# Analysis I
# Theorems & Lemmas

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# Contents

# 1 Functions

**Definition 1.1: Functions/Maps/Transformations**

A **function** f from a set $X$ to a set $Y$ is an assignment of an element of $Y$ to each element of $X$. The element $y \in Y$ to which $x \in X$ is assigned to is denoted $f(x)$. We write $f : X \to Y$ and sometimes also speak of a **map**, **mapping** or a **transformation**. The set $X$ is the **domain** and the set $Y$ is the **codomain**. We refer to the set $X$ as **domain** or **domain of definition**, and the set $Y$ as **domain of values** or **codomain**. The set

$$\{(x, f(x)) \mid x \in X\} \subseteq X \times Y$$

is called the **graph** of f. In the context of a function $f : X \to Y$, an element of the domain of definition is also called **argument**, and an element $y = f(x) \in Y$ assumed by the function, is also called **value** of the function. If $f : X \to Y$ is a function, one also writes

$$\begin{aligned} f : X &\to Y \\ x &\mapsto f(x), \end{aligned}$$

where $f(X)$ could be a concrete formula. We pronounce ´$\mapsto$´ as ´is mapped to'. Two functions $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are said to be equal if $X_1 = X_2$, $Y_1 = Y_2$ and $f_1(x) = f_2(x) \quad \forall x \in X_1$.

**Definition 1.2: Injective, Surjective and Bijective Functions**

Let $f : X \to Y$ be a function. We call $f$:

1. **injective** (or an **injection**) if

$$\forall x_1, x_2 \in X \;:\; x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2);$$

2. **surjective** (or a **sujection**) if

$$\forall y \in Y \;\exists x \in X \;:\; f(x) = y;$$

3. **bijective** (or a **bijection**) if $f$ is both injective and surjective.

Thus, a function $f : X \to Y$ is *not* injective if there exists distinct $x_1 \neq x_2 \in X$ such that $f(x_1) = f(x_2)$, and *not* surjective if there exists $y \in Y$ such that $f(x) \neq y$ for all $x \in X$.

**Definition 1.3: Image and Preimage of a Function**

For $f : X \to Y$ and $A \subseteq X$, define the **image** of $A$ under the function $f$ as

$$f(A) := \{y \in Y \mid \exists x \in X : f(x) = y\}.$$

For $B \subseteq Y$, define the **preimage** of $B$ under the function $f$ as

$$f^{-1}(B) := \{x \in X \mid f(x) \in B\}.$$

**Remark 1.4.** *Saying that $f : X \to Y$ is surjective is equivalent to $f(X) = Y$. Equivalently, $f$ is surjective if $f^{-1}(\{y\}) \neq \emptyset$ for all $y \in Y$.*

### Definition 1.5: Composition of Functions

Let $f : X \to Y$ and $g : Y \to Z$. The **composition** is $g \circ f : X \to Z$, defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.
**Associativity:** If $f : W \to X, g : X \to Y$ and $h : Y \to Z$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Indeed, for all $w \in W$, we have

$$h \circ (g \circ f)(w) = h((g \circ f)(w)) = h(g(f(w))) = (h \circ g)(f(w)) = ((h \circ g) \circ f)(w).$$

Therefore, we may omit parentheses and write $h \circ g \circ f : W \to Z$.

### Definition 1.6: Identity and Inverse Function

Given a set $X$, the **identity function** $\mathrm{id}_X : X \to X$ is defined by

$$\mathrm{id}_X(x) = x \qquad \forall x \in X.$$

If $f : X \to Y$ is bijective, then there exists a unique function $g : Y \to X$ such that, for each $y \in Y$, the value $g(y)$ is the unique element $x \in X$ satisfying $f(x) = y$. With this definition,

$$g \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ g = \mathrm{id}_Y\,.$$

The function $g$ is called the **inverse function** (or **inverse mapping**) of $f$, and is denoted by $f^{-1}$.

**Remark 1.7.** *A function $f : X \to Y$ is bijective if and only if there exists a function $g : Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$.*

# 2 The Real Numbers

## 2.1 Groups, Rings, Fields

**Definition 2.1: Groups**

A **group** is a non-empty set $G$ together with a rule (called an *operation*) denoted by $\star : G \times G \to G$ that combines any two elements of G into another element of G. This operation must satisfy three conditions:

- **Associativity:** No matter how you place parentheses, the result is the same for all $a, b, c \in G$,
$$(a \star b) \star c = a \star (b \star c).$$
- **Neutral element:** There is a special element $e \in G$ such that combining it with any $a \in G$ leaves $a$ unchanged, i.e.,
$$\forall a \in G : a \star e = e \star a = a.$$
- **Inverse element:** Every $a \in G$ has a 'partner' $a^{-1} \in G$ that 'cancels it out', giving the neutral element, i.e.,
$$a \star a^{-1} = a^{-1} \star a = e.$$

Note that, in general, one does not require that $a \star b = b \star a$. If the order of the operation does not matter, i.e., $a \star b = b \star a$ for all $a, b \in G$, the group is called **commutative** or **abelian**.

**Lemma 2.2: Basic Properties of Groups**

*Let $(G, \star)$ be a group. Then:*

1. *The neutral element is unique.*
2. *The inverse of an element is unique.*
3. *The inverse of the inverse of an element is the element itself, namely $(a^{-1})^{-1} = a$ for all $a \in G$.*

*Proof.* 1. Assume that, in addition to $e \in G$, we have a second element $e'$ with the property that $e' \star a = a \star e' = a$ for all elements $a \in G$. Then, we can choose $a = e$ to obtain
$$e \star e' = e.$$
Similarly, since $e$ is a neutral element, we have
$$e \star e' = e'.$$
Combining the two identities, we get
$$e = e \star e' = e'.$$
This proves that $e' = e$, so we speak of *the* neutral element of a group.

2. Assume that for an element $a \in G$, there exists two elements $b, c \in G$ that are both the inverse

of $a$, namely
$$a \star b = b \star a = e, \qquad a \star c = c \star a = e.$$
Then, using associativity, we observe that
$$b = b \star e = b \star (a \star c) = (b \star a) \star c = e \star c = c.$$
This proves that the inverse of an element $a$ is unique, so we can speak of *the* inverse element, and the notation $a^{-1}$ makes sense.

3. Since $a \star a^{-1} = e$, we deduce that $a$ is the inverse element of $a^{-1}$, thus
$$(a^{-1})^{-1} = a. \tag{2.1}$$
□

Groups capture the idea of combining elements with a single operation. But to describe the arithmetic of numbers more faithfully, we also need a second operation (as we do with addition and multiplication). This leads us to the notion of *rings* and *fields*.

### Definition 2.3: Rings and Fields

A **ring** is a non-empty set $R$ in which we can both 'add' and 'multiply' elements with two operations '+' and '$\cdot$'. Also, these two operations are compatible with each other. More precisely:

- $(R, +)$ is a **commutative group**, with neutral element denoted 0.
- Multiplication $\cdot$ is **associative**, has a **neutral element** (usually written as 1), and **distributes over addition**, i.e.,
$$a \cdot (b + c) = a \cdot b + a \cdot c, \qquad (b + c) \cdot a = b \cdot a + c \cdot a \qquad \forall a, b, c \in R.$$

If multiplication is also commutative, we call $(R, +, \cdot)$ a **commutative ring**. Note that, unlike addition, we do not require that every element has an inverse for multiplication. A **field** is a special kind of commutative ring, i.e. every non-zero element has an inverse for multiplication. In other words, if $(R, +, \cdot)$ is a commutative ring, then $(R, +, \cdot)$ is a field if $R \setminus \{0\}$ forms a commutative group under multiplication. Traditionally, we use the letter $F$ to denote a field. We also write $F^* = F \setminus \{0\}$ for the set of all invertible elements of $F$.

### Lemma 2.4: Basic Properties of Fields

*Let $(F, +, \cdot)$ be a field and let $a, b \in F$. Then:*

1. $0 \cdot a = a \cdot 0 = 0$.
2. $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$. *In particular* $(-1) \cdot a = -a$.
3. $(-a) \cdot (-b) = a \cdot b$. *In particular,* $(-a)^{-1} = -(a^{-1})$ *whenever* $a \neq 0$.

*Proof.* 1. Since 0 is the neutral element for the addition, we have $0 + 0 = 0$. Hence, using distributivity, we get
$$0 \cdot a = (0 + 0) \cdot a = (0 \cdot a) + (0 \cdot a).$$

Adding $-0 \cdot a$ (i.e., the inverse of $0 \cdot a$), we deduce that $0 \cdot a = 0$. The case of $a \cdot 0$ is analogous.

2. By the distributive law,

$$a \cdot b + a\dot{(} - b) = a \cdot (b + (-b)) = a \cdot 0 = 0.$$

So $a \cdot (-b)$ is the additive inverse of $a \cdot b$, i.e., $-(a \cdot b) = a \cdot (-b)$. Taking $b = 1$ gives $-a = (-1) \cdot a$. The validity of $(-a) \cdot b = -(a \cdot b)$ follows by exchanging $a$ and $b$ in the argument above.

3. By 2. we know that $-(a \cdot b) = a \cdot (-b)$. Hence, recalling Equation 2.1,

$$a \cdot b = -(a \cdot (-b)).$$

On the other hand, applying 2. with $(-b)$ instead of $b$, we also have

$$-(a \cdot (-b)) = (-a) \cdot (-b).$$

Combining the two identities above, we conclude that $(-a) \cdot (-b) = a \cdot b$. Finally, taking $b = a^{-1}$ yields $(-a) \cdot (-(a^{-1})) = a \cdot a^{-1} = 1$, which gives the second assertion. □

## 2.2 Order Relation

**Definition 2.5: Cartesian Product**

Let $X$ and $Y$ be two sets. The **cartesian product** $X \times Y$ is the set of ordered pairs of elements of $X$ and $Y$, i.e.,

$$X \times Y := \{(x, y) \mid x \in X, y \in Y\}.$$

**Definition 2.6: Subsets**

Let $P$ and $Q$ be sets. Then

- $P$ is a **subset** of $Q$, written $P \subset Q$ (or $P \subseteq Q$), if every element of $P$ also belongs to $Q$.
- $P$ is a **proper subset** of $Q$, written $P \subsetneq Q$, if $P$ is a subset of $Q$ but $P \neq Q$.
- We write $P \nsubseteq Q$ if $P$ is not a subset of $Q$.

**Definition 2.7: Relations**

Let $X$ be a set. A **relation** on $X$ is a subset $\mathcal{R} \subseteq X \times X$, that is, a collection of ordered pairs of elements of $X$. If $(x, y) \in \mathcal{R}$ we write $x\mathcal{R}y$. Common symbols for relations include $<, \leq, \sim, \equiv, \cong$. If $\sim$ is a relation on $X$, we write $x \nsim y$ if $x \sim y$ does not hold. A realtion $\sim$ may have the following properties:

1. **Reflexive:** if $x \sim x \qquad \forall x \in X$.
2. **Transitive:** if $x \sim y$ and $y \sim z$, then $x \sim z$.
3. **Symmetric:** if $x \sim y$, then $y \sim x$.
4. **Antisymmetric:** if $x \sim y$ and $y \sim x$, then $x = y$.

A relation is an **equivalence relation** if it is reflexive, transitive and symmetric. It is an **order relation** if it is reflexive, transitive and antisymmetric.

## 2.3 Ordered Fields

**Definition 2.8: Ordered Field**

Let $F$ be a field, and let $\leq$ be an order relation on $F$. We call $(F, \leq)$, or simply $F$, an **ordered field** if the following hold:

1. **Linearity of order:** for all $x, y \in F$, at least one of $x \leq y$ or $y \leq x$ holds.
2. **Compatibility with addition:** for all $x, y, z \in F$,
$$x \leq y \Rightarrow x + z \leq y + z.$$
3. **Compatibility with multiplication:** for all $x, y \in F$,
$$0 \leq x \ \wedge \ 0 \leq y \Rightarrow 0 \leq x \cdot y.$$

**Lemma 2.9: Ordered Field: Basic Consequences**

*Let $(F, \leq)$ be and ordered field, and let $x, y, z, w \in F$. Then:*

*(a) (Trichotomy) Either $x < y$, or $x = y$, or $x > y$.*

*(b) If $x < y$ and $y \leq z$, then $x < z$. (Analogously, $x \leq yandy < zimplyx < z$.)*

*(c) (Addition of inequalities) If $x \leq y$ and $z \leq w$, then $x + z \leq y + w$. (Analogously, $x < z$ and $z \leq w$ imply $x + z < y + w$.)*

*(d) $x \leq y$ if and only if $0 \leq y - x$.*

*(e) $x \leq 0$ if and only if $0 \leq -x$.*

*(f) $x^2 \geq 0$, and $x^2 > 0$ if $x \neq 0$.*

*(g) 0 < 1.*

*(h) If $0 \leq x$ and $y \leq z$, then $xy \leq xz$.*

*(i) If $x \leq 0$ and $y \leq z$, then $xy \geq xz$.*

*(j) If $0 < x \leq y$, then $0 < y^{-1} \leq x^{-1}$.*

*(k) If $0 \leq x \leq y$ and $0 \leq z \leq w$, then $0 \leq xz \leq yw$.*

*(l) If $x + y \leq x + z$, then $y \leq z$.*

*(m) If $xy \leq xz$ and $x > 0$, then $y \leq z$.*

**Lemma 2.10: Integers and Rationals Inside an Ordered Field**

*Let $(F, \leq)$ be and ordered field, and denote by 0 and 1 the neutral elements for addition and multiplication, respectively. Then:*

*(i) The elements $\ldots, -2, -1, 0, 1, 2, \ldots$ defined by*

$$2 = 1 + 1, \quad 3 = 2 + 1, \ldots, \quad -n = (-1) \cdot n$$

*are all distinct and satisfy*

$$\ldots < -2 < -1 < 0 < 1 < 2 < 3 < \ldots.$$

*We denote this set of elements by $\mathbb{Z}$, and we call them 'integers'*

*(ii) Every fraction $pq^{-1}$ with $p, q \in \mathbb{Z}$, $q \neq 0$, lies in $F$ and the set of all such elements is denoted by $\mathbb{Q}$. Also,*

$$\mathbb{Z} \subsetneq \mathbb{Q} \subseteq F.$$

*Proof.* (i) By Lemma 2.9(g), we have that $0 < 1$. Then Lemma 2.9(c) yields $0 < 1 < 2 < 3 < \ldots$, and taking negatives gives $\ldots < -2 < -1 < 0$. Hence all these elements are distinct.

(ii) For $q \neq 0$, q is invertible in $F$; define $\frac{p}{q} = pq^{-1}$. The set of such fractions is a field contained in $F$, which we denote by $\mathbb{Q}$.

To show that $\mathbb{Q}$ strictly contains $\mathbb{Z}$, consider $\frac{1}{2}$ (the inverse of 2). Since $2 > 1$, it follows from Lemma 2.9(j) that $0 < \frac{1}{2} < 1$, so $\frac{1}{2} \notin \mathbb{Z}$. □

**Definition 2.11: Absolute Value and Sign**

Let $(F, \leq)$ be and ordered field.

- The **absolute value** (or **modulus**) is the function $|\cdot| : F \to F$ defined by

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

- The **sign** is the function $\operatorname{sgn} : F \to \{-1, 0, 1\}$ defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

**Lemma 2.12: Absolute Value and Sign: Basic Properties**

*Let $(F, \leq)$ be an ordered field and let $x, y \in F$. Then:*

*(a)* $x = \operatorname{sgn}(x)|x|, \quad |-x| = |x|, \quad \operatorname{sgn}(-x) = -\operatorname{sgn}(x)$.

*(b)* $|x| \geq 0$, *and* $|x| = 0$ *if and only if* $x = 0$ *(by Trichotomy Lemma* **??***).*

*(c) (Multiplicativity)* $\operatorname{sgn}(xy) = \operatorname{sgn}(x)\operatorname{sgn}(y)$ *and* $|xy| = |x||y|$.

*(d) If* $x \neq 0$, *then* $|x^{-1}| = |x|^{-1}$.

*(e)* $|x| \leq y$ *iff* $-y \leq x \leq y$.

*(f)* $|x| < y$ *iff* $-y < x < y$.

*(g) (Triangle inequality)* $|x + y| \leq |x| + |y|$.

*(h) (Inverse triangle inequality)* $||x| - |y|| \leq |x - y|$.

*Proof.* (g) Thanks to (e) we have $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding these two inequalities we get

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

Applying (e) again yields the result.

(h) From (g) we have $|x| \leq |x - y| + |y|$, therefore

$$|x| - |y| \leq |x - y|.$$

Exchanging the roles of $x$ and $y$, we also have $|y| - |x| leq |y - x| = |x - y|$. Combining these two inequalities yields

$$-|x - y| \leq |x| - |y| \leq |x - y|,$$

and the result follows by applying (e) again. □

## 2.4 Completeness Axiom

**Definition 2.13: Completeness Axiom**

Let $(K, \leq)$ be an ordered field. We say that $(K, \leq)$ is **complete** (or a **completely ordered field**) if the following statement holds:

> Let $X, Y$ be non-empty subsets of $K$ such that $x \leq y$ for all $x \in X$ and $y \in Y$. Then there exists $c \in K$ lying between $X$ and $Y$, in the sense that $x \leq c \leq y$ for all $x \in X$ and $y \in Y$.

The statement above is called the **completeness axiom**.

**Definition 2.14: Real Numbers**

We call **the field of real numbers**, any completely ordered field and denote it by $\mathbb{R}$.

## 2.5 Intervals

**Definition 2.15: Intervals**

Let $a, b \in \mathbb{R}$. We define:

- The **closed interval**
$$[a, b] := \{x \in R \mid a \leq x \leq b\};$$
- The **open interval**
$$(a, b) := \{x \in R \mid a < x < b\};$$
- The **half-open intervals**
$$[a, b) := \{x \in R \mid a \leq x < b\} \quad \text{and} \quad (a, b] := \{x \in R \mid a < x \leq b\};$$
- The **unbounded closed intervals**
$$[a, \infty) := \{x \in R \mid a \leq x\} \quad \text{and} \quad (-\infty, b] := \{x \in R \mid x \leq b\};$$
- The **unbounded open intervals**
$$(a, \infty) := \{x \in R \mid a < x\} \quad \text{and} \quad (-\infty, b) := \{x \in R \mid x < b\};$$

**Definition 2.16: Set Operations**

Let $P, Q$ be sets. The **intersection** $P \cap Q$, the **union** $P \cup Q$, the **relative complement** $P \setminus Q$ and the **symmetric difference** $P \triangle Q$ are defined by

$$\begin{aligned}
P \cap Q &= \{x \mid x \in P \text{and} x \in Q\}, \\
P \cup Q &= \{x \mid x \in P \text{or} x \in Q\}, \\
P \setminus Q &= \{x \mid x \in P \text{and} x \notin Q\}, \\
P \triangle Q &= (P \setminus Q) \cup (Q \setminus P) = (P \cup Q) \setminus (P \cap Q).
\end{aligned}$$

**Definition 2.17: Union and Intersection of several Sets**

Let $\mathcal{A}$ be a family of sets (i.e., a set whose elements are sets). We define the **union** and **intersection** of the sets in $\mathcal{A}$ as

$$\bigcup_{A \in \mathcal{A}} A = \{x \mid \exists A \in A : x \in A\}, \qquad \bigcap_{A \in \mathcal{A}} A = \{x \mid \forall A \in \mathcal{A} : x \in A\}.$$

If $\mathcal{A} = \{A_1, A_2, \ldots\}$, we also write

$$\bigcup_{i=1}^{\infty} A_i = \{x \mid \exists i \geq 1 : x \in A_i\}, \qquad \bigcap_{i=1}^{\infty} A_i = \{x \mid \forall i \geq 1 : x \in A_i\}.$$

**Definition 2.18: Neighborhoods**

Let $x \in \mathbb{R}$. A **neighborhood** of $x$ is a set containing an interval $I$ such that $x \in I$. Given $\delta > 0$, the open interval $(x-\delta, x+\delta)$ is called the $\delta$-**neighborhood** of $x$.

**Definition 2.19: Open and Closed Sets**

A subset $U \subseteq \mathbb{R}$ is called **open** in $\mathbb{R}$ if for every $x \in U$ there exists open interval $I$ such that $x \in I$ and $I \subseteq U$. A subset $F \subseteq \mathbb{R}$ is called **closed** in $\mathbb{R}$ if its complement $\mathbb{R} \setminus F$ is open.

**Remark 2.20.** *The sets $\emptyset$ and $\mathbb{R}$ are both open in $\mathbb{R}$. Hence, they are also closed since $\emptyset^c = \mathbb{R}$ and $\mathbb{R}^c = \emptyset$. We note that $\mathbb{Q} \subseteq \mathbb{R}$ and $[a, b) \subseteq \mathbb{R}$ are neither open nor closed.*

**Remark 2.21.** *Let $\mathcal{U}$ be a family of open sets, and $\mathcal{F}$ be a family of closed subsets of $\mathbb{R}$. Then the union and intersection*

$$\bigcup_{U \in \mathcal{U}} U, \qquad \bigcap_{F \in \mathcal{F}} F$$

*Are open and closed, respectively.*

## 2.6 Complex Numbers

Starting from the field of real numbers $\mathbb{R}$, we define the set of **complex numbers** as

$$\mathbb{C} = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

We denote the elements $z = (x, y) \in \mathbb{C}$ in the form $z = x + iy$, where $i$ is the **imaginary unit**. Here $x \in \mathbb{R}$ is the **real part** of $z$, written as $x = \mathrm{Re}(z)$, and $y \in \mathbb{R}$ is the **imaginary part**, written as $y = \mathrm{Im}(z)$. Elements with $\mathrm{Im}(z) = 0$ are called **real**, while those with $\mathrm{Re}(z) = 0$ are **purely imaginary**. Via the injective map $\mathbb{R} \ni x \mapsto x + i \cdot 0 \in \mathbb{C}$, we identify $\mathbb{R}$ with the subset of real numbers inside $\mathbb{C}$.

As you may expect from previous knowledge, we want to satisfy $i^2 = -1$. To achieve this, we define addition and multiplication on $\mathbb{C}$ so that it becomes a field. Additionally, we want these operations to coincide with the usual addition and multiplication when considering real numbers.

Since $i^2 = -1$, using commutativity and distributivity we get

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_2x_1 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

This motivates the following definition

**Definition 2.22: Addition and Multiplication on $\mathbb{C}$**

On $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ we define **addition** and **multiplication** as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2),$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1).$$

**Proposition 2.23: $\mathbb{C}$ is a Field**

*With the operation of Definition 2.22, together with the zero element $(0, 0)$ and the unit element $(1, 0)$, the set $\mathbb{C}$ is a field.*

**Definition 2.24: Complex Conjugation**

For $z = x + iy \in \mathbb{C}$ we define its **conjugate** as $\bar{z} = x - iy$. The mapping $\mathbb{C} \ni z \mapsto \bar{z} \in \mathbb{C}$ is called **complex conjugation**.

**Lemma 2.25: Properties of Complex Conjugation**

*For all $z, w \in \mathbb{C}$:*

*(i)* $z\bar{z} = x^2 + y^2 \in \mathbb{R}_{\geq 0}$. *In particular,* $z\bar{z} = 0$ *if and only if* $z = 0$.

*(ii)* $\overline{z + w} = \bar{z} + \bar{w}$.

*(iii)* $\overline{zw} = \bar{z}\bar{w}$.

*Proof.* Property (i) follows from the fact that, for $z = x + iy$, $(x + iy)(x - iy) = x^2 + y^2$. Also, $x^2 + y^2 = 0$ if and only if $x + iy = 0$. Properties (ii) and (iii) follow from a direct computation, writing $z = x_+ + iy_1$ and $w = x_2 + iy_2$, which yields

$$\begin{aligned}
\overline{z + w} &= \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_2) + (x_2 - iy_2) = \bar{z} + \bar{w}, \\
\overline{z \cdot w} &= \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)} = (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2) \\
&= (x_1 - iy_1) \cdot (x_2 - iy_2) = \bar{z} \cdot \bar{w}.
\end{aligned}$$

□

**Definition 2.26: Absolute Value**

The **absolute value** (or **norm**) on $\mathbb{C}$ is the map $|\cdot| : \mathbb{C} \to \mathbb{R}$ given by

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}, \qquad z = x + iy \in \mathbb{C}.$$

**Lemma 2.27: Cauchy-Schwart Inequality**

*If $z = x_1 + iy_1$, and $w = x_2 + iy_2$, then*

$$x_1x_2 + y_1y_2 \leq |z||w|. \tag{2.2}$$

*Proof.* We observe that

$$\begin{aligned}
|z|^2|w|^2 - (x_1x_2 + y_1y_2)^2 &= (x_1^2 + y_1^2)(x_2^2 + y_2^2) - (x_1x_2 + y_1y_2)^2 \\
&= x_1^2x_2^2 + y_1^2y_2^2 + y_1^2x_2^2 + x_+^2y_2^2 - (x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2) \\
&= y_1^2x_2^2 + x_+^2y_2^2 - 2x_1x_2y_1y_2 \\
&= (y_1x_2 - x_1y_2)^2 \geq 0.
\end{aligned}$$

This proves that $(x_1x_2 + y_1y_2)^2 \leq |z|^2|w|^2$, so it follows that

$$|x_1x_2 + y_1y_2| \leq |z||w|.$$

Since $x \leq |x|$ for all $x \in \mathbb{R}$, we obtain Equation 2.2. □

**Proposition 2.28: Trianlge Inequality**

*For all* $z, w \in \mathbb{C}$, *one has*

$$|z + w| \leq |z| + |w|.$$

*Proof.* For $z = x_1 + iy_1$ and $w = x_2 + iy_2$, using Lemma 2.27, we have

$$\begin{aligned} |z + w|^2 &= (x_1 + x_2)^2 + (y_1 + y_2)^2 \\ &= |z|^2 + |w|^2 + 2(x_1 x_2 + y_1 y_2) \\ &\leq |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2. \end{aligned}$$

Taking square roots proves the result. □

**Definition 2.29: Cicular Disks**

For $z \in \mathbb{C}$ and $r > 0$, we define the **open disk** with radius $r > 0$ around $z$ as

$$B(z, r) := \{w \in \mathbb{C} \mid |z - w| < r\},$$

and the **closed disk** with radius $r > 0$ around $z$ as

$$\overline{B(z, r)} := \{w \in \mathbb{C} \mid |z - w| \leq r\}.$$

In other words, the open disk $B(z, r)$ is the set of points at distance strictly less than $r$ form $z$. We note that this definition is compatible with the one of neighborhoods in $\mathbb{R}$: if $x \in \mathbb{R}$ and $r > 0$, then

$$B(x, r) \cap \mathbb{R} = (x - r, x + r).$$

**Definition 2.30: Open and Closed Sets**

A set $U \subseteq \mathbb{C}$ is **open** if for every $z \in U$ there exists $r > 0$ such that $B(z, r) \subseteq U$. A set $C \subseteq \mathbb{C}$ is **closed** if its complement $\mathbb{C} \setminus C$ is open.

## 2.7 Maximum and Supremum

### 2.7.1 Existence of the Supremum

**Definition 2.31: Bounded Sets, Maxima and Minima**

et $X \subseteq \mathbb{R}$ be a subset of real numbers.

- $X$ is **bounded from above** if there exists $s \in \mathbb{R}$ such that $x \leq s$ for all $x \in X$. Such a number $s$ is called an **upper bound** of $X$. If $s$ is an upper bound and also an element of $X$, we say that $s$ is the **maximum** of $X$ and write
$$s = \max(X).$$
- Analogously, $X$ is **bounded from below** if there exists $r \in \mathbb{R}$ such that $r \leq x$ for all $x \in X$. Such a number $r$ is called a **lower bound** of $X$. If $r$ is a lower bound and also an element of $X$, we say that $r$ is the **minimum** of $X$ and write
$$r = \min(X).$$
- $X$ is called **bounded** if it is both bounded from above and bounded from below.

**Remark 2.32.** *If a set $X \subseteq \mathbb{R}$ has a maximum, then it is unique. Indeed, if $x_1, x_2 \in X$ are both maxima, then $x_1 \leq x_2$ (since $x_2$ is a maximum) and $x_2 \leq x_1$ (since $x_1$ is a maximum), so $x_1 = x_2$.*

A closed interval $[a, b]$ with $a < b$ has both a minimum and maximum, i.e., $a = \min([a, b])$ and $b = \max([a, b])$. But not all sets have a maximum. For instance, the open interval $(a, b)$ does not have a maximum because the endpoint $b$, though an upper bound, is not contained in the set. Similarly $\mathbb{R}$ and unbounded intervals such as $[a, \infty)$ or $(a, \infty)$ have no maximum.

**Definition 2.33: Supremum**

Let $X \subseteq \mathbb{R}$ be a subset and let
$$A := \{a \in \mathbb{R} \mid x \leq a \quad \forall x \in X\}$$
be the set of all upper bounds of $X$. If $A$ has a minimum, we call this minimum the **supremum** of $X$ and write
$$\sup(X) = \min(A).$$
The **infimum** is defined analogously using the maximum of the set of all lower bounds.

In other words, the supremum of $X$ is the smallest real number that is greater than or equal to every element of $X$. Note that we can describe the supremum $s = \sup(X)$ as follows
$$x \leq s \qquad \forall x \in X, \qquad \text{and} \qquad \text{if } t < s, \text{the } t \text{ is not an upper bound of } X. \tag{2.3}$$
This means that for every $t < s$, there exists some $x \in X$ such that $x > t$. That is,
$$x \leq s \qquad \forall x \in X, \qquad \text{and} \qquad \forall t < s\, \exists x \in X : x > t. \tag{2.4}$$
The two characterizations 2.3 and 2.4 are equivalent.

Note that not every set has a supremum. If $X = \emptyset$ or if $X$ is unbounded form above, then $\sup(X)$ does not exist. However, for any non-empty and bounded-above subset of $\mathbb{R}$, the supremum always exists.

**Remark 2.34.** *If a set $X$ has a maximum, then this element is also the supremum. Indeed, the maximum is an upper bound of $X$, and since it lies in $X$, no smaller upper bound can exist.*

### Theorem 2.35: Existence of Supremum

*Let $X \subseteq \mathbb{R}$ be non-empty and bounded from above. Then $\sup(X)$ exists and is a real number.*

*Proof.* Since $X$ is bounded from above, the set $A := \{a \in \mathbb{R} \mid x \leq a \quad \forall x \in X\}$ of upper bounds is non-empty. Since $x \leq a$ for any $x \in X$ and $a \in A$, we can apply the completeness axiom (Definition 2.13) to find $c \in \mathbb{R}$ such that

$$x \leq c \leq a \qquad \forall x \in X, \forall a \in A.$$

The first inequality implies that $c$ is itself an upper bound (so $c \in A$), while the second inequality tells us that $c$ is smaller than or equal to every upper bound. Hence, $c = \min(A) = \sup(X)$. □

### Proposition 2.36: Supremum and Set Operations

*Let $X$ and $Y$ be non-empty subsets of $\mathbb{R}$ that are bounded from above. Define*

$$X + Y := \{x + y \mid x \in X, y \in Y\} \quad \textit{and} \quad X \cdot Y := \{x \cdot y \mid x \in X, y \in Y\}.$$

*The the sets $X \cup Y, X \cap Y$, and $X + Y$ are also bounded from above. Moreover, if $X, Y \subseteq \mathbb{R}_{\geq 0}$ (i.e., $x \geq 0$ and $y \geq 0$ for all $x \in X$ and $y \in Y$), then $X \cdot Y$ is bounded from above as well. In these cases, the following formulas hold:*

*(1)* $\sup(X \cup Y) = \max\{\sup(X), \sup(Y)\}$,

*(2) If $X \cap Y \neq \emptyset$, then* $\sup(X \cap Y) \leq \min\{\sup(X), \sup(Y)\}$,

*(3)* $\sup(X + Y) = \sup(X) + \sup(Y)$,

*(4) If $X, Y \subseteq \mathbb{R}_{\geq 0}$, then* $\sup(X \cdot Y) = \sup(X) \cdot \sup(Y)$.

*Proof.* (3) Let $x_0 = \sup(X)$ and $y_0 = \sup(Y)$. For any $z \in X + Y$, there exists $x \in X$ and $y \in Y$ such that $z = x + y$. Since $x \leq x_0$ and $y \leq y_0$, we have

$$z = x + y \leq x_0 + y_0,$$

so $x_0 + y_0$ is an upper bound form $X + Y$. We now want to show that $x_0 + y_0 = \sup(X + Y)$.

Let $z_0 = \sup(X + Y)$ and suppose, by contradiction, that

$$\varepsilon := x_0 + y_0 - z_0 > 0.$$

Since $x_0 = \sup(X)$, by the characterization 2.4 there exists $x \in X$ such that $x > x_0 - \varepsilon/2$. Likewise, there exists $y \in Y$ such that $y > y_0 - \varepsilon/2$. Setting $z = x + y$, we obtain

$$z > x_0 - \frac{\varepsilon}{2} + y_0 - \frac{\varepsilon}{2} = x_0 + y_0 - \varepsilon = z_0,$$

contradicting the assumption that $z_0$ is an upper bound for $X + Y$. Therefore, $z_0 = x_0 + y_0$.

(4) The proof is analogous. If all elements of $X$ and $Y$ are non-negative, and we set $x_0 = \sup(X)$ and $y_0 = \sup(Y)$, then for any $z = x \cdot y \in X \cdot Y$, we have

$$z = x \cdot y \leq x_0 \cdot y_0,$$

which shows that $x_0 \cdot y_0$ is an upper bound for $X \cdot Y$. Using a similar '$\varepsilon$-argument' as done above, when proving (3), one shows that this upper bound is sharp, i.e., $x_0 \cdot y_0$ is the least upper bound. $\square$

## 2.8 Two-Point Compactification

In this section, we extend the notions of **supremum** and **infimum** to arbitrary subsets of $\mathbb{R}$. To do so, we introduce two formal symbols

$$+\infty \quad \text{and} \quad -\infty,$$

which are not real numbers. We define the **extended real numbers line** (also called the **two-point compactification** of $\mathbb{R}$) by

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}.$$

We extend the usual order relation $\leq$ on $\mathbb{R}$ to $\overline{\mathbb{R}}$ by requiring that

$$-\infty < x < +\infty \qquad \forall x \in \mathbb{R}.$$

For simplicity, we often write $\infty$ instead of $+\infty$.

We now introduce some standard (but informal) computation rules involving these symbols. For all $x \in \mathbb{R}$, we adopt the conventions:

$$\infty + x = \infty + \infty = \infty, \qquad -\infty + x = -\infty - \infty = -\infty.$$

If $x > 0$, then

$$x \cdot \infty = \infty \cdot \infty = \infty, \qquad x \cdot (-\infty) = \infty \cdot (-\infty) = -\infty,$$

while for $x < 0$ we have

$$x \cdot \infty = -\infty \cdot \infty = -\infty, \qquad x \cdot (-\infty) = -\infty \cdot (-\infty) = \infty.$$

These rules are widely used as notational shorthand, but one must handle them with care. Expressions like

$$\infty - \infty, \quad 0 \cdot \infty, \quad \text{or similar}$$

are undefined and should be avoided.

**Definition 2.37: Supremum and Infimum in the Extended Line**

Let $X \subseteq \mathbb{R}$.

- If $X$ is not bounded from above, we define $\sup(X) = \infty$.
- If $X = \emptyset$, we define $\sup(\emptyset) = -\infty$.
- If $X$ is not bounded from below, we define $\inf(X) = -\infty$.
- If $X = \emptyset$, we define $\inf(\emptyset) = \infty$.

In this context, we refer to $\infty$ and $-\infty$ as **indefinite values**.

In other words:

- Saying $\sup(X) = \infty$ means that $X$ is not bounded from above, i.e.,

$$\forall x_0 \in X \; \exists x \in X : x > x_0.$$

- Saying $\sup(X) = -\infty$ means that $X$ is empty.
- Similarly, $\inf(X) = -\infty$ means that $X$ is not bounded from below, and $\inf(X) = \infty$ means $X$ is empty.

## 2.9 Consequences of Completeness

### 2.9.1 Archimedean Principle

The archimedean principle states that for every real number $x \in \mathbb{R}$ there exists and integers $n$ greater than $x$. The following theorem, proved using the existence of suprema (and implicitly the completeness axiom), gives a precise formulation of this principle.

**Theorem 2.38: Archimedean Principle**

*For every $x \in \mathbb{R}$ there exists exactly one $n \in \mathbb{Z}$ such that*

$$n \leq x < n + 1.$$

*Proof.* We first treat the case $x \geq 0$. Fix $\mathbb{R} \ni x \geq 0$ and define

$$E = \{n \in \mathbb{Z} \mid n \leq x\}.$$

Since $0 \in E$ and $x$ is an upper bound, $E$ is a non-empty subset of $\mathbb{R}$ bounded from above. Hence, by Theorem 2.35, the supremum $s_0 = \sup(E)$ exists. From the definition of supremum we deduce:

(i) $s_0 \leq x$ (because $x$ is an upper bound);

(ii) there exists $n_0 \in E$ with $s_0 - 1 < n_0$ (otherwise $s_0 - 1$ would also be an upper bound).

From (ii) we obtain $s_0 < n_0 + 1$, which implies

(iii) $n_0 + 1 \notin E$ (otherwise $s_0$ would not be an upper bound for $E$).

Moreover, since $m \leq s_0$ for every $m \in E$, we have $m < n_0 + 1$ for all $m \in E$. As all elements of $E$ are integers,

$$m < n_0 + 1 \quad \Leftrightarrow \quad m - n_0 < 1 \quad \Leftrightarrow \quad m - n_0 \leq 0 \quad \Leftrightarrow \quad m \leq n_0.$$

Thus, every $m \in E$ is less than or equal to $n_0$, and since $n_0 \in E$, we conclude that $n_0 = \max(E)$. In particular, by Remark 2.34, the maximum is also the supremum, so $s_0 = n_0$.

Finally, recalling (iii) and the definition of $E$, we have $n_0 + 1 > x$. Together with (i), this shows

$$n_0 = s_0 \leq x < n_0 + 1,$$

establishing the claim for any $x \geq 0$.

Now, if $x < 0$, apply the previous argument to $-x > 0$. Then there exists $m \in \mathbb{Z}$ such that

$$m \leq -x < m + 1,$$

which is equivalent to

$$-m - 1 < x \leq -m.$$

If $x = -m$, then set $n = -m$. If $x < -m$, set $n = -m - 1$. In both cases, we obtain

$$n \leq x < n + 1.$$

Finally, for uniqueness, assume that $n_1, n_2 \in \mathbb{Z}$ both satisfy $n_i \leq x < n_i + 1$. From $n_1 \leq x < n_2 + 1$ we deduce that $n_1 < n_2 + 1$, and therefore $n_1 \leq n_2$. Reversing the roles of $n_1$ and $n_2$ gives $n_2 \leq n_1$. Hence, $n_1 = n_2$. □

**Definition 2.39: Integer and Fractional Parts**

The **integer part** $\lfloor x \rfloor$ of $x \in \mathbb{R}$ is the integer $n \in \mathbb{Z}$ uniquely determined by Theorem 2.38 such that $n \leq x < n + 1$. The map $x \mapsto \lfloor x \rfloor$ from $\mathbb{R}$ to $\mathbb{Z}$ is called the **rounding function**. The **fractional part** of $x$ is defined as

$$\{x\} = x - \lfloor x \rfloor \in [0, 1).$$

**Corollary 2.40: $\frac{1}{n}$ is Arbitrarily Small**

*For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$, with $n \geq 1$, such that*

$$\frac{1}{n} < \varepsilon.$$

*Proof.* Applying Theorem 2.38 to $x = \frac{1}{\varepsilon} > 0$, we find $m \in \mathbb{Z}$ such that

$$m \leq \frac{1}{\varepsilon} < m + 1.$$

Set $n := m + 1$. In this way we have $0 < \frac{1}{\varepsilon} < n$, which is equivalent to $n > 0$ (therefore, $n \geq 1$) and $\frac{1}{n} < \varepsilon$. □

**Definition 2.41: Dense Sets**

A subset $X \subseteq \mathbb{R}$ is called **dense** in $\mathbb{R}$ if every open non-empty interval contains an element of $X$.

**Corollary 2.42: Density of $\mathbb{Q}$**

*For every $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.*

*Proof.* Set $\varepsilon = b - a$. By Corollary 2.40, there exists $m \in \mathbb{N}$ with $\frac{1}{m} < \varepsilon$. Then, by Theorem 2.38 applied with $x = ma$, there exists $n \in \mathbb{Z}$ with

$$n \leq ma < n + 1,$$

or equivalently,

$$\frac{n}{m} \leq a < \frac{n+1}{m}.$$

Since $\frac{1}{m} < \varepsilon$, by the two inequalities above, we obtain

$$a < \frac{n+1}{m} \leq a + \frac{1}{m} < a + \varepsilon = b.$$

Thus $r = \frac{n+1}{m}$ is a rational number between $a$ and $b$. □

**Corollary 2.43: Density of $\mathbb{R} \setminus \mathbb{Q}$**

*For every $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < r < b$.*

*Proof.* We want to show that for every $x \in \mathbb{R}$ and $\delta > 0$, there exists an $a \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$a \in (x - \delta, x + \delta).$$

By Corollary 2.42, we find a $q \in \mathbb{Q}$ such that $q \in (x - \delta, x + \delta)$. By Corollary 2.40 we find an $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \frac{(x + \delta) - q}{\sqrt{2}} \quad \Rightarrow \quad \frac{\sqrt{2}}{N} < (x + \delta) - q.$$

This implies that

$$x - \delta < q < \frac{\sqrt{2}}{N} + q < x + \delta.$$

Choosing $r = \frac{\sqrt{2}}{N} + q$ proves the statement. □

### 2.9.2 Uncountability

**Definition 2.44: Cardinality**

Let $X$ and $Y$ be sets.

- We say $X$ and $Y$ have the **same cardinality**, written $X \sim Y$, if there is a bijection $f : X \to Y$.
- We write $X \preceq Y$ if there exists an injection $f : X \to Y$.
- The empty set has cardinality 0.
- A set $X$ has **finite cardinality** $|X| = n$ if there exists a bijection with $\{1, \dots, n\}$.
- A set is **infinite** it it is not finite.
- A set is **countable** if it has a bijection to $\mathbb{N}$. Its cardinality is denoted $\aleph_0$, pronounced Aleph-0.
- A set is **uncountable** if it is infinite but not countable.

If $X \preceq Y$ and $Y \preceq X$, then $X \sim Y$. In other words, if there exists an injective map $f : X \to Y$ and an injective map $g : Y \to X$, then one can find a bijective map $h : X \to Y$. This non-trivial statement is the **Schröder-Bernstein Theorem**.

We will now list some statements about different sets of numbers from the lecture:

1. $\mathbb{N}$ and the even numbers have the same cardinality.
2. $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.
3. $\mathbb{Q}$ is countable, i.e., $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{Z}$.

**Proposition 2.45: Uncountability of $\mathbb{R}$**

*The set $\mathbb{R}$ is uncountable.*

#### Extra Material

**Definition 2.46: Power Set**

Let $X$ be a set. The **power set** $\mathcal{P}(X)$ of $X$ is the set of all subsets of $X$, i.e.,

$$\mathcal{P}(X) := \{A \subseteq X\}.$$

**Theorem 2.47: Cantor's Theorem**

*For any set $X$, the power set $\mathcal{P}(X)$ has strictly larger cardinality than $X$.*

**Proposition 2.48: The Reals have the same cardinality as $\mathcal{P}(\mathbb{N})$**

$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

# 3 Sequences of Real Numbers

## 3.1 Convergence of Sequences

**Definition 3.1: Sequences**

A **sequence** is a function $a : \mathbb{N} \to \mathbb{R}$. The image $a(n)$ of $n \in \mathbb{N}$ is also written as $a_n$ and is called the $n$-th element of a. Instead of $a : \mathbb{N} \to \mathbb{R}$ one often writes $(a_n)_{n\in\mathbb{N}}, (a_n)_{n=0}^{\infty}, (a_n)_{n\geq 0}$.

**Definition 3.2: (Eventually) Constant Sequences**

A sequence $(x_n)_{n=0}^{\infty}$ is **constant** if $x_n = x_m \forall n, m \in \mathbb{N}$. It is **eventually constant** if there exists $N \in \mathbb{N}$ such that $x_n = x_m \forall n, m \geq N$.

**Definition 3.3: Convergence of Sequences**

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. We say that $(x_n)_{n=0}^{\infty}$ **converges** (or is **convergent**) if $\exists A \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : |x_n - A| < \varepsilon \quad \forall n \geq N.$$

In this case we write

$$\lim_{n\to\infty} x_n = A \tag{3.1}$$

and call $A$ the **limit** of $(x_n)_{n=0}^{\infty}$.

**Lemma 3.4: Uniqueness of the Limit**

*A convergent sequence $(x_n)_{n=0}^{\infty}$ has exactly one limit.*

*Proof.* Let $A, B \in \mathbb{R}$ be limits of $(x_n)_{n=0}^{\infty}$. Fix $\varepsilon > 0$. Then there exists $N_A, N_B \in \mathbb{N}$ such that $|x_n - A| < \varepsilon$ for all $n \geq N_A$ and $|x_n - B| < \varepsilon$ for all $n \geq N_B$. We define $N := \max\{N_A, N_B\}$. Then it holds that

$$|A - B| \leq |A - x_N| + |x_N - B| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $A = B$. □

## 3.2 Convergent Subsequences and Acculmulation Points

**Definition 3.5: Subsequences**

Let $(x_n)_{n=0}^{\infty}$ be a sequence. A **subsequence** is of the form $(x_{n_k})_{k=0}^{\infty}$, where $(n_k)_{k=0}^{\infty}$ is a strictly increasing sequence of non-negative integers, i.e., $n_{k+1} > n_k \ \forall k \in \mathbb{N}$.

**Remark 3.6.** *Since $n_{k+1} > n_k$ for all $k \in \mathbb{N}$ it follows by induction that $n_k \geq k$ for all $k \in \mathbb{N}$.*

*Proof.* For $k = 0$ we have that $n_0 \geq 0$, because $(n_k)_{k=0}^{\infty}$ is a sequence of non-negative integers. So the condition is fulfilled. For the inductive step we want to show that the condition holds for $k+1$ under the assumption that the condition is true for $k$. Because $(n_k)_{k=0}^{\infty}$ is also a strictly increasing sequence, we have that $n_{k+1} > n_k \geq k$. Additionally since $n_k \in \mathbb{N}$, we have that $n_{k+1} \geq n_k + 1$. So it follows that $n_{k+1} \geq n_k + 1 \geq k + 1$, which proofs the condition for $k + 1$. □

### Lemma 3.7: Subsequences of Convergent Sequences are Convergent

*Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Then every subsequence $(x_{n_k})_{k=0}^{\infty}$ also converges to $A$.*

*Proof.* Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^{\infty}$ converges to $A$, there exists $N \in \mathbb{N}$ such that $|x_n - A| < \varepsilon \ \forall n \geq N$. As by Remark 3.6 we know that $n_k \geq k$ for all $k \in \mathbb{N}$. Therefore for all $k \geq N$ it holds that $|x_{n_k} - A| < \varepsilon$. ☐

### Definition 3.8: Accumulation Points of Sequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. A point $A \in \mathbb{R}$ is an **accumulation point** of $(x_n)_{n=0}^{\infty}$ if

$$\forall \varepsilon > 0 \, \forall N \in \mathbb{N} \, \exists n \geq N : |x_n - A| < \varepsilon.$$

### Proposition 3.9: Subsequences and Accumulation Points

*Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. A point $A$ is and accumulation point of $(x_n)_{n=0}^{\infty}$ if and only if there exists a convergent subsequence of $(x_n)_{n=0}^{\infty}$ with limit $A$.*

*Proof.* First assume that $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^{\infty}$. We construct $(n_k)_{k\geq 0}$ recursively:

- first, apply the definition of accumulation point with $N = 1$ and $\varepsilon = 1 = 2^0$ to find $n_0 \geq 1$ with $|x_{n_0} - A| \leq 2^0$,
- second, apply the definition the definition of accumulation point with $N = n_0 + 1$ and $\varepsilon = 2^{-1}$ to find $n_1 \geq n_0 + 1$ with $|x_{n_1} - A| \leq 2^{-1}$,
- more in general given $n_{k-1}$, we apply the definition of accumulation point with $N = n_{k-1} + 1$ and $\varepsilon = 2^{-k}$ to find $n_k \geq n_{k-1} + 1$ with $|x_{n_k} - A| \leq 2^{-k}$.

Now given $\varepsilon > 0$ choose $N$ such that $2^{-N} < \varepsilon$. Then for all $k \geq N$ we have that

$$|x_{n_k} - A| \leq 2^{-k} \leq 2^{-N} < \varepsilon,$$

so $\lim_{k\to\infty} x_{n_k} = A$.

Conversely, assume that there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ converging to $A$. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. Since $\lim_{k\to\infty} x_{n_k} = A$, there exists $N_0$ such that $|x_{n_k} - A| < \varepsilon$ for all $k \geq N_0$. Hence if we choose $k = \max\{N_0, N\}$, because $n_k \geq n$ (recall Remark 3.6) we have that $n_k \geq N$ and $|x_{n_k} - A| < \varepsilon$. Thus $A$ is an accumulation point. ☐

### Corollary 3.10: Infinitely Many Terms Near an Accumulation Point

*If $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^{\infty}$, then for every $\varepsilon > 0$ there are infinitely many $n$ with $x_n \in (A - \varepsilon, A + \varepsilon)$.*

*Proof.* By Proposition 3.9, there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ with $\lim_{k\to\infty} = A$. Hence for every $\varepsilon > 0$ there exists $K$ such that $x_{n_k} \in (A - \varepsilon, A + \varepsilon)$ for all $k \geq K$, providing infinitely many elements of the sequence inside the interval $(A - \varepsilon, A + \varepsilon)$. ☐

**Corollary 3.11: Accumulation Points of Convergent Sequences**

*convergent sequence has exactly one accumulation point, namely its limit.*

## 3.3 Addition, Multiplication and Inequalities

**Proposition 3.12: Limits and Operations**

*Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be sequences converging to $A, B \in \mathbb{R}$ respectively. Then:*

1. *The sequence $(x_n + y_n)_{n=0}^{\infty}$ converges to $A + B$.*
2. *The sequence $(x_n y_n)_{n=0}^{\infty}$ converges to $AB$.*
3. *Given $\alpha \in \mathbb{R}$, the sequence $(\alpha x_n)_{n=0}^{\infty}$ converges to $\alpha A$.*
4. *Suppose $x_n \neq 0$ for all $n \in \mathbb{N}$ and $A \neq 0$. Then the sequence $(x_n^{-1})_{n=0}^{\infty}$ converges to $A^{-1}$.*

**Proposition 3.13: Limits and Inequalities**

*Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be sequences converging to $A, B \in \mathbb{R}$ respectively.*

1. *If $A < B$, then there exists $N \in \mathbb{N}$ such that $x_n < y_n$ for all $n \geq \mathbb{N}$.*
2. *If there exists $N \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \geq N$, then $A \leq B$.*

**Remark 3.14.** *In Propostion 3.13 even if we assume that $x_n < y_n$ for all $n \in \mathbb{N}$, we cannot conclude that $A < B$. for example take*

$$x_n = \frac{1}{n}, \quad y_n = \frac{1}{n}.$$

*Then we have that $x_n < y_n$ for all $n \in \mathbb{N}$ but $A = B = 0$.*

**Lemma 3.15: Sandwich Lemma**

*Let $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty}$, $(z_n)_{n=0}^{\infty}$ be sequences such that for some $N \in \mathbb{N}$, we have that*

$$x_n \leq y_n \leq z_n \quad \forall n \geq N.$$

*Suppose that both $(x_n)_{n=0}^{\infty}$ and $(z_n)_{n=0}^{\infty}$ converge to the same limit. Then $(y_n)_{n=0}^{\infty}$ also converges, and we have that*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n.$$

*Proof.* Let $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty}$, $(z_n)_{n=0}^{\infty}$ be sequences such that for some $N_0 \in \mathbb{N}$, we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N_0.$$

Additionally suppose that $(x_n)_{n=0}^{\infty}$ and $(z_n)_{n=0}^{\infty}$ converge to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^{\infty}$, $(z_n)_{n=0}^{\infty}$ converge to $A$ there exists $N_x, N_z \in \mathbb{N}$ such that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N_x$$
$$A - \varepsilon < z_n < A + \varepsilon \quad \forall n_z \geq N_z.$$

So we choose $N := \max\{N_0, N_x, N_z\}$. Then we have that

$$A - \varepsilon < x_n \le y_n \le z_n < A + \varepsilon \quad \forall n \ge N,$$

which shows that $\lim_{n\to\infty} y_n = A$. □

### Definition 3.16: Bounded Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called **bounded** if there exists a real number $M \ge 0$ such that

$$|x_n| \le M \quad \forall n \in \mathbb{N}.$$

### Lemma 3.17: Convergent Sequences are Bounded

*Every convergent sequence is bounded.*

*Proof.* Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Let $\varepsilon = 1$. Then, by convergence of $(x_n)_{n=0}^{\infty}$, there exists $N$ such that $|x_n - A| \le 1$ for all $n \ge N$. So we have that

$$|x_n| = |x_n - A + A| \le |x_n - A| + |A| \le 1 + |A| \qquad \forall n \ge N.$$

We choose

$$M = \max(|x_0|, |x_1|, \ldots, |x_{N-1}|, 1 + |A|).$$

Then $|x_n| \le M$ for all $n \in \mathbb{N}$ as desired. □

### Definition 3.18: Monotone Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called:

- **(monotonically) increasing** if $m > n \Rightarrow x_m \ge x_n$,
- **strictly (monotonically) increasing** if $m > n \Rightarrow x_m > x_n$,
- **(monotonically) decreasing** if $m > n \Rightarrow x_m \le x_n$,
- **strictly (monotonically) decreasing** if $m > n \Rightarrow x_m < x_n$.

If a sequence is decreasing or increasing we call it monotone. If a sequence is strictly increasing or strictly decreasing then we call it strictly monotone.

**Remark 3.19.** *An equivalent formulation of monotone sequences can be given using only successive terms:*

- $(x_n)_{n=0}^{\infty}$ *is increasing if* $x_{n+1} \ge x_n$ *for all* $n$*,*
- $(x_n)_{n=0}^{\infty}$ *is strictly increasing if* $x_{n+1} > x_n$ *for all* $n$*,*
- $(x_n)_{n=0}^{\infty}$ *is decreasing if* $x_{n+1} \le x_n$ *for all* $n$*,*
- $(x_n)_{n=0}^{\infty}$ *is strictly decreasing if* $x_{n+1} < x_n$ *for all* $n$*.*

**Theorem 3.20: Convergence of Monotone Sequences**

*A monotone sequence $(x_n)_{n=0}^{\infty}$ converges if and only if it is bounded. More precisely, let $X = \{x_n \mid n \in \mathbb{N}\}$ denote the set of points in the sequence.*

- *If $(x_n)_{n=0}^{\infty}$ is increasing, then* $\lim_{n\to\infty} x_n = \sup(X)$,
- *if $(x_n)_{n=0}^{\infty}$ decreasing, then* $\lim_{n\to\infty} x_n = \inf(X)$.

*Proof.* If $(x_n)_{n=0}^{\infty}$ converges Lemma 3.17 says that its bounded.

Conversely, let $(x_n)_{n=0}^{\infty}$ be a bounded monotone sequence. Wlog assume that $(x_n)_{n=0}^{\infty}$ is increasing (otherwise consider $(-x_n)_{n=0}^{\infty}$). Since $(x_n)_{n=0}^{\infty}$ is bounded from above, the set $X = \{x_n \mid n \in \mathbb{N}\}$ has a supremum, that we'll call $A = \sup(X)$.

By definiton of $A$:

(i) $x_n \leq A \quad \forall n \in \mathbb{N}$,

(ii) $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_N > A - \varepsilon$.

Then, for all $n \geq N$ using (ii) and monotonicity, we have that $x_n \geq x_N > A - \varepsilon$. Then using (i), we conclude that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N.$$

□

## 3.4 Superior and Inferior Limits

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence. To study its behavior for large $n$ its is useful to look at its tails

$$X_{\geq n} = \{x_k \mid k \geq n\} \subseteq \mathbb{R}.$$

The concept of limits can be restated using the tails of a sequence, i.e., the sequence $(x_n)_{n=0}^{\infty}$ converges to $A \in \mathbb{R}$ if and only if, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $X_N \subseteq (A - \varepsilon, A + \varepsilon)$.

However, since not every sequence has a limit we now introduce a related notion (the **superior** and **inferior limits**), which always exist for bounded sequences.

For each $n \in \mathbb{N}$, define

$$s_n = \sup(X_{\geq n}) = \sup_{k \geq n} x_k, \qquad i_n = \inf(X_{\geq n}) = \inf_{k \geq n} x_k.$$

Since $X_{\geq m} \subset X_{\geq n}$, whenever $m > n$, we have that

$$i_n \leq i_m \leq s_m \leq s_n \qquad \forall m > n.$$

Thus, $(s_n)_{n=0}^{\infty}$ is a monotonically decreasing sequence, while $(i_n)_{n=0}^{\infty}$ is a monotonically increasing sequence. Moreover, since $(x_n)_{n=0}^{\infty}$ is bounded both $(s_n)_{n=0}^{\infty}$ and $(i_n)_{n=0}^{\infty}$ are bounded as well. Hence by Theorem 3.20, both sequences converge. Their limits will be called the *superior* and the *inferior limit* of $(x_n)_{n=0}^{\infty}$ respectively.

Note that, since $x_n \in X_{\geq n}$, we have that

$$i_n \leq x_n \leq s_n \qquad \forall n \in \mathbb{N}. \tag{3.2}$$

### Definition 3.21: Superior and Inferior Limits

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence in $\mathbb{R}$. The numbers

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\big(\sup_{k\geq n} x_k\big), \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty}\big(\inf_{k\geq n} x_k\big)$$

are called the **superior** and **inferior limit** of $(x_n)_{n=0}^{\infty}$ respectively. From Equation 3.2 and Proposition 3.13, we have

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n.$$

### Lemma 3.22: Convergence and Superior/Inferior Limits

*A bounded sequence $(x_n)_{n=0}^{\infty}$ in $\mathbb{R}$ converges if and only if*

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n.$$

*Proof.* For every $n \in \mathbb{N}$, define

$$i_n = \inf_{k\geq n} x_k, \qquad s_n = \sup_{k\geq n} x_k,$$

and set

$$I = \lim_{n\to\infty} i_n = \liminf_{n\to\infty} x_n, \qquad S = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} x_n.$$

First suppose that $I = S$. Since $i_n \leq x_n \leq s_n$ (see Equation 3.2), the Sandwich Lemma 3.15 implies that the sequence $(x_n)_{n=0}^{\infty}$ converges, and its limit equals $I = S$.

Conversely, assume that $(x_n)_{n=0}^{\infty}$ converges to $A \in \mathbb{R}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$A - \varepsilon < x_n < A + \varepsilon \qquad \forall n \geq N.$$

Then for all $n \geq N$, the same inequalities holds for $i_n$ and $s_n$, i.e.,

$$A - \varepsilon \leq i_n \leq s_n \leq A + \varepsilon.$$

Taking limits and using Proposition 3.13, we obtain

$$A - \varepsilon \leq I \leq S \leq A + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $A = I = S$, which proves the result. □

### Theorem 3.23: Superior and Inferior Limits are Accumulation Points

*Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence and let $A = \limsup_{n\to\infty} x_n$. Then $A$ is an accumulation point of $(x_n)_{n=0}^{\infty}$, and for every $\varepsilon > 0$ the following hold:*

1. *only finitely many elements satisfy $x_n \geq A + \varepsilon$;*
2. *infinitely many elements satisfy $A - \varepsilon < x_n < A + \varepsilon$.*

*An analogous statement holds for the inferior limit.*

*Proof.* Since the sequence $(s_n)_{n=0}^{\infty}$ is monotonically decreasing and converges to $A$, given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$A \leq s_n < A + \varepsilon \qquad \forall n \geq N_0. \tag{3.3}$$

We first prove that $A$ is an accumulation point.

Fix $N \in \mathbb{N}$ and set $N_1 = \max\{N, N_0\}$. Since $s_{N_1} = \sup_{k \geq N_1} x_k$, there exists $n_1 \geq N_1 \geq N_0$ such that

$$s_{N_1} - \varepsilon < x_{n_1} \leq s_{N_1}.$$

Thus, combining this bound with Equation 3.3 we obtain

$$A - \varepsilon < s_{N_1} - \varepsilon < x_{n_1} \leq s_{N_1} < A + \varepsilon.$$

This construct shows that for any $\varepsilon > 0$ and any $N \in \mathbb{N}$, there exists $n_1 \geq N$ such that $A - \varepsilon < x_{n_1} < A + \varepsilon$. Thus $A$ is an accumulation point for $(x_n)_{n=0}^{\infty}$.

We now prove1. and 2.. From Equation 3.3 we have $x_n < A + \varepsilon$ for all $n \geq N_0$, so only finitely many terms satisfy $x_n \geq A + \varepsilon$. This shows 1..

Also since $A$ is an accumulation point, it follows from Corollary 3.10 that infinitely many terms of the sequence lie within any interval $(A - \varepsilon, A + \varepsilon)$. □

**Corollary 3.24: Bounded Sequences have Convergent Subsequences**

*Every bounded sequence has at least one accumulation point and therefore possesses a convergent subsequence.*

*Proof.* By Theorem 3.23, the number

$$A = \limsup_{n \to \infty} x_n$$

is always an accumulation point of $(x_n)_{n=0}^{\infty}$. Moreover, by Proposition 3.9, every accumulation point is the limit of a convergent subsequence. Hence every bounded sequence admits at least one convergent subsequence. □

## 3.5 Cauchy Sequences

**Definition 3.25: Cauchy Sequences**

A sequence $(x_n)_{n=0}^{\infty}$ is called a **Cauchy sequence** if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \qquad \forall n, m \geq N.$$

**Lemma 3.26: Cauchy Sequences are Bounded**

*Every Cauchy sequence is bounded.*

*Proof.* By definition, there exists $N \in \mathbb{N}$ such that

$$|x_n - x_N| \leq 1 \qquad \forall n \geq N.$$

Hence, for $n \geq N$, we have $|x_n| \leq 1 + |x_N|$. Now, define

$$M = \max\{|x_0|, |x_1|, \ldots, |x_{N-1}|, 1 + |x_N|\}.$$

Then, $|x_n| \leq M$ for all $n \in \mathbb{N}$, so $(x_n)_{n=0}^{\infty}$ is bounded. □

**Theorem 3.27: Convergence and Cauchy Sequences**

*A sequence $(x_n)_{n=0}^{\infty}$ of real numbers converges if and only if it is a Cauchy sequence.*

*Proof.* Suppose first that $(x_n)_{n=0}^{\infty}$ converges to some $A \in \mathbb{R}$, and let us prove that $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|x_n - A| < \frac{\varepsilon}{2} \qquad \forall n \geq N.$$

Then for all $n, m \geq N$, we have that

$$|x_n - x_m| \leq |x_n - A| + |x_m - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence.

Viceversa, let $(x_n)_{n=0}^{\infty}$ be a Cauchy sequence. Since it is bounded (by Lemma 3.26), Corollary 3.24 implies that there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ converging to some $A \in \mathbb{R}$. Given $\varepsilon > 0$, choose $N_0 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \qquad \forall n, m \geq N_0,$$

and choose $N_1 \in \mathbb{N}$ such that

$$|x_{n_k} - A| < \frac{\varepsilon}{2} \qquad \forall k \geq N_1.$$

Let $N = \max\{N_0, N_1\}$. Since $n_N \geq N$ (see Remark 3.6), for all $n \geq N$ we have

$$|x_n - A| \leq |x_n - x_{n_N}| + |x_{n_N} - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $(x_n)_{n=0}^{\infty}$ converges to $A$. □

## 3.6 Improper Limits

We now extend the notion of limit to allow the **improper limit values** $+\infty$ (often abbreviated as $\infty$) and $-\infty$.

**Definition 3.28: Improper Limits**

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. We say $(x_n)_{n=0}^{\infty}$ **diverges to** $+\infty$, and we write

$$\lim_{n\to\infty} x_n = +\infty,$$

if for every $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n > M$ for all $n \geq N$.
Similarly, $(x_n)_{n=0}^{\infty}$ **diverges to** $-\infty$ if for every $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n < -M$ for all $n \geq N$. In both cases, we say that $(x_n)_{n=0}^{\infty}$ has an **improper limit**.

An unbounded sequence doesn't need to diverge to $+\infty$ or $-\infty$. For instance, the sequence $x_n = (-1)^n n$, is unbounded but neither diverges to $+\infty$ nor to $-\infty$.

The notion of improper limit allows us to extend the definitions of superior and inferior limits to *unbounded* sequences. If $(x_n)_{n=0}^{\infty}$ is not bounded from above, then

$$\sup_{k\geq n} x_k = +\infty \qquad \forall n \in \mathbb{N},$$

and we write

$$\limsup_{n\to\infty} x_n = +\infty.$$

If $(x_n)_{n=0}^{\infty}$ is bounded from above but not from below, then we define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} (\sup_{k\geq n} x_k),$$

where the right-hand side is a real limit if the decreasing sequence $\sup_{k\geq n} x_k$ is bounded, and the improper limit $-\infty$ otherwise. The definition of the inferior limit extends analogously.

### 3.7 Sequences of Complex Numbers

Informally, a **sequence of complex numbers** is just like a sequence of real numbers, except that each term is a complex number instead of a real one. Thus, we study ordered lists $(z_0, z_1, \ldots)$, where $z_n : \mathbb{N} \to \mathbb{C}$. As in the real case, we are mainly interested in their convergence, divergence and limit behavior.

To analyze sequences in $\mathbb{C}$, it is often sufficient to consider separately the corresponding sequences of real and imaginary parts in $\mathbb{R}$.

**Definition 3.29: Sequences of Complex Numbers**

A sequence of complex numbers $(z_n)_{n=0}^{\infty}$, where

$$z_n = x_n + iy_n,$$

is said to **converge** to a limit $A + iB \in \mathbb{C}$ if the two sequences of real numbers $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ converge to $A$ and $B$, respectively. In this case, we write

$$\lim_{n\to\infty} z_n = A + iB.$$

We say that $(z_n)_{n=0}^{\infty}$ **diverges to** $\infty$ if the sequence of moduli $(|z_n|)_{n=0}^{\infty}$ diverges to $+\infty$, i.e.,

$$\lim_{n\to\infty} |z_n| = \lim_{n\to\infty} \sqrt{x_n^2 + y_n^2} = +\infty.$$

**Remark 3.30.** *As for sequences of real numbers, one can consider subsequences of sequences $\mathbb{C}$. Given a strictly increasing sequence of non-negative integers $(n_k)_{k=0}^{\infty}$, the corresponding subsequence is*

$$(z_{n_k})_{k=0}^{\infty} = (x_{n_k} + iy_{n_k})_{n=0}^{\infty}.$$

## 4 Functions of one Real Variable

In this chapter we study real-valued functions defined on subsets of $\mathbb{R}$, typically intervals. The central concept is *continuity*.

## 4.1 Real valued functions

### 4.1.1 Boundedness and Monotonitcity

For a non-empty set $D \subseteq \mathbb{R}$, the set of **real-valued** functions on $D$ is

$$\mathcal{F}(D) = \{f \mid f : D \to \mathbb{R}\}.$$

For $f_1, f_2 \in \mathcal{F}(D)$, $\alpha \in \mathbb{R}$, and $x \in D$ we define

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \qquad (\alpha f_1)(x) = \alpha f_1(x), \qquad (f_1 f_2)(x) = f_1(x) f_2(x).$$

Given $\alpha \in \mathbb{R}$, we write $f \equiv \alpha$ for the constant function $x \mapsto \alpha$ on $D$.

**Remark 4.1.** *With the operations above, $\mathcal{F}(D)$ is a commutative ring (the additive identity is $f \equiv 0$ and the multiplicative identity is $f \equiv 1$).*

A point $x \in D$ is a **zero** of $f \in \mathcal{F}(D)$ if $f(x) = 0$. The **zero set** of $f$ is $\{x \in D \mid f(x) = 0\}$. We order $\mathcal{F}(D)$ pointwise: for $f_1, f_2 \in \mathcal{F}(D)$,

$$\begin{aligned} f_1 \leq f_2 \quad &\Leftrightarrow \quad f_1(x) \leq f_2(x) \quad \forall x \in D, \\ f_1 < f_2 \quad &\Leftrightarrow \quad f_1(x) < f_2(x) \quad \forall x \in D. \end{aligned}$$

We say that $f \in \mathcal{F}(D)$ is **non-negative** if $f \geq 0$, and **positive** if $f > 0$.

**Definition 4.2: Bounded Functions**

Let $D \neq \emptyset$ and $f : D \to \mathbb{R}$. We say that $f$ is **bounded from above** if there exists $M > 0$ such that

$$f(x) \leq M \qquad \forall x \in D.$$

We say that $f$ is **bounded from below** if there exists $M > 0$ such that

$$f(x) \geq -M \qquad \forall x \in D.$$

We say that $f$ is **bounded** if it is both bounded from above and from below. Equivalently, $f$ if bounded if there exists $M > 0$ such that

$$|f(x)| \leq M \qquad \forall x \in D.$$

**Definition 4.3: Monotone Functions**

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. The function $f$ is:

1. **increasing** if $x < y \quad \Rightarrow \quad f(x) \leq f(y) \quad \forall x, y \in D$;
2. **strictly increasing** if $x < y \quad \Rightarrow \quad f(x) < f(y) \quad \forall x, y \in D$;
3. **decreasing** if $x < y \quad \Rightarrow \quad f(x) \geq f(y) \quad \forall x, y \in D$;
4. **strictly decreasing** if $x < y \quad \Rightarrow \quad f(x) > f(y) \quad \forall x, y \in D$.

We call $f$ **monotone** if it is increasing or decreasing, and **strictly monotone** if it is strictly increasing or strictly decreasing.

### 4.1.2 Continuity

**Definition 4.4: Continuous Functions**

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. We say that $f$ is **continuous at** $x_0 \in D$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

We say that $f$ is **continuous on** $D$ if it is continuous at every point of $D$.

**Remark 4.5.** *It suffices to verify the implication above for small $\varepsilon$. Precisely, assume there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ there is a $\delta > 0$ such that*

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

*Then $f$ is continuous at $x_0$.*

*Indeed, for $\varepsilon_0 > \varepsilon$ we can choose the number $\delta > 0$ corresponding to $\varepsilon$ to get*

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon < \varepsilon_0.$$

*In other words, if $\delta$ works for $\varepsilon$, then it works for all $\varepsilon_0 > \varepsilon$.*

**Definition 4.6: Restriction**

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. For any $D' \subseteq D$ the **restriction** of $f$ to $D'$ is the function $f|_{D'} : D' \to \mathbb{R}$ defined by

$$f|_{D'}(x) = f(x) \qquad \forall x \in D'.$$

We regard $f|_{D'}$ and $f$ as different functions unless $D' = D$.

**Proposition 4.7: Combination of Continuous Functions**

*Let $D \subseteq \mathbb{R}$, and let $f_1, f_2 : D \to \mathbb{R}$ be continuous at $x_0 \in D$. Then $f_1 + f_2$, $f_1 f_2$, and $\alpha f_1$ (for any $\alpha \in \mathbb{R}$) are continuous at $x_0$.*

*Proof.* We first prove the result for the sum. Let $\varepsilon > 0$. Since $f_1$ and $f_2$ are continuous at $x_0$, there exists $\delta_1, \delta_2 > 0$ such that for all $x \in D$,

$$|x - x_0| < \delta_1 \;\Rightarrow\; |f_1(x) - f_1(x_0)| < \frac{\varepsilon}{2}, \quad |x - x_0| < \delta_2 \;\Rightarrow\; |f_2(x) - f_2(x_0)| < \frac{\varepsilon}{2}.$$

So, choosing $\delta = \min \delta_1, \delta_2$, for $|x - x_0| < \delta$ we get

$$|(f_1 + f_2)(x) - (f_1 + f_2)(x_0)| \le |f_1(x) - f_1(x_0)| + |f_2(x) - f_2(x_0)| < \varepsilon,$$

which shows that $f_1 + f_2$ is continuous at $x_0$.

For the product, note that

$$\begin{aligned} |f_1(x)f_2(x) - f_1(x_0)f_2(x_0)| &= |f_1(x)f_2(x) - f_1(x_0)f_2(x) + f_1(x_0)f_2(x) - f_1(x_0)f_2(x_0)| \\ &\le |f_1(x)f_2(x) - f_1(x_0)f_2(x)| + |f_1(x_0)f_2(x) - f_1(x_0)f_2(x_0)| \\ &= |f_2(x)||f_1(x) - f_1(x_0)| + |f_1(x_0)||f_2(x) - f_2(x_0)|. \end{aligned}$$

Now, first choose $\delta_0 > 0$ such that $|x - x_0| < \delta_0$ implies $|f_2(x) - f_2(x_0)| < 1$, so that

$$|x - x_0| < \delta_0 \quad \Rightarrow \quad |f_2(x)| < 1 + |f_2(x_0)|.$$

Then choose $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} |x - x_0| < \delta_1 \quad &\Rightarrow \quad |f_1(x) - f_1(x_0)| < \frac{\varepsilon}{2(1 + |f_2(x_0)|)}, \\ |x - x_0| < \delta_2 \quad &\Rightarrow \quad |f_2(x) - f_2(x_0)| < \frac{\varepsilon}{2(1 + |f_1(x_0)|)}. \end{aligned}$$

So choosing $\delta = \min \delta_0, \delta_1, \delta_2$, for $|x - x_0| < \delta$ we get

$$\begin{aligned} |f_1(x)f_2(x) - f_1(x_0)f_2(x_0)| &< |f_2(X)| \frac{\varepsilon}{2(1 + |f_2(x_0)|)} + |f_1(x_0)| \frac{\varepsilon}{2(1 + |f_1(x_0)|)} \\ &< (1 + |f_2(x_0)|) \frac{\varepsilon}{2(1 + |f_2(x_0)|)} + |f_1(x_0)| \frac{\varepsilon}{2(1 + |f_1(x_0)|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

thus $f_1 f_2$ is continuous at $x_0$.

Finally, the statement about $\alpha f_1$ follows by choosing $f_2 \equiv \alpha$ (a constant function) and using the product case proved above: since $f_1$ and $f_2$ are continuous at $x_0$, their product $f_1 f_2 = \alpha f_1$ is continuous at $x_0$. □

### Definition 4.8: Sum and Product Notation

Let $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n \in \mathbb{R}$. We use the notation

$$\sum_{j=0}^{n} a_j = a_0 + a_1 + \ldots + a_n, \qquad \prod_{j=0}^{n} a_0 \cdot a_1 \cdot \ldots \cdot a_n.$$

Here $a_j$ is a **summand** in the sum and a **factor** in the product; $j$ is the **index** (or **running variable**). If $J$ is a finite set and numbers $(a_j)_{j \in J}$ are given, we write

$$\sum_{j \in J} a_j, \qquad \prod_{j \in J} a_j.$$

By convention, for the empty index set $\emptyset$,

$$\sum_{j \in \emptyset} a_j = 0, \qquad \prod_{j \in \emptyset} a_j = 1.$$

### Proposition 4.9: Composition of Continuous Functions

*Let $D_1, D_2 \subseteq \mathbb{R}, x_0 \in D_1$ and $f : D_1 \to D_2$ be continuous at $x_0$. If $g : D_2 \to \mathbb{R}$ is continuous at $f(x_0)$, then $g \circ f : D_1 \to \mathbb{R}$ is continuous at $x_0$. In particular, the composition of continuous functions is continuous.*

*Proof.* Let $\varepsilon > 0$. By continuity of $g$ at $f(x_0)$, there exists $\eta > 0$ such that

$$\forall y \in D_2, \quad |y - f(x_0)| < \eta \quad \Rightarrow \quad |g(y) - g(f(x_0))| < \varepsilon.$$

By continuity of $f$ at $x_0$, there exists $\delta > 0$ such that

$$\forall x \in D_1, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \eta.$$

Combining the implications gives, for any $x \in D_1$,

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \eta \quad \Rightarrow \quad |g(f(x)) - g(f(x_0))| < \varepsilon.$$ □

**Remark 4.10.** *Applying Proposition 4.9 with $g(y) = |y|$, we see that if $f : D \to \mathbb{R}$ is continuous, then $x \mapsto |f(x)|$ is continuous.*

### 4.1.3 Sequential Continuity

**Definition 4.11: Notation for Limits of Sequences**

Let $(x_n)_{n=0}^{\infty} \subseteq \mathbb{R}$ and $\bar{x} \in \mathbb{R}$. We write

$$x_n \to \bar{x} \qquad \text{or} \qquad x_n \underset{n\to\infty}{\longrightarrow} \bar{x}$$

to mean

$$\lim_{n\to\infty} x_n = \bar{x}.$$

**Theorem 4.12: Continuity = Sequential Continuity**

*Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and $\bar{x} \in D$. Then $f$ is continuous at $\bar{x}$ if and only if for every sequence $(x_n)_{n=0}^{\infty} \subseteq D$ with $x_n \to \bar{x}$ we have $f(x_n) \to f(\bar{x})$.*

*Proof.* '⇒': First Assume that $f$ is continuous at $\bar{x}$. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall x \in D, \quad |x - \bar{x}| < \delta \quad \Rightarrow \quad |f(x) - f(\bar{x})| < \varepsilon.$$

Also, since $x_n \to \bar{x}$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad |x_n - \bar{x}| < \delta.$$

Thus,

$$n \geq N \quad \Rightarrow \quad |f(x_n) - f(\bar{x})| < \varepsilon,$$

which implies that the sequence $(f(x_n))_{n=0}^{\infty}$ converges to $f(\bar{x})$.

'⇐': To prove the converse, assume that $f$ is not continuous at $x_0$. This means that there exists $\varepsilon > 0$ such that, for every $\delta > 0$, there is $x \in D$ with

$$|x - \bar{x}| < \delta \quad \text{and} \quad |f(x) - f(\bar{x})| \geq \varepsilon.$$

Now, for every $n \in \mathbb{N}$, we apply this property with $\delta = 2^{-n}$ to find a point $x_n \in D$ such that

$$|x_n - \bar{x}| < 2^{-n} \quad \text{and} \quad |f(x_n) - f(\bar{x})| \geq \varepsilon$$

Then the sequence constructed in this way satisfies $x_n \to \bar{x}$ but $f(x_n) \not\to f(\bar{x})$. □

**Remark 4.13.** *The proof above shows that if $f : D \to \mathbb{R}$ is not continuous at $\bar{x}$, then there exists $\varepsilon > 0$ and a sequence $(x_n)_{n=0}^{\infty} \subseteq D$ with $x_n \to \bar{x}$ such that $|f(x_n) - f(\bar{x})| \geq \varepsilon$ for all $n \in \mathbb{N}$. This is useful to show that a function $f$ is not continuous at $\bar{x}$.*

## 4.2 Continuous Functions

### 4.2.1 Intermediate Value Theorem

In this section we prove a fundamental theorem that formalizes the idea that the graph of a continuous function on an interval is a continuous curve, and thus cannot make any jumps.

**Theorem 4.14: Intermediate Value Theorem**

*Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) \leq f(b)$. Then, for every real number $c$ with $f(a) \leq c \leq f(b)$, there exists $\bar{x} \in [a, b]$ such that $f(\bar{x}) = c$.*

*Proof.* Fix $c \in [f(a), f(b)]$. Then define

$$X = \{x \in [a, b] \mid f(x) \leq c\}.$$

Since $a \in X$ and $X \subseteq [a, b]$, the set is non-empty and bounded from above. By Theorem 2.35, its supremum

$$\bar{x} = sup(X) \in [a, b]$$

exists. We now use the continuity of $f$ at $x_0$ to show that $f(\bar{x}) = c$.

Since $\bar{x}$ is the supremum of $X$, for each $n \geq 0$, we can find a point $x_n \in [\bar{x} - 2^{-n}, \bar{x}]$. Then $|x_n - \bar{x}| \leq 2^{-n}$, hence $x_n \longrightarrow \bar{x}$. Also, by the definition of $X$, we have $f(x_n) \leq c$. Thus, by Theorem 4.12 (continuity of $f$ along sequences),

$$\lim_{n\to\infty} f(x_n) = f(\bar{x}).$$

And Proposition 3.13 yields $\lim_{n\to\infty} f(x_n) \leq c$. Therefore, $f(\bar{x}) \leq c$.

Suppose, by contradiction, $f(\bar{x}) < c$ and set $\varepsilon := c - f(\bar{x}) > 0$. By continuity at $\bar{x}$, there exists $\delta > 0$ such that for all $x \in [a, b]$

$$|x - \bar{x}| < \delta \quad \Rightarrow \quad |f(x) - f(\bar{x})| < \varepsilon,$$

hence $f(x) < f(\bar{x}) + \varepsilon = c$. Therefore, by the definition of $X$,

$$(\bar{x} - \delta, \bar{x} + \delta) \cap [a, b] \subseteq X.$$

Moreover, since $f(\bar{x}) < c \leq f(b)$, we cannot have $\bar{x} = b$; hence $\bar{x} < b$. Because $\bar{x} < b$, the interval $(\bar{x}, \bar{x} + \delta) \cap [a, b] \subseteq X$ is non-empty. Pick

$$y \in (\bar{x}, \bar{x} + \delta) \cap [a, b] \subseteq X.$$

Then $y \in X$ and $y > \bar{x}$, which contradicts the defining property of the supremum: $\bar{x}$ is an upper bound of $X$, and $X$ cannot contain elements larger than $\bar{x}$. This contradiction shows that $f(\bar{x}) \geq c$. Together with $f(\bar{x}) \leq c$ proved above, we conclude that $f(\bar{x}) = c$, as desired. $\square$

**Theorem 4.15: Inverse Function Theorem**

*Let $I$ be an interval and $f : I \to \mathbb{R}$ a continuous strictly monotone function. Then $f(I)$ is an interval, and the mapping $f : I \to f(I)$ has a continuous strictly monotone inverse function $f^{-1} : f(I) \to I$.*

*Proof.* We may assume that $I$ is non-empty and not a single point. Also, w.l.o.g, suppose $f$ is strictly increasing (otherwise replace $f$ with $-f$).

Let $J = f(I)$. Since $f$ is strictly monotone it is injective. Also, since by definition $J = f(I)$, it is surjective, hence bijective. Therefore there exists a unique inverse $g = f^{-1} : J \to I$.

Because $f$ is strictly increasing, we have

$$x_1 < x_2 \quad \Leftrightarrow \quad f(x_1) < f(x_2) \qquad \forall x_1, x_2 \in I. \tag{4.1}$$

(Note: here we have equivalence in the statements because $f$ is both injective and strictly increasing) Defining $y_1 = f(x_1)$ and $y_2 = f(x_2)$, this is equivalent to

$$y_1 < y_2 \quad \Leftrightarrow \quad g(y_1) < g(y_2) \qquad \forall y_1, y_2 \in J$$

Thus, $g$ is strictly increasing.

To show that $J$ is an interval, $y_1, y_2 \in J$, and assume w.l.o.g that $y_1 < y_2$. Since, $J = f(I)$, Equation 4.1 implies that $y_1 = f(x_1), y_2 = f(x_2)$ for some $x_1, x_2 \in I$ with $x_1 < x_2$. Now by the Intermediate Value Theorem 4.14 applied to $f : [x_1, x_2] \to \mathbb{R}$, we have that all values $c \in [y_1, y_2]$ are in the image of $f : [x_1, x_2] \to \mathbb{R}$, i.e.,

$$[y_1, y_2] \subseteq f([x_1, x_2]) \subseteq J.$$

Since, $y_1, y_2$ were two arbitrary points in $J$, this proves that $J$ is an interval.

It remains to show that $g = f^{-1}$ is continuous. Fix $\bar{y} \in J$ and suppose, by contradiction, that $g$ is not continuous at $\bar{y}$. Then by Remark 4.13, there exists $\varepsilon > 0$ and a sequence $(y_n)_{n=0}^{\infty} \subseteq J$ such that

$$y_n \longrightarrow \bar{y} \qquad \text{but} \qquad |g(y_n) - g(\bar{y})| \geq \varepsilon \quad \forall n \in \mathbb{N}. \tag{4.2}$$

Set $x_n = g(y_n) \in I$ and $\bar{x} = g(\bar{y}) \in I$. Then for every $n \in \mathbb{N}$, either $x_n \leq \bar{x} - \varepsilon$ or $x_n \geq \bar{x} + \varepsilon$. In particular, at least one of these cases must occur infinitely often. W.l.o.g, assume $x_n \leq \bar{x} - \varepsilon$ for infinitely many $n$, and extract a subsequence $(x_{n_k})_{k=0}^{\infty}$ with $x_{n_k} \leq \bar{x} - \varepsilon$ for all $k$. Since, $I$ is an interval, $\bar{x} - \varepsilon \in I$, and by strict monotonicity of $f$ we obtain

$$y_{n_k} = f(x_{n_k}) \leq f(\bar{x} - \varepsilon) < f(\bar{x}) = \bar{y}.$$

Then Proposition 3.13 gives (recall $y_n \longrightarrow \bar{y}$, see 4.2)

$$\bar{y} = \lim_{k \to \infty} y_{n_k} \leq f(\bar{x} - \varepsilon) < f(\bar{x}) = \bar{y},$$

a contradiction. Hence, $g$ is continuous. □

## 4.3 Continuous Functions on Compact Intervals

In this section we show that continuous functions on **bounded closed** intervals, called **compact intervals**, enjoy special properties.

### 4.3.1 Boundedness and Extrema

**Lemma 4.16: Compactness**

*Let $[a, b]$ be a compact interval, and let $(x_n)_{n=0}^{\infty}$ be a sequence contained in $[a, b]$. Then there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ such that*

$$\lim_{k\to\infty} x_{n_k} = \bar{x} \qquad \text{for some } \bar{x} \in [a, b].$$

*Proof.* Since $(x_n)_{n=0}^{\infty}$ is bounded (as it lies in $[a, b]$), Corollary 3.24 ensures the existence of a convergent subsequence $(x_{n_k})_{k=0}^{\infty}$. Let $\bar{x}$ denote its limit. Because $a \leq x_{n_k} \leq b$ for all k, Proposition 3.13 yields $a \leq \bar{x} \leq b$. □

**Theorem 4.17: Boundedness**

*Let $[a, b]$ be compact interval, and let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f$ is bounded.*

*Proof.* Assume by contradiction that $f$ is unbounded. Then, for every $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|f(x_n)| \geq n$. By Lemma 4.16, there is a subsequence $(x_{n_k})_{k=0}^{\infty}$ converging to some $\bar{x} \in [a, b]$.

Since $f$ is continuous, so is $|f|$ (recall Remark 4.10), therefore $|f(x_{n_k})| \longrightarrow |f(\bar{x})| \in \mathbb{R}$. This contradicts $|f(x_{n_k})| \geq n_k \longrightarrow \infty$, so $f$ must be bounded. □

**Exercise 4.18.** *Find examples of:*

1. *a continuous but unbounded function on a bounded open interval.*

$$f : (0, 1) \to \mathbb{R},\ x \mapsto \frac{1}{x}.$$

2. *a continuous but unbounded function on an unbounded closed interval.*

$$f : [0, \infty) \to \mathbb{R},\ x \mapsto x.$$

3. *an unbounded function on a compact interval but discontinuous at only one point.*

$$f : [0, 1] \to \mathbb{R},\ x \mapsto \begin{cases} \frac{1}{x}, & \text{for } x \neq 0 \\ a \in \mathbb{R}, & \text{for } x = 0. \end{cases}$$

**Definition 4.19: Extreme Values**

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$.

- We say that $f$ takes its **maximum value** at $x_0 \in D$ if $f(x) \leq f(x_0)$ for all $x \in D$. Then $f(x_0)$ is the **maximum** of $f$.
- We say that $f$ takes its **minimum value** at $x_0 \in D$ if $f(x) \geq f(x_0)$ for all $x \in D$. Then $f(x_0)$ is the **minimum** of $f$.

Maxima and minima ar called **extreme values** or **extrema**.

**Theorem 4.20: Extreme Value Theorem**

*Let $[a, b]$ be a compact interval, and let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f$ attains both its minimum and its maximum.*

*Proof.* Theorem 4.17 guarantees that $f$ is bounded, or equivalently, that $f([a, b]) \subseteq \mathbb{R}$ is a bounded subset of $\mathbb{R}$. Thus, Theorem 2.35 implies that

$$S := \sup f([a, b])$$

exists. By definition of the supremum, for each $n \in \mathbb{N}$ there exists $y_n \in f([a, b])$ such that $S - 2^{-n} \leq y_n \leq S$. Hence, $y_n \to S$. Also, since $y_n \in f([a, b])$, there exists $x_n \in [a, b]$ such that $f(x_n) = y_n$.

Now, by Lemma 4.16, we can find a subsequence $(x_{n_k})_{k=0}^{\infty}$ such that $x_{n_k} \to \bar{x} \in [a, b]$. By continuity of $f$, we have that

$$f(\bar{x}) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = S,$$

so $f$ attains its maximum at $\bar{x}$.

Applying the same reasoning to $-f$ shows that $f$ also attains its minimum. □

### 4.3.2 Uniform Continuity

**Definition 4.21: Uniform Continuity**

Let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ is **uniformly continuous** if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon \qquad \forall x, y \in D.$$

**Remark 4.22.** *The difference between the usual definition of continuity and the one of uniform continuity lies in how the choice of $\delta$ depends on the points considered.*

*For a function that is continuous at each $x_0 \in D$, the $\delta$ in the definition may depend on both $\varepsilon$* ***and*** *$x_0$: for every $\varepsilon > 0$ and each $x_0$, we can find a $\delta = \delta(\varepsilon, x_0)$ that works near $x_0$.*

*Uniform continuity is stronger: there exists a single $\delta = \delta(\varepsilon)$ that works* ***simultaneously*** *for all $x, y \in D$. In other words, the control on the variation of $f$ does not deteriorate as we move along the domain. This property is automatically satisfied on compact intervals for continuous functions, as we will prove below.*

**Theorem 4.23: Uniform Continuity on Compact Intervals**

*Let $[a, b]$ be a compact interval, and $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$. Then $f$ is uniformly continuous.*

*Proof.* Assume, by contradiction, that $f$ is not uniformly continuous on $[a, b]$. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$ one can find $x, y \in [a, b]$ with

$$|x - y| < \delta \quad \text{and} \quad |f(x) - f(y)| \geq \varepsilon.$$

Taking $\delta = 2^{-n}$ for each $n \in \mathbb{N}$, we obtain sequences $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ in $[a, b]$ with

$$|x_n - y_n| < 2^{-n} \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \varepsilon. \tag{4.3}$$

By Lemma 4.16, the sequence $(x_n)_{n=0}^{\infty}$ has a subsequence $(x_{n_k})_{k=0}^{\infty}$ converging to some $\bar{x} \in [a, b]$. Then

$$|y_{n_k} - \bar{x}| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - \bar{x}| < 2^{-n_k} + |x_{n_k} - \bar{x}| \underset{k\to\infty}{\longrightarrow} 0,$$

so $y_{n_k} \to \bar{x}$ as well. Thus, by continuity of $f$ and Theorem 4.12, we have that

$$\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k}) = f(\bar{x}),$$

therefore,

$$|f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(\bar{x})| + |f(\bar{x}) - f(y_{n_k})| \underset{k\to\infty}{\longrightarrow} 0,$$

which contradicts Equation 4.3. Hence, $f$ is uniformly continuous on $[a, b]$. □

**Definition 4.24: Lipschitz Continuity**

Let $D \subseteq \mathbb{R}$, and $f : D \to \mathbb{R}$. We say that $f$ is **Lipschitz continuous** if there exists $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y| \qquad \forall x, y \in D.$$

**Lemma 4.25: Lipschitz Continuity ⇒ Uniform Continuity**

*Let $D \subseteq \mathbb{R}$, and $f : D \to \mathbb{R}$ be a Lipschitz continuous function. Then $f$ is uniformly continuous.*

*Proof.* Let $D \subseteq \mathbb{R}$ and assume that $f : D \to \mathbb{R}$ is a Lipschitz continuous function. Then there exists $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y| \qquad \forall x, y \in D.$$

Now, fix $\varepsilon > 0$. We assume that $L \neq 0$ (otherwise the result follows immediately) and choose $\delta = \frac{\varepsilon}{L}$. Because of the Lipschitz continuity of $f$, we have that for all $x, y \in D$ it holds that

$$\begin{aligned} |x - y| < \delta = \frac{\varepsilon}{L} \quad &\Leftrightarrow \quad L|x - y| < \varepsilon \\ \Rightarrow \quad &|f(x) - f(y)| \leq L|x - y| < \varepsilon, \end{aligned}$$

which shows that $f$ is also uniformly continuous. □

## 4.4 Example: Expential and Logarithmic Functions

### 4.4.1 Definition of the Exponential Function

**Lemma 4.26: Bernoulli's Inequality**

*For all $a \in \mathbb{R}$ with $a \geq -1$ and all $n \in \mathbb{N}$ with $n \geq 1$, it holds that*

$$(1 + a)^n \geq 1 + na.$$

*Proof.* We proceed by induction. For $n = 1$ we have $(1 + a)^1 = 1 + a = 1 + 1 \cdot a$.

Now assume that the inequality holds for some $n \geq 1$. Since $1 + a \geq 0$ by assumption, we find

$$(1 + a)^{n+1} = (1 + a)^n(1 + a) \geq (1 + na)(1 + a) = 1 + na + a + na^2 \geq 1 + (n + 1)a,$$

which establishes the induction step and completes the proof. □

**Proposition 4.27: Existence of the Exponential**

*Let $x \in \mathbb{R}$. The sequence $(a_n)_{n=1}^{\infty}$ defined by*

$$a_n = \left(1 + \frac{x}{n}\right)^n$$

*is convergent, and its limit is a positive real number.*

**Lemma 4.28: Monotonicity**

*Given $x \in \mathbb{R}$, let $n_0 \in \mathbb{N}$ satisfy $n_0 \geq 1$ and $n_0 > -x$. Then the sequence $(a_n)_{n=n_0}^{\infty}$ defined in Proposition 4.27 is increasing.*

**Definition 4.29: Exponential Function**

The **exponential function** $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is defined by

$$\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n \qquad \forall x \in \mathbb{R}.$$

**Corollary 4.30: Growth of the Exponential**

*Given $n \in \mathbb{N}$ with $n \geq 1$, the exponential function satisfies*

$$\exp(x) \geq \left(1 + \frac{x}{n}\right)^n \qquad \forall x > -n.$$

*Proof.* By Lemma 4.28 and Definition 4.29, for $x > -n$ we have

$$a_n \leq a_{n+1} \leq \ldots \leq \exp(x). \qquad \square$$

### 4.4.2 Properties of the Exponential Function

**Theorem 4.31: Properties of the Exponential Function**

*The exponential function $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is bijective, strictly increasing, and continuous. Moreover,*

$$\begin{aligned}\exp(0) &= 1,\\ \exp(-x) &= \exp(x)^{-1},\\ \exp(x+y) &= \exp(x)\exp(y),\end{aligned}$$

*for all $x, y \in \mathbb{R}$.*

### 4.4.3 The Natural Logarithm

**Definition 4.32: Logarithm**

The unique inverse function

$$\log : \mathbb{R}_{>0} \to \mathbb{R}$$

of the bijective map *exp* $: \mathbb{R} \to \mathbb{R}_{>0}$ is called the **logarithm**.

**Corollary 4.33: Properties of the Logarithm**

*The logarithm* $\log : \mathbb{R}_{>0} \to \mathbb{R}$ *is strictly increasing, continuous, and bijective. Moreover,*

$$\begin{aligned} \log(1) &= 0, \\ \log(a^{-1}) &= -\log(a), \\ \log(ab) &= \log(a) + \log(b), \end{aligned}$$

*for all* $a, b \in \mathbb{R}_{>0}$.

The logarithm defined here is also called the **natural logarithm** to distinguish it from logarithms with another **base** $a > 1$ (for instance $a = 10$ or $a = 2$). For any $a > 1$, we define

$$\log_a(x) = \frac{\log(x)}{\log(a)} \qquad \forall x > 0.$$

Unless stated otherwise, $\log(x)$ always denotes the natural logarithm, i.e., the logarithm to base $e$.

We can now define powers with arbitrary real exponents. For $a > 0$ and $x \in \mathbb{R}$ we set

$$a^x = \exp(x \log(a)).$$

## 4.5 Limits of Functions

We consider functions $f : D \to \mathbb{R}$ defined on a subset $D \subseteq \mathbb{R}$, and we wish to define the limit of $f(x)$ as $x \in D$ approaches a point $x_0 \in \mathbb{R}$. Typical examples include $D = \mathbb{R}, D = [0, 1]$ or $D = (0, 1)$, with $x_0 = 0$ in each case.

### 4.5.1 Limit in the Vicinity of a Point

Let $D \subseteq \mathbb{R}$ be non-empty, and let $x_0 \in \mathbb{R}$ be such that

$$D \cap (x_0 - \delta, x_0 + \delta) \neq \emptyset \tag{4.4}$$

for all $\delta > 0$. Whenever this holds, we say that $x_0$ is an **accumulation point** of $D$. Note that if $x_0 \in D$, then Equation 4.4 is automatically satisfied.

Condition 4.4 ensures that there exists a sequence of points in $D$ converging to $x_0$.

**Definition 4.34: Limit of a Function**

Let $f : D \to \mathbb{R}$, and $x_0$ be an accumulation point of $D$. A number $L \in \mathbb{R}$ is called the **limit of** $f(x)$ **as** $x \to x_0$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon \qquad \forall x \in D.$$

In general, the limit of $f(x)$ as $x \to x_0$ may not exist. However, if it exists, it is uniquely determined. Hence we speak of *the* limit and write

$$\lim_{x \to x_0} f(x) = L$$

to indicate the limit exists and is equal to $L$. Informally, this means that the function values $f(x)$ are arbitrarily close to $L$ whenever $x \in D$ is sufficiently close to $x_0$.

The limit of a function satisfies properties analogous to those of Proposition 3.13. More precisely, if $f, g$ are functions on $D$ such that

$$\lim_{x\to x_0} f(x) = L_1 \qquad \text{and} \qquad \lim_{x\to x_0} g(x) = L_2,$$

then

$$\lim_{x\to x_0} (f+g)(x) = L_1 + L_2, \qquad \lim_{x\to x_0} (f\cdot g)(x) = L_1 \cdot L_2.$$

Moreover, $f \le g$ implies $L_1 \le L_2$, and the sandwich lemma holds: if $f \le h \le g$ and $L_1 = L_2$ then $\lim_{x\to x_0} h(x) = L_1 = L_2$.

**Remark 4.35.** *Let $f : D \to \mathbb{R}$ be a function. If $x_0 \in D$, then $f$ is continuous at $x_0$ if and only if $\lim_{x\to x_0} f(x) = f(x_0)$.*

Suppose that $x_0 \in D$ is an accumulation point of $D \setminus \{x_0\}$. Let $f : D \to \mathbb{R}$, and consider the restriction $f|_{D\setminus\{x_0\}}$. It may happen that $f$ is discontinuous at $x_0$, but the limit

$$L = \lim_{x\to x_0} f|_{D\setminus\{x_0\}}(x) \tag{4.5}$$

nevertheless exists. In this case, the point $x_0$ is called a **removable discontinuity** of $f$, and one also writes

$$L = \lim_{\substack{x\to x_0 \\ x\neq x_0}} f(x). \tag{4.6}$$

If we now define

$$\tilde{f}(x) = \begin{cases} f(x), & x \in D \setminus \{x_0\}, \\ L, & x = x_0, \end{cases} \tag{4.7}$$

then $\tilde{f}$ is continuous at $x_0$. In other words, we can remove the discontinuity of $f$ by redefining its value at $x_0$ to be $L$.

If instead $x_0 \notin D$ but the limit in Equation 4.6 exists, we call the function $\tilde{f}$ defined in Equation 4.7 the **continuous extension** of $f$ to $D \cup \{x_0\}$.

Arguing as in the proof of Theorem 4.12, we obtain the following result.

**Lemma 4.36: Limit and Sequences**

*Let $f : D \to \mathbb{R}$. Then $L = \lim_{x\to\bar{x}} f(x)$ if and only if, for every sequence $(x_n)_{n=0}^{\infty} \subseteq D$ converging to $\bar{x}$, one has $\lim_{n\to\infty} f(x_n) = L$.*

We now state a result describing the behaviour of limits under composition with a continuous function.

**Proposition 4.37: Limit and Composition**

*Let $E \subseteq \mathbb{R}$, and let $f : D \to E$ be such that the limit $L = \lim_{x\to\bar{x}} f(x)$ exists and belongs to $E$. If $g : E \to \mathbb{R}$ is continuous at $L$, then*

$$\lim_{x\to\bar{x}} g(f(x)) = g(L).$$

*Proof.* Let $(x_n)_{n=0}^{\infty} \subseteq D$ be a sequence converging to $\bar{x}$. By Lemma 4.36, we have $\lim_{n\to\infty} f(x_n) = L$. Since $g$ is continuous at $L$, Theorem 4.12 gives $\lim_{n\to\infty} g(f(x_n)) = g(L)$. Because $(x_n)_{n=0}^{\infty}$ was arbitrary, using Lemma 4.36 again, we conclude that $\lim_{x\to\bar{x}} g(f(x)) = g(L)$. $\square$

We now introduce conventions for improper limits of functions, in analogy with improper limits for sequences.

**Definition 4.38: Improper Limits**

Let $f : D \to \mathbb{R}$, and let $x_0$ be an accumulation point of $D$. We say that $f$ **diverges to** $+\infty$ **as** $x \to x_0$, and write

$$\lim_{x \to x_0} f(x) = +\infty,$$

if for every $M > 0$, there exists $\delta > 0$ such that

$$\forall x \in D : \quad |x - x_0| < \delta \quad \Rightarrow \quad f(x) \geq M.$$

Analogously, $f$ **diverges to** $-\infty$ **as** $x \to x_0$ and we write $\lim_{x \to x_0} f(x) = -\infty$, if for every $M > 0$, there exists $\delta > 0$ such that

$$\forall x \in D : \quad |x - x_0| < \delta \quad \Rightarrow \quad f(x) \leq -M.$$

### 4.5.2 One-Sided Limits

It is often useful to consider limits taken form one side only and to allow $x_0$ to be $\pm\infty$ as well. To this end, let $x_0 \in \mathbb{R}$ be such that

$$D \cap (x_0, x_0 + \delta) \neq \emptyset \tag{4.8}$$

for every $\delta > 0$. In this case, we say that $x_0$ is a **right-hand accumulation point** of $D$. Analogously, if

$$D \cap (x_0 - \delta, x_0) \neq \emptyset \tag{4.9}$$

for every $\delta > 0$, we say that $x_0$ is a **left-hand accumulation point** of $D$.

**Definition 4.39: One-Sided Limits**

Let $f : D \to \mathbb{R}$, and let $x_0 \in \mathbb{R}$ be a right-hand accumulation point of $D$. A number $L \in \mathbb{R}$ is called the **right-hand limit** of $f$ at $x_0$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in D \cap (x_0, x_0 + \delta) \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

In this case we write $L = \lim_{x \to x_0^+} f(x)$. We also allow improper one-sided limits. We say that

$$\lim_{x \to x_0^+} f(x) = +\infty$$

if for every $M > 0$, there exists $\delta > 0$ such that

$$x \in D \cap (x_0, x_0 + \delta) \quad \Rightarrow \quad f(x) \geq M.$$

Similarly, $\lim_{x \to x_0^+} f(x) = -\infty$ means that, for every $M > 0$, there exists $\delta > 0$ such that

$$x \in D \cap (x_0, x_0 + \delta) \quad \Rightarrow \quad f(x) \leq -M.$$

The **left-hand limit** is defined analogously, considering a left-hand accumulation point of $D$ and writing $\lim_{x \to x_0^-} f(x)$.

Next, we define the notion of limit at infinity.

### Definition 4.40: Limits at Infinity

Let $f : D \to \mathbb{R}$, and assume that $D \cap (R, \infty) \neq \emptyset$ for every $R > 0$. A number $L \in \mathbb{R}$ is called the **limit of $f$ as $x \to +\infty$** if, for every $\varepsilon > 0$, there exists $R > 0$ such that

$$x \in D \cap (R, \infty) \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

We say that $f$ **diverges to $+\infty$ as $x \to +\infty$** if, for every $M > 0$, there exists $R > 0$ such that

$$x \in D \cap (R, \infty) \quad \Rightarrow \quad f(x) \geq M.$$

The corresponding definition for $x \to -\infty$ and diverges to $-\infty$ are analogous.

Limits at $+\infty$ can be converted into right-hand limits at 0 via inversion. Given $f : D \to \mathbb{R}$ as above, define

$$E = \{x > 0 \mid x^{-1} \in D\}, \qquad g : E \to \mathbb{R}, \qquad g(x) = f(x^{-1}).$$

Then

$$\lim_{x \to +\infty} f(x) = \lim_{x \to 0^+} g(x),$$

so one limit exists if and only if the other does.

### Definition 4.41: One-Sided Continuity and Jumps

Let $f : D \to \mathbb{R}$ and $x_0 \in D$. If $\lim_{x \to x_0^+} f(x)$ exists and equals $f(x_0)$, then $f$ is **continuous from the right** at $x_0$. **Continuity form the left** is defined similarly. We call $x_0$ a **jump point** if both one-sided limits exist but are different, i.e.,

$$L_- := \lim_{x \to x_0^-} f(x) \in \mathbb{R}, \qquad L_+ := \lim_{x \to x_0^+} f(x) \in \mathbb{R}, \qquad L_- \neq L_+.$$