Analysis I
Theorems & Lemmas

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November 19, 2025

Contents

1 Sequences of Real Numbers

1.1 Convergence of Sequences

Definition 1.1: Sequences

A **sequence** is a function $a : \mathbb{N} \to \mathbb{R}$. The image $a(n)$ of $n \in \mathbb{N}$ is also written as a_n and is called the n-th element of a. Instead of $a : \mathbb{N} \to \mathbb{R}$ one often writes $(a_n)_{n\in\mathbb{N}}, (a_n)_{n=0}^{\infty}, (a_n)_{n\geq 0}$.

Definition 1.2: (Eventually) Constant Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is **constant** if $x_n = x_m \forall n, m \in \mathbb{N}$. It is **eventually constant** if there exists $N \in \mathbb{N}$ such that $x_n = x_m \forall n, m \geq N$.

Definition 1.3: Convergence of Sequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. We say that $(x_n)_{n=0}^{\infty}$ **converges** (or is **convergent**) if $\exists A \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} : |x_n - A| < \varepsilon \quad \forall n \geq N.$$

In this case we write

$$\lim_{n\to\infty} x_n = A \tag{1.1}$$

and call A the **limit** of $(x_n)_{n=0}^{\infty}$.

Lemma 1.4: Uniqueness of the Limit

A convergent sequence $(x_n)_{n=0}^{\infty}$ has exactly one limit.

Proof. Let $A, B \in \mathbb{R}$ be limits of $(x_n)_{n=0}^{\infty}$. Fix $\varepsilon > 0$. Then there exists $N_A, N_B \in \mathbb{N}$ such that $|x_n - A| < \varepsilon$ for all $n \geq N_A$ and $|x_n - B| < \varepsilon$ for all $n \geq N_B$. We define $N := \max\{N_A, N_B\}$. Then it holds that

$$|A - B| \leq |A - x_N| + |x_N - B| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $A = B$. □

1.2 Convergent Subsequences and Acculmulation Points

Definition 1.5: Subsequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence. A **subsequence** is of the form $(x_{n_k})_{k=0}^{\infty}$, where $(n_k)_{k=0}^{\infty}$ is a strictly increasing sequence of non-negative integers, i.e., $n_{k+1} > n_k \; \forall k \in \mathbb{N}$.

Remark 1.6. *Since $n_{k+1} > n_k$ for all $k \in \mathbb{N}$ it follows by induction that $n_k \geq k$ for all $k \in \mathbb{N}$.*

Proof. For $k = 0$ we have that $n_0 \geq 0$, because $(n_k)_{k=0}^{\infty}$ is a sequence of non-negative integers. So the condition is fulfilled. For the inductive step we want to show that the condition holds for $k+1$ under the assumption that the condition is true for k. Because $(n_k)_{k=0}^{\infty}$ is also a strictly increasing sequence, we have that $n_{k+1} > n_k \geq k$. Additionally since $n_k \in \mathbb{N}$, we have that $n_{k+1} \geq n_k + 1$. So it follows that $n_{k+1} \geq n_k + 1 \geq k + 1$, which proofs the condition for $k + 1$. □

Lemma 1.7: Subsequences of Convergent Sequences are Convergent

Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Then every subsequence $(x_{n_k})_{k=0}^{\infty}$ also converges to A.

Proof. Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^{\infty}$ converges to A, there exists $N \in \mathbb{N}$ such that $|x_n - A| < \varepsilon \ \forall n \geq N$. As by Remark 1.6 we know that $n_k \geq k$ for all $k \in \mathbb{N}$. Therefore for all $k \geq N$ it holds that $|x_{n_k} - A| < \varepsilon$. □

Definition 1.8: Accumulation Points of Sequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. A point $A \in \mathbb{R}$ is an **accumulation point** of $(x_n)_{n=0}^{\infty}$ if

$$\forall \varepsilon > 0 \ \forall N \in \mathbb{N} \ \exists n \geq N : |x_n - A| < \varepsilon.$$

Proposition 1.9: Subsequences and Accumulation Points

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. A point A is and accumulation point of $(x_n)_{n=0}^{\infty}$ if and only if there exists a convergent subsequence of $(x_n)_{n=0}^{\infty}$ with limit A.

Proof. First assume that $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^{\infty}$. We construct $(n_k)_{k\geq 0}$ recursively:

- first, apply the definition of accumulation point with $N = 1$ and $\varepsilon = 1 = 2^0$ to find $n_0 \geq 1$ with $|x_{n_0} - A| \leq 2^0$,
- second, apply the definition the definition of accumulation point with $N = n_0 + 1$ and $\varepsilon = 2^{-1}$ to find $n_1 \geq n_0 + 1$ with $|x_{n_1} - A| \leq 2^{-1}$,
- more in general given n_{k-1}, we apply the definition of accumulation point with $N = n_{k-1} + 1$ and $\varepsilon = 2^{-k}$ to find $n_k \geq n_{k-1} + 1$ with $|x_{n_k} - A| \leq 2^{-k}$.

Now given $\varepsilon > 0$ choose N such that $2^{-N} < \varepsilon$. Then for all $k \geq N$ we have that

$$|x_{n_k} - A| \leq 2^{-k} \leq 2^{-N} < \varepsilon,$$

so $\lim_{k\to\infty} x_{n_k} = A$.

Conversely, assume that there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ converging to A. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. Since $\lim_{k\to\infty} x_{n_k} = A$, there exists N_0 such that $|x_{n_k} - A| < \varepsilon$ for all $k \geq N_0$. Hence if we choose $k = \max\{N_0, N\}$, because $n_k \geq n$ (recall Remark 1.6) we have that $n_k \geq N$ and $|x_{n_k} - A| < \varepsilon$. Thus A is an accumulation point. □

Corollary 1.10: Infinitely Many Terms Near an Accumulation Point

If $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^{\infty}$, then for every $\varepsilon > 0$ there are infinitely many n with $x_n \in (A - \varepsilon, A + \varepsilon)$.

Proof. By Proposition 1.9, there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ with $\lim_{k\to\infty} = A$. Hence for every $\varepsilon > 0$ there exists K such that $x_{n_k} \in (A - \varepsilon, A + \varepsilon)$ for all $k \geq K$, providing infinitely many elements of the sequence inside the interval $(A - \varepsilon, A + \varepsilon)$. □

Corollary 1.11: Accumulation Points of Convergent Sequences

convergent sequence has exactly one accumulation point, namely its limit.

1.3 Addition, Multiplication and Inequalities

Proposition 1.12: Limits and Operations

Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be sequences converging to $A, B \in \mathbb{R}$ respectively. Then:

1. *The sequence $(x_n + y_n)_{n=0}^{\infty}$ converges to $A + B$.*
2. *The sequence $(x_n y_n)_{n=0}^{\infty}$ converges to AB.*
3. *Given $\alpha \in \mathbb{R}$, the sequence $(\alpha x_n)_{n=0}^{\infty}$ converges to αA.*
4. *Suppose $x_n \neq 0$ for all $n \in \mathbb{N}$ and $A \neq 0$. Then the sequence $(x_n^{-1})_{n=0}^{\infty}$ converges to A^{-1}.*

Proposition 1.13: Limits and Inequalities

Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be sequences converging to $A, B \in \mathbb{R}$ respectively.

1. *If $A < B$, then there exists $N \in \mathbb{N}$ such that $x_n < y_n$ for all $n \geq \mathbb{N}$.*
2. *If there exists $N \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \geq N$, then $A \leq B$.*

Remark 1.14. *In Propostion 1.13 even if we assume that $x_n < y_n$ for all $n \in \mathbb{N}$, we cannot conclude that $A < B$. for example take*

$$x_n = \frac{1}{n}, \quad y_n = \frac{1}{n}.$$

Then we have that $x_n < y_n$ for all $n \in \mathbb{N}$ but $A = B = 0$.

Lemma 1.15: Sandwich Lemma

Let $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty}$, $(z_n)_{n=0}^{\infty}$ be sequences such that for some $N \in \mathbb{N}$, we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N.$$

Suppose that both $(x_n)_{n=0}^{\infty}$ and $(z_n)_{n=0}^{\infty}$ converge to the same limit. Then $(y_n)_{n=0}^{\infty}$ also converges, and we have that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n.$$

Proof. Let $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty}$, $(z_n)_{n=0}^{\infty}$ be sequences such that for some $N_0 \in \mathbb{N}$, we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N_0.$$

Additionally suppose that $(x_n)_{n=0}^{\infty}$ and $(z_n)_{n=0}^{\infty}$ converge to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^{\infty}$, $(z_n)_{n=0}^{\infty}$ converge to A there exists $N_x, N_z \in \mathbb{N}$ such that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N_x$$
$$A - \varepsilon < z_n < A + \varepsilon \quad \forall n_z \geq N_z.$$

So we choose $N := \max\{N_0, N_x, N_z\}$. Then we have that

$$A - \varepsilon < x_n \leq y_n \leq z_n < A + \varepsilon \quad \forall n \geq N,$$

which shows that $\lim_{n\to\infty} y_n = A$. ☐

Definition 1.16: Bounded Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called **bounded** if there exists a real number $M \geq 0$ such that

$$|x_n| \leq M \quad \forall n \in \mathbb{N}.$$

Lemma 1.17: Convergent Sequences are Bounded

Every convergent sequence is bounded.

Proof. Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Let $\varepsilon = 1$. Then, by convergence of $(x_n)_{n=0}^{\infty}$, there exists N such that $|x_n - A| < 1$ for all $n \geq N$. So we have that

$$|x_n| = |x_n - A + A| \leq |x_n - A| + |A| < 1 + |A|.$$

We choose

$$M = \max(|x_0|, |x_1|, \ldots, |x_{N-1}|, 1 + |A|).$$

Then $(x_n)_{n=0}^{\infty} \leq M$ for all $n \in \mathbb{N}$ as desired. ☐

Definition 1.18: Monotone Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called:

- **(monotonically) increasing** if $m > n \;\Rightarrow\; x_m \geq x_n$,
- **strictly (monotonically) increasing** if $m > n \;\Rightarrow\; x_m > x_n$,
- **(monotonically) decreasing** if $m > n \;\Rightarrow\; x_m \leq x_n$,
- **strictly (monotonically) decreasing** if $m > n \;\Rightarrow\; x_m < x_n$.

If a sequence is decreasing or increasing we call it monotone. If a sequence is strictly increasing or strictly decreasing then we call it strictly monotone.

Remark 1.19. *An equivalent formulation of monotone sequences can be given using only successive terms:*

- $(x_n)_{n=0}^{\infty}$ *is increasing if* $x_{n+1} \geq x_n$ *for all* n,
- $(x_n)_{n=0}^{\infty}$ *is strictly increasing if* $x_{n+1} > x_n$ *for all* n,
- $(x_n)_{n=0}^{\infty}$ *is decreasing if* $x_{n+1} \leq x_n$ *for all* n,
- $(x_n)_{n=0}^{\infty}$ *is strictly decreasing if* $x_{n+1} < x_n$ *for all* n.

Theorem 1.20: Convergence of Monotone Sequences

A monotone sequence $(x_n)_{n=0}^{\infty}$ *converges if and only if it is bounded. More precisely, let* $X = \{x_n \mid n \in \mathbb{N}\}$ *denote the set of points in the sequence.*

- *If* $(x_n)_{n=0}^{\infty}$ *is increasing, then* $\lim_{n\to\infty} x_n = \sup(X)$,
- *if* $(x_n)_{n=0}^{\infty}$ *decreasing, then* $\lim_{n\to\infty} x_n = \inf(X)$.

Proof. If $(x_n)_{n=0}^{\infty}$ converges Lemma 1.17 says that its bounded.

Conversely, let $(x_n)_{n=0}^{\infty}$ be a bounded monotone sequence. Wlog assume that $(x_n)_{n=0}^{\infty}$ is increasing (otherwise consider $(-x_n)_{n=0}^{\infty}$). Since $(x_n)_{n=0}^{\infty}$ is bounded from above, the set $X = \{x_n \mid n \in \mathbb{N}\}$ has a supremum, that we'll call $A = \sup(X)$.

By definiton of A:

(i) $x_n \leq A \quad \forall n \in \mathbb{N}$,

(ii) $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_N > A - \varepsilon$.

Then, for all $n \geq N$ using (ii) and monotonicity, we have that $x_n \geq x_N > A - \varepsilon$. Then using (i), we conclude that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N.$$

□

1.4 Superior and Inferior Limits

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence. To study its behavior for large n its is useful to look at its tails

$$X_{\geq n} = \{x_k \mid k \geq n\} \subseteq \mathbb{R}.$$

The concept of limits can be restated using the tails of a sequence, i.e., the sequence $(x_n)_{n=0}^{\infty}$ converges to $A \in \mathbb{R}$ if and only if, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $X_N \subseteq (A - \varepsilon, A + \varepsilon)$.

However, since not every sequence has a limit we now introduce a related notion (the **superior** and **inferior limits**), which always exist for bounded sequences.

For each $n \in \mathbb{N}$, define

$$s_n = \sup(X_{\geq n}) = \sup_{k \geq n} x_k, \qquad i_n = \inf(X_{\geq n}) = \inf_{k \geq n} x_k.$$

Since $X_{\geq m} \subset X_{\geq n}$, whenever $m > n$, we have that

$$i_n \leq i_m \leq s_m \leq s_n \qquad \forall m > n.$$

Thus, $(s_n)_{n=0}^{\infty}$ is a monotonically decreasing sequence, while $(i_n)_{n=0}^{\infty}$ is a monotonically increasing sequence. Moreover, since $(x_n)_{n=0}^{\infty}$ is bounded both $(s_n)_{n=0}^{\infty}$ and $(i_n)_{n=0}^{\infty}$ are bounded as well. Hence by Theorem 1.20, both sequences converge. Their limits will be called the *superior* and the *inferior limit* of $(x_n)_{n=0}^{\infty}$ respectively.

Note that, since $x_n \in X_{\geq n}$, we have that

$$i_n \leq x_n \leq s_n \qquad \forall n \in \mathbb{N}. \tag{1.2}$$

Definition 1.21: Superior and Inferior Limits

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence in $\mathbb{R}$. The numbers

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \big(\sup_{k\geq n} x_k\big), \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty} \big(\inf_{k\geq n} x_k\big)$$

are called the **superior** and **inferior limit** of $(x_n)_{n=0}^{\infty}$ respectively. From Equation 1.2 and Proposition 1.13, we have

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n.$$

Lemma 1.22: Convergence and Superior/Inferior Limits

A bounded sequence $(x_n)_{n=0}^{\infty}$ in $\mathbb{R}$ converges if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n.$$

Proof. For every $n \in \mathbb{N}$, define

$$i_n = \inf_{k\geq n} x_k, \qquad s_n = \sup_{k\geq n} x_k,$$

and set

$$I = \lim_{n\to\infty} i_n = \liminf_{n\to\infty} x_n, \qquad S = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} x_n.$$

First suppose that $I = S$. Since $i_n \leq x_n \leq s_n$ (see Equation 1.2), the Sandwich Lemma 1.15 implies that the sequence $(x_n)_{n=0}^{\infty}$ converges, and its limit equals $I = S$.

Conversely, assume that $(x_n)_{n=0}^{\infty}$ converges to $A \in \mathbb{R}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$A - \varepsilon < x_n < A + \varepsilon \qquad \forall n \geq N.$$

Then for all $n \geq N$, the same inequalities holds for i_n and s_n, i.e.,

$$A - \varepsilon \leq i_n \leq s_n \leq A + \varepsilon.$$

Taking limits and using Proposition 1.13, we obtain

$$A - \varepsilon \leq I \leq S \leq A + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $A = I = S$, which proves the result. $\square$