Analysis I
Theorems & Lemmas

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Contents

1 Functions

Definition 1.1: Functions/Maps/Transformations

A **function** f from a set X to a set Y is an assignment of an element of Y to each element of X. The element $y \in Y$ to which $x \in X$ is assigned to is denoted $f(x)$. We write $f : X \to Y$ and sometimes also speak of a **map**, **mapping** or a **transformation**. The set X is the **domain** and the set Y is the **codomain**. We refer to the set X as **domain** or **domain of definition**, and the set Y as **domain of values** or **codomain**. The set

$$\{(x, f(x)) \mid x \in X\} \subseteq X \times Y$$

is called the **graph** of f. In the context of a function $f : X \to Y$, an element of the domain of definition is also called **argument**, and an element $y = f(x) \in Y$ assumed by the function, is also called **value** of the function. If $f : X \to Y$ is a function, one also writes

$$f : X \to Y$$
$$x \mapsto f(x),$$

where $f(X)$ could be a concrete formula. We pronounce $'\mapsto'$ as $'$is mapped to$'$. Two functions $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are said to be equal if $X_1 = X_2$, $Y_1 = Y_2$ and $f_1(x) = f_2(x) \quad \forall x \in X_1$.

Definition 1.2: Injective, Surjective and Bijective Functions

Let $f : X \to Y$ be a function. We call f:

1. **injective** (or an **injection**) if

$$\forall x_1, x_2 \in X \,:\, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2);$$

2. **surjective** (or a **sujection**) if

$$\forall y \in Y \,\exists x \in X \,:\, f(x) = y;$$

3. **bijective** (or a **bijection**) if f is both injective and surjective.

Thus, a function $f : X \to Y$ is *not* injective if there exists distinct $x_1 \neq x_2 \in X$ such that $f(x_1) = f(x_2)$, and *not* surjective if there exists $y \in Y$ such that $f(x) \neq y$ for all $x \in X$.

Definition 1.3: Image and Preimage of a Function

For $f : X \to Y$ and $A \subseteq X$, define the **image** of A under the function f as

$$f(A) := \{y \in Y \mid \exists x \in X : f(x) = y\}.$$

For $B \subseteq Y$, define the **preimage** of B under the function f as

$$f^{-1}(B) := \{x \in X \mid f(x) \in B\}.$$

Remark 1.4. *Saying that $f : X \to Y$ is surjective is equivalent to $f(X) = Y$. Equivalently, f is surjective if $f^{-1}(\{y\}) \neq \emptyset$ for all $y \in Y$.*

2   The Real Numbers

2.1   Groups, Rings, Fields

Definition 2.1: Groups

A **group** is a non-empty set G together with a rule (called an *operation*) denoted by $\star : G \times G \to G$ that combines any two elements of G into another element of G. This operation must satisfy three conditions:

- **Associativity:** No matter how you place parentheses, the result is the same for all $a, b, c \in G$,
$$(a \star b) \star c = a \star (b \star c).$$

- **Neutral element:** There is a special element $e \in G$ such that combining it with any $a \in G$ leaves a unchanged, i.e.,

$$\forall a \in G : a \star e = e \star a = a.$$

- **Inverse element:** Every $a \in G$ has a 'partner' $a^{-1} \in G$ that 'cancels it out', giving the neutral element, i.e.,
$$a \star a^{-1} = a^{-1} \star a = e.$$

Note that, in general, one does not require that $a \star b = b \star a$. If the order of the operation does not matter, i.e., $a \star b = b \star a$ for all $a, b \in G$, the group is called **commutative** or **abelian**.

Lemma 2.2: Basic Properties of Groups

Let $(G, \star)$ be a group. Then:

1. *The neutral element is unique.*

2. *The inverse of an element is unique.*

3. *The inverse of the inverse of an element is the element itself, namely $(a^{-1})^{-1} = a$ for all $a \in G$.*

Proof. 1. Assume that, in addition to $e \in G$, we have a second element e' with the property that $e' \star a = a \star e' = a$ for all elements $a \in G$. Then, we can choose $a = e$ to obtain

$$e \star e' = e.$$

Similarly, since e is a neutral element, we have

$$e \star e' = e'.$$

Combining the two identities, we get

$$e = e \star e' = e'.$$

This proves that $e' = e$, so we speak of *the* neutral element of a group.

2. Assume that for an element $a \in G$, there exists two elements $b, c \in G$ that are both the inverse of a, namely

$$a \star b = b \star a = e, \qquad a \star c = c \star a = e.$$

Then, using associativity, we observe that

$$b = b \star e = b \star (a \star c) = (b \star a) \star c = e \star c = c.$$

This proves that the inverse of an element a is unique, so we can speak of *the* inverse element, and the notation a^{-1} makes sense.

3. Since $a \star a^{-1} = e$, we deduce that a is the inverse element of a^{-1}, thus

$$(a^{-1})^{-1} = a. \tag{2.1}$$

$\square$

Groups capture the idea of combining elements with a single operation. But to describe the arithmetic of numbers more faithfully, we also need a second operation (as we do with addition and multiplication). This leads us to the notion of *rings* and *fields*.

Definition 2.3: Rings and Fields

A **ring** is a non-empty set R in which we can both 'add' and 'multiply' elements with two operations '+' and '·'. Also, these two operations are compatible with each other. More precisely:

- $(R, +)$ is a **commutative group**, with neutral element denoted 0.

- Multiplication · is **associative**, has a **neutral element** (usually written as 1), and **distributes over addition**, i.e.,

$$a \cdot (b + c) = a \cdot b + a \cdot c, \qquad (b + c) \cdot a = b \cdot a + c \cdot a \qquad \forall a, b, c \in R.$$

If multiplication is also commutative, we call $(R, +, \cdot)$ a **commutative ring**. Note that, unlike addition, we do not require that every element has an inverse for multiplication. A **field** is a special kind of commutative ring, i.e. every non-zero element has an inverse for multiplication. In other words, if $(R, +, \cdot)$ is a commutative ring, then $(R, +, \cdot)$ is a field if $R \setminus \{0\}$ forms a commutative group under multiplication. Traditionally, we use the letter F to denote a field. We also write $F^* = F \setminus \{0\}$ for the set of all invertible elements of F.

Lemma 2.4: Basic Properties of Fields

Let $(F, +, \cdot)$ be a field and let $a, b \in F$. Then:

1. $0 \cdot a = a \cdot 0 = 0$.

2. $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$. In particular $(-1) \cdot a = -a$.

3. $(-a) \cdot (-b) = a \cdot b$. In particular, $(-a)^{-1} = -(a^{-1})$ whenever $a \neq 0$.

Proof. 1. Since 0 is the neutral element for the addition, we have $0 + 0 = 0$. Hence, using distributivity,

we get

$$0 \cdot a = (0 + 0) \cdot a = (0 \cdot a) + (0 \cdot a).$$

Adding $-0 \cdot a$ (i.e., the inverse of $0 \cdot a$), we deduce that $0 \cdot a = 0$. The case of $a \cdot 0$ is analogous.

2. By the distributive law,

$$a \cdot b + a(-b) = a \cdot (b + (-b)) = a \cdot 0 = 0.$$

So $a \cdot (-b)$ is the additive inverse of $a \cdot b$, i.e., $-(a \cdot b) = a \cdot (-b)$. Taking $b = 1$ gives $-a = (-1) \cdot a$. The validity of $(-a) \cdot b = -(a \cdot b)$ follows by exchanging a and b in the argument above.

3. By 2. we know that $-(a \cdot b) = a \cdot (-b)$. Hence, recalling Equation 2.1,

$$a \cdot b = -(a \cdot (-b)).$$

On the other hand, applying 2. with $(-b)$ instead of b, we also have

$$-(a \cdot (-b)) = (-a) \cdot (-b).$$

Combining the two identities above, we conclude that $(-a) \cdot (-b) = a \cdot b$. Finally, taking $b = a^{-1}$ yields $(-a) \cdot (-(a^{-1})) = a \cdot a^{-1} = 1$, which gives the second assertion. $\qquad\square$

2.2   Order Relation

Definition 2.5: Cartesian Product

Let X and Y be two sets. The **cartesian product** $X \times Y$ is the set of ordered pairs of elements of X and Y, i.e.,

$$X \times Y := \{(x, y) \mid x \in X, y \in Y\}.$$

Definition 2.6: Subsets

Let P and Q be sets. Then

- P is a **subset** of Q, written $P \subset Q$ (or $P \subseteq Q$), if every element of P also belongs to Q.

- P is a **proper subset** of Q, written $P \subsetneq Q$, if P is a subset of Q but $P \neq Q$.

- We write $P \nsubseteq Q$ if P is not a subset of Q.

Definition 2.7: Relations

Let X be a set. A **relation** on X is a subset $\mathcal{R} \subseteq X \times X$, that is, a collection of ordered pairs of elements of X. If $(x, y) \in \mathcal{R}$ we write $x\mathcal{R}y$. Common symbols for relations include $<, \leq, \sim, \equiv, \cong$. If $\sim$ is a relation on X, we write $x \nsim y$ if $x \sim y$ does not hold. A realtion $\sim$ may have the following properties:

1. **Reflexive:** if $x \sim x \qquad \forall x \in X$.

2. **Transitive:** if $x \sim y$ and $y \sim z$, then $x \sim z$.

3. **Symmetric:** if $x \sim y$, then $y \sim x$.

4. **Antisymmetric:** if $x \sim y$ and $y \sim x$, then $x = y$.

A relation is an **equivalence relation** if it is reflexive, transitive and symmetric. It is an **order relation** if it is reflexive, transitive and antisymmetric.

2.3   Ordered Fields

Definition 2.8: Ordered Field

Let F be a field, and let $\leq$ be an order relation on F. We call $(F, \leq)$, or simply F, an **ordered field** if the following hold:

1. **Linearity of order:** for all $x, y \in F$, at least one of $x \leq y$ or $y \leq x$ holds.

2. **Compatibility with addition:** for all $x, y, z \in F$,

$$x \leq y \Rightarrow x + z \leq y + z.$$

3. **Compatibility with multiplication:** for all $x, y \in F$,

$$0 \leq x \,\wedge\, 0 \leq y \Rightarrow 0 \leq x \cdot y.$$

Lemma 2.9: Ordered Field: Basic Consequences

Let $(F, \leq)$ be and ordered field, and let $x, y, z, w \in F$. Then:

(a) *(Trichotomy) Either $x < y$, or $x = y$, or $x > y$.*

(b) *If $x < y$ and $y \leq z$, then $x < z$. (Analogously, $x \leq y$ and $y < z$ imply $x < z$.)*

(c) *(Addition of inequalities) If $x \leq y$ and $z \leq w$, then $x + z \leq y + w$. (Analogously, $x < z$ and $z \leq w$ imply $x + z < y + w$.)*

(d) *$x \leq y$ if and only if $0 \leq y - x$.*

(e) *$x \leq 0$ if and only if $0 \leq -x$.*

(f) *$x^2 \geq 0$, and $x^2 > 0$ if $x \neq 0$.*

(g) *$0 < 1$.*

(h) *If $0 \leq x$ and $y \leq z$, then $xy \leq xz$.*

(i) *If $x \leq 0$ and $y \leq z$, then $xy \geq xz$.*

(j) *If $0 < x \leq y$, then $0 < y^{-1} \leq x^{-1}$.*

(k) *If $0 \leq x \leq y$ and $0 \leq z \leq w$, then $0 \leq xz \leq yw$.*

(l) *If $x + y \leq x + z$, then $y \leq z$.*

(m) *If $xy \leq xz$ and $x > 0$, then $y \leq z$.*

Lemma 2.10: Integers and Rationals Inside an Ordered Field

Let $(F, \leq)$ be and ordered field, and denote by 0 and 1 the neutral elements for addition and multiplication, respectively. Then:

(i) *The elements $\ldots, -2, -1, 0, 1, 2, \ldots$ defined by*

$$2 = 1 + 1, \quad 3 = 2 + 1, \ldots, \quad -n = (-1) \cdot n$$

are all distinct and satisfy

$$\ldots < -2 < -1 < 0 < 1 < 2 < 3 < \ldots.$$

We denote this set of elements by $\mathbb{Z}$, and we call them 'integers'

(ii) *Every fraction pq^{-1} with $p, q \in \mathbb{Z}$, $q \neq 0$, lies in F and the set of all such elements is denoted by $\mathbb{Q}$. Also,*

$$\mathbb{Z} \subsetneq \mathbb{Q} \subseteq F.$$

Proof. (i) By Lemma 2.9(g), we have that $0 < 1$. Then Lemma 2.9(c) yields $0 < 1 < 2 < 3 < \ldots$, and taking negatives gives $\ldots < -2 < -1 < 0$. Hence all these elements are distinct.

(ii) For $q \neq 0$, q is invertible in F; define $\frac{p}{q} = pq^{-1}$. The set of such fractions is a field contained in F, which we denote by $\mathbb{Q}$.

To show that $\mathbb{Q}$ strictly contains $\mathbb{Z}$, consider $\frac{1}{2}$ (the inverse of 2). Since $2 > 1$, it follows from

Lemma 2.9(j) that $0 < \frac{1}{2} < 1$, so $\frac{1}{2} \notin \mathbb{Z}$.   □

Definition 2.11: Absolute Value and Sign

Let $(F, \leq)$ be and ordered field.

- The **absolute value** (or **modulus**) is the function $|\cdot| : F \to F$ defined by

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

- The **sign** is the function $\mathrm{sgn} : F \to \{-1, 0, 1\}$ defined by

$$\mathrm{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Lemma 2.12: Absolute Value and Sign: Basic Properties

Let $(F, \leq)$ be an ordered field and let $x, y \in F$. Then:

(a) $x = \mathrm{sgn}(x)|x|, \quad |-x| = |x|, \quad \mathrm{sgn}(-x) = -\mathrm{sgn}(x)$.

(b) $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$ (by Trichotomy Lemma ??).

(c) (Multiplicativity) $\mathrm{sgn}(xy) = \mathrm{sgn}(x)\,\mathrm{sgn}(y)$ and $|xy| = |x||y|$.

(d) If $x \neq 0$, then $|x^{-1}| = |x|^{-1}$.

(e) $|x| \leq y$ iff $-y \leq x \leq y$.

(f) $|x| < y$ iff $-y < x < y$.

(g) (Triangle inequality) $|x + y| \leq |x| + |y|$.

(h) (Inverse triangle inequality) $||x| - |y|| \leq |x - y|$.

Proof. (g) Thanks to (e) we have $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding these two inequalities we get

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

Applying (e) again yields the result.

(h) From (g) we have $|x| \leq |x - y| + |y|$, therefore

$$|x| - |y| \leq |x - y|.$$

Exchanging the roles of x and y, we also have $|y| - |x| leq |y - x| = |x - y|$. Combining these two inequalities yields

$$-|x - y| \leq |x| - |y| \leq |x - y|,$$

and the result follows by applying (e) again.   □

2.4   Completeness Axiom

> ### Definition 2.13: Completeness Axiom
>
> Let $(K, \leq)$ be an ordered field. We say that $(K, \leq)$ is **complete** (or a **completely ordered field**) if the following statement holds:
>
> > Let X, Y be non-empty subsets of K such that $x \leq y$ for all $x \in X$ and $y \in Y$. Then there exists $c \in K$ lying between X and Y, in the sense that $x \leq c \leq y$ for all $x \in X$ and $y \in Y$.
>
> The statement above is called the **completeness axiom**.

> ### Definition 2.14: Real Numbers
>
> We call **the field of real numbers**, any completely ordered field and denote it by $\mathbb{R}$.

2.5   Intervals

> ### Definition 2.15: Intervals
>
> Let $a, b \in \mathbb{R}$. We define:
>
> - The **closed interval**
> $$[a, b] := \{x \in R \mid a \leq x \leq b\};$$
>
> - The **open interval**
> $$(a, b) := \{x \in R \mid a < x < b\};$$
>
> - The **half-open intervals**
> $$[a, b) := \{x \in R \mid a \leq x < b\} \quad \text{and} \quad (a, b] := \{x \in R \mid a < x \leq b\};$$
>
> - The **unbounded closed intervals**
> $$[a, \infty) := \{x \in R \mid a \leq x\} \quad \text{and} \quad (-\infty, b] := \{x \in R \mid x \leq b\};$$
>
> - The **unbounded open intervals**
> $$(a, \infty) := \{x \in R \mid a < x\} \quad \text{and} \quad (-\infty, b) := \{x \in R \mid x < b\};$$

Definition 2.16: Set Operations

Let P, Q be sets. The **intersection** $P \cap Q$, the **union** $P \cup Q$, the **relative complement** $P \setminus Q$ and the **symmetric difference** $P \triangle Q$ are defined by

$$P \cap Q = \{x \mid x \in P \text{ and } x \in Q\},$$
$$P \cup Q = \{x \mid x \in P \text{ or } x \in Q\},$$
$$P \setminus Q = \{x \mid x \in P \text{ and } x \notin Q\},$$
$$P \triangle Q = (P \setminus Q) \cup (Q \setminus P) = (P \cup Q) \setminus (P \cap Q).$$

Definition 2.17: Union and Intersection of several Sets

Let $\mathcal{A}$ be a family of sets (i.e., a set whose elements are sets). We define the **union** and **intersection** of the sets in $\mathcal{A}$ as

$$\bigcup_{A \in \mathcal{A}} A = \{x \mid \exists A \in A : x \in A\}, \quad \bigcap_{A \in \mathcal{A}} A = \{x \mid \forall A \in \mathcal{A} : x \in A\}.$$

If $\mathcal{A} = \{A_1, A_2, \ldots\}$, we also write

$$\bigcup_{i=1}^{\infty} A_i = \{x \mid \exists i \geq 1 : x \in A_i\}, \quad \bigcap_{i=1}^{\infty} A_i = \{x \mid \forall i \geq 1 : x \in A_i\}.$$

Definition 2.18: Neighborhoods

Let $x \in \mathbb{R}$. A **neighborhood** of x is a set containing an interval I such that $x \in I$. Given $\delta > 0$, the open interval $(x - \delta, x + \delta)$ is called the δ-**neighborhood** of x.

Definition 2.19: Open and Closed Sets

A subset $U \subseteq \mathbb{R}$ is called **open** in $\mathbb{R}$ if for every $x \in U$ there exists open interval I such that $x \in I$ and $I \subseteq U$. A subset $F \subseteq \mathbb{R}$ is called **closed** in $\mathbb{R}$ if its complement $\mathbb{R} \setminus F$ is open.

Remark 2.20. *The sets $\emptyset$ and $\mathbb{R}$ are both open in $\mathbb{R}$. Hence, they are also closed since $\emptyset^c = \mathbb{R}$ and $\mathbb{R}^c = \emptyset$. We note that $\mathbb{Q} \subseteq \mathbb{R}$ and $[a, b] \subseteq \mathbb{R}$ are neither open nor closed.*

Remark 2.21. *Let $\mathcal{U}$ be a family of open sets, and $\mathcal{F}$ be a family of closed subsets of $\mathbb{R}$. Then the union and intersection*

$$\bigcup_{U \in \mathcal{U}} U, \quad \bigcap_{F \in \mathcal{F}} F$$

Are open and closed, respectively.

2.6   Complex Numbers

Starting from the field of real numbers $\mathbb{R}$, we define the set of **complex numbers** as

$$\mathbb{C} = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

We denote the elements $z = (x, y) \in \mathbb{C}$ in the form $z = x + iy$, where i is the **imaginary unit**. Here $x \in \mathbb{R}$ is the **real part** of z, written as $x = \operatorname{Re}(z)$, and $y \in \mathbb{R}$ is the **imaginary part**, written

as $y = \text{Im}(z)$. Elements with $\text{Im}(z) = 0$ are called **real**, while those with $\text{Re}(z) = 0$ are **purely imaginary**. Via the injective map $\mathbb{R} \ni x \mapsto x + i \cdot 0 \in \mathbb{C}$, we identify $\mathbb{R}$ with the subset of real numbers inside $\mathbb{C}$.

As you may expect from previous knowledge, we want to satisfy $i^2 = -1$. To achieve this, we define addition and multiplication on $\mathbb{C}$ so that it becomes a field. Additionally, we want these operations to coincide with the usual addition and multiplication when considering real numbers.

Since $i^2 = -1$, using commutativity and distributivity we get

$$(x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + ix_1 y_2 + iy_2 x_1 + i^2 y_1 y_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2).$$

This motivates the following definition

Definition 2.22: Addition and Multiplication on $\mathbb{C}$

On $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ we define **addition** and **multiplication** as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \, y_1 + y_2),$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, \, x_1 y_2 + x_2 y_1).$$

Proposition 2.23: $\mathbb{C}$ is a Field

With the operation of Definition 2.22, together with the zero element $(0,0)$ and the unit element $(1,0)$, the set $\mathbb{C}$ is a field.

Definition 2.24: Complex Conjugation

For $z = x + iy \in \mathbb{C}$ we define its **conjugate** as $\bar{z} = x - iy$. The mapping $\mathbb{C} \ni z \mapsto \bar{z} \in \mathbb{C}$ is called **complex conjugation**.

Lemma 2.25: Properties of Complex Conjugation

For all $z, w \in \mathbb{C}$:

(i) $z\bar{z} = x^2 + y^2 \in \mathbb{R}_{\geq 0}$. *In particular, $z\bar{z} = 0$ if and only if $z = 0$.*

(ii) $\overline{z + w} = \bar{z} + \bar{w}$.

(iii) $\overline{zw} = \bar{z}\bar{w}$.

Proof. Property (i) follows from the fact that, for $z = x + iy$, $(x + iy)(x - iy) = x^2 + y^2$. Also, $x^2 + y^2 = 0$ if and only if $x + iy = 0$. Properties (ii) and (iii) follow from a direct computation, writing $z = x_+ + iy_1$ and $w = x_2 + iy_2$, which yields

$$\overline{z + w} = \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_2) + (x_2 - iy_2) = \bar{z} + \bar{w},$$
$$\overline{z \cdot w} = \overline{(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)} = (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + y_1 x_2)$$
$$= (x_1 - iy_1) \cdot (x_2 - iy_2) = \bar{z} \cdot \bar{w}. \qquad \square$$

> **Definition 2.26: Absolute Value**
>
> The **absolute value** (or **norm**) on $\mathbb{C}$ is the map $|\cdot| : \mathbb{C} \to \mathbb{R}$ given by
>
> $$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}, \qquad z = x + iy \in \mathbb{C}.$$

> **Lemma 2.27: Cauchy-Schwart Inequality**
>
> *If $z = x_1 + iy_1$, and $w = x_2 + iy_2$, then*
>
> $$x_1 x_2 + y_1 y_2 \le |z||w|. \tag{2.2}$$

Proof. We observe that

$$
\begin{aligned}
|z|^2|w|^2 - (x_1 x_2 + y_1 y_2)^2 &= (x_1^2 + y_1^2)(x_2^2 + y_2^2) - (x_1 x_2 + y_1 y_2)^2 \\
&= x_1^2 x_2^2 + y_1^2 y_2^2 + y_1^2 x_2^2 + x_+^2 y_2^2 - (x_1^2 x_2^2 + y_1^2 y_2^2 + 2x_1 x_2 y_1 y_2) \\
&= y_1^2 x_2^2 + x_+^2 y_2^2 - 2x_1 x_2 y_1 y_2 \\
&= (y_1 x_2 - x_1 y_2)^2 \ge 0.
\end{aligned}
$$

This proves that $(x_1 x_2 + y_1 y_2)^2 \le |z|^2|w|^2$, so it follows that

$$|x_1 x_2 + y_1 y_2| \le |z||w|.$$

Since $x \le |x|$ for all $x \in \mathbb{R}$, we obtain Equation 2.2. $\qquad\square$

> **Proposition 2.28: Trianlge Inequality**
>
> *For all $z, w \in \mathbb{C}$, one has*
> $$|z + w| \le |z| + |w|.$$

Proof. For $z = x_1 + iy_1$ and $w = x_2 + iy_2$, using Lemma 2.27, we have

$$
\begin{aligned}
|z + w|^2 &= (x_1 + x_2)^2 + (y_1 + y_2)^2 \\
&= |z|^2 + |w|^2 + 2(x_1 x_2 + y_1 y_2) \\
&\le |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2.
\end{aligned}
$$

Taking square roots proves the result. $\qquad\square$

> **Definition 2.29: Cicular Disks**
>
> For $z \in \mathbb{C}$ and $r > 0$, we define the **open disk** with radius $r > 0$ around z as
>
> $$B(z, r) := \{w \in \mathbb{C} \mid |z - w| < r\},$$
>
> and the **closed disk** with radius $r > 0$ around z as
>
> $$\overline{B(z, r)} := \{w \in \mathbb{C} \mid |z - w| \le r\}.$$

In other words, the open disk $B(z, r)$ is the set of points at distance strictly less than r form z. We note that this definition is compatible with the one of neighborhoods in $\mathbb{R}$: if $x \in \mathbb{R}$ and $r > 0$,

then
$$B(x, r) \cap \mathbb{R} = (x - r, x + r).$$

Definition 2.30: Open and Closed Sets

A set $U \subseteq \mathbb{C}$ is **open** if for every $z \in U$ there exists $r > 0$ such that $B(z, r) \subseteq U$. A set $C \subseteq \mathbb{C}$ is **closed** if its complement $\mathbb{C} \setminus C$ is open.

2.7   Consequences of Completeness

3   Sequences of Real Numbers

3.1   Convergence of Sequences

Definition 3.1: Sequences

A **sequence** is a function $a : \mathbb{N} \to \mathbb{R}$. The image $a(n)$ of $n \in \mathbb{N}$ is also written as a_n and is called the n-th element of a. Instead of $a : \mathbb{N} \to \mathbb{R}$ one often writes $(a_n)_{n \in \mathbb{N}}, (a_n)_{n=0}^{\infty}, (a_n)_{n \geq 0}$.

Definition 3.2: (Eventually) Constant Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is **constant** if $x_n = x_m \forall n, m \in \mathbb{N}$. It is **eventually constant** if there exists $N \in \mathbb{N}$ such that $x_n = x_m \forall n, m \geq N$.

Definition 3.3: Convergence of Sequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. We say that $(x_n)_{n=0}^{\infty}$ **converges** (or is **convergent**) if $\exists A \in \mathbb{R}$ such that
$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : |x_n - A| < \varepsilon \quad \forall n \geq N.$$

In this case we write
$$\lim_{n \to \infty} x_n = A \tag{3.1}$$

and call A the **limit** of $(x_n)_{n=0}^{\infty}$.

Lemma 3.4: Uniqueness of the Limit

A convergent sequence $(x_n)_{n=0}^{\infty}$ has exactly one limit.

Proof. Let $A, B \in \mathbb{R}$ be limits of $(x_n)_{n=0}^{\infty}$. Fix $\varepsilon > 0$. Then there exists $N_A, N_B \in \mathbb{N}$ such that $|x_n - A| < \varepsilon$ for all $n \geq N_A$ and $|x_n - B| < \varepsilon$ for all $n \geq N_B$. We define $N := \max\{N_A, N_B\}$. Then it holds that
$$|A - B| \leq |A - x_N| + |x_N - B| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $A = B$. $\qquad\qquad\square$

3.2 Convergent Subsequences and Acculmulation Points

> **Definition 3.5: Subsequences**
>
> Let $(x_n)_{n=0}^\infty$ be a sequence. A **subsequence** is of the form $(x_{n_k})_{k=0}^\infty$, where $(n_k)_{k=0}^\infty$ is a strictly increasing sequence of non-negative integers, i.e., $n_{k+1} > n_k \; \forall k \in \mathbb{N}$.

Remark 3.6. *Since $n_{k+1} > n_k$ for all $k \in \mathbb{N}$ it follows by induction that $n_k \geq k$ for all $k \in \mathbb{N}$.*

Proof. For $k = 0$ we have that $n_0 \geq 0$, because $(n_k)_{k=0}^\infty$ is a sequence of non-negative integers. So the condition is fulfilled. For the inductive step we want to show that the condition holds for $k + 1$ under the assumption that the condition is true for k. Because $(n_k)_{k=0}^\infty$ is also a strictly increasing sequence, we have that $n_{k+1} > n_k \geq k$. Additionally since $n_k \in \mathbb{N}$, we have that $n_{k+1} \geq n_k + 1$. So it follows that $n_{k+1} \geq n_k + 1 \geq k + 1$, which proofs the condition for $k + 1$. $\qquad\square$

> **Lemma 3.7: Subsequences of Convergent Sequences are Convergent**
>
> *Let $(x_n)_{n=0}^\infty$ be a sequence converging to $A \in \mathbb{R}$. Then every subsequence $(x_{n_k})_{k=0}^\infty$ also converges to A.*

Proof. Let $(x_n)_{n=0}^\infty$ be a sequence converging to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^\infty$ converges to A, there exists $N \in \mathbb{N}$ such that $|x_n - A| < \varepsilon \; \forall n \geq N$. As by Remark 3.6 we know that $n_k \geq k$ for all $k \in \mathbb{N}$. Therefore for all $k \geq N$ it holds that $|x_{n_k} - A| < \varepsilon$. $\qquad\square$

> **Definition 3.8: Accumulation Points of Sequences**
>
> Let $(x_n)_{n=0}^\infty$ be a sequence in $\mathbb{R}$. A point $A \in \mathbb{R}$ is an **accumulation point** of $(x_n)_{n=0}^\infty$ if
>
> $$\forall \varepsilon > 0 \; \forall N \in \mathbb{N} \; \exists n \geq N : |x_n - A| < \varepsilon.$$

> **Proposition 3.9: Subsequences and Accumulation Points**
>
> *Let $(x_n)_{n=0}^\infty$ be a sequence in $\mathbb{R}$. A point A is and accumulation point of $(x_n)_{n=0}^\infty$ if and only if there exists a convergent subsequence of $(x_n)_{n=0}^\infty$ with limit A.*

Proof. First assume that $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^\infty$. We construct $(n_k)_{k \geq 0}$ recursively:

- first, apply the definition of accumulation point with $N = 1$ and $\varepsilon = 1 = 2^0$ to find $n_0 \geq 1$ with $|x_{n_0} - A| \leq 2^0$,

- second, apply the definition the definition of accumulation point with $N = n_0 + 1$ and $\varepsilon = 2^{-1}$ to find $n_1 \geq n_0 + 1$ with $|x_{n_1} - A| \leq 2^{-1}$,

- more in general given n_{k-1}, we apply the definition of accumulation point with $N = n_{k-1} + 1$ and $\varepsilon = 2^{-k}$ to find $n_k \geq n_{k-1} + 1$ with $|x_{n_k} - A| \leq 2^{-k}$.

Now given $\varepsilon > 0$ choose N such that $2^{-N} < \varepsilon$. Then for all $k \geq N$ we have that

$$|x_{n_k} - A| \leq 2^{-k} \leq 2^{-N} < \varepsilon,$$

so $\lim_{k \to \infty} x_{n_k} = A$.

Conversely, assume that there exists a subsequence $(x_{n_k})_{k=0}^\infty$ converging to A. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. Since $\lim_{k \to \infty} x_{n_k} = A$, there exists N_0 such that $|x_{n_k} - A| < \varepsilon$ for all $k \geq N_0$. Hence if we

choose $k = \max\{N_0, N\}$, because $n_k \geq n$ (recall Remark 3.6) we have that $n_k \geq N$ and $|x_{n_k} - A| < \varepsilon$. Thus A is an accumulation point. $\qquad\square$

Corollary 3.10: Infinitely Many Terms Near an Accumulation Point

If $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^{\infty}$, then for every $\varepsilon > 0$ there are infinitely many n with $x_n \in (A - \varepsilon, A + \varepsilon)$.

Proof. By Proposition 3.9, there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ with $\lim_{k \to \infty} = A$. Hence for every $\varepsilon > 0$ there exists K such that $x_{n_k} \in (A - \varepsilon, A + \varepsilon)$ for all $k \geq K$, providing infinitely many elements of the sequence inside the interval $(A - \varepsilon, A + \varepsilon)$. $\qquad\square$

Corollary 3.11: Accumulation Points of Convergent Sequences

convergent sequence has exactly one accumulation point, namely its limit.

3.3   Addition, Multiplication and Inequalities

Proposition 3.12: Limits and Operations

Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be sequences converging to $A, B \in \mathbb{R}$ respectively. Then:

1. *The sequence $(x_n + y_n)_{n=0}^{\infty}$ converges to $A + B$.*

2. *The sequence $(x_n y_n)_{n=0}^{\infty}$ converges to AB.*

3. *Given $\alpha \in \mathbb{R}$, the sequence $(\alpha x_n)_{n=0}^{\infty}$ converges to αA.*

4. *Suppose $x_n \neq 0$ for all $n \in \mathbb{N}$ and $A \neq 0$. Then the sequence $(x_n^{-1})_{n=0}^{\infty}$ converges to A^{-1}.*

Proposition 3.13: Limits and Inequalities

Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be sequences converging to $A, B \in \mathbb{R}$ respectively.

1. *If $A < B$, then there exists $N \in \mathbb{N}$ such that $x_n < y_n$ for all $n \geq \mathbb{N}$.*

2. *If there exists $N \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \geq N$, then $A \leq B$.*

Remark 3.14. *In Propostion 3.13 even if we assume that $x_n < y_n$ for all $n \in \mathbb{N}$, we cannot conclude that $A < B$. for example take*

$$x_n = \frac{1}{n}, \quad y_n = \frac{1}{n}.$$

Then we have that $x_n < y_n$ for all $n \in \mathbb{N}$ but $A = B = 0$.

Lemma 3.15: Sandwich Lemma

Let $(x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}, (z_n)_{n=0}^{\infty}$ be sequences such that for some $N \in \mathbb{N}$, we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N.$$

Suppose that both $(x_n)_{n=0}^{\infty}$ and $(z_n)_{n=0}^{\infty}$ converge to the same limit. Then $(y_n)_{n=0}^{\infty}$ also converges, and we have that

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \lim_{n \to \infty} z_n.$$

Proof. Let $(x_n)_{n=0}^\infty$, $(y_n)_{n=0}^\infty$, $(z_n)_{n=0}^\infty$ be sequences such that for some $N_0 \in \mathbb{N}$, we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N_0.$$

Additionally suppose that $(x_n)_{n=0}^\infty$ and $(z_n)_{n=0}^\infty$ converge to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^\infty$, $(z_n)_{n=0}^\infty$ converge to A there exists $N_x, N_z \in \mathbb{N}$ such that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N_x$$
$$A - \varepsilon < z_n < A + \varepsilon \quad \forall n_z \geq N_z.$$

So we choose $N := \max\{N_0, N_x, N_z\}$. Then we have that

$$A - \varepsilon < x_n \leq y_n \leq z_n < A + \varepsilon \quad \forall n \geq N,$$

which shows that $\lim_{n\to\infty} y_n = A$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 3.16: Bounded Sequences

A sequence $(x_n)_{n=0}^\infty$ is called **bounded** if there exists a real number $M \geq 0$ such that

$$|x_n| \leq M \quad \forall n \in \mathbb{N}.$$

Lemma 3.17: Convergent Sequences are Bounded

Every convergent sequence is bounded.

Proof. Let $(x_n)_{n=0}^\infty$ be a sequence converging to $A \in \mathbb{R}$. Let $\varepsilon = 1$. Then, by convergence of $(x_n)_{n=0}^\infty$, there exists N such that $|x_n - A| \leq 1$ for all $n \geq N$. So we have that

$$|x_n| = |x_n - A + A| \leq |x_n - A| + |A| \leq 1 + |A| \qquad \forall n \geq N.$$

We choose
$$M = \max(|x_0|, |x_1|, \ldots, |x_{N-1}|, 1 + |A|).$$

Then $|x_n| \leq M$ for all $n \in \mathbb{N}$ as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Definition 3.18: Monotone Sequences

A sequence $(x_n)_{n=0}^\infty$ is called:

- **(monotonically) increasing** if $m > n \Rightarrow x_m \geq x_n$,

- **strictly (monotonically) increasing** if $m > n \Rightarrow x_m > x_n$,

- **(monotonically) decreasing** if $m > n \Rightarrow x_m \leq x_n$,

- **strictly (monotonically) decreasing** if $m > n \Rightarrow x_m < x_n$.

If a sequence is decreasing or increasing we call it monotone. If a sequence is strictly increasing or strictly decreasing then we call it strictly monotone.

Remark 3.19. *An equivalent formulation of monotone sequences can be given using only successive terms:*

– $(x_n)_{n=0}^{\infty}$ is increasing if $x_{n+1} \geq x_n$ for all n,

– $(x_n)_{n=0}^{\infty}$ is strictly increasing if $x_{n+1} > x_n$ for all n,

– $(x_n)_{n=0}^{\infty}$ is decreasing if $x_{n+1} \leq x_n$ for all n,

– $(x_n)_{n=0}^{\infty}$ is strictly decreasing if $x_{n+1} < x_n$ for all n.

> ### Theorem 3.20: Convergence of Monotone Sequences
>
> A monotone sequence $(x_n)_{n=0}^{\infty}$ converges if and only if it is bounded. More precisely, let $X = \{x_n \mid n \in \mathbb{N}\}$ denote the set of points in the sequence.
>
>   – If $(x_n)_{n=0}^{\infty}$ is increasing, then $\lim_{n \to \infty} x_n = \sup(X)$,
>
>   – if $(x_n)_{n=0}^{\infty}$ decreasing, then $\lim_{n \to \infty} x_n = \inf(X)$.

Proof. If $(x_n)_{n=0}^{\infty}$ converges Lemma 3.17 says that its bounded.

Conversely, let $(x_n)_{n=0}^{\infty}$ be a bounded monotone sequence. Wlog assume that $(x_n)_{n=0}^{\infty}$ is increasing (otherwise consider $(-x_n)_{n=0}^{\infty}$). Since $(x_n)_{n=0}^{\infty}$ is bounded from above, the set $X = \{x_n \mid n \in \mathbb{N}\}$ has a supremum, that we'll call $A = \sup(X)$.

By definiton of A:

(i) $x_n \leq A \quad \forall n \in \mathbb{N}$,

(ii) $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_N > A - \varepsilon$.

Then, for all $n \geq N$ using (ii) and monotonicity, we have that $x_n \geq x_N > A - \varepsilon$. Then using (i), we conclude that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N.$$

$\square$

3.4 Superior and Inferior Limits

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence. To study its behavior for large n its is useful to look at its tails

$$X_{\geq n} = \{x_k \mid k \geq n\} \subseteq \mathbb{R}.$$

The concept of limits can be restated using the tails of a sequence, i.e., the sequence $(x_n)_{n=0}^{\infty}$ converges to $A \in \mathbb{R}$ if and only if, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $X_N \subseteq (A - \varepsilon, A + \varepsilon)$.

However, since not every sequence has a limit we now introduce a related notion (the **superior** and **inferior limits**), which always exist for bounded sequences.

For each $n \in \mathbb{N}$, define

$$s_n = \sup(X_{\geq n}) = \sup_{k \geq n} x_k, \qquad i_n = \inf(X_{\geq n}) = \inf_{k \geq n} x_k.$$

Since $X_{\geq m} \subset X_{\geq n}$, whenever $m > n$, we have that

$$i_n \leq i_m \leq s_m \leq s_n \qquad \forall m > n.$$

Thus, $(s_n)_{n=0}^{\infty}$ is a monotonically decreasing sequence, while $(i_n)_{n=0}^{\infty}$ is a monotonically increasing sequence. Moreover, since $(x_n)_{n=0}^{\infty}$ is bounded both $(s_n)_{n=0}^{\infty}$ and $(i_n)_{n=0}^{\infty}$ are bounded as well. Hence

by Theorem 3.20, both sequences converge. Their limits will be called the *superior* and the *inferior limit* of $(x_n)_{n=0}^\infty$ respectively.

Note that, since $x_n \in X_{\geq n}$, we have that

$$i_n \leq x_n \leq s_n \qquad \forall n \in \mathbb{N}. \tag{3.2}$$

Definition 3.21: Superior and Inferior Limits

Let $(x_n)_{n=0}^\infty$ be a bounded sequence in $\mathbb{R}$. The numbers

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \left(\sup_{k\geq n} x_k \right), \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty} \left(\inf_{k\geq n} x_k \right)$$

are called the **superior** and **inferior limit** of $(x_n)_{n=0}^\infty$ respectively. From Equation 3.2 and Proposition 3.13, we have

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n.$$

Lemma 3.22: Convergence and Superior/Inferior Limits

A bounded sequence $(x_n)_{n=0}^\infty$ in $\mathbb{R}$ converges if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n.$$

Proof. For every $n \in \mathbb{N}$, define

$$i_n = \inf_{k\geq n} x_k, \qquad s_n = \sup_{k\geq n} x_k,$$

and set

$$I = \lim_{n\to\infty} i_n = \liminf_{n\to\infty} x_n, \qquad S = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} x_n.$$

First suppose that $I = S$. Since $i_n \leq x_n \leq s_n$ (see Equation 3.2), the Sandwich Lemma 3.15 implies that the sequence $(x_n)_{n=0}^\infty$ converges, and its limit equals $I = S$.

Conversely, assume that $(x_n)_{n=0}^\infty$ converges to $A \in \mathbb{R}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$A - \varepsilon < x_n < A + \varepsilon \qquad \forall n \geq N.$$

Then for all $n \geq N$, the same inequalities holds for i_n and s_n, i.e.,

$$A - \varepsilon \leq i_n \leq s_n \leq A + \varepsilon.$$

Taking limits and using Proposition 3.13, we obtain

$$A - \varepsilon \leq I \leq S \leq A + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $A = I = S$, which proves the result. $\qquad\square$

Theorem 3.23: Superior and Inferior Limits are Accumulation Points

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence and let $A = \limsup_{n\to\infty} x_n$. Then A is an accumulation point of $(x_n)_{n=0}^{\infty}$, and for every $\varepsilon > 0$ the following hold:

1. *only finitely many elements satisfy $x_n \geq A + \varepsilon$;*

2. *infinitely many elements satisfy $A - \varepsilon < x_n < A + \varepsilon$.*

An analogous statement holds for the inferior limit.

Proof. Since the sequence $(s_n)_{n=0}^{\infty}$ is monotonically decreasing and converges to A, given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$A \leq s_n < A + \varepsilon \qquad \forall n \geq N_0. \tag{3.3}$$

We first prove that A is an accumulation point.

Fix $N \in \mathbb{N}$ and set $N_1 = \max\{N, N_0\}$. Since $s_{N_1} = \sup_{k \geq N_1} x_k$, there exists $n_1 \geq N_1 \geq N_0$ such that

$$s_{N_1} - \varepsilon < x_{n_1} \leq s_{N_1}.$$

Thus, combining this bound with Equation 3.3 we obtain

$$A - \varepsilon < s_{N_1} - \varepsilon < x_{n_1} \leq s_{N_1} < A + \varepsilon.$$

This construct shows that for any $\varepsilon > 0$ and any $N \in \mathbb{N}$, there exists $n_1 \geq N$ such that $A - \varepsilon < x_{n_1} < A + \varepsilon$. Thus A is an accumulation point for $(x_n)_{n=0}^{\infty}$.

We now prove 1. and 2.. From Equation 3.3 we have $x_n < A + \varepsilon$ for all $n \geq N_0$, so only finitely many terms satisfy $x_n \geq A + \varepsilon$. This shows 1..

Also since A is an accumulation point, it follows from Corollary 3.10 that infinitely many terms of the sequence lie within any interval $(A - \varepsilon, A + \varepsilon)$. $\qquad\square$

Corollary 3.24: Bounded Sequences have Convergent Subsequences

Every bounded sequence has at least one accumulation point and therefore possesses a convergent subsequence.

Proof. By Theorem 3.23, the number

$$A = \limsup_{n\to\infty} x_n$$

is always an accumulation point of $(x_n)_{n=0}^{\infty}$. Moreover, by Proposition 3.9, every accumulation point is the limit of a convergent subsequence. Hence every bounded sequence admits at least one convergent subsequence. $\qquad\square$

3.5   Cauchy Sequences

Definition 3.25: Cauchy Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called a **Cauchy sequence** if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \qquad \forall n, m \geq N.$$

> **Lemma 3.26: Cauchy Sequences are Bounded**
>
> *Every Cauchy sequence is bounded.*

Proof. By definition, there exists $N \in \mathbb{N}$ such that

$$|x_n - x_N| \leq 1 \qquad \forall n \geq N.$$

Hence, for $n \geq N$, we have $|x_n| \leq 1 + |x_N|$. Now, define

$$M = \max\{|x_0|, |x_1|, \ldots, |x_{N-1}|, 1 + |x_N|\}.$$

Then, $|x_n| \leq M$ for all $n \in \mathbb{N}$, so $(x_n)_{n=0}^{\infty}$ is bounded. $\qquad\square$

> **Theorem 3.27: Convergence and Cauchy Sequences**
>
> *A sequence $(x_n)_{n=0}^{\infty}$ of real numbers converges if and only if it is a Cauchy sequence.*

Proof. Suppose first that $(x_n)_{n=0}^{\infty}$ converges to some $A \in \mathbb{R}$, and let us prove that $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|x_n - A| < \frac{\varepsilon}{2} \qquad \forall n \geq N.$$

Then for all $n, m \geq N$, we have that

$$|x_n - x_m| \leq |x_n - A| + |x_m - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence.

Viceversa, let $(x_n)_{n=0}^{\infty}$ be a Cauchy sequence. Since it is bounded (by Lemma 3.26), Corollary 3.24 implies that there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ converging to some $A \in \mathbb{R}$. Given $\varepsilon > 0$, choose $N_0 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \qquad \forall n, m \geq N_0,$$

and choose $N_1 \in \mathbb{N}$ such that

$$|x_{n_k} - A| < \frac{\varepsilon}{2} \qquad \forall k \geq N_1.$$

Let $N = \max\{N_0, N_1\}$. Since $n_N \geq N$ (see Remark 3.6), for all $n \geq N$ we have

$$|x_n - A| \leq |x_n - x_{n_N}| + |x_{n_N} - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $(x_n)_{n=0}^{\infty}$ converges to A. $\qquad\square$

3.6   Improper Limits

We now extend the notion of limit to allow the **improper limit values** $+\infty$ (often abbreviated as ∞) and $-\infty$.

> **Definition 3.28: Improper Limits**
>
> Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. We say $(x_n)_{n=0}^{\infty}$ **diverges to** $+\infty$, and we write
>
> $$\lim_{n \to \infty} x_n = +\infty,$$
>
> if for every $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n > M$ for all $n \geq N$.
> Similarly, $(x_n)_{n=0}^{\infty}$ **diverges to** $-\infty$ if for every $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n < -M$ for all $n \geq N$. In both cases, we say that $(x_n)_{n=0}^{\infty}$ has an **improper limit**.

An unbounded sequence doesn't need to diverge to $+\infty$ or $-\infty$. For instance, the sequence $x_n = (-1)^n n$, is unbounded but neither diverges to $+\infty$ nor to $-\infty$.

The notion of improper limit allows us to extend the definitions of superior and inferior limits to *unbounded* sequences. If $(x_n)_{n=0}^{\infty}$ is not bounded from above, then

$$\sup_{k \geq n} x_k = +\infty \qquad \forall n \in \mathbb{N},$$

and we write

$$\limsup_{n \to \infty} x_n = +\infty.$$

If $(x_n)_{n=0}^{\infty}$ is bounded from above but not from below, then we define

$$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} (\sup_{k \geq n} x_k),$$

where the right-hand side is a real limit if the decreasing sequence $\sup_{k \geq n} x_k$ is bounded, and the improper limit $-\infty$ otherwise. The definition of the inferior limit extends analogously.

3.7   Sequences of Complex Numbers

Informally, a **sequence of complex numbers** is just like a sequence of real numbers, except that each term is a complex number instead of a real one. Thus, we study ordered lists $(z_0, z_1, \ldots)$, where $z_n : \mathbb{N} \to \mathbb{C}$. As in the real case, we are mainly interested in their convergence, divergence and limit behavior.

To analyze sequences in $\mathbb{C}$, it is often sufficient to consider separately the corresponding sequences of real and imaginary parts in $\mathbb{R}$.

> **Definition 3.29: Sequences of Complex Numbers**
>
> A sequence of complex numbers $(z_n)_{n=0}^{\infty}$, where
>
> $$z_n = x_n + iy_n,$$
>
> is said to **converge** to a limit $A + iB \in \mathbb{C}$ if the two sequences of real numbers $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ converge to A and B, respectively. In this case, we write
>
> $$\lim_{n \to \infty} z_n = A + iB.$$
>
> We say that $(z_n)_{n=0}^{\infty}$ **diverges to** ∞ if the sequence of moduli $(|z_n|)_{n=0}^{\infty}$ diverges to $+\infty$, i.e.,
>
> $$\lim_{n \to \infty} |z_n| = \lim_{n \to \infty} \sqrt{x_n^2 + y_n^2} = +\infty.$$

Remark 3.30. *As for sequences of real numbers, one can consider subsequences of sequences $\mathbb{C}$. Given a strictly increasing sequence of non-negative integers $(n_k)_{k=0}^{\infty}$, the corresponding subsequence is*

$$(z_{n_k})_{k=0}^{\infty} = (x_{n_k} + iy_{n_k})_{n=0}^{\infty}.$$