frac{p}{q}$ is not an output of the function. Then, we can say that there does not exist a

$m_1 \in \mathbb{N} \bigcup \{0\}$ and $p_1, q_1 \in \mathbb{N}$, and $p_1 < q_1$ such that $f(n) = m_1 + \frac{p_1}{q_1} = \frac{p}{q}$.
Thus, there does not exist m_1, p_1, q_1 such that

$$\frac{1}{m_1 + 1 - \frac{p_1}{q_1}} = \frac{q_1}{q_1(m_1+1) - p_1} = \frac{p}{q}$$

Since, as previously proven, the fractions are always in simplest terms. Therefore, we can equate $p = q_1$ and $q = q_1(m_1+1) - p_1$. Substituting, we get $q = p(m_1+1) - 1(p_1)$. By division algorithm, we can find a m_1 and p_1 fulfilling this equation such that $p_1 < p = q_1$. However, we claimed that there are no such m_1, p_1, so this is a contradiction. Thus, there are no $\frac{p}{q}$ that cannot be expressed by the function.

This is an incomplete and incorrect proof. Note that the proof only states that there exists m_1, p_1, q_1 *that satisfies the statement, not that this certain combination exists.*

Note: There may be a proof method that is similar to this poof by ordering the rationals under the sum of numerator and denominator, applying WOP since the numerator and denominator are naturals, then arise at a contradiction when we assume that $\frac{p}{q}$ is the smallest (not necessarily unique) fraction as $p_1 + q_1 < p + q$.

d. Connecting the Function and the Tree

Conjecture 4. If $f(n)$ is a node on the tree, then $f(2n)$ would be the right child of $f(n)$. Thus, we can say that, $f(n) = \frac{p}{q}$, then $f(2n) = \frac{p}{q+p}$ and $f(2n+1) = \frac{q+p}{q}$.

Proposition 2. If $f(2n) = \frac{p}{q+p}$ then $f(2n+1) = \frac{q+p}{q}$.

Proof. Assuming the first part of conjecture 4, we can say that for all $n, \exists p, q$ such that $f(2n) = \frac{p}{q+p}$. Since $2n$ is even, $f(2n) < 1$ by Proposition 1, so

$$f(2n+1) = \frac{1}{0 + 1 - \frac{p}{q+p}} = \frac{1}{\frac{q+p-p}{q+p}} = \frac{q+p}{q}$$

□

Conjecture 5. The n in $f(n)$ corresponds to a node on the tree in such a way. Let $n = 2^k + w$ where k is the greatest integer such that $2^k \leq n$ and w is a positive integer or 0. Then, $f(n)$ corresponds to the node that is in the $k+1$ row and w from the right (where if $w = 0$, then $f(n)$ is the rightmost index).

e. Binary Tree

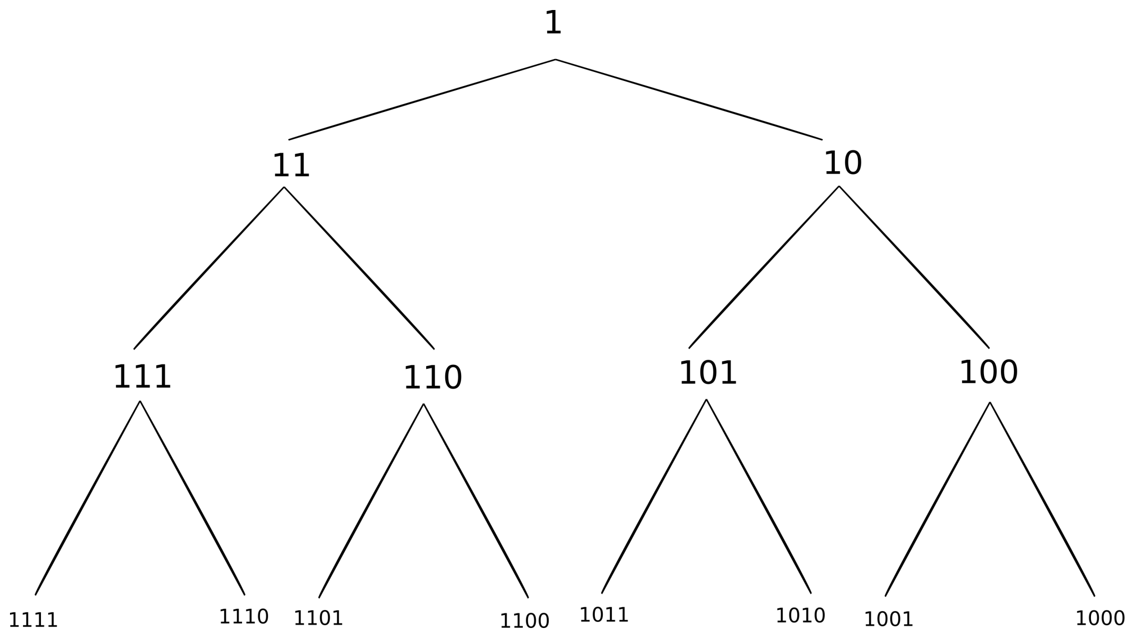

Figure 6. Binary Tree

The binary tree in Figure 6 numbers the nodes of the trees. Observe how from one node to its two children, either 0 or 1 gets added after the digit. Expressed as an equation, if n is the parent node, its children nodes are $2n$ to the right and $2n+1$ to the left, which confirms a pattern with $f(n)$ that we have observed earlier.

Proposition 3. We can use the binary tree to find $f(n)$ given n

Assuming conjecture 5, we can say the following: given the node of the position of $\frac{p}{q}$ on the tree, then we can represent n in binary, such that $n = (x_m x_{m-1} x_{m-2} \dots x_0)_2$ where x_i are 1 or 0.
Thus,

$$f(n) = f(x_m \cdot 2^m + x_{m-1} \cdot 2^{m-1} + \cdots + x_0 \cdot 2^0)$$

If we let $h(x) = x_m 2^{m-1} + x_{m-1} 2^{m-2} + \cdots + x_1$. Then, we can say that $f(n) = f(2h(x) + x_0)$. We can repeat this process, so that $f(n) = f(2(2h'(x) + x_1) + x_0$. Thus, we can say that

$$\begin{aligned} &f(2(x_m \cdot 2^{m-1} + x_{m-1} 2^{m-2} + x_{m-2} 2^{m-3} + \cdots + x_1) + x_0) \\ &= f(2(2(2(\dots 2(x_m) + x_{m-1}) + x_{m-3} + \dots) + x_1) + x_0) \end{aligned}$$

Thus, we have shown a recursive way of getting $f(n)$ with the binary tree. As stated in Conjecture 5, moving down and to the right on the tree is the same as taking $f(2n)$ and moving down and to the left on the tree is the same as taking $f(2n+1)$. The left and right movement are determined by the $x_0, x_1, \dots x_{m-1}$. Then, starting with $f(1) = \frac{1}{1}$, we can compute the continued fraction of the value of a function using its binary representation.

Example 1. Compute $f(11)$.
The binary representation for $11_2 = 1011$, so from $f(1)$ we move right, left, and left. From the formula of the tree T, given a node such that the value associated on the tree is $\frac{p}{q}$, the left child is $\frac{p+q}{q} = 1 + \frac{p}{q}$ and the right child is $\frac{p}{p+q} = \frac{1}{1+\frac{q}{p}}$. Thus, given $f(1) = \frac{1}{1}$, we can use the above to find $f(2) = \frac{1}{1+\frac{1}{1}}$, $f(5) = 1 + \frac{1}{1+\frac{1}{1}}$, and finally $f(11) = 2 + \frac{1}{1+\frac{1}{1}} = \frac{5}{2}$.

f. Miscellaneous Work

Conjecture 6. Given a fraction $\frac{p}{q}$, we conjecture that we can restrict the number of potential n such that $f(n) = \frac{p}{a}$ for some denominator a to a set of $\varphi(p)$ odd numbers such that the product of a power of two with it as n gives all the fractions with a numerator of p.

Consider the following table:

$$f(n) = \frac{1}{p} \qquad f(n+1) = \frac{p}{p-1} = 1 + \frac{1}{p-1} \qquad f(n+2) = \frac{p-1}{2p-3}$$

$$f(n+3) = \frac{2p-3}{p-2} = 2 + \frac{1}{p-2} \qquad f(n+4) = \frac{p-2}{2p-7} \qquad f(n+4) = \frac{p-2}{3p-7}$$

$$f(n+5) = \frac{3p-7}{2p-5} = 1 + \frac{p-2}{2p-5} \qquad f(n+6) = \frac{2p-5}{3p-8} \qquad f(n+7) = \frac{3p-8}{p-3} = 3 + \frac{1}{p-3}$$

$$f(n+8) = \frac{p-3}{4p-13} \qquad f(n+9) = \frac{4p-13}{3p-10} = 1 + \frac{p-3}{3p-10} \qquad f(n+10) = \frac{3p-10}{2p-7}$$

Observation: Observe the pattern of the domain restrictions of p in the above table. The list of the domain restrictions of the above would be $1, \frac{3}{2}, 2, \frac{7}{3}, \frac{5}{2}, \frac{8}{3}, \frac{7}{2}, 3, \frac{13}{4}, \frac{10}{3}, \frac{7}{2}$. Now consider only the fractional part of each fraction (leaving the integers alone): $1, \frac{1}{2}, 2, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 3, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$. Notice that the fractions between each integer is the Farey sequence of order n, where n is the next integer after the Farey sequence. As $n+k$ approaches infinity, then amount of different domain

g. Conclusion

We were able to show that:

- The nodes of the Binary Tree T is always in lowest terms
- The function f is one-to-one and onto
- The function alternates between greater than 1 and less than 1 depending on parity

and conjecture that:

- The function is connected to the binary tree
- The function and the tree can also be used to connect to other number theory topics

Our main focus right now is to find a non-recursive formula or expression of f and a formula for $f^{-1}(n)$. We made several attempts such as exploiting symmetry in the tree and connecting it to the function inductively, using the binary tree, and studying patterns within the values. Each path we followed was fascinating and complex - however, we were unable to unify our explorations into a single sufficient proof and thus felt each exploration required more time before formalization. Some further work we could do is:

- Creating a formula for $f^{-1}(\frac{p}{q})$ through a formula for the set of all odd numbers in Conjecture 6, usage of the connection between the tree and function, once fully proven, to justify inverse operations concretely, and more.
- An extension to enumeration of $\mathbb{Q}^+$ by $\mathbb{Q}^+$ so we can explore the ternary tree and
- Exploring the connections to φ

h. Acknowledgements

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