Analysis I
Theorems & Lemmas

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1.1 Convergence of Sequences

Definition 1.1: Sequences

A **sequence** is a function $a : \mathbb{N} \to \mathbb{R}$. The image $a(n)$ of $n \in \mathbb{N}$ is also written as a_n and is called the n-th element of a. Instead of $a : \mathbb{N} \to \mathbb{R}$ one often writes $(a_n)_{n\in\mathbb{N}}, (a_n)_{n=0}^{\infty}, (a_n)_{n\geq 0}$.

Definition 1.2: (Eventually) Constant Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is **constant** if $x_n = x_m \forall n, m \in \mathbb{N}$. It is **eventually constant** if there exists $N \in \mathbb{N}$ such that $x_n = x_m \forall n, m \geq N$.

Definition 1.3: Convergence of Sequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. We say that $(x_n)_{n=0}^{\infty}$ **converges** (or is **convergent**) if $\exists A \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} : |x_n - A| < \varepsilon \quad \forall n \geq N.$$

In this case we write

$$\lim_{n\to\infty} x_n = A \tag{1.1}$$

and call A the **limit** of $(x_n)_{n=0}^{\infty}$.

Lemma 1.4: Uniqueness of the Limit

A convergent sequence $(x_n)_{n=0}^{\infty}$ has exactly one limit.

Proof. Let $A, B \in \mathbb{R}$ be limits of $(x_n)_{n=0}^{\infty}$. Fix $\varepsilon > 0$. Then there exists $N_A, N_B \in \mathbb{N}$ such that $|x_n - A| < \varepsilon$ for all $n \geq N_A$ and $|x_n - B| < \varepsilon$ for all $n \geq N_B$. We define $N := \max\{N_A, N_B\}$. Then it holds that

$$|A - B| \leq |A - x_N| + |x_N - B| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $A = B$. $\square$

1.2 Convergent Subsequences and Acculmulation Points

Definition 1.5: Subsequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence. A **subsequence** is of the form $(x_{n_k})_{k=0}^{\infty}$, where $(n_k)_{k=0}^{\infty}$ is a strictly increasing sequence of non-negative integers, i.e., $n_{k+1} > n_k \; \forall k \in \mathbb{N}$.

Remark 1.6. *Since $n_{k+1} > n_k$ for all $k \in \mathbb{N}$ it follows by induction that $n_k \geq k$ for all $k \in \mathbb{N}$.*

Proof. For $k = 0$ we have that $n_0 \geq 0$, because $(n_k)_{k=0}^{\infty}$ is a sequence of non-negative integers. So the condition is fulfilled. For the inductive step we want to show that the condition holds for $k+1$ under the assumption that the condition is true for k. Because $(n_k)_{k=0}^{\infty}$ is also a strictly increasing sequence, we have that $n_{k+1} > n_k \geq k$. Additionally since $n_k \in \mathbb{N}$, we have that $n_{k+1} \geq n_k + 1$. So it follows that $n_{k+1} \geq n_k + 1 \geq k + 1$, which proofs the condition for $k + 1$. $\square$

Lemma 1.7: Subsequences of Convergent Sequences are Convergent

Let $(x_n)_{n=0}^{\infty}$ *be a sequence converging to* $A \in \mathbb{R}$. *Then every subsequence* $(x_{n_k})_{k=0}^{\infty}$ *also converges to* A.

Proof. Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^{\infty}$ converges to A, there exists $N \in \mathbb{N}$ such that $|x_n - A| < \varepsilon \ \forall n \geq N$. As by Remark 1.6 we know that $n_k \geq k$ for all $k \in \mathbb{N}$. Therefore for all $k \geq N$ it holds that $|x_{n_k} - A| < \varepsilon$. □

Definition 1.8: Accumulation Points of Sequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. A point $A \in \mathbb{R}$ is an **accumulation point** of $(x_n)_{n=0}^{\infty}$ if

$$\forall \varepsilon > 0 \ \forall N \in \mathbb{N} \ \exists n \geq N : |x_n - A| < \varepsilon.$$

Proposition 1.9: Subsequences and Accumulation Points

Let $(x_n)_{n=0}^{\infty}$ *be a sequence in* $\mathbb{R}$. *A point* A *is and accumulation point of* $(x_n)_{n=0}^{\infty}$ *if and only if there exists a convergent subsequence of* $(x_n)_{n=0}^{\infty}$ *with limit* A.

Corollary 1.10: Infinitely Many Terms Near an Accumulation Point

If $A \in \mathbb{R}$ *is an accumulation point of* $(x_n)_{n=0}^{\infty}$, *then for every* $\varepsilon > 0$ *there are infinitely many* n *with* $x_n \in (A - \varepsilon, A + \varepsilon)$.

Proof. By Proposition 1.9, there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ with $\lim_{k \to \infty} = A$. Hence for every $\varepsilon > 0$ there exists K such that $x_{n_k} \in (A - \varepsilon, A + \varepsilon)$ for all $k \geq K$, providing infinitely many elements of the sequence inside the interval $(A - \varepsilon, A + \varepsilon)$. □

Corollary 1.11: Accumulation Points of Convergent Sequences

convergent sequence has exactly one accumulation point, namely its limit.

1.3 Addition, Multiplication and Inequalities

Proposition 1.12: Limits and Operations

Let $(x_n)_{n=0}^{\infty}$ *and* $(y_n)_{n=0}^{\infty}$ *be sequences converging to* $A, B \in \mathbb{R}$ *respectively. Then:*

1. *The sequence* $(x_n + y_n)_{n=0}^{\infty}$ *converges to* $A + B$.
2. *The sequence* $(x_n y_n)_{n=0}^{\infty}$ *converges to* AB.
3. *Given* $\alpha \in \mathbb{R}$, *the sequence* $(\alpha x_n)_{n=0}^{\infty}$ *converges to* αA.
4. *Suppose* $x_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $A \neq 0$. *Then the sequence* $(x_n^{-1})_{n=0}^{\infty}$ *converges to* A^{-1}.

Proposition 1.13: Limits and Inequalities

Let $(x_n)_{n=0}^\infty$ and $(y_n)_{n=0}^\infty$ be sequences converging to $A, B \in \mathbb{R}$ respectively.

1. *If $A < B$, then there exists $N \in \mathbb{N}$ such that $x_n < y_n$ for all $n \geq \mathbb{N}$.*

2. *If there exists $N \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \geq N$, then $A \leq B$.*

Remark 1.14. *In Propostion 1.13 even if we assume that $x_n < y_n$ for all $n \in \mathbb{N}$, we cannot conclude that $A < B$. for example take*

$$x_n = \frac{1}{n}, \quad y_n = \frac{1}{n}.$$

Then we have that $x_n < y_n$ for all $n \in \mathbb{N}$ but $A = B = 0$.

Lemma 1.15: Sandwich Lemma

Let $(x_n)_{n=0}^\infty$, $(y_n)_{n=0}^\infty$, $(z_n)_{n=0}^\infty$ be sequences such that for some $N \in \mathbb{N}$, we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N.$$

Suppose that both $(x_n)_{n=0}^\infty$ and $(z_n)_{n=0}^\infty$ converge to the same limit. Then $(y_n)_{n=0}^\infty$ also converges, and we have that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n.$$

Proof. Let $(x_n)_{n=0}^\infty$, $(y_n)_{n=0}^\infty$, $(z_n)_{n=0}^\infty$ be sequences such that for some $N_0 \in \mathbb{N}$, we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N_0.$$

Additionally suppose that $(x_n)_{n=0}^\infty$ and $(z_n)_{n=0}^\infty$ converge to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^\infty$, $(z_n)_{n=0}^\infty$ converge to A there exists $N_x, N_z \in \mathbb{N}$ such that

$$\begin{aligned} A - \varepsilon < x_n < A + \varepsilon \quad &\forall n \geq N_x \\ A - \varepsilon < z_n < A + \varepsilon \quad &\forall n_z \geq N_z. \end{aligned}$$

So we choose $N := \max\{N_0, N_x, N_z\}$. Then we have that

$$A - \varepsilon < x_n \leq y_n \leq z_n < A + \varepsilon \quad \forall n \geq N,$$

which shows that $\lim_{n\to\infty} y_n = A$. □

Definition 1.16: Bounded Sequences

A sequence $(x_n)_{n=0}^\infty$ is called **bounded** if there exists a real number $M \geq 0$ such that

$$|x_n| \leq M \quad \forall n \in \mathbb{N}.$$

Lemma 1.17: Convergent Sequences are Bounded

Every convergent sequence is bounded.

Proof. Let $(x_n)_{n=0}^\infty$ be a sequence converging to $A \in \mathbb{R}$. Let $\varepsilon = 1$. Then, by convergence of $(x_n)_{n=0}^\infty$, there exists N such that $|x_n - A| < 1$ for all $n \geq N$. So we have that

$$|x_n| = |x_n - A + A| \leq |x_n - A| + |A| < 1 + |A|.$$

We choose

$$M = \max(|x_0|, |x_1|, \dots, |x_{N-1}|, 1 + |A|).$$

Then $(x_n)_{n=0}^{\infty} \leq M$ for all $n \in \mathbb{N}$ as desired. □

Definition 1.18: Monotone Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called:

- **(monotonically) increasing** if $m > n \Rightarrow x_m \geq x_n$,
- **strictly (monotonically) increasing** if $m > n \Rightarrow x_m > x_n$,
- **(monotonically) decreasing** if $m > n \Rightarrow x_m \leq x_n$,
- **strictly (monotonically) decreasing** if $m > n \Rightarrow x_m < x_n$.

If a sequence is decreasing or increasing we call it monotone. If a sequence is strictly increasing or strictly decreasing then we call it strictly monotone.

Remark 1.19. *An equivalent formulation of monotone sequences can be given using only successive terms:*

- $(x_n)_{n=0}^{\infty}$ *is increasing if* $x_{n+1} \geq x_n$ *for all* n,
- $(x_n)_{n=0}^{\infty}$ *is strictly increasing if* $x_{n+1} > x_n$ *for all* n,
- $(x_n)_{n=0}^{\infty}$ *is decreasing if* $x_{n+1} \leq x_n$ *for all* n,
- $(x_n)_{n=0}^{\infty}$ *is strictly decreasing if* $x_{n+1} < x_n$ *for all* n.

Theorem 1.20: Convergence of Monotone Sequences

A monotone sequence $(x_n)_{n=0}^{\infty}$ *converges if and only if it is bounded. More precisely, let* $X = \{x_n \mid n \in \mathbb{N}\}$ *denote the set of points in the sequence.*

- *If* $(x_n)_{n=0}^{\infty}$ *is increasing, then* $\lim_{n\to\infty} x_n = \sup(X)$,
- *if* $(x_n)_{n=0}^{\infty}$ *decreasing, then* $\lim_{n\to\infty} x_n = \inf(X)$.

Proof. If $(x_n)_{n=0}^{\infty}$ converges Lemma 1.17 says that its bounded.

Conversely, let $(x_n)_{n=0}^{\infty}$ be a bounded monotone sequence. Wlog assume that $(x_n)_{n=0}^{\infty}$ is increasing (otherwise consider $(-x_n)_{n=0}^{\infty}$). Since $(x_n)_{n=0}^{\infty}$ is bounded from above, the set $X = \{x_n \mid n \in \mathbb{N}\}$ has a supremum, that we'll call $A = \sup(X)$.

By definiton of A:

(i) $x_n \leq A \quad \forall n \in \mathbb{N}$,

(ii) $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_N > A - \varepsilon$.

Then, for all $n \geq N$ using (ii) and monotonicity, we have that $x_n \geq x_N > A - \varepsilon$. Then using (i), we conclude that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N.$$

□