Analysis I
Theorems & Lemmas

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1 Functions

Definition 1.1: Functions/Maps/Transformations

A **function** f from a set X to a set Y is an assignment of an element of Y to each element of X. The element $y \in Y$ to which $x \in X$ is assigned to is denoted $f(x)$. We write $f : X \to Y$ and sometimes also speak of a **map**, **mapping** or a **transformation**. The set X is the **domain** and the set Y is the **codomain**. We refer to the set X as **domain** or **domain of definition**, and the set Y as **domain of values** or **codomain**. The set

$$\{(x, f(x)) \mid x \in X\} \subseteq X \times Y$$

is called the **graph** of f. In the context of a function $f : X \to Y$, an element of the domain of definition is also called **argument**, and an element $y = f(x) \in Y$ assumed by the function, is also called **value** of the function. If $f : X \to Y$ is a function, one also writes

$$\begin{aligned} f : X &\to Y \\ x &\mapsto f(x), \end{aligned}$$

where $f(X)$ could be a concrete formula. We pronounce ´$\mapsto$´ as ´is mapped to'. Two functions $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are said to be equal if $X_1 = X_2$, $Y_1 = Y_2$ and $f_1(x) = f_2(x) \quad \forall x \in X_1$.

Definition 1.2: Injective, Surjective and Bijective Functions

Let $f : X \to Y$ be a function. We call f:

1. **injective** (or an **injection**) if

$$\forall x_1, x_2 \in X \, : \, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2);$$

2. **surjective** (or a **sujection**) if

$$\forall y \in Y \; \exists x \in X \, : \, f(x) = y;$$

3. **bijective** (or a **bijection**) if f is both injective and surjective.

Thus, a function $f : X \to Y$ is *not* injective if there exists distinct $x_1 \neq x_2 \in X$ such that $f(x_1) = f(x_2)$, and *not* surjective if there exists $y \in Y$ such that $f(x) \neq y$ for all $x \in X$.

Definition 1.3: Image and Preimage of a Function

For $f : X \to Y$ and $A \subseteq X$, define the **image** of A under the function f as

$$f(A) := \{y \in Y \mid \exists x \in X : f(x) = y\}.$$

For $B \subseteq Y$, define the **preimage** of B under the function f as

$$f^{-1}(B) := \{x \in X \mid f(x) \in B\}.$$

Remark 1.4. *Saying that $f : X \to Y$ is surjective is equivalent to $f(X) = Y$. Equivalently, f is surjective if $f^{-1}(\{y\}) \neq \emptyset$ for all $y \in Y$.*

Definition 1.5: Composition of Functions

Let $f : X \to Y$ and $g : Y \to Z$. The **composition** is $g \circ f : X \to Z$, defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.
Associativity: If $f : W \to X, g : X \to Y$ and $h : Y \to Z$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Indeed, for all $w \in W$, we have

$$h \circ (g \circ f)(w) = h((g \circ f)(w)) = h(g(f(w))) = (h \circ g)(f(w)) = ((h \circ g) \circ f)(w).$$

Therefore, we may omit parentheses and write $h \circ g \circ f : W \to Z$.

Definition 1.6: Identity and Inverse Function

Given a set X, the **identity function** $\mathrm{id}_X : X \to X$ is defined by

$$\mathrm{id}_X(x) = x \qquad \forall x \in X.$$

If $f : X \to Y$ is bijective, then there exists a unique function $g : Y \to X$ such that, for each $y \in Y$, the value $g(y)$ is the unique element $x \in X$ satisfying $f(x) = y$. With this definition,

$$g \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ g = \mathrm{id}_Y .$$

The function g is called the **inverse function** (or **inverse mapping**) of f, and is denoted by f^{-1}.

Remark 1.7. *A function $f : X \to Y$ is bijective if and only if there exists a function $g : Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$.*

Definition 1.8: Affine Functions

An **affine function** is a function of the form $x \mapsto sx + r$, for $s, r \in \mathbb{R}$. The graph of an affine function is a non-vertical liine in $\mathbb{R}^2$. The parameter s in the equation $y = sx + r$ is called the **slope** of the line.

Definition 1.9: Even and Odd Functions

Let $D \subseteq \mathbb{R}$ be a set satisfying $D = -D$, i.e.,

$$\forall x \in D: \ x \in D \quad \Rightarrow \quad -x \in D.$$

A function $f : D \to \mathbb{R}$ is called

- **even** if $f(-x) = f(x)$ for all $x \in D$,
- **odd** if $f(-x) = -f(x)$ for all $x \in D$.

Geometrically, and even function is symmetric with respect to the y-axis, while an odd function is symmetric with respect to the origin.

2 The Real Numbers

2.1 Groups, Rings, Fields

Definition 2.1: Groups

A **group** is a non-empty set G together with a rule (called an *operation*) denoted by $\star : G \times G \to G$ that combines any two elements of G into another element of G. This operation must satisfy three conditions:

- **Associativity:** No matter how you place parentheses, the result is the same for all $a, b, c \in G$,
$$(a \star b) \star c = a \star (b \star c).$$
- **Neutral element:** There is a special element $e \in G$ such that combining it with any $a \in G$ leaves a unchanged, i.e.,
$$\forall a \in G : a \star e = e \star a = a.$$
- **Inverse element:** Every $a \in G$ has a 'partner' $a^{-1} \in G$ that 'cancels it out', giving the neutral element, i.e.,
$$a \star a^{-1} = a^{-1} \star a = e.$$

Note that, in general, one does not require that $a \star b = b \star a$. If the order of the operation does not matter, i.e., $a \star b = b \star a$ for all $a, b \in G$, the group is called **commutative** or **abelian**.

Lemma 2.2: Basic Properties of Groups

Let $(G, \star)$ be a group. Then:

1. *The neutral element is unique.*
2. *The inverse of an element is unique.*
3. *The inverse of the inverse of an element is the element itself, namely $(a^{-1})^{-1} = a$ for all $a \in G$.*

Proof. 1. Assume that, in addition to $e \in G$, we have a second element e' with the property that $e' \star a = a \star e' = a$ for all elements $a \in G$. Then, we can choose $a = e$ to obtain

$$e \star e' = e.$$

Similarly, since e is a neutral element, we have

$$e \star e' = e'.$$

Combining the two identities, we get

$$e = e \star e' = e'.$$

This proves that $e' = e$, so we speak of *the* neutral element of a group.

2. Assume that for an element $a \in G$, there exists two elements $b, c \in G$ that are both the inverse of a, namely

$$a \star b = b \star a = e, \qquad a \star c = c \star a = e.$$

Then, using associativity, we observe that

$$b = b \star e = b \star (a \star c) = (b \star a) \star c = e \star c = c.$$

This proves that the inverse of an element a is unique, so we can speak of *the* inverse element, and the notation a^{-1} makes sense.

3. Since $a \star a^{-1} = e$, we deduce that a is the inverse element of a^{-1}, thus

$$(a^{-1})^{-1} = a. \tag{2.1}$$

□

Groups capture the idea of combining elements with a single operation. But to describe the arithmetic of numbers more faithfully, we also need a second operation (as we do with addition and multiplication). This leads us to the notion of *rings* and *fields*.

Definition 2.3: Rings and Fields

A **ring** is a non-empty set R in which we can both 'add' and 'multiply' elements with two operations '+' and '$\cdot$'. Also, these two operations are compatible with each other. More precisely:

– $(R, +)$ is a **commutative group**, with neutral element denoted 0.

– Multiplication $\cdot$ is **associative**, has a **neutral element** (usually written as 1), and **distributes over addition**, i.e.,

$$a \cdot (b + c) = a \cdot b + a \cdot c, \qquad (b + c) \cdot a = b \cdot a + c \cdot a \qquad \forall a, b, c \in R.$$

If multiplication is also commutative, we call $(R, +, \cdot)$ a **commutative ring**. Note that, unlike addition, we do not require that every element has an inverse for multiplication. A **field** is a special kind of commutative ring, i.e. every non-zero element has an inverse for multiplication. In other words, if $(R, +, \cdot)$ is a commutative ring, then $(R, +, \cdot)$ is a field if $R \setminus \{0\}$ forms a commutative group under multiplication. Traditionally, we use the letter F to denote a field. We also write $F^* = F \setminus \{0\}$ for the set of all invertible elements of F.

Lemma 2.4: Basic Properties of Fields

Let $(F, +, \cdot)$ be a field and let $a, b \in F$. Then:

1. $0 \cdot a = a \cdot 0 = 0$.
2. $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$. *In particular* $(-1) \cdot a = -a$.
3. $(-a) \cdot (-b) = a \cdot b$. *In particular,* $(-a)^{-1} = -(a^{-1})$ *whenever* $a \neq 0$.

Proof. 1. Since 0 is the neutral element for the addition, we have $0+0 = 0$. Hence, using distributivity, we get

$$0 \cdot a = (0 + 0) \cdot a = (0 \cdot a) + (0 \cdot a).$$

Adding $-0 \cdot a$ (i.e., the inverse of $0 \cdot a$), we deduce that $0 \cdot a = 0$. The case of $a \cdot 0$ is analogous.

2. By the distributive law,

$$a \cdot b + a \cdot (-b) = a \cdot (b + (-b)) = a \cdot 0 = 0.$$

So $a \cdot (-b)$ is the additive inverse of $a \cdot b$, i.e., $-(a \cdot b) = a \cdot (-b)$. Taking $b = 1$ gives $-a = (-1) \cdot a$. The validity of $(-a) \cdot b = -(a \cdot b)$ follows by exchanging a and b in the argument above.

3. By 2. we know that $-(a \cdot b) = a \cdot (-b)$. Hence, recalling Equation 2.1,

$$a \cdot b = -(a \cdot (-b)).$$

On the other hand, applying 2. with $(-b)$ instead of b, we also have

$$-(a \cdot (-b)) = (-a) \cdot (-b).$$

Combining the two identities above, we conclude that $(-a) \cdot (-b) = a \cdot b$. Finally, taking $b = a^{-1}$ yields $(-a) \cdot (-(a^{-1})) = a \cdot a^{-1} = 1$, which gives the second assertion. $\square$

2.2 Order Relation

Definition 2.5: Cartesian Product

Let X and Y be two sets. The **cartesian product** $X \times Y$ is the set of ordered pairs of elements of X and Y, i.e.,

$$X \times Y := \{(x, y) \mid x \in X, y \in Y\}.$$

Definition 2.6: Subsets

Let P and Q be sets. Then

- P is a **subset** of Q, written $P \subset Q$ (or $P \subseteq Q$), if every element of P also belongs to Q.
- P is a **proper subset** of Q, written $P \subsetneq Q$, if P is a subset of Q but $P \neq Q$.
- We write $P \not\subseteq Q$ if P is not a subset of Q.

Definition 2.7: Relations

Let X be a set. A **relation** on X is a subset $\mathcal{R} \subseteq X \times X$, that is, a collection of ordered pairs of elements of X. If $(x, y) \in \mathcal{R}$ we write $x\mathcal{R}y$. Common symbols for relations include $<, \leq, \sim, \equiv, \cong$. If $\sim$ is a relation on X, we write $x \nsim y$ if $x \sim y$ does not hold. A realtion $\sim$ may have the following properties:

1. **Reflexive:** if $x \sim x \qquad \forall x \in X$.
2. **Transitive:** if $x \sim y$ and $y \sim z$, then $x \sim z$.
3. **Symmetric:** if $x \sim y$, then $y \sim x$.
4. **Antisymmetric:** if $x \sim y$ and $y \sim x$, then $x = y$.

A relation is an **equivalence relation** if it is reflexive, transitive and symmetric. It is an **order relation** if it is reflexive, transitive and antisymmetric.

2.3 Ordered Fields

Definition 2.8: Ordered Field

Let F be a field, and let $\leq$ be an order relation on F. We call $(F, \leq)$, or simply F, an **ordered field** if the following hold:

1. **Linearity of order:** for all $x, y \in F$, at least one of $x \leq y$ or $y \leq x$ holds.
2. **Compatibility with addition:** for all $x, y, z \in F$,

$$x \leq y \Rightarrow x + z \leq y + z.$$

3. **Compatibility with multiplication:** for all $x, y \in F$,

$$0 \leq x \ \wedge \ 0 \leq y \Rightarrow 0 \leq x \cdot y.$$

Lemma 2.9: Ordered Field: Basic Consequences

Let $(F, \leq)$ be and ordered field, and let $x, y, z, w \in F$. Then:

(a) (Trichotomy) Either $x < y$, or $x = y$, or $x > y$.

(b) If $x < y$ and $y \leq z$, then $x < z$. (Analogously, $x \leq yandy < zimplyx < z$.)

(c) (Addition of inequalities) If $x \leq y$ and $z \leq w$, then $x + z \leq y + w$. (Analogously, $x < z$ and $z \leq w$ imply $x + z < y + w$.)

(d) $x \leq y$ if and only if $0 \leq y - x$.

(e) $x \leq 0$ if and only if $0 \leq -x$.

(f) $x^2 \geq 0$, and $x^2 > 0$ if $x \neq 0$.

(g) 0 < 1.

(h) If $0 \leq x$ and $y \leq z$, then $xy \leq xz$.

(i) If $x \leq 0$ and $y \leq z$, then $xy \geq xz$.

(j) If $0 < x \leq y$, then $0 < y^{-1} \leq x^{-1}$.

(k) If $0 \leq x \leq y$ and $0 \leq z \leq w$, then $0 \leq xz \leq yw$.

(l) If $x + y \leq x + z$, then $y \leq z$.

(m) If $xy \leq xz$ and $x > 0$, then $y \leq z$.

Lemma 2.10: Integers and Rationals Inside an Ordered Field

Let $(F, \leq)$ be and ordered field, and denote by 0 and 1 the neutral elements for addition and multiplication, respectively. Then:

(i) The elements $\ldots, -2, -1, 0, 1, 2, \ldots$ defined by

$$2 = 1 + 1, \quad 3 = 2 + 1, \ldots, \quad -n = (-1) \cdot n$$

are all distinct and satisfy

$$\ldots < -2 < -1 < 0 < 1 < 2 < 3 < \ldots.$$

We denote this set of elements by $\mathbb{Z}$, and we call them 'integers'

(ii) Every fraction pq^{-1} with $p, q \in \mathbb{Z}$, $q \neq 0$, lies in F and the set of all such elements is denoted by $\mathbb{Q}$. Also,

$$\mathbb{Z} \subsetneq \mathbb{Q} \subseteq F.$$

Proof. (i) By Lemma 2.9(g), we have that 0 < 1. Then Lemma 2.9(c) yields $0 < 1 < 2 < 3 < \ldots$, and taking negatives gives $\ldots < -2 < -1 < 0$. Hence all these elements are distinct.

(ii) For $q \neq 0$, q is invertible in F; define $\frac{p}{q} = pq^{-1}$. The set of such fractions is a field contained in F, which we denote by $\mathbb{Q}$.

To show that $\mathbb{Q}$ strictly contains $\mathbb{Z}$, consider $\frac{1}{2}$ (the inverse of 2). Since 2 > 1, it follows from

Lemma 2.9(j) that $0 < \frac{1}{2} < 1$, so $\frac{1}{2} \notin \mathbb{Z}$. □

Definition 2.11: Absolute Value and Sign

Let $(F, \leq)$ be and ordered field.

- The **absolute value** (or **modulus**) is the function $|\cdot| : F \to F$ defined by

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

- The **sign** is the function $\operatorname{sgn} : F \to \{-1, 0, 1\}$ defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Lemma 2.12: Absolute Value and Sign: Basic Properties

Let $(F, \leq)$ be an ordered field and let $x, y \in F$. Then:

(a) $x = \operatorname{sgn}(x)|x|, \quad |-x| = |x|, \quad \operatorname{sgn}(-x) = -\operatorname{sgn}(x)$.

(b) $|x| \geq 0$, *and* $|x| = 0$ *if and only if* $x = 0$ *(by Trichotomy Lemma* **??***).*

(c) (Multiplicativity) $\operatorname{sgn}(xy) = \operatorname{sgn}(x)\operatorname{sgn}(y)$ *and* $|xy| = |x||y|$.

(d) If $x \neq 0$, *then* $|x^{-1}| = |x|^{-1}$.

(e) $|x| \leq y$ *iff* $-y \leq x \leq y$.

(f) $|x| < y$ *iff* $-y < x < y$.

(g) (Triangle inequality) $|x + y| \leq |x| + |y|$.

(h) (Inverse triangle inequality) $||x| - |y|| \leq |x - y|$.

Proof. (g) Thanks to (e) we have $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding these two inequalities we get

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

Applying (e) again yields the result.

(h) From (g) we have $|x| \leq |x - y| + |y|$, therefore

$$|x| - |y| \leq |x - y|.$$

Exchanging the roles of x and y, we also have $|y| - |x| leq |y - x| = |x - y|$. Combining these two inequalities yields

$$-|x - y| \leq |x| - |y| \leq |x - y|,$$

and the result follows by applying (e) again. □

2.4 Completeness Axiom

Definition 2.13: Completeness Axiom

Let $(K, \leq)$ be an ordered field. We say that $(K, \leq)$ is **complete** (or a **completely ordered field**) if the following statement holds:

> Let X, Y be non-empty subsets of K such that $x \leq y$ for all $x \in X$ and $y \in Y$. Then there exists $c \in K$ lying between X and Y, in the sense that $x \leq c \leq y$ for all $x \in X$ and $y \in Y$.

The statement above is called the **completeness axiom**.

Definition 2.14: Real Numbers

We call **the field of real numbers**, any completely ordered field and denote it by $\mathbb{R}$.

2.5 Intervals

Definition 2.15: Intervals

Let $a, b \in \mathbb{R}$. We define:

- The **closed interval**
$$[a, b] := \{x \in R \mid a \leq x \leq b\};$$
- The **open interval**
$$(a, b) := \{x \in R \mid a < x < b\};$$
- The **half-open intervals**
$$[a, b) := \{x \in R \mid a \leq x < b\} \quad \text{and} \quad (a, b] := \{x \in R \mid a < x \leq b\};$$
- The **unbounded closed intervals**
$$[a, \infty) := \{x \in R \mid a \leq x\} \quad \text{and} \quad (-\infty, b] := \{x \in R \mid x \leq b\};$$
- The **unbounded open intervals**
$$(a, \infty) := \{x \in R \mid a < x\} \quad \text{and} \quad (-\infty, b) := \{x \in R \mid x < b\};$$

Definition 2.16: Set Operations

Let P, Q be sets. The **intersection** $P \cap Q$, the **union** $P \cup Q$, the **relative complement** $P \setminus Q$ and the **symmetric difference** $P \triangle Q$ are defined by

$$\begin{aligned} P \cap Q &= \{x \mid x \in P \text{ and } x \in Q\}, \\ P \cup Q &= \{x \mid x \in P \text{ or } x \in Q\}, \\ P \setminus Q &= \{x \mid x \in P \text{ and } x \notin Q\}, \\ P \triangle Q &= (P \setminus Q) \cup (Q \setminus P) = (P \cup Q) \setminus (P \cap Q). \end{aligned}$$

Definition 2.17: Union and Intersection of several Sets

Let $\mathcal{A}$ be a family of sets (i.e., a set whose elements are sets). We define the **union** and **intersection** of the sets in $\mathcal{A}$ as

$$\bigcup_{A \in \mathcal{A}} A = \{x \mid \exists A \in A : x \in A\}, \qquad \bigcap_{A \in \mathcal{A}} A = \{x \mid \forall A \in \mathcal{A} : x \in A\}.$$

If $\mathcal{A} = \{A_1, A_2, \ldots\}$, we also write

$$\bigcup_{i=1}^{\infty} A_i = \{x \mid \exists i \geq 1 : x \in A_i\}, \qquad \bigcap_{i=1}^{\infty} A_i = \{x \mid \forall i \geq 1 : x \in A_i\}.$$

Definition 2.18: Neighborhoods

Let $x \in \mathbb{R}$. A **neighborhood** of x is a set containing an interval I such that $x \in I$. Given $\delta > 0$, the open interval $(x - \delta, x + \delta)$ is called the **δ-neighborhood** of x.

Definition 2.19: Open and Closed Sets

A subset $U \subseteq \mathbb{R}$ is called **open** in $\mathbb{R}$ if for every $x \in U$ there exists open interval I such that $x \in I$ and $I \subseteq U$. A subset $F \subseteq \mathbb{R}$ is called **closed** in $\mathbb{R}$ if its complement $\mathbb{R} \setminus F$ is open.

Remark 2.20. *The sets $\emptyset$ and $\mathbb{R}$ are both open in $\mathbb{R}$. Hence, they are also closed since $\emptyset^c = \mathbb{R}$ and $\mathbb{R}^c = \emptyset$. We note that $\mathbb{Q} \subseteq \mathbb{R}$ and $[a, b) \subseteq \mathbb{R}$ are neither open nor closed.*

Remark 2.21. *Let $\mathcal{U}$ be a family of open sets, and $\mathcal{F}$ be a family of closed subsets of $\mathbb{R}$. Then the union and intersection*

$$\bigcup_{U \in \mathcal{U}} U, \qquad \bigcap_{F \in \mathcal{F}} F$$

Are open and closed, respectively.

2.6 Complex Numbers

Starting from the field of real numbers $\mathbb{R}$, we define the set of **complex numbers** as

$$\mathbb{C} = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

We denote the elements $z = (x, y) \in \mathbb{C}$ in the form $z = x + iy$, where i is the **imaginary unit**. Here $x \in \mathbb{R}$ is the **real part** of z, written as $x = \mathrm{Re}(z)$, and $y \in \mathbb{R}$ is the **imaginary part**, written as $y = \mathrm{Im}(z)$. Elements with $\mathrm{Im}(z) = 0$ are called **real**, while those with $\mathrm{Re}(z) = 0$ are **purely imaginary**. Via the injective map $\mathbb{R} \ni x \mapsto x + i \cdot 0 \in \mathbb{C}$, we identify $\mathbb{R}$ with the subset of real numbers inside $\mathbb{C}$.

As you may expect from previous knowledge, we want to satisfy $i^2 = -1$. To achieve this, we define addition and multiplication on $\mathbb{C}$ so that it becomes a field. Additionally, we want these operations to coincide with the usual addition and multiplication when considering real numbers.

Since $i^2 = -1$, using commutativity and distributivity we get

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_2x_1 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

This motivates the following definition

Definition 2.22: Addition and Multiplication on $\mathbb{C}$

On $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ we define **addition** and **multiplication** as follows:

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2,\, y_1 + y_2),\\ (x_1, y_1) \cdot (x_2, y_2) &= (x_1x_2 - y_1y_2,\, x_1y_2 + x_2y_1).\end{aligned}$$

Proposition 2.23: $\mathbb{C}$ is a Field

With the operation of Definition 2.22, together with the zero element $(0, 0)$ and the unit element $(1, 0)$, the set $\mathbb{C}$ is a field.

Definition 2.24: Complex Conjugation

For $z = x + iy \in \mathbb{C}$ we define its **conjugate** as $\bar{z} = x - iy$. The mapping $\mathbb{C} \ni z \mapsto \bar{z} \in \mathbb{C}$ is called **complex conjugation**.

Lemma 2.25: Properties of Complex Conjugation

For all $z, w \in \mathbb{C}$:

(i) $z\bar{z} = x^2 + y^2 \in \mathbb{R}_{\geq 0}$. *In particular,* $z\bar{z} = 0$ *if and only if* $z = 0$.

(ii) $\overline{z + w} = \bar{z} + \bar{w}$.

(iii) $\overline{zw} = \bar{z}\bar{w}$.

Proof. Property (i) follows from the fact that, for $z = x + iy$, $(x + iy)(x - iy) = x^2 + y^2$. Also, $x^2 + y^2 = 0$ if and only if $x + iy = 0$. Properties (ii) and (iii) follow from a direct computation, writing $z = x_+ + iy_1$ and $w = x_2 + iy_2$, which yields

$$\begin{aligned}\overline{z + w} &= \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_2) + (x_2 - iy_2) = \bar{z} + \bar{w},\\ \overline{z \cdot w} &= \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)} = (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2)\\ &= (x_1 - iy_1) \cdot (x_2 - iy_2) = \bar{z} \cdot \bar{w}.\end{aligned}$$

□

Definition 2.26: Absolute Value

The **absolute value** (or **norm**) on $\mathbb{C}$ is the map $|\cdot|:\mathbb{C}\to\mathbb{R}$ given by

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2+y^2}, \qquad z = x+iy \in \mathbb{C}.$$

Lemma 2.27: Cauchy-Schwart Inequality

If $z = x_1 + iy_1$, and $w = x_2 + iy_2$, then

$$x_1x_2 + y_1y_2 \leq |z||w|. \tag{2.2}$$

Proof. We observe that

$$\begin{aligned}
|z|^2|w|^2 - (x_1x_2+y_1y_2)^2 &= (x_1^2+y_1^2)(x_2^2+y_2^2) - (x_1x_2+y_1y_2)^2 \\
&= x_1^2x_2^2 + y_1^2y_2^2 + y_1^2x_2^2 + x_+^2y_2^2 - (x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2) \\
&= y_1^2x_2^2 + x_+^2y_2^2 - 2x_1x_2y_1y_2 \\
&= (y_1x_2 - x_1y_2)^2 \geq 0.
\end{aligned}$$

This proves that $(x_1x_2+y_1y_2)^2 \leq |z|^2|w|^2$, so it follows that

$$|x_1x_2 + y_1y_2| \leq |z||w|.$$

Since $x \leq |x|$ for all $x \in \mathbb{R}$, we obtain Equation 2.2. □

Proposition 2.28: Trianlge Inequality

For all $z, w \in \mathbb{C}$, one has

$$|z+w| \leq |z| + |w|.$$

Proof. For $z = x_1 + iy_1$ and $w = x_2 + iy_2$, using Lemma 2.27, we have

$$\begin{aligned}
|z+w|^2 &= (x_1+x_2)^2 + (y_1+y_2)^2 \\
&= |z|^2 + |w|^2 + 2(x_1x_2 + y_1y_2) \\
&\leq |z|^2 + |w|^2 + 2|z||w| = (|z|+|w|)^2.
\end{aligned}$$

Taking square roots proves the result. □

Definition 2.29: Cicular Disks

For $z \in \mathbb{C}$ and $r > 0$, we define the **open disk** with radius $r > 0$ around z as

$$B(z,r) := \{w \in \mathbb{C} \mid |z-w| < r\},$$

and the **closed disk** with radius $r > 0$ around z as

$$\overline{B(z,r)} := \{w \in \mathbb{C} \mid |z-w| \leq r\}.$$

In other words, the open disk $B(z,r)$ is the set of points at distance strictly less than r form z. We note that this definition is compatible with the one of neighborhoods in $\mathbb{R}$: if $x \in \mathbb{R}$ and $r > 0$, then

$$B(x,r) \cap \mathbb{R} = (x-r, x+r).$$

Definition 2.30: Open and Closed Sets

A set $U \subseteq \mathbb{C}$ is **open** if for every $z \in U$ there exists $r > 0$ such that $B(z,r) \subseteq U$. A set $C \subseteq \mathbb{C}$ is **closed** if its complement $\mathbb{C} \setminus C$ is open.

2.7 Maximum and Supremum

2.7.1 Existence of the Supremum

Definition 2.31: Bounded Sets, Maxima and Minima

et $X \subseteq \mathbb{R}$ be a subset of real numbers.

- X is **bounded from above** if there exists $s \in \mathbb{R}$ such that $x \leq s$ for all $x \in X$. Such a number s is called an **upper bound** of X. If s is an upper bound and also an element of X, we say that s is the **maximum** of X and write

$$s = \max(X).$$

- Analogously, X is **bounded from below** if there exists $r \in \mathbb{R}$ such that $r \leq x$ for all $x \in X$. Such a number r is called a **lower bound** of X. If r is a lower bound and also an element of X, we say that r is the **minimum** of X and write

$$r = \min(X).$$

- X is called **bounded** if it is both bounded from above and bounded from below.

Remark 2.32. *If a set $X \subseteq \mathbb{R}$ has a maximum, then it is unique. Indeed, if $x_1, x_2 \in X$ are both maxima, then $x_1 \leq x_2$ (since x_2 is a maximum) and $x_2 \leq x_1$ (since x_1 is a maximum), so $x_1 = x_2$.*

A closed interval $[a,b]$ with $a < b$ has both a minimum and maximum, i.e., $a = \min([a,b])$ and $b = \max([a,b])$. But not all sets have a maximum. For instance, the open interval (a,b) does not have a maximum because the endpoint b, though an upper bound, is not contained in the set. Similarly $\mathbb{R}$ and unbounded intervals such as $[a,\infty)$ or (a,∞) have no maximum.

Definition 2.33: Supremum

Let $X \subseteq \mathbb{R}$ be a subset and let

$$A := \{a \in \mathbb{R} \mid x \leq a \quad \forall x \in X\}$$

be the set of all upper bounds of X. If A has a minimum, we call this minimum the **supremum** of X and write

$$\sup(X) = \min(A).$$

The **infimum** is defined analogously using the maximum of the set of all lower bounds.

In other words, the supremum of X is the smallest real number that is greater than or equal to every element of X. Note that we can describe the supremum $s = \sup(X)$ as follows

$$x \leq s \qquad \forall x \in X, \qquad \text{and} \qquad \text{if } t < s, \text{the } t \text{ is not an upper bound of } X. \tag{2.3}$$

This means that for every $t < s$, there exists some $x \in X$ such that $x > t$. That is,

$$x \leq s \qquad \forall x \in X, \qquad \text{and} \qquad \forall t < s\, \exists x \in X : x > t. \tag{2.4}$$

The two characterizations 2.3 and 2.4 are equivalent.

Note that not every set has a supremum. If $X = \emptyset$ or if X is unbounded form above, then $\sup(X)$ does not exist. However, for any non-empty and bounded-above subset of $\mathbb{R}$, the supremum always exists.

Remark 2.34. *If a set X has a maximum, then this element is also the supremum. Indeed, the maximum is an upper bound of X, and since it lies in X, no smaller upper bound can exist.*

Theorem 2.35: Existence of Supremum

Let $X \subseteq \mathbb{R}$ be non-empty and bounded from above. Then $\sup(X)$ exists and is a real number.

Proof. Since X is bounded from above, the set $A := \{a \in \mathbb{R} \mid x \leq a \quad \forall x \in X\}$ of upper bounds is non-empty. Since $x \leq a$ for any $x \in X$ and $a \in A$, we can apply the completeness axiom (Definition 2.13) to find $c \in \mathbb{R}$ such that

$$x \leq c \leq a \qquad \forall x \in X, \forall a \in A.$$

The first inequality implies that c is itself an upper bound (so $c \in A$), while the second inequality tells us that c is smaller than or equal to every upper bound. Hence, $c = \min(A) = \sup(X)$. □

Proposition 2.36: Supremum and Set Operations

Let X and Y be non-empty subsets of $\mathbb{R}$ that are bounded from above. Define

$$X + Y := \{x + y \mid x \in X, y \in Y\} \quad \text{and} \quad X \cdot Y := \{x \cdot y \mid x \in X, y \in Y\}.$$

The the sets $X \cup Y, X \cap Y$, and $X + Y$ are also bounded from above. Moreover, if $X, Y \subseteq \mathbb{R}_{\geq 0}$ (i.e., $x \geq 0$ and $y \geq 0$ for all $x \in X$ and $y \in Y$), then $X \cdot Y$ is bounded from above as well. In these cases, the following formulas hold:

(1) $\sup(X \cup Y) = \max\{\sup(X), \sup(Y)\}$,

(2) If $X \cap Y \neq \emptyset$, then $\sup(X \cap Y) \leq \min\{\sup(X), \sup(Y)\}$,

(3) $\sup(X + Y) = \sup(X) + \sup(Y)$,

(4) If $X, Y \subseteq \mathbb{R}_{\geq 0}$, then $\sup(X \cdot Y) = \sup(X) \cdot \sup(Y)$.

Proof. (3) Let $x_0 = \sup(X)$ and $y_0 = \sup(Y)$. For any $z \in X + Y$, there exists $x \in X$ and $y \in Y$ such that $z = x + y$. Since $x \leq x_0$ and $y \leq y_0$, we have

$$z = x + y \leq x_0 + y_0,$$

so $x_0 + y_0$ is an upper bound form $X + Y$. We now want to show that $x_0 + y_0 = \sup(X + Y)$.

Let $z_0 = \sup(X + Y)$ and suppose, by contradiction, that

$$\varepsilon := x_0 + y_0 - z_0 > 0.$$

Since $x_0 = \sup(X)$, by the characterization 2.4 there exists $x \in X$ such that $x > x_0 - \varepsilon/2$. Likewise, there exists $y \in Y$ such that $y > y_0 - \varepsilon/2$. Setting $z = x + y$, we obtain

$$z > x_0 - \frac{\varepsilon}{2} + y_0 - \frac{\varepsilon}{2} = x_0 + y_0 - \varepsilon = z_0,$$

contradicting the assumption that z_0 is an upper bound for $X + Y$. Therefore, $z_0 = x_0 + y_0$.

(4) The proof is analogous. If all elements of X and Y are non-negative, and we set $x_0 = \sup(X)$ and $y_0 = \sup(Y)$, then for any $z = x \cdot y \in X \cdot Y$, we have

$$z = x \cdot y \leq x_0 \cdot y_0,$$

which shows that $x_0 \cdot y_0$ is an upper bound for $X \cdot Y$. Using a similar 'ε-argument' as done above, when proving (3), one shows that this upper bound is sharp, i.e., $x_0 \cdot y_0$ is the least upper bound. □

2.8 Two-Point Compactification

In this section, we extend the notions of **supremum** and **infimum** to arbitrary subsets of $\mathbb{R}$. To do so, we introduce two formal symbols

$$+\infty \quad \text{and} \quad -\infty,$$

which are not real numbers. We define the **extended real numbers line** (also called the **two-point compactification** of $\mathbb{R}$) by

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}.$$

We extend the usual order relation $\leq$ on $\mathbb{R}$ to $\overline{\mathbb{R}}$ by requiring that

$$-\infty < x < +\infty \qquad \forall x \in \mathbb{R}.$$

For simplicity, we often write ∞ instead of $+\infty$.

We now introduce some standard (but informal) computation rules involving these symbols. For all $x \in \mathbb{R}$, we adopt the conventions:

$$\infty + x = \infty + \infty = \infty, \qquad -\infty + x = -\infty - \infty = -\infty.$$

If $x > 0$, then

$$x \cdot \infty = \infty \cdot \infty = \infty, \qquad x \cdot (-\infty) = \infty \cdot (-\infty) = -\infty,$$

while for $x < 0$ we have

$$x \cdot \infty = -\infty \cdot \infty = -\infty, \qquad x \cdot (-\infty) = -\infty \cdot (-\infty) = \infty.$$

These rules are widely used as notational shorthand, but one must handle them with care. Expressions like

$$\infty - \infty, \quad 0 \cdot \infty, \quad \text{or similar}$$

are undefined and should be avoided.

Definition 2.37: Supremum and Infimum in the Extended Line

Let $X \subseteq \mathbb{R}$.

- If X is not bounded from above, we define $\sup(X) = \infty$.
- If $X = \emptyset$, we define $\sup(\emptyset) = -\infty$.
- If X is not bounded from below, we define $\inf(X) = -\infty$.
- If $X = \emptyset$, we define $\inf(\emptyset) = \infty$.

In this context, we refer to ∞ and $-\infty$ as **indefinite values**.

In other words:

- Saying $\sup(X) = \infty$ means that X is not bounded from above, i.e.,

$$\forall x_0 \in X \; \exists x \in X : x > x_0.$$

- Saying $\sup(X) = -\infty$ means that X is empty.
- Similarly, $\inf(X) = -\infty$ means that X is not bounded from below, and $\inf(X) = \infty$ means X is empty.

2.9 Consequences of Completeness

2.9.1 Archimedean Principle

The archimedean principle states that for every real number $x \in \mathbb{R}$ there exists and integers n greater than x. The following theorem, proved using the existence of suprema (and implicitly the completeness axiom), gives a precise formulation of this principle.

Theorem 2.38: Archimedean Principle

For every $x \in \mathbb{R}$ there exists exactly one $n \in \mathbb{Z}$ such that

$$n \leq x < n + 1.$$

Proof. We first treat the case $x \geq 0$. Fix $\mathbb{R} \ni x \geq 0$ and define

$$E = \{n \in \mathbb{Z} \mid n \leq x\}.$$

Since $0 \in E$ and x is an upper bound, E is a non-empty subset of $\mathbb{R}$ bounded from above. Hence, by Theorem 2.35, the supremum $s_0 = \sup(E)$ exists. From the definition of supremum we deduce:

(i) $s_0 \leq x$ (because x is an upper bound);

(ii) there exists $n_0 \in E$ with $s_0 - 1 < n_0$ (otherwise $s_0 - 1$ would also be an upper bound).

From (ii) we obtain $s_0 < n_0 + 1$, which implies

(iii) $n_0 + 1 \notin E$ (otherwise s_0 would not be an upper bound for E).

Moreover, since $m \le s_0$ for every $m \in E$, we have $m < n_0 + 1$ for all $m \in E$. As all elements of E are integers,

$$m < n_0 + 1 \quad \Leftrightarrow \quad m - n_0 < 1 \quad \Leftrightarrow \quad m - n_0 \le 0 \quad \Leftrightarrow \quad m \le n_0.$$

Thus, every $m \in E$ is less than or equal to n_0, and since $n_0 \in E$, we conclude that $n_0 = \max(E)$. In particular, by Remark 2.34, the maximum is also the supremum, so $s_0 = n_0$.

Finally, recalling (iii) and the definition of E, we have $n_0 + 1 > x$. Together with (i), this shows

$$n_0 = s_0 \le x < n_0 + 1,$$

establishing the claim for any $x \ge 0$.

Now, if $x < 0$, apply the previous argument to $-x > 0$. Then there exists $m \in \mathbb{Z}$ such that

$$m \le -x < m + 1,$$

which is equivalent to

$$-m - 1 < x \le -m.$$

If $x = -m$, then set $n = -m$. If $x < -m$, set $n = -m - 1$. In both cases, we obtain

$$n \le x < n + 1.$$

Finally, for uniqueness, assume that $n_1, n_2 \in \mathbb{Z}$ both satisfy $n_i \le x < n_i + 1$. From $n_1 \le x < n_2 + 1$ we deduce that $n_1 < n_2 + 1$, and therefore $n_1 \le n_2$. Reversing the roles of n_1 and n_2 gives $n_2 \le n_1$. Hence, $n_1 = n_2$. □

Definition 2.39: Integer and Fractional Parts

The **integer part** $\lfloor x \rfloor$ of $x \in \mathbb{R}$ is the integer $n \in \mathbb{Z}$ uniquely determined by Theorem 2.38 such that $n \le x < n + 1$. The map $x \mapsto \lfloor x \rfloor$ from $\mathbb{R}$ to $\mathbb{Z}$ is called the **rounding function**. The **fractional part** of x is defined as

$$\{x\} = x - \lfloor x \rfloor \in [0, 1).$$

Corollary 2.40: $\frac{1}{n}$ is Arbitrarily Small

For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$, with $n \ge 1$, such that

$$\frac{1}{n} < \varepsilon.$$

Proof. Applying Theorem 2.38 to $x = \frac{1}{\varepsilon} > 0$, we find $m \in \mathbb{Z}$ such that

$$m \le \frac{1}{\varepsilon} < m + 1.$$

Set $n := m + 1$. In this way we have $0 < \frac{1}{\varepsilon} < n$, which is equivalent to $n > 0$ (therefore, $n \ge 1$) and $\frac{1}{n} < \varepsilon$. □

Definition 2.41: Dense Sets

A subset $X \subseteq \mathbb{R}$ is called **dense** in $\mathbb{R}$ if every open non-empty interval contains an element of X.

Corollary 2.42: Density of $\mathbb{Q}$

For every $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Proof. Set $\varepsilon = b - a$. By Corollary 2.40, there exists $m \in \mathbb{N}$ with $\frac{1}{m} < \varepsilon$. Then, by Theorem 2.38 applied with $x = ma$, there exists $n \in \mathbb{Z}$ with

$$n \leq ma < n + 1,$$

or equivalently,

$$\frac{n}{m} \leq a < \frac{n+1}{m}.$$

Since $\frac{1}{m} < \varepsilon$, by the two inequalities above, we obtain

$$a < \frac{n+1}{m} \leq a + \frac{1}{m} < a + \varepsilon = b.$$

Thus $r = \frac{n+1}{m}$ is a rational number between a and b. $\square$

Corollary 2.43: Density of $\mathbb{R} \setminus \mathbb{Q}$

For every $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < r < b$.

Proof. We want to show that for every $x \in \mathbb{R}$ and $\delta > 0$, there exists an $a \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$a \in (x - \delta, x + \delta).$$

By Corollary 2.42, we find a $q \in \mathbb{Q}$ such that $q \in (x - \delta, x + \delta)$. By Corollary 2.40 we find an $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \frac{(x + \delta) - q}{\sqrt{2}} \quad \Rightarrow \quad \frac{\sqrt{2}}{N} < (x + \delta) - q.$$

This implies that

$$x - \delta < q < \frac{\sqrt{2}}{N} + q < x + \delta.$$

Choosing $r = \frac{\sqrt{2}}{N} + q$ proves the statement. $\square$

2.9.2 Uncountability

Definition 2.44: Cardinality

Let X and Y be sets.

- We say X and Y have the **same cardinality**, written $X \sim Y$, if there is a bijection $f : X \to Y$.
- We write $X \preceq Y$ if there exists an injection $f : X \to Y$.
- The empty set has cardinality 0.
- A set X has **finite cardinality** $|X| = n$ if there exists a bijection with $\{1, \dots, n\}$.
- A set is **infinite** it it is not finite.
- A set is **countable** if it has a bijection to $\mathbb{N}$. Its cardinality is denoted $\aleph_0$, pronounced Aleph-0.
- A set is **uncountable** if it is infinite but not countable.

If $X \preceq Y$ and $Y \preceq X$, then $X \sim Y$. In other words, if there exists an injective map $f : X \to Y$ and an injective map $g : Y \to X$, then one can find a bijective map $h : X \to Y$. This non-trivial statement is the **Schröder-Bernstein Theorem**.

We will now list some statements about different sets of numbers from the lecture:

1. $\mathbb{N}$ and the even numbers have the same cardinality.
2. $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.
3. $\mathbb{Q}$ is countable, i.e., $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{Z}$.

Proposition 2.45: Uncountability of $\mathbb{R}$

The set $\mathbb{R}$ is uncountable.

Extra Material

Definition 2.46: Power Set

Let X be a set. The **power set** $\mathcal{P}(X)$ of X is the set of all subsets of X, i.e.,

$$\mathcal{P}(X) := \{A \subseteq X\}.$$

Theorem 2.47: Cantor's Theorem

For any set X, the power set $\mathcal{P}(X)$ has strictly larger cardinality than X.

Proposition 2.48: The Reals have the same cardinality as $\mathcal{P}(\mathbb{N})$

$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

3 Sequences of Real Numbers

3.1 Convergence of Sequences

Definition 3.1: Sequences

A **sequence** is a function $a : \mathbb{N} \to \mathbb{R}$. The image $a(n)$ of $n \in \mathbb{N}$ is also written as a_n and is called the n-th element of a. Instead of $a : \mathbb{N} \to \mathbb{R}$ one often writes $(a_n)_{n\in\mathbb{N}}, (a_n)_{n=0}^{\infty}, (a_n)_{n\geq 0}$.

Definition 3.2: (Eventually) Constant Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is **constant** if $x_n = x_m \forall n, m \in \mathbb{N}$. It is **eventually constant** if there exists $N \in \mathbb{N}$ such that $x_n = x_m \forall n, m \geq N$.

Definition 3.3: Convergence of Sequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. We say that $(x_n)_{n=0}^{\infty}$ **converges** (or is **convergent**) if $\exists A \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} : |x_n - A| < \varepsilon \quad \forall n \geq N.$$

In this case we write

$$\lim_{n\to\infty} x_n = A \tag{3.1}$$

and call A the **limit** of $(x_n)_{n=0}^{\infty}$.

Lemma 3.4: Uniqueness of the Limit

A convergent sequence $(x_n)_{n=0}^{\infty}$ has exactly one limit.

Proof. Let $A, B \in \mathbb{R}$ be limits of $(x_n)_{n=0}^{\infty}$. Fix $\varepsilon > 0$. Then there exists $N_A, N_B \in \mathbb{N}$ such that $|x_n - A| < \varepsilon$ for all $n \geq N_A$ and $|x_n - B| < \varepsilon$ for all $n \geq N_B$. We define $N := \max\{N_A, N_B\}$. Then it holds that

$$|A - B| \leq |A - x_N| + |x_N - B| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $A = B$. □

3.2 Convergent Subsequences and Acculmulation Points

Definition 3.5: Subsequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence. A **subsequence** is of the form $(x_{n_k})_{k=0}^{\infty}$, where $(n_k)_{k=0}^{\infty}$ is a strictly increasing sequence of non-negative integers, i.e., $n_{k+1} > n_k \; \forall k \in \mathbb{N}$.

Remark 3.6. *Since $n_{k+1} > n_k$ for all $k \in \mathbb{N}$ it follows by induction that $n_k \geq k$ for all $k \in \mathbb{N}$.*

Proof. For $k = 0$ we have that $n_0 \geq 0$, because $(n_k)_{k=0}^{\infty}$ is a sequence of non-negative integers. So the condition is fulfilled. For the inductive step we want to show that the condition holds for $k+1$ under the assumption that the condition is true for k. Because $(n_k)_{k=0}^{\infty}$ is also a strictly increasing sequence, we have that $n_{k+1} > n_k \geq k$. Additionally since $n_k \in \mathbb{N}$, we have that $n_{k+1} \geq n_k + 1$. So it follows that $n_{k+1} \geq n_k + 1 \geq k + 1$, which proves the condition for $k + 1$. □

Lemma 3.7: Subsequences of Convergent Sequences are Convergent

Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Then every subsequence $(x_{n_k})_{k=0}^{\infty}$ also converges to A.

Proof. Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^{\infty}$ converges to A, there exists $N \in \mathbb{N}$ such that $|x_n - A| < \varepsilon \; \forall n \geq N$. As by Remark 3.6 we know that $n_k \geq k$ for all $k \in \mathbb{N}$. Therefore for all $k \geq N$ it holds that $|x_{n_k} - A| < \varepsilon$. ☐

Definition 3.8: Accumulation Points of Sequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. A point $A \in \mathbb{R}$ is an **accumulation point** of $(x_n)_{n=0}^{\infty}$ if

$$\forall \varepsilon > 0 \; \forall N \in \mathbb{N} \; \exists n \geq N : |x_n - A| < \varepsilon.$$

Proposition 3.9: Subsequences and Accumulation Points

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. A point $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^{\infty}$ if and only if there exists a convergent subsequence of $(x_n)_{n=0}^{\infty}$ with limit A.

Proof. First assume that $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^{\infty}$. We construct $(n_k)_{k\geq 0}$ recursively:

– first, apply the definition of accumulation point with $N = 1$ and $\varepsilon = 1 = 2^0$ to find $n_0 \geq 1$ with $|x_{n_0} - A| \leq 2^0$,

– second, apply the definition the definition of accumulation point with $N = n_0 + 1$ and $\varepsilon = 2^{-1}$ to find $n_1 \geq n_0 + 1$ with $|x_{n_1} - A| \leq 2^{-1}$,

– more in general given n_{k-1}, we apply the definition of accumulation point with $N = n_{k-1} + 1$ and $\varepsilon = 2^{-k}$ to find $n_k \geq n_{k-1} + 1$ with $|x_{n_k} - A| \leq 2^{-k}$.

Now given $\varepsilon > 0$ choose N such that $2^{-N} < \varepsilon$. Then for all $k \geq N$ we have that

$$|x_{n_k} - A| \leq 2^{-k} \leq 2^{-N} < \varepsilon,$$

so $\lim_{k\to\infty} x_{n_k} = A$.

Conversely, assume that there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ converging to A. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. Since $\lim_{k\to\infty} x_{n_k} = A$, there exists N_0 such that $|x_{n_k} - A| < \varepsilon$ for all $k \geq N_0$. Hence if we choose $k = \max\{N_0, N\}$, because $n_k \geq n$ (recall Remark 3.6) we have that $n_k \geq N$ and $|x_{n_k} - A| < \varepsilon$. Thus A is an accumulation point. ☐

Corollary 3.10: Infinitely Many Terms Near an Accumulation Point

If $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^{\infty}$, then for every $\varepsilon > 0$ there are infinitely many n with $x_n \in (A - \varepsilon, A + \varepsilon)$.

Proof. By Proposition 3.9, there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ with $\lim_{k\to\infty} = A$. Hence for every $\varepsilon > 0$ there exists K such that $x_{n_k} \in (A - \varepsilon, A + \varepsilon)$ for all $k \geq K$, providing infinitely many elements of the sequence inside the interval $(A - \varepsilon, A + \varepsilon)$. ☐

Corollary 3.11: Accumulation Points of Convergent Sequences

convergent sequence has exactly one accumulation point, namely its limit.

3.3 Addition, Multiplication and Inequalities

Proposition 3.12: Limits and Operations

Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be sequences converging to $A, B \in \mathbb{R}$ respectively. Then:

1. *The sequence $(x_n + y_n)_{n=0}^{\infty}$ converges to $A + B$.*
2. *The sequence $(x_n y_n)_{n=0}^{\infty}$ converges to AB.*
3. *Given $\alpha \in \mathbb{R}$, the sequence $(\alpha x_n)_{n=0}^{\infty}$ converges to αA.*
4. *Suppose $x_n \neq 0$ for all $n \in \mathbb{N}$ and $A \neq 0$. Then the sequence $(x_n^{-1})_{n=0}^{\infty}$ converges to A^{-1}.*

Proposition 3.13: Limits and Inequalities

Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be sequences converging to $A, B \in \mathbb{R}$ respectively.

1. *If $A < B$, then there exists $N \in \mathbb{N}$ such that $x_n < y_n$ for all $n \geq \mathbb{N}$.*
2. *If there exists $N \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \geq N$, then $A \leq B$.*

Remark 3.14. *In Propostion 3.13 even if we assume that $x_n < y_n$ for all $n \in \mathbb{N}$, we cannot conclude that $A < B$. for example take*

$$x_n = \frac{1}{n}, \quad y_n = \frac{1}{n}.$$

Then we have that $x_n < y_n$ for all $n \in \mathbb{N}$ but $A = B = 0$.

Lemma 3.15: Sandwich Lemma

Let $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty}$, $(z_n)_{n=0}^{\infty}$ be sequences such that for some $N \in \mathbb{N}$, we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N.$$

Suppose that both $(x_n)_{n=0}^{\infty}$ and $(z_n)_{n=0}^{\infty}$ converge to the same limit. Then $(y_n)_{n=0}^{\infty}$ also converges, and we have that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n.$$

Proof. Let $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty}$, $(z_n)_{n=0}^{\infty}$ be sequences such that for some $N_0 \in \mathbb{N}$, we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N_0.$$

Additionally suppose that $(x_n)_{n=0}^{\infty}$ and $(z_n)_{n=0}^{\infty}$ converge to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^{\infty}$, $(z_n)_{n=0}^{\infty}$ converge to A there exists $N_x, N_z \in \mathbb{N}$ such that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N_x$$
$$A - \varepsilon < z_n < A + \varepsilon \quad \forall n_z \geq N_z.$$

So we choose $N := \max\{N_0, N_x, N_z\}$. Then we have that

$$A - \varepsilon < x_n \le y_n \le z_n < A + \varepsilon \quad \forall n \ge N,$$

which shows that $\lim_{n\to\infty} y_n = A$. □

Definition 3.16: Bounded Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called **bounded** if there exists a real number $M \ge 0$ such that

$$|x_n| \le M \quad \forall n \in \mathbb{N}.$$

Lemma 3.17: Convergent Sequences are Bounded

Every convergent sequence is bounded.

Proof. Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Let $\varepsilon = 1$. Then, by convergence of $(x_n)_{n=0}^{\infty}$, there exists N such that $|x_n - A| \le 1$ for all $n \ge N$. So we have that

$$|x_n| = |x_n - A + A| \le |x_n - A| + |A| \le 1 + |A| \qquad \forall n \ge N.$$

We choose

$$M = \max(|x_0|, |x_1|, \ldots, |x_{N-1}|, 1 + |A|).$$

Then $|x_n| \le M$ for all $n \in \mathbb{N}$ as desired. □

Definition 3.18: Monotone Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called:

- **(monotonically) increasing** if $m > n \Rightarrow x_m \ge x_n$,
- **strictly (monotonically) increasing** if $m > n \Rightarrow x_m > x_n$,
- **(monotonically) decreasing** if $m > n \Rightarrow x_m \le x_n$,
- **strictly (monotonically) decreasing** if $m > n \Rightarrow x_m < x_n$.

If a sequence is decreasing or increasing we call it monotone. If a sequence is strictly increasing or strictly decreasing then we call it strictly monotone.

Remark 3.19. *An equivalent formulation of monotone sequences can be given using only successive terms:*

- $(x_n)_{n=0}^{\infty}$ *is increasing if* $x_{n+1} \ge x_n$ *for all* n*,*
- $(x_n)_{n=0}^{\infty}$ *is strictly increasing if* $x_{n+1} > x_n$ *for all* n*,*
- $(x_n)_{n=0}^{\infty}$ *is decreasing if* $x_{n+1} \le x_n$ *for all* n*,*
- $(x_n)_{n=0}^{\infty}$ *is strictly decreasing if* $x_{n+1} < x_n$ *for all* n*.*

Theorem 3.20: Convergence of Monotone Sequences

A monotone sequence $(x_n)_{n=0}^{\infty}$ converges if and only if it is bounded. More precisely, let $X = \{x_n \mid n \in \mathbb{N}\}$ denote the set of points in the sequence.

- *If $(x_n)_{n=0}^{\infty}$ is increasing, then $\lim_{n\to\infty} x_n = \sup(X)$,*
- *if $(x_n)_{n=0}^{\infty}$ decreasing, then $\lim_{n\to\infty} x_n = \inf(X)$.*

Proof. If $(x_n)_{n=0}^{\infty}$ converges Lemma 3.17 says that its bounded.

Conversely, let $(x_n)_{n=0}^{\infty}$ be a bounded monotone sequence. Wlog assume that $(x_n)_{n=0}^{\infty}$ is increasing (otherwise consider $(-x_n)_{n=0}^{\infty}$). Since $(x_n)_{n=0}^{\infty}$ is bounded from above, the set $X = \{x_n \mid n \in \mathbb{N}\}$ has a supremum, that we'll call $A = \sup(X)$.

By definiton of A:

(i) $x_n \leq A \quad \forall n \in \mathbb{N}$,

(ii) $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_N > A - \varepsilon$.

Then, for all $n \geq N$ using (ii) and monotonicity, we have that $x_n \geq x_N > A - \varepsilon$. Then using (i), we conclude that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N.$$

□

3.4 Superior and Inferior Limits

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence. To study its behavior for large n its is useful to look at its tails

$$X_{\geq n} = \{x_k \mid k \geq n\} \subseteq \mathbb{R}.$$

The concept of limits can be restated using the tails of a sequence, i.e., the sequence $(x_n)_{n=0}^{\infty}$ converges to $A \in \mathbb{R}$ if and only if, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $X_N \subseteq (A - \varepsilon, A + \varepsilon)$.

However, since not every sequence has a limit we now introduce a related notion (the **superior** and **inferior limits**), which always exist for bounded sequences.

For each $n \in \mathbb{N}$, define

$$s_n = \sup(X_{\geq n}) = \sup_{k \geq n} x_k, \qquad i_n = \inf(X_{\geq n}) = \inf_{k \geq n} x_k.$$

Since $X_{\geq m} \subset X_{\geq n}$, whenever $m > n$, we have that

$$i_n \leq i_m \leq s_m \leq s_n \qquad \forall m > n.$$

Thus, $(s_n)_{n=0}^{\infty}$ is a monotonically decreasing sequence, while $(i_n)_{n=0}^{\infty}$ is a monotonically increasing sequence. Moreover, since $(x_n)_{n=0}^{\infty}$ is bounded both $(s_n)_{n=0}^{\infty}$ and $(i_n)_{n=0}^{\infty}$ are bounded as well. Hence by Theorem 3.20, both sequences converge. Their limits will be called the *superior* and the *inferior limit* of $(x_n)_{n=0}^{\infty}$ respectively.

Note that, since $x_n \in X_{\geq n}$, we have that

$$i_n \leq x_n \leq s_n \qquad \forall n \in \mathbb{N}. \tag{3.2}$$

Definition 3.21: Superior and Inferior Limits

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence in $\mathbb{R}$. The numbers

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\big(\sup_{k\geq n} x_k\big), \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty}\big(\inf_{k\geq n} x_k\big)$$

are called the **superior** and **inferior limit** of $(x_n)_{n=0}^{\infty}$ respectively. From Equation 3.2 and Proposition 3.13, we have

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n.$$

Lemma 3.22: Convergence and Superior/Inferior Limits

A bounded sequence $(x_n)_{n=0}^{\infty}$ in $\mathbb{R}$ converges if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n.$$

Proof. For every $n \in \mathbb{N}$, define

$$i_n = \inf_{k\geq n} x_k, \qquad s_n = \sup_{k\geq n} x_k,$$

and set

$$I = \lim_{n\to\infty} i_n = \liminf_{n\to\infty} x_n, \qquad S = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} x_n.$$

First suppose that $I = S$. Since $i_n \leq x_n \leq s_n$ (see Equation 3.2), the Sandwich Lemma 3.15 implies that the sequence $(x_n)_{n=0}^{\infty}$ converges, and its limit equals $I = S$.

Conversely, assume that $(x_n)_{n=0}^{\infty}$ converges to $A \in \mathbb{R}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$A - \varepsilon < x_n < A + \varepsilon \qquad \forall n \geq N.$$

Then for all $n \geq N$, the same inequalities holds for i_n and s_n, i.e.,

$$A - \varepsilon \leq i_n \leq s_n \leq A + \varepsilon.$$

Taking limits and using Proposition 3.13, we obtain

$$A - \varepsilon \leq I \leq S \leq A + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $A = I = S$, which proves the result. □

Theorem 3.23: Superior and Inferior Limits are Accumulation Points

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence and let $A = \limsup_{n\to\infty} x_n$. Then A is an accumulation point of $(x_n)_{n=0}^{\infty}$, and for every $\varepsilon > 0$ the following hold:

1. *only finitely many elements satisfy $x_n \geq A + \varepsilon$;*
2. *infinitely many elements satisfy $A - \varepsilon < x_n < A + \varepsilon$.*

An analogous statement holds for the inferior limit.

Proof. Since the sequence $(s_n)_{n=0}^{\infty}$ is monotonically decreasing and converges to A, given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$A \le s_n < A + \varepsilon \qquad \forall n \ge N_0. \tag{3.3}$$

We first prove that A is an accumulation point.

Fix $N \in \mathbb{N}$ and set $N_1 = \max\{N, N_0\}$. Since $s_{N_1} = \sup_{k \ge N_1} x_k$, there exists $n_1 \ge N_1 \ge N_0$ such that

$$s_{N_1} - \varepsilon < x_{n_1} \le s_{N_1}.$$

Thus, combining this bound with Equation 3.3 we obtain

$$A - \varepsilon < s_{N_1} - \varepsilon < x_{n_1} \le s_{N_1} < A + \varepsilon.$$

This construct shows that for any $\varepsilon > 0$ and any $N \in \mathbb{N}$, there exists $n_1 \ge N$ such that $A - \varepsilon < x_{n_1} < A + \varepsilon$. Thus A is an accumulation point for $(x_n)_{n=0}^{\infty}$.

We now prove1. and 2.. From Equation 3.3 we have $x_n < A + \varepsilon$ for all $n \ge N_0$, so only finitely many terms satisfy $x_n \ge A + \varepsilon$. This shows 1..

Also since A is an accumulation point, it follows from Corollary 3.10 that infinitely many terms of the sequence lie within any interval $(A - \varepsilon, A + \varepsilon)$. □

Corollary 3.24: Bounded Sequences have Convergent Subsequences

Every bounded sequence has at least one accumulation point and therefore possesses a convergent subsequence.

Proof. By Theorem 3.23, the number

$$A = \limsup_{n \to \infty} x_n$$

is always an accumulation point of $(x_n)_{n=0}^{\infty}$. Moreover, by Proposition 3.9, every accumulation point is the limit of a convergent subsequence. Hence every bounded sequence admits at least one convergent subsequence. □

3.5 Cauchy Sequences

Definition 3.25: Cauchy Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called a **Cauchy sequence** if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \qquad \forall n, m \ge N.$$

Lemma 3.26: Cauchy Sequences are Bounded

Every Cauchy sequence is bounded.

Proof. By definition, there exists $N \in \mathbb{N}$ such that

$$|x_n - x_N| \le 1 \qquad \forall n \ge N.$$

Hence, for $n \ge N$, we have $|x_n| \le 1 + |x_N|$. Now, define

$$M = \max\{|x_0|, |x_1|, \ldots, |x_{N-1}|, 1 + |x_N|\}.$$

Then, $|x_n| \le M$ for all $n \in \mathbb{N}$, so $(x_n)_{n=0}^{\infty}$ is bounded. □

Theorem 3.27: Convergence and Cauchy Sequences

A sequence $(x_n)_{n=0}^{\infty}$ of real numbers converges if and only if it is a Cauchy sequence.

Proof. Suppose first that $(x_n)_{n=0}^{\infty}$ converges to some $A \in \mathbb{R}$, and let us prove that $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|x_n - A| < \frac{\varepsilon}{2} \qquad \forall n \ge N.$$

Then for all $n, m \ge N$, we have that

$$|x_n - x_m| \le |x_n - A| + |x_m - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence.

Viceversa, let $(x_n)_{n=0}^{\infty}$ be a Cauchy sequence. Since it is bounded (by Lemma 3.26), Corollary 3.24 implies that there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ converging to some $A \in \mathbb{R}$. Given $\varepsilon > 0$, choose $N_0 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \qquad \forall n, m \ge N_0,$$

and choose $N_1 \in \mathbb{N}$ such that

$$|x_{n_k} - A| < \frac{\varepsilon}{2} \qquad \forall k \ge N_1.$$

Let $N = \max\{N_0, N_1\}$. Since $n_N \ge N$ (see Remark 3.6), for all $n \ge N$ we have

$$|x_n - A| \le |x_n - x_{n_N}| + |x_{n_N} - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $(x_n)_{n=0}^{\infty}$ converges to A. □

3.6 Improper Limits

We now extend the notion of limit to allow the **improper limit values** $+\infty$ (often abbreviated as ∞) and $-\infty$.

Definition 3.28: Improper Limits

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. We say $(x_n)_{n=0}^{\infty}$ **diverges to** $+\infty$, and we write

$$\lim_{n \to \infty} x_n = +\infty,$$

if for every $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n > M$ for all $n \ge N$.
Similarly, $(x_n)_{n=0}^{\infty}$ **diverges to** $-\infty$ if for every $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n < -M$ for all $n \ge N$. In both cases, we say that $(x_n)_{n=0}^{\infty}$ has an **improper limit**.

An unbounded sequence doesn't need to diverge to $+\infty$ or $-\infty$. For instance, the sequence $x_n = (-1)^n n$, is unbounded but neither diverges to $+\infty$ nor to $-\infty$.

The notion of improper limit allows us to extend the definitions of superior and inferior limits to *unbounded* sequences. If $(x_n)_{n=0}^{\infty}$ is not bounded from above, then

$$\sup_{k\geq n} x_k = +\infty \qquad \forall n \in \mathbb{N},$$

and we write

$$\limsup_{n\to\infty} x_n = +\infty.$$

If $(x_n)_{n=0}^{\infty}$ is bounded from above but not from below, then we define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} (\sup_{k\geq n} x_k),$$

where the right-hand side is a real limit if the decreasing sequence $\sup_{k\geq n} x_k$ is bounded, and the improper limit $-\infty$ otherwise. The definition of the inferior limit extends analogously.

3.7 Sequences of Complex Numbers

Informally, a **sequence of complex numbers** is just like a sequence of real numbers, except that each term is a complex number instead of a real one. Thus, we study ordered lists $(z_0, z_1, \ldots)$, where $z_n : \mathbb{N} \to \mathbb{C}$. As in the real case, we are mainly interested in their convergence, divergence and limit behavior.

To analyze sequences in $\mathbb{C}$, it is often sufficient to consider separately the corresponding sequences of real and imaginary parts in $\mathbb{R}$.

Definition 3.29: Sequences of Complex Numbers

A sequence of complex numbers $(z_n)_{n=0}^{\infty}$, where

$$z_n = x_n + iy_n,$$

is said to **converge** to a limit $A + iB \in \mathbb{C}$ if the two sequences of real numbers $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ converge to A and B, respectively. In this case, we write

$$\lim_{n\to\infty} z_n = A + iB.$$

We say that $(z_n)_{n=0}^{\infty}$ **diverges to** ∞ if the sequence of moduli $(|z_n|)_{n=0}^{\infty}$ diverges to $+\infty$, i.e.,

$$\lim_{n\to\infty} |z_n| = \lim_{n\to\infty} \sqrt{x_n^2 + y_n^2} = +\infty.$$

Remark 3.30. *As for sequences of real numbers, one can consider subsequences of sequences* $\mathbb{C}$*. Given a strictly increasing sequence of non-negative integers* $(n_k)_{k=0}^{\infty}$*, the corresponding subsequence is*

$$(z_{n_k})_{k=0}^{\infty} = (x_{n_k} + iy_{n_k})_{n=0}^{\infty}.$$

4 Functions of one Real Variable

In this chapter we study real-valued functions defined on subsets of $\mathbb{R}$, typically intervals. The central concept is *continuity.*

4.1 Real valued functions

4.1.1 Boundedness and Monotonitcity

For a non-empty set $D \subseteq \mathbb{R}$, the set of **real-valued** functions on D is

$$\mathcal{F}(D) = \{f \mid f : D \to \mathbb{R}\}.$$

For $f_1, f_2 \in \mathcal{F}(D)$, $\alpha \in \mathbb{R}$, and $x \in D$ we define

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \qquad (\alpha f_1)(x) = \alpha f_1(x), \qquad (f_1 f_2)(x) = f_1(x) f_2(x).$$

Given $\alpha \in \mathbb{R}$, we write $f \equiv \alpha$ for the constant function $x \mapsto \alpha$ on D.

Remark 4.1. *With the operations above, $\mathcal{F}(D)$ is a commutative ring (the additive identity is $f \equiv 0$ and the multiplicative identity is $f \equiv 1$).*

A point $x \in D$ is a **zero** of $f \in \mathcal{F}(D)$ if $f(x) = 0$. The **zero set** of f is $\{x \in D \mid f(x) = 0\}$. We order $\mathcal{F}(D)$ pointwise: for $f_1, f_2 \in \mathcal{F}(D)$,

$$\begin{aligned} f_1 \leq f_2 \quad &\Leftrightarrow \quad f_1(x) \leq f_2(x) \quad \forall x \in D, \\ f_1 < f_2 \quad &\Leftrightarrow \quad f_1(x) < f_2(x) \quad \forall x \in D. \end{aligned}$$

We say that $f \in \mathcal{F}(D)$ is **non-negative** if $f \geq 0$, and **positive** if $f > 0$.

Definition 4.2: Bounded Functions

Let $D \neq \emptyset$ and $f : D \to \mathbb{R}$. We say that f is **bounded from above** if there exists $M > 0$ such that

$$f(x) \leq M \qquad \forall x \in D.$$

We say that f is **bounded from below** if there exists $M > 0$ such that

$$f(x) \geq -M \qquad \forall x \in D.$$

We say that f is **bounded** if it is both bounded from above and from below. Equivalently, f if bounded if there exists $M > 0$ such that

$$|f(x)| \leq M \qquad \forall x \in D.$$

Definition 4.3: Monotone Functions

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. The function f is:

1. **increasing** if $x < y \quad \Rightarrow \quad f(x) \leq f(y) \quad \forall x, y \in D$;
2. **strictly increasing** if $x < y \quad \Rightarrow \quad f(x) < f(y) \quad \forall x, y \in D$;
3. **decreasing** if $x < y \quad \Rightarrow \quad f(x) \geq f(y) \quad \forall x, y \in D$;
4. **strictly decreasing** if $x < y \quad \Rightarrow \quad f(x) > f(y) \quad \forall x, y \in D$.

We call f **monotone** if it is increasing or decreasing, and **strictly monotone** if it is strictly increasing or strictly decreasing.

4.1.2 Continuity

Definition 4.4: Continuous Functions

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. We say that f is **continuous at** $x_0 \in D$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

We say that f is **continuous on** D if it is continuous at every point of D.

Remark 4.5. *It suffices to verify the implication above for small ε. Precisely, assume there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ there is a $\delta > 0$ such that*

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

Then f is continuous at x_0.

Indeed, for $\varepsilon_0 > \varepsilon$ we can choose the number $\delta > 0$ corresponding to ε to get

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon < \varepsilon_0.$$

In other words, if δ works for ε, then it works for all $\varepsilon_0 > \varepsilon$.

Definition 4.6: Restriction

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. For any $D' \subseteq D$ the **restriction** of f to D' is the function $f|_{D'} : D' \to \mathbb{R}$ defined by

$$f|_{D'}(x) = f(x) \qquad \forall x \in D'.$$

We regard $f|_{D'}$ and f as different functions unless $D' = D$.

Proposition 4.7: Combination of Continuous Functions

Let $D \subseteq \mathbb{R}$, and let $f_1, f_2 : D \to \mathbb{R}$ be continuous at $x_0 \in D$. Then $f_1 + f_2$, $f_1 f_2$, and αf_1 (for any $\alpha \in \mathbb{R}$) are continuous at x_0.

Proof. We first prove the result for the sum. Let $\varepsilon > 0$. Since f_1 and f_2 are continuous at x_0, there exists $\delta_1, \delta_2 > 0$ such that for all $x \in D$,

$$|x - x_0| < \delta_1 \;\Rightarrow\; |f_1(x) - f_1(x_0)| < \frac{\varepsilon}{2}, \quad |x - x_0| < \delta_2 \;\Rightarrow\; |f_2(x) - f_2(x_0)| < \frac{\varepsilon}{2}.$$

So, choosing $\delta = \min \delta_1, \delta_2$, for $|x - x_0| < \delta$ we get

$$|(f_1 + f_2)(x) - (f_1 + f_2)(x_0)| \leq |f_1(x) - f_1(x_0)| + |f_2(x) - f_2(x_0)| < \varepsilon,$$

which shows that $f_1 + f_2$ is continuous at x_0.

For the product, note that

$$\begin{aligned} |f_1(x)f_2(x) - f_1(x_0)f_2(x_0)| &= |f_1(x)f_2(x) - f_1(x_0)f_2(x) + f_1(x_0)f_2(x) - f_1(x_0)f_2(x_0)| \\ &\leq |f_1(x)f_2(x) - f_1(x_0)f_2(x)| + |f_1(x_0)f_2(x) - f_1(x_0)f_2(x_0)| \\ &= |f_2(x)||f_1(x) - f_1(x_0)| + |f_1(x_0)||f_2(x) - f_2(x_0)|. \end{aligned}$$

Now, first choose $\delta_0 > 0$ such that $|x - x_0| < \delta_0$ implies $|f_2(x) - f_2(x_0)| < 1$, so that

$$|x - x_0| < \delta_0 \quad \Rightarrow \quad |f_2(x)| < 1 + |f_2(x_0)|.$$

Then choose $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned}
|x - x_0| < \delta_1 \quad &\Rightarrow \quad |f_1(x) - f_1(x_0)| < \frac{\varepsilon}{2(1 + |f_2(x_0)|)}, \\
|x - x_0| < \delta_2 \quad &\Rightarrow \quad |f_2(x) - f_2(x_0)| < \frac{\varepsilon}{2(1 + |f_1(x_0)|)}.
\end{aligned}$$

So choosing $\delta = \min \delta_0, \delta_1, \delta_2$, for $|x - x_0| < \delta$ we get

$$\begin{aligned}
|f_1(x)f_2(x) - f_1(x_0)f_2(x_0)| &< |f_2(X)| \frac{\varepsilon}{2(1 + |f_2(x_0)|)} + |f_1(x_0)| \frac{\varepsilon}{2(1 + |f_1(x_0)|)} \\
&< (1 + |f_2(x_0)|) \frac{\varepsilon}{2(1 + |f_2(x_0)|)} + |f_1(x_0)| \frac{\varepsilon}{2(1 + |f_1(x_0)|)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

thus $f_1 f_2$ is continuous at x_0.

Finally, the statement about αf_1 follows by choosing $f_2 \equiv \alpha$ (a constant function) and using the product case proved above: since f_1 and f_2 are continuous at x_0, their product $f_1 f_2 = \alpha f_1$ is continuous at x_0. □

Definition 4.8: Sum and Product Notation

Let $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n \in \mathbb{R}$. We use the notation

$$\sum_{j=0}^{n} a_j = a_0 + a_1 + \ldots + a_n, \qquad \prod_{j=0}^{n} a_0 \cdot a_1 \cdot \ldots \cdot a_n.$$

Here a_j is a **summand** in the sum and a **factor** in the product; j is the **index** (or **running variable**). If J is a finite set and numbers $(a_j)_{j \in J}$ are given, we write

$$\sum_{j \in J} a_j, \qquad \prod_{j \in J} a_j.$$

By convention, for the empty index set $\emptyset$,

$$\sum_{j \in \emptyset} a_j = 0, \qquad \prod_{j \in \emptyset} a_j = 1.$$

Proposition 4.9: Composition of Continuous Functions

Let $D_1, D_2 \subseteq \mathbb{R}, x_0 \in D_1$ and $f : D_1 \to D_2$ be continuous at x_0. If $g : D_2 \to \mathbb{R}$ is continuous at $f(x_0)$, then $g \circ f : D_1 \to \mathbb{R}$ is continuous at x_0. In particular, the composition of continuous functions is continuous.

Proof. Let $\varepsilon > 0$. By continuity of g at $f(x_0)$, there exists $\eta > 0$ such that

$$\forall y \in D_2, \quad |y - f(x_0)| < \eta \quad \Rightarrow \quad |g(y) - g(f(x_0))| < \varepsilon.$$

By continuity of f at x_0, there exists $\delta > 0$ such that

$$\forall x \in D_1, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \eta.$$

Combining the implications gives, for any $x \in D_1$,

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \eta \quad \Rightarrow \quad |g(f(x)) - g(f(x_0))| < \varepsilon. \qquad \square$$

Remark 4.10. *Applying Proposition 4.9 with $g(y) = |y|$, we see that if $f : D \to \mathbb{R}$ is continuous, then $x \mapsto |f(x)|$ is continuous.*

4.1.3 Sequential Continuity

Definition 4.11: Notation for Limits of Sequences

Let $(x_n)_{n=0}^{\infty} \subseteq \mathbb{R}$ and $\bar{x} \in \mathbb{R}$. We write

$$x_n \to \bar{x} \qquad \text{or} \qquad x_n \underset{n\to\infty}{\longrightarrow} \bar{x}$$

to mean

$$\lim_{n\to\infty} x_n = \bar{x}.$$

Theorem 4.12: Continuity = Sequential Continuity

Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and $\bar{x} \in D$. Then f is continuous at $\bar{x}$ if and only if for every sequence $(x_n)_{n=0}^{\infty} \subseteq D$ with $x_n \to \bar{x}$ we have $f(x_n) \to f(\bar{x})$.

Proof. '$\Rightarrow$': First Assume that f is continuous at $\bar{x}$. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall x \in D, \quad |x - \bar{x}| < \delta \quad \Rightarrow \quad |f(x) - f(\bar{x})| < \varepsilon.$$

Also, since $x_n \to \bar{x}$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad |x_n - \bar{x}| < \delta.$$

Thus,

$$n \geq N \quad \Rightarrow \quad |f(x_n) - f(\bar{x})| < \varepsilon,$$

which implies that the sequence $(f(x_n))_{n=0}^{\infty}$ converges to $f(\bar{x})$.

'$\Leftarrow$': To prove the converse, assume that f is not continuous at x_0. This means that there exists $\varepsilon > 0$ such that, for every $\delta > 0$, there is $x \in D$ with

$$|x - \bar{x}| < \delta \quad \text{and} \quad |f(x) - f(\bar{x})| \geq \varepsilon.$$

Now, for every $n \in \mathbb{N}$, we apply this property with $\delta = 2^{-n}$ to find a point $x_n \in D$ such that

$$|x_n - \bar{x}| < 2^{-n} \quad \text{and} \quad |f(x_n) - f(\bar{x})| \geq \varepsilon$$

Then the sequence constructed in this way satisfies $x_n \to \bar{x}$ but $f(x_n) \not\to f(\bar{x})$. $\square$

Remark 4.13. *The proof above shows that if $f : D \to \mathbb{R}$ is not continuous at $\bar{x}$, then there exists $\varepsilon > 0$ and a sequence $(x_n)_{n=0}^{\infty} \subseteq D$ with $x_n \to \bar{x}$ such that $|f(x_n) - f(\bar{x})| \geq \varepsilon$ for all $n \in \mathbb{N}$. This is useful to show that a function f is not continuous at $\bar{x}$.*

4.2 Continuous Functions

4.2.1 Intermediate Value Theorem

In this section we prove a fundamental theorem that formalizes the idea that the graph of a continuous function on an interval is a continuous curve, and thus cannot make any jumps.

Theorem 4.14: Intermediate Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) \leq f(b)$. Then, for every real number c with $f(a) \leq c \leq f(b)$, there exists $\bar{x} \in [a, b]$ such that $f(\bar{x}) = c$.

Proof. Fix $c \in [f(a), f(b)]$. Then define

$$X = \{x \in [a, b] \mid f(x) \leq c\}.$$

Since $a \in X$ and $X \subseteq [a, b]$, the set is non-empty and bounded from above. By Theorem 2.35, its supremum

$$\bar{x} = sup(X) \in [a, b]$$

exists. We now use the continuity of f at x_0 to show that $f(\bar{x}) = c$.

Since $\bar{x}$ is the supremum of X, for each $n \geq 0$, we can find a point $x_n \in [\bar{x} - 2^{-n}, \bar{x}]$. Then $|x_n - \bar{x}| \leq 2^{-n}$, hence $x_n \longrightarrow \bar{x}$. Also, by the definition of X, we have $f(x_n) \leq c$. Thus, by Theorem 4.12 (continuity of f along sequences),

$$\lim_{n\to\infty} f(x_n) = f(\bar{x}).$$

And Proposition 3.13 yields $\lim_{n\to\infty} f(x_n) \leq c$. Therefore, $f(\bar{x}) \leq c$.

Suppose, by contradiction, $f(\bar{x}) < c$ and set $\varepsilon := c - f(\bar{x}) > 0$. By continuity at $\bar{x}$, there exists $\delta > 0$ such that for all $x \in [a, b]$

$$|x - \bar{x}| < \delta \quad \Rightarrow \quad |f(x) - f(\bar{x})| < \varepsilon,$$

hence $f(x) < f(\bar{x}) + \varepsilon = c$. Therefore, by the definition of X,

$$(\bar{x} - \delta, \bar{x} + \delta) \cap [a, b] \subseteq X.$$

Moreover, since $f(\bar{x}) < c \leq f(b)$, we cannot have $\bar{x} = b$; hence $\bar{x} < b$. Because $\bar{x} < b$, the interval $(\bar{x}, \bar{x} + \delta) \cap [a, b] \subseteq X$ is non-empty. Pick

$$y \in (\bar{x}, \bar{x} + \delta) \cap [a, b] \subseteq X.$$

Then $y \in X$ and $y > \bar{x}$, which contradicts the defining property of the supremum: $\bar{x}$ is an upper bound of X, and X cannot contain elements larger than $\bar{x}$. This contradiction shows that $f(\bar{x}) \geq c$. Together with $f(\bar{x}) \leq c$ proved above, we conclude that $f(\bar{x}) = c$, as desired. $\square$

Theorem 4.15: Inverse Function Theorem

Let I be an interval and $f : I \to \mathbb{R}$ a continuous strictly monotone function. Then $f(I)$ is an interval, and the mapping $f : I \to f(I)$ has a continuous strictly monotone inverse function $f^{-1} : f(I) \to I$.

Proof. We may assume that I is non-empty and not a single point. Also, w.l.o.g, suppose f is strictly increasing (otherwise replace f with $-f$).

Let $J = f(I)$. Since f is strictly monotone it is injective. Also, since by definition $J = f(I)$, it is surjective, hence bijective. Therefore there exists a unique inverse $g = f^{-1} : J \to I$.

Because f is strictly increasing, we have

$$x_1 < x_2 \quad \Leftrightarrow \quad f(x_1) < f(x_2) \qquad \forall x_1, x_2 \in I. \tag{4.1}$$

(Note: here we have equivalence in the statements because f is both injective and strictly increasing) Defining $y_1 = f(x_1)$ and $y_2 = f(x_2)$, this is equivalent to

$$y_1 < y_2 \quad \Leftrightarrow \quad g(y_1) < g(y_2) \qquad \forall y_1, y_2 \in J$$

Thus, g is strictly increasing.

To show that J is an interval, $y_1, y_2 \in J$, and assume w.l.o.g that $y_1 < y_2$. Since, $J = f(I)$, Equation 4.1 implies that $y_1 = f(x_1), y_2 = f(x_2)$ for some $x_1, x_2 \in I$ with $x_1 < x_2$. Now by the Intermediate Value Theorem 4.14 applied to $f : [x_1, x_2] \to \mathbb{R}$, we have that all values $c \in [y_1, y_2]$ are in the image of $f : [x_1, x_2] \to \mathbb{R}$, i.e.,

$$[y_1, y_2] \subseteq f([x_1, x_2]) \subseteq J.$$

Since, y_1, y_2 were two arbitrary points in J, this proves that J is an interval.

It remains to show that $g = f^{-1}$ is continuous. Fix $\bar{y} \in J$ and suppose, by contradiction, that g is not continuous at $\bar{y}$. Then by Remark 4.13, there exists $\varepsilon > 0$ and a sequence $(y_n)_{n=0}^{\infty} \subseteq J$ such that

$$y_n \longrightarrow \bar{y} \qquad \text{but} \qquad |g(y_n) - g(\bar{y})| \geq \varepsilon \quad \forall n \in \mathbb{N}. \tag{4.2}$$

Set $x_n = g(y_n) \in I$ and $\bar{x} = g(\bar{y}) \in I$. Then for every $n \in \mathbb{N}$, either $x_n \leq \bar{x} - \varepsilon$ or $x_n \geq \bar{x} + \varepsilon$. In particular, at least one of these cases must occur infinitely often. W.l.o.g, assume $x_n \leq \bar{x} - \varepsilon$ for infinitely many n, and extract a subsequence $(x_{n_k})_{k=0}^{\infty}$ with $x_{n_k} \leq \bar{x} - \varepsilon$ for all k. Since, I is an interval, $\bar{x} - \varepsilon \in I$, and by strict monotonicity of f we obtain

$$y_{n_k} = f(x_{n_k}) \leq f(\bar{x} - \varepsilon) < f(\bar{x}) = \bar{y}.$$

Then Proposition 3.13 gives (recall $y_n \longrightarrow \bar{y}$, see 4.2)

$$\bar{y} = \lim_{k \to \infty} y_{n_k} \leq f(\bar{x} - \varepsilon) < f(\bar{x}) = \bar{y},$$

a contradiction. Hence, g is continuous. □

4.3 Continuous Functions on Compact Intervals

In this section we show that continuous functions on **bounded closed** intervals, called **compact intervals**, enjoy special properties.

4.3.1 Boundedness and Extrema

Lemma 4.16: Compactness

Let $[a, b]$ be a compact interval, and let $(x_n)_{n=0}^{\infty}$ be a sequence contained in $[a, b]$. Then there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ such that

$$\lim_{k\to\infty} x_{n_k} = \bar{x} \qquad \textit{for some } \bar{x} \in [a, b].$$

Proof. Since $(x_n)_{n=0}^{\infty}$ is bounded (as it lies in $[a, b]$), Corollary 3.24 ensures the existence of a convergent subsequence $(x_{n_k})_{k=0}^{\infty}$. Let $\bar{x}$ denote its limit. Because $a \le x_{n_k} \le b$ for all k, Proposition 3.13 yields $a \le \bar{x} \le b$. □

Theorem 4.17: Boundedness

Let $[a, b]$ be compact interval, and let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is bounded.

Proof. Assume by contradiction that f is unbounded. Then, for every $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|f(x_n)| \ge n$. By Lemma 4.16, there is a subsequence $(x_{n_k})_{k=0}^{\infty}$ converging to some $\bar{x} \in [a, b]$.

Since f is continuous, so is $|f|$ (recall Remark 4.10), therefore $|f(x_{n_k})| \longrightarrow |f(\bar{x})| \in \mathbb{R}$. This contradicts $|f(x_{n_k})| \ge n_k \longrightarrow \infty$, so f must be bounded. □

Exercise 4.18. *Find examples of:*

1. *a continuous but unbounded function on a bounded open interval.*

$$f : (0, 1) \to \mathbb{R},\ x \mapsto \frac{1}{x}.$$

2. *a continuous but unbounded function on an unbounded closed interval.*

$$f : [0, \infty) \to \mathbb{R},\ x \mapsto x.$$

3. *an unbounded function on a compact interval but discontinuous at only one point.*

$$f : [0, 1] \to \mathbb{R},\ x \mapsto \begin{cases} \frac{1}{x}, & \textit{for } x \neq 0 \\ a \in \mathbb{R}, & \textit{for } x = 0. \end{cases}$$

Definition 4.19: Extreme Values

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$.

- We say that f takes its **maximum value** at $x_0 \in D$ if $f(x) \le f(x_0)$ for all $x \in D$. Then $f(x_0)$ is the **maximum** of f.
- We say that f takes its **minimum value** at $x_0 \in D$ if $f(x) \ge f(x_0)$ for all $x \in D$. Then $f(x_0)$ is the **minimum** of f.

Maxima and minima ar called **extreme values** or **extrema**.

Theorem 4.20: Extreme Value Theorem

Let $[a,b]$ be a compact interval, and let $f : [a,b] \to \mathbb{R}$ be continuous. Then f attains both its minimum and its maximum.

Proof. Theorem 4.17 guarantees that f is bounded, or equivalently, that $f([a,b]) \subseteq \mathbb{R}$ is a bounded subset of $\mathbb{R}$. Thus, Theorem 2.35 implies that

$$S := \sup f([a,b])$$

exists. By definition of the supremum, for each $n \in \mathbb{N}$ there exists $y_n \in f([a,b])$ such that $S - 2^{-n} \leq y_n \leq S$. Hence, $y_n \to S$. Also, since $y_n \in f([a,b])$, there exists $x_n \in [a,b]$ such that $f(x_n) = y_n$.

Now, by Lemma 4.16, we can find a subsequence $(x_{n_k})_{k=0}^{\infty}$ such that $x_{n_k} \to \bar{x} \in [a,b]$. By continuity of f, we have that

$$f(\bar{x}) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = S,$$

so f attains its maximum at $\bar{x}$.

Applying the same reasoning to $-f$ shows that f also attains its minimum. □

4.3.2 Uniform Continuity

Definition 4.21: Uniform Continuity

Let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ is **uniformly continuous** if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon \qquad \forall x, y \in D.$$

Remark 4.22. *The difference between the usual definition of continuity and the one of uniform continuity lies in how the choice of δ depends on the points considered.*

*For a function that is continuous at each $x_0 \in D$, the δ in the definition may depend on both ε **and** x_0: for every $\varepsilon > 0$ and each x_0, we can find a $\delta = \delta(\varepsilon, x_0)$ that works near x_0.*

*Uniform continuity is stronger: there exists a single $\delta = \delta(\varepsilon)$ that works **simultaneously** for all $x, y \in D$. In other words, the control on the variation of f does not deteriorate as we move along the domain. This property is automatically satisfied on compact intervals for continuous functions, as we will prove below.*

Theorem 4.23: Uniform Continuity on Compact Intervals

Let $[a,b]$ be a compact interval, and $f : [a,b] \to \mathbb{R}$ continuous on $[a,b]$. Then f is uniformly continuous.

Proof. Assume, by contradiction, that f is not uniformly continuous on $[a,b]$. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$ one can find $x, y \in [a,b]$ with

$$|x - y| < \delta \quad \text{and} \quad |f(x) - f(y)| \geq \varepsilon.$$

Taking $\delta = 2^{-n}$ for each $n \in \mathbb{N}$, we obtain sequences $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ in $[a,b]$ with

$$|x_n - y_n| < 2^{-n} \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \varepsilon. \tag{4.3}$$

By Lemma 4.16, the sequence $(x_n)_{n=0}^{\infty}$ has a subsequence $(x_{n_k})_{k=0}^{\infty}$ converging to some $\bar{x} \in [a,b]$. Then

$$|y_{n_k} - \bar{x}| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - \bar{x}| < 2^{-n_k} + |x_{n_k} - \bar{x}| \underset{k\to\infty}{\longrightarrow} 0,$$

so $y_{n_k} \to \bar{x}$ as well. Thus, by continuity of f and Theorem 4.12, we have that

$$\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k}) = f(\bar{x}),$$

therefore,

$$|f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(\bar{x})| + |f(\bar{x}) - f(y_{n_k})| \underset{k\to\infty}{\longrightarrow} 0,$$

which contradicts Equation 4.3. Hence, f is uniformly continuous on $[a,b]$. □

Definition 4.24: Lipschitz Continuity

Let $D \subseteq \mathbb{R}$, and $f : D \to \mathbb{R}$. We say that f is **Lipschitz continuous** if there exists $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x-y| \qquad \forall x, y \in D.$$

Lemma 4.25: Lipschitz Continuity ⇒ Uniform Continuity

Let $D \subseteq \mathbb{R}$, and $f : D \to \mathbb{R}$ be a Lipschitz continuous function. Then f is uniformly continuous.

Proof. Let $D \subseteq \mathbb{R}$ and assume that $f : D \to \mathbb{R}$ is a Lipschitz continuous function. Then there exists $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x-y| \qquad \forall x, y \in D.$$

Now, fix $\varepsilon > 0$. We assume that $L \neq 0$ (otherwise the result follows immediately) and choose $\delta = \frac{\varepsilon}{L}$. Because of the Lipschitz continuity of f, we have that for all $x, y \in D$ it holds that

$$\begin{aligned} |x-y| < \delta = \frac{\varepsilon}{L} \quad &\Leftrightarrow \quad L|x-y| < \varepsilon \\ \Rightarrow \qquad &|f(x) - f(y)| \leq L|x-y| < \varepsilon, \end{aligned}$$

which shows that f is also uniformly continuous. □

4.4 Example: Exponential and Logarithmic Functions

4.4.1 Definition of the Exponential Function

Lemma 4.26: Bernoulli's Inequality

For all $a \in \mathbb{R}$ with $a \geq -1$ and all $n \in \mathbb{N}$ with $n \geq 1$, it holds that

$$(1+a)^n \geq 1 + na.$$

Proof. We proceed by induction. For $n = 1$ we have $(1+a)^1 = 1 + a = 1 + 1 \cdot a$.

Now assume that the inequality holds for some $n \geq 1$. Since $1 + a \geq 0$ by assumption, we find

$$(1+a)^{n+1} = (1+a)^n(1+a) \geq (1+na)(1+a) = 1 + na + a + na^2 \geq 1 + (n+1)a,$$

which establishes the induction step and completes the proof. □

Proposition 4.27: Existence of the Exponential

Let $x \in \mathbb{R}$. The sequence $(a_n)_{n=1}^{\infty}$ defined by

$$a_n = \left(1 + \frac{x}{n}\right)^n$$

is convergent, and its limit is a positive real number.

Lemma 4.28: Monotonicity

Given $x \in \mathbb{R}$, let $n_0 \in \mathbb{N}$ satisfy $n_0 \geq 1$ and $n_0 > -x$. Then the sequence $(a_n)_{n=n_0}^{\infty}$ defined in Proposition 4.27 is increasing.

Definition 4.29: Exponential Function

The **exponential function** $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is defined by

$$\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n \qquad \forall x \in \mathbb{R}.$$

Corollary 4.30: Growth of the Exponential

Given $n \in \mathbb{N}$ with $n \geq 1$, the exponential function satisfies

$$\exp(x) \geq \left(1 + \frac{x}{n}\right)^n \qquad \forall x > -n.$$

Proof. By Lemma 4.28 and Definition 4.29, for $x > -n$ we have

$$a_n \leq a_{n+1} \leq \ldots \leq \exp(x).$$

□

4.4.2 Properties of the Exponential Function

Theorem 4.31: Properties of the Exponential Function

The exponential function $\exp : \mathbb{R} \to \mathbb{R}_{>0}$ is bijective, strictly increasing, and continuous. Moreover,

$$\begin{aligned} \exp(0) &= 1, \\ \exp(-x) &= \exp(x)^{-1}, \\ \exp(x+y) &= \exp(x)\exp(y), \end{aligned}$$

for all $x, y \in \mathbb{R}$.

4.4.3 The Natural Logarithm

Definition 4.32: Logarithm

The unique inverse function

$$\log : \mathbb{R}_{>0} \to \mathbb{R}$$

of the bijective map $exp : \mathbb{R} \to \mathbb{R}_{>0}$ is called the **logarithm**.

Corollary 4.33: Properties of the Logarithm

The logarithm $\log : \mathbb{R}_{>0} \to \mathbb{R}$ *is strictly increasing, continuous, and bijective. Moreover,*

$$\begin{aligned}\log(1) &= 0,\\ \log(a^{-1}) &= -\log(a),\\ \log(ab) &= \log(a) + \log(b),\end{aligned}$$

for all $a, b \in \mathbb{R}_{>0}$.

The logarithm defined here is also called the **natural logarithm** to distinguish it from logarithms with another **base** $a > 1$ (for instance $a = 10$ or $a = 2$). For any $a > 1$, we define

$$\log_a(x) = \frac{\log(x)}{\log(a)} \qquad \forall x > 0.$$

Unless stated otherwise, $\log(x)$ always denotes the natural logarithm, i.e., the logarithm to base e.

We can now define powers with arbitrary real exponents. For $a > 0$ and $x \in \mathbb{R}$ we set

$$a^x = \exp(x \log(a)).$$

4.5 Limits of Functions

We consider functions $f : D \to \mathbb{R}$ defined on a subset $D \subseteq \mathbb{R}$, and we wish to define the limit of $f(x)$ as $x \in D$ approaches a point $x_0 \in \mathbb{R}$. Typical examples include $D = \mathbb{R}, D = [0, 1]$ or $D = (0, 1)$, with $x_0 = 0$ in each case.

4.5.1 Limit in the Vicinity of a Point

Let $D \subseteq \mathbb{R}$ be non-empty, and let $x_0 \in \mathbb{R}$ be such that

$$D \cap (x_0 - \delta, x_0 + \delta) \neq \emptyset \tag{4.4}$$

for all $\delta > 0$. Whenever this holds, we say that x_0 is an **accumulation point** of D. Note that if $x_0 \in D$, then Equation 4.4 is automatically satisfied.

Condition 4.4 ensures that there exists a sequence of points in D converging to x_0.

Definition 4.34: Limit of a Function

Let $f : D \to \mathbb{R}$, and x_0 be an accumulation point of D. A number $L \in \mathbb{R}$ is called the **limit of** $f(x)$ **as** $x \to x_0$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon \qquad \forall x \in D.$$

In general, the limit of $f(x)$ as $x \to x_0$ may not exist. However, if it exists, it is uniquely determined. Hence we speak of *the* limit and write

$$\lim_{x \to x_0} f(x) = L$$

to indicate the limit exists and is equal to L. Informally, this means that the function values $f(x)$ are arbitrarily close to L whenever $x \in D$ is sufficiently close to x_0.

The limit of a function satisfies properties analogous to those of Proposition 3.13. More precisely, if f, g are functions on D such that

$$\lim_{x\to x_0} f(x) = L_1 \qquad \text{and} \qquad \lim_{x\to x_0} g(x) = L_2,$$

then

$$\lim_{x\to x_0} (f+g)(x) = L_1 + L_2, \qquad \lim_{x\to x_0} (f\cdot g)(x) = L_1 \cdot L_2.$$

Moreover, $f \le g$ implies $L_1 \le L_2$, and the sandwich lemma holds: if $f \le h \le g$ and $L_1 = L_2$ then $\lim_{x\to x_0} h(x) = L_1 = L_2$.

Remark 4.35. *Let $f : D \to \mathbb{R}$ be a function. If $x_0 \in D$, then f is continuous at x_0 if and only if* $\lim_{x\to x_0} f(x) = f(x_0)$.

Suppose that $x_0 \in D$ is an accumulation point of $D \setminus \{x_0\}$. Let $f : D \to \mathbb{R}$, and consider the restriction $f|_{D\setminus\{x_0\}}$. It may happen that f is discontinuous at x_0, but the limit

$$L = \lim_{x\to x_0} f|_{D\setminus\{x_0\}}(x) \tag{4.5}$$

nevertheless exists. In this case, the point x_0 is called a **removable discontinuity** of f, and one also writes

$$L = \lim_{\substack{x\to x_0\\ x\neq x_0}} f(x). \tag{4.6}$$

If we now define

$$\tilde{f}(x) = \begin{cases} f(x), & x \in D \setminus \{x_0\}, \\ L, & x = x_0, \end{cases} \tag{4.7}$$

then $\tilde{f}$ is continuous at x_0. In other words, we can remove the discontinuity of f by redefining its value at x_0 to be L.

If instead $x_0 \notin D$ but the limit in Equation 4.6 exists, we call the function $\tilde{f}$ defined in Equation 4.7 the **continuous extension** of f to $D \cup \{x_0\}$.

Arguing as in the proof of Theorem 4.12, we obtain the following result.

Lemma 4.36: Limit and Sequences

Let $f : D \to \mathbb{R}$. Then $L = \lim_{x\to \bar{x}} f(x)$ if and only if, for every sequence $(x_n)_{n=0}^{\infty} \subseteq D$ converging to $\bar{x}$, one has $\lim_{n\to\infty} f(x_n) = L$.

We now state a result describing the behaviour of limits under composition with a continuous function.

Proposition 4.37: Limit and Composition

Let $E \subseteq \mathbb{R}$, and let $f : D \to E$ be such that the limit $L = \lim_{x\to\bar{x}} f(x)$ exists and belongs to E. If $g : E \to \mathbb{R}$ is continuous at L, then

$$\lim_{x\to\bar{x}} g(f(x)) = g(L).$$

Proof. Let $(x_n)_{n=0}^{\infty} \subseteq D$ be a sequence converging to $\bar{x}$. By Lemma 4.36, we have $\lim_{n\to\infty} f(x_n) = L$. Since g is continuous at L, Theorem 4.12 gives $\lim_{n\to\infty} g(f(x_n)) = g(L)$. Because $(x_n)_{n=0}^{\infty}$ was arbitrary, using Lemma 4.36 again, we conclude that $\lim_{x\to\bar{x}} g(f(x)) = g(L)$. □

We now introduce conventions for improper limits of functions, in analogy with improper limits for sequences.

Definition 4.38: Improper Limits

Let $f : D \to \mathbb{R}$, and let x_0 be an accumulation point of D. We say that f **diverges to** $+\infty$ **as** $x \to x_0$, and write

$$\lim_{x \to x_0} f(x) = +\infty,$$

if for every $M > 0$, there exists $\delta > 0$ such that

$$\forall x \in D: \quad |x - x_0| < \delta \quad \Rightarrow \quad f(x) \geq M.$$

Analogously, f **diverges to** $-\infty$ **as** $x \to x_0$ and we write $\lim_{x \to x_0} f(x) = -\infty$, if for every $M > 0$, there exists $\delta > 0$ such that

$$\forall x \in D: \quad |x - x_0| < \delta \quad \Rightarrow \quad f(x) \leq -M.$$

4.5.2 One-Sided Limits

It is often useful to consider limits taken form one side only and to allow x_0 to be $\pm\infty$ as well. To this end, let $x_0 \in \mathbb{R}$ be such that

$$D \cap (x_0, x_0 + \delta) \neq \emptyset \tag{4.8}$$

for every $\delta > 0$. In this case, we say that x_0 is a **right-hand accumulation point** of D. Analogously, if

$$D \cap (x_0 - \delta, x_0) \neq \emptyset \tag{4.9}$$

for every $\delta > 0$, we say that x_0 is a **left-hand accumulation point** of D.

Definition 4.39: One-Sided Limits

Let $f : D \to \mathbb{R}$, and let $x_0 \in \mathbb{R}$ be a right-hand accumulation point of D. A number $L \in \mathbb{R}$ is called the **right-hand limit** of f at x_0 if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in D \cap (x_0, x_0 + \delta) \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

In this case we write $L = \lim_{x \to x_0^+} f(x)$. We also allow improper one-sided limits. We say that

$$\lim_{x \to x_0^+} f(x) = +\infty$$

if for every $M > 0$, there exists $\delta > 0$ such that

$$x \in D \cap (x_0, x_0 + \delta) \quad \Rightarrow \quad f(x) \geq M.$$

Similarly, $\lim_{x \to x_0^+} f(x) = -\infty$ means that, for every $M > 0$, there exists $\delta > 0$ such that

$$x \in D \cap (x_0, x_0 + \delta) \quad \Rightarrow \quad f(x) \leq -M.$$

The **left-hand limit** is defined analogously, considering a left-hand accumulation point of D and writing $\lim_{x \to x_0^-} f(x)$.

Next, we define the notion of limit at infinity.

Definition 4.40: Limits at Infinity

Let $f : D \to \mathbb{R}$, and assume that $D \cap (R, \infty) \neq \emptyset$ for every $R > 0$. A number $L \in \mathbb{R}$ is called the **limit of f as $x \to +\infty$** if, for every $\varepsilon > 0$, there exists $R > 0$ such that

$$x \in D \cap (R, \infty) \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

We say that f **diverges to $+\infty$ as $x \to +\infty$** if, for every $M > 0$, there exists $R > 0$ such that

$$x \in D \cap (R, \infty) \quad \Rightarrow \quad f(x) \geq M.$$

The corresponding definition for $x \to -\infty$ and diverges to $-\infty$ are analogous.

Limits at $+\infty$ can be converted into right-hand limits at 0 via inversion. Given $f : D \to \mathbb{R}$ as above, define

$$E = \{x > 0 \mid x^{-1} \in D\}, \qquad g : E \to \mathbb{R}, \qquad g(x) = f(x^{-1}).$$

Then

$$\lim_{x \to +\infty} f(x) = \lim_{x \to 0^+} g(x),$$

so one limit exists if and only if the other does.

Definition 4.41: One-Sided Continuity and Jumps

Let $f : D \to \mathbb{R}$ and $x_0 \in D$. If $\lim_{x \to x_0^+} f(x)$ exists and equals $f(x_0)$, then f is **continuous from the right** at x_0. **Continuity form the left** is defined similarly. We call x_0 a **jump point** if both one-sided limits exist but are different, i.e.,

$$L_- := \lim_{x \to x_0^-} f(x) \in \mathbb{R}, \qquad L_+ := \lim_{x \to x_0^+} f(x) \in \mathbb{R}, \qquad L_- \neq L_+.$$

4.5.3 Landau Notation

We introduce two standard notations that compare the asymptotic behaviour of a function to that of another function. (often called *relative* asymptotics).

Definition 4.42: Big-O at a Point

Let $f, g : D \to \mathbb{R}$, and let x_0 be an accumulation point of D. We write

$$f(x) = O(g(x)) \quad \text{as } x \to x_0$$

if there exists $M > 0$ and $\delta > 0$ such that

$$x \in D \cap (x_0 - \delta, x_0 + \delta) \quad \Rightarrow \quad |f(x)| \leq M|g(x)|.$$

We then say that f is a **Big-O** of g as $x \to x_0$.

If $g(x) \neq 0$ for all x sufficiently close to x_0 (with $x \in D$), then

$$f(x) = O(g(x)) \quad \text{as } x \to x_0 \qquad \Leftrightarrow \qquad \frac{f(x)}{g(x)} \text{ is bounded near } x_0.$$

Definition 4.43: Big-O at Infinity

Let $f, g : D \to \mathbb{R}$, and assume $D \cap (R, \infty) \neq \emptyset$ for every $R > 0$. We write

$$f(x) = O(g(x)) \quad \text{as } x \to +\infty$$

if there exists $M > 0$ and $R > 0$ such that

$$x \in D \cap (R, \infty) \quad \Rightarrow \quad |f(x)| \leq M|g(x)|.$$

The definition for $x \to -\infty$ is analogous.

The big-O notation hides the precise bound by an *implicit constant* M, which is often irrelevant for the argument one is interested in.

Example

- if f and g are bounded and continuous near x_0 with $g(x_0) \neq 0$, then $f(x) = O(g(x))$ as $x \to x_0$.
- As $x \to 0$, one has $x^2 = O(x)$, but $x \neq O(x^2)$ (since x/x^2 is unbounded near 0).
- As $x \to +\infty$, $\frac{3x^3}{x^3+3} = O(1)$, but $\frac{3x^3}{x^3+3} \neq O(x^\alpha)$ for $\alpha < 0$.

As discussed above, the big-O means that f is bounded by a multiple of g. One may also consider a stronger condition, namely that f is asymptotically negligible with respect to g. This leads to the following definition.

Definition 4.44: Little-O at a Point

Let $f, g : D \to \mathbb{R}$, and let x_0 be an accumulation point of D. We write

$$f(x) = o(g(x)) \quad \text{as } x \to x_0$$

if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x \in D \cap (x_0 - \delta, x_0 + \delta) \quad \Rightarrow \quad |f(x)| \leq \varepsilon|g(x)|.$$

We then say that f is a **little-o** of g as $x \to x_0$.

If $g(x) \neq 0$ for all x near x_0 (with $x \in D$), then

$$f(x) = o(g(x)) \quad \text{as } x \to x_0 \qquad \Leftrightarrow \qquad \lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

Moreover, $f(x) = o(g(x)) \;\Rightarrow\; f(x) = O(g(x))$.

Definition 4.45: Little-o at Infinity

Let $f, g : D \to \mathbb{R}$, and assume that $D \cap (R, \infty) \neq \emptyset$ for every $R > 0$. We write

$$f(x) = o(g(x)) \quad \text{as } x \to +\infty$$

if, for every $\varepsilon > 0$ there exists $R > 0$ such that

$$x \in D \cap (R, \infty) \quad \Rightarrow \quad |f(x)| \leq \varepsilon |g(x)|.$$

The definition for $x \to -\infty$ is analogous.

Example

- $x = o(x^2)$ as $x \to +\infty$, and $x^2 = o(x)$ as $x \to 0$
- For any $\alpha < 1$,
$$\frac{3x^3}{2x^2 + x^10} = o(|x|^\alpha) \quad \text{as } x \to 0,$$
but not for $\alpha \geq 1$. Indeed,
$$|\frac{3x^3}{|x|^\alpha 2x^2 + x^10}| = |x|^{1-\alpha} \frac{3}{2 + x^8} \longrightarrow 0 \quad \text{as } x \to 0,$$
whenever $\alpha < 1$.

In computations, one often uses Landau symbols as placeholders. Writing

$$f(x) = o(g(x)) \quad \text{as } x \to x_0$$

means there is a function $h : D \to \mathbb{R}$ with $h(x) = o(g(x))$ as $x \to x_0$. Similarly for big-O.

Example

Polynomial division gives, as $x \to +\infty$,

$$\frac{x^3 - 7x^2 + 6x + 2}{x^2} = x - 7 + O\left(\frac{1}{x}\right) = x - 7 + o(1) = x + O(1) = x + o(x).$$

4.6 Sequences of Functions

4.6.1 Pointwise Convergence

Definition 4.46: Sequences of Functions

A **sequence** of real-valued on a subset $D \subseteq \mathbb{R}$ is a family of functions $f_n : D \to \mathbb{R}$ indexed by $\mathbb{N}$. The function f_n is called the n-th **element** of the sequence. One often writes $(f_n)_{n \in \mathbb{N}}$, $(f_n)_{n=0}^\infty$, or $(f_n)n \geq 0$ for a sequence of functions.

Definition 4.47: Pointwise Convergence

et $D \subseteq \mathbb{R}$, and let $(f_n)_{n=0}^{\infty}$ be a sequence of functions $f_n : D \to \mathbb{R}$. Let $f : D \to \mathbb{R}$ be another function. We say that $(f_n)_{n=0}^{\infty}$ **converges pointwise** to f, if for every $x \in D$, the sequence of real numbers $(f_n(x))_{n=0}^{\infty}$ converges to $f(x)$, i.e., for every $x \in D$ and for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall n \geq N.$$

In this case, f is called the **pointwise limit** of the sequence $(f_n)_{n=0}^{\infty}$.

Remark 4.48. *Note that, in the definition of pointwise convergence the index N may depend on both x and ε. In other words, for each point $x \in D$ we examine the convergence of $(f_n)_{n=0}^{\infty}$ to f separately.*

4.6.2 Uniform Convergence

Definition 4.49: Uniform Convergence

Let $D \subseteq \mathbb{R}$, and let $(f_n)_{n=0}^{\infty}$ be a sequence of functions $f_n : D \to \mathbb{R}$. Let $f : D \to \mathbb{R}$ be another function. We say that $(f_n)_{n=0}^{\infty}$ **converges uniformly** to f on D if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall n \geq N,\ \forall x \in D.$$

Remark 4.50. *Note that, in the definition of uniform convergence the index N may only depend on ε, and therefore the condition has to hold for all $x \in D$ for the sequence of functions $(f_n)_{n=0}^{\infty}$ to converge uniformly to f on D.*

Remark 4.51. *Let $D \subseteq \mathbb{R}$, and $(f_n)_{n=0}^{\infty}$ be a sequence of functions $f_n : D \to \mathbb{R}$ converging uniformly to $f : D \to \mathbb{R}$ on D. Then the sequence of functions $(f_n)_{n=0}^{\infty}$ also converges pointwise to f.*

Theorem 4.52: Continuity under Uniform Convergence

Let $D \subseteq \mathbb{R}$, and let $(f_n)_{n=0}^{\infty}$ be a sequence of continuous functions converging uniformly to $f : D \to \mathbb{R}$. Then f is continuous.

Proof. To prove that f is continuous, we fix $\bar{x} \in D$ and show that f is continuous at $\bar{x}$. Given $\varepsilon > 0$, the uniform convergence of f_N to f provides $N \in \mathbb{N}$ such that

$$|f_N(y) - f(y)| < \frac{\varepsilon}{3} \qquad \forall y \in D.$$

Also, since f_N is continuous at $\bar{x}$, there exists $\delta > 0$ such that

$$|x - \bar{x}| < \delta \quad \Rightarrow \quad |f_N(x) - f_N(\bar{x})| < \frac{\varepsilon}{3}.$$

Then, for $|x - \bar{x}| < \delta$, we have

$$|f(x) - f(\bar{x})| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(\bar{x})| + |f_N(\bar{x}) - f(\bar{x})| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Which shows that f is continuous at $\bar{x}$. Since, $\bar{x}$ is arbitrary, f is continuous on D. $\square$

Intuitively, uniform convergence allows us to *exchange* the order of taking limits. More precisely, assume $(f_n)_{n=0}^{\infty}$ is a sequence of continuous functions converging pointwise to f. Then, by the pointwise convergence and the continuity of the functions f_n we have

$$f(\bar{x}) = \lim_{n\to\infty} f_n(\bar{x}), \qquad f_n(\bar{x}) = \lim_{x\to\bar{x}} f_n(x), \qquad f(x) = \lim_{n\to\infty} f_n(x) \quad \forall x \in D.$$

Hence,

$$f(\bar{x}) = \lim_{n\to\infty} f_n(\bar{x}) = \lim_{n\to\infty} \left(\lim_{x\to\bar{x}} f_n(x) \right), \qquad \lim_{x\to\bar{x}} f(x) = \lim_{x\to\bar{x}} \left(\lim_{n\to\infty} f_n(x) \right).$$

Note that the function f is continuous at $\bar{x}$ if and only if $f(\bar{x}) = \lim_{x\to\bar{x}} f(x)$, which by the identities above is equivalent to

$$\lim_{x\to\bar{x}} \left(\lim_{n\to\infty} f_n(x) \right) = \lim_{n\to\infty} \left(\lim_{x\to\bar{x}} f_n(x) \right).$$

As we have seen, for pointwise convergent this interchange may fail because f need not be continuous. However, Theorem 4.52 ensures that this equality holds under uniform convergence.

5 Series and Power Series

In this chapter we study series (infinite sums). They provide a framework to define many classical functions; in particular, we will use series to define trigonometric functions.

5.1 Series of Real Numbers

Definition 5.1: Convergent and Divergent Series

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let $A \in \mathbb{R}$. We say that the series $\sum_{k=0}^{\infty} a_k$ **converges** to A if

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k = A.$$

In other words, computing the infinite sum $\sum_{k=0}^{\infty} a_k$ means finding (if it exists) the limit of the **partial sums**

$$s_n = \sum_{k=0}^{n} a_k, \qquad n \in \mathbb{N}.$$

We call a_n the **n-th term** (or **n-th summand**) of the series. If the limit exists, its value A is the **sum of the series**. If the limit does not exist, the series is said to be **not convergent**. In particular, if the sequence of partial sums $(s_n)_{n=0}^{\infty}$ diverges to $+\infty$ (respectively, to $-\infty$), we say that the series **diverges to** $+\infty$ (respectively, **to** $-\infty$). This situation is therefore a specific case of a series that does not converge.

Proposition 5.2: Necessary Condition for Convergence

Is the series $\sum_{k=0}^{\infty} a_k$ converges, then $a_n \to 0$ as $n \to \infty$.

Proof. By assumption the partial sums $s_n = \sum_{k=0}^{n} a_k$ satisfy $s_n \to A \in \mathbb{R}$. Then for $n \geq 1$, we have

$$a_n = s_n - s_{n-1} \underset{n\to\infty}{\longrightarrow} A - A = 0.$$

□

Geometric Series

For $q \in \mathbb{R}$, the geometric series $\sum_{k=0}^{\infty} q^k$ converges if and only if $|q| < 1$, and in this case

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}.$$

Indeed, if the series converges, then by Proposition 5.2 we must have $q^n \to 0$ as $n \to \infty$, hence $|q| < 1$. Conversely, for $|q| < 1$ one provides by induction that

$$s_n = \sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \qquad \forall n \in \mathbb{N}, q \neq 1.$$

Also since $|q| < 1$, $q^{n+1} \to 0$ as $n \to \infty$. Thus,

$$s_n = \frac{1-q^{n+1}}{1-q} \xrightarrow[n\to\infty]{} \frac{1}{1-q}.$$

Harmonic Series

The converse of Proposition 5.2 fails: the **harmonic series** $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge. To see this, consider $n = 2^\ell$ with $\ell \in \mathbb{N}$. Grouping terms gives

$$\begin{aligned}
\sum_{k=1}^{2^\ell} \frac{1}{k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \ldots + \frac{1}{8}\right) + \ldots + \left(\frac{1}{2^{\ell-1}+1} + \ldots + \frac{1}{2^\ell}\right) \\
&\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{=\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{=\frac{1}{2}} + \ldots + \underbrace{\frac{1}{2^\ell} + \frac{1}{2^\ell}}_{=\frac{1}{2}} \\
&= 1 + \underbrace{\frac{1}{2} + \ldots + \frac{1}{2}}_{\ell\text{ - times}} = 1 + \frac{\ell}{2},
\end{aligned}$$

which is unbounded as $\ell \to \infty$.

Lemma 5.3: Convergence of the Tail

Let $\sum_{k=0}^{\infty} a_k$ be a series and fix $N \in \mathbb{N}$. Then $\sum_{k=0}^{\infty} a_k$ is convergent if and only if $\sum_{k=N}^{\infty} a_k$ is convergent, and in that case

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{N-1} a_k + \sum_{k=N}^{\infty} a_k.$$

The same equivalence holds for divergence to $+\infty$ or $-\infty$.

Proof. For every $n \geq N$,

$$\sum_{k=0}^{n} a_k = \sum_{k=0}^{N-1} a_k + \sum_{k=N}^{n} a_k.$$

Thus, the partial sums of $\sum_{k=0}^{\infty} a_k$ converge if and only if those of $\sum_{k=N}^{\infty} a_k$ do, and the identity in the statement follows by letting $n \to \infty$. The divergence case is analogous. □

5.1.1 Series with Non-negative Elements

Proposition 5.4: Non-negative Series: Convergence vs. Divergence

Let $\sum_{k=0}^{\infty} a_k$ be a series with non-negative terms $a_k \geq 0$ for all $k \in \mathbb{N}$. Then the partial sums $s_n = \sum_{k=0}^{n} a_k$ form an increasing sequence. If $(s_n)_{n=0}^{\infty}$ is bounded, the series $\sum_{k=0}^{\infty} a_k$ converges; otherwise it diverges to $+\infty$.

Proof. Since $a_{n+1} \geq 0$, we have $s_{n+1} = s_n + a_{n+1} \geq s_n$ for all $n \in \mathbb{N}$, so $(s_n)_{n=0}^{\infty}$ is increasing.

If the sequence $(s_n)_{n=0}^{\infty}$ is bounded, then it converges by Theorem 3.20. If the partial sums are not bounded, then they diverge to $+\infty$. □

Remark 5.5. *If $\sum_{k=0}^{\infty} a_k$ has non-negative terms, then $(s_n)_{n=0}^{\infty}$ is bounded if and only if it has a bounded subsequence $(s_{n_k})_{k=0}^{\infty}$.*

Corollary 5.6: Comparison Test (Majorant/Minorant)

Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be series with $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Then

$$0 \leq \sum_{k=0}^{\infty} a_k \leq \sum_{k=0}^{\infty} b_k,$$

and in particular

$$\begin{aligned} \sum_{k=0}^{\infty} b_k \text{ convergent} \quad &\Rightarrow \quad \sum_{k=0}^{\infty} a_k \text{ convergent}, \\ \sum_{k=0}^{\infty} a_k \text{ divergent to } +\infty \quad &\Rightarrow \quad \sum_{k=0}^{\infty} b_k \text{ divergent to } +\infty. \end{aligned}$$

These implications remain true if the inequalities $0 \leq a_n \leq b_n$ hold only for all $n \geq N$, for some $N \in \mathbb{N}$.

Proof. From $a_k \leq b_k$ we get $\sum_{k=0}^{n} a_k \leq \sum_{k=0}^{n} b_k$ for all $n \in \mathbb{N}$. Therefore,

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \leq \lim_{n\to\infty} \sum_{k=0}^{n} b_k = \sum_{k=0}^{\infty} b_k.$$

The last part of the statement follows form Lemma 5.3. □

Under the assumptions of the corollary, $\sum_{k=0}^{\infty} b_k$ is called a *majorant* of $\sum_{k=0}^{\infty} a_k$, and $\sum_{k=0}^{\infty} a_k$ a *minorant* of $\sum_{k=0}^{\infty} b_k$. Hence the names **majorant** and **minorant criterion**.

Proposition 5.7: Cauchy Condensation Test

Let $(a_k)_{k=0}^{\infty}$ be a decreasing sequence of non-negative numbers. Then

$$\sum_{k=0}^{\infty} a_k \text{ converges} \quad \Leftrightarrow \quad \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ converges}.$$

Proof. Consider the partial sums of the series $\sum_{k=0}^{\infty} a_k$ starting form $k = 2$ up to an index that is a power of 2. Since the terms a_k are decreasing, the following inequalities hold:

$$\begin{aligned}
\sum_{k=2}^{2^{n+1}} a_k &= a_2 + (a_3 + a_4) + (a_5 + \ldots + a_8) + \ldots + (a_{2^n+1} + \ldots + a_{2^{n+1}}) \\
&\le \underbrace{a_1}_{=1\cdot a_1} + \underbrace{(a_2 + a_2)}_{=2\cdot a_2} + \underbrace{(a_4 + \ldots + a_4)}_{=4\cdot a_4} + \ldots + \underbrace{(a_{2^n}) + \ldots + a_{2^n}}_{=2^n\cdot a_{2^n}} \\
&= a_1 + 2a_2 + 4a_4 + \ldots + 2^n a_{2^n} = \sum_{k=0}^{n} 2^k a_{2^k},
\end{aligned}$$

and similarly,

$$\begin{aligned}
\sum_{k=2}^{2^{n+1}} a_k &= a_2 + (a_3 + a_4) + (a_5 + \ldots + a_8) + \ldots + (a_{2^n+1} + \ldots + a_{2^{n+1}}) \\
&\ge \underbrace{a_2}_{=1\cdot a_2} + \underbrace{(a_4 + a_4)}_{=2\cdot a_4} + \underbrace{(a_8 + \ldots + a_8)}_{=4\cdot a_8} + \ldots + \underbrace{(a_{2^{n+1}}) + \ldots + a_{2^{n+1}}}_{=2^n\cdot a_{2^{n+1}}} \\
&= \frac{1}{2}(2a_2 + 4a_4 + \ldots + 2^{n+1} a_{2^{n+1}}) = \frac{1}{2}\sum_{k=1}^{n+1} 2^k a_{2^k}.
\end{aligned}$$

In other words,

$$\sum_{k=0}^{n} 2^k a_{2^k} \ge \sum_{j=2}^{2^{n+1}} a_j \ge \frac{1}{2}\sum_{k=1}^{n+1} 2^k a_{2^k}.$$

By Remark 5.5 and Corollary 5.6, the partial sums of one series are bounded if and only if those of the other are. Hence, the two series converge or diverge together. □

5.1.2 Conditional Convergence

Definition 5.8: Absolute and Conditional Convergence

A series $\sum_{k=0}^{\infty} a_k$ is **absolutely convergent** if $\sum_{k=0}^{\infty} |a_k|$ converges. It is **conditionally convergent** if $\sum_{k=0}^{\infty} a_k$ converges but $\sum_{k=0}^{\infty} |a_k|$ diverges.

A striking feature of conditionally convergent series is that their terms can be rearranged to obtain any prescribed limit.

Theorem 5.9: Riemann Rearrangement Theorem

Let $\sum_{n=0}^{\infty} a_n$ be a conditionally convergent series and let $A \in \mathbb{R}$. Then there exists a bijection $\varphi : \mathbb{N} \to \mathbb{N}$ such that

$$A = \sum_{n=0}^{\infty} a_{\varphi(n)}.$$

The proof of this Theorem is extra material.

5.1.3 Leibniz Criterion for Alternating Series

Definition 5.10: Alternating Series

If $(a_k)_{k=0}^{\infty}$ is a sequence of non-negative numbers, the series

$$\sum_{k=0}^{\infty}(-1)^k a_k$$

is called the **alternating series** associated with the sequence $(a_k)_{k=0}^{\infty}$.

Proposition 5.11: Leibniz Criterion

Let $(a_k)_{k=0}^{\infty}$ be a monotonically decreasing sequence of non-negative numbers with $a_k \to 0$. Then the alternating series $\sum_{k=0}^{\infty}(-1)^k a_k$ converges, and for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{2n+1}(-1)^k a_k \leq \sum_{k=0}^{\infty}(-1)^k a_k \leq \sum_{k=0}^{2n}(-1)^k a_k. \tag{5.1}$$

Proof. Let $s_n = \sum_{k=0}^{n}(-1)^k a_k$. Since the sequence $(a_n)_{n=0}^{\infty}$ is decreasing and non-negative, we have

$$s_{2n+2} = s_{2n} \underbrace{-a_{2n+1} + a_{2n+2}}_{\leq 0} \leq s_{2n},$$

$$s_{2n+1} = s2n - 1 \underbrace{+a_{2n} - a2n + 1}_{\geq 0} \geq s_{2n-1},$$

$$s_{2n+2} = s_{2n+1} \underbrace{+a2n + 2}_{\geq 0} \geq s_{2n+1}$$

for all $n \in \mathbb{N}$. In other words,

$$s_1 \leq s_3 \leq \ldots \leq s_{2n-1} \leq s_{2n+1} \leq \ldots \leq s_{2n+2} \leq s_{2n} \leq \ldots \leq s_2 \leq s_0.$$

This implies that the sequence $(s_{2n})_{n=0}^{\infty}$ is decreasing and bounded below, while the sequence $(s_{2n+1})_{n=0}^{\infty}$ is increasing and bounded form above. Thus, both limits $A = \lim_{n\to\infty} s_{n+1}$ and $B = \lim_{n\to\infty} s_{2n}$ exist and satify

$$s_1 \leq s_3 \leq \ldots \leq s_{2n-1} \leq s_{2n+1} \leq A \leq B \leq s_{2n+2} \leq s_{2n} \leq \ldots \leq s_2 \leq s_0. \tag{5.2}$$

In particular,

$$0 \leq B - A \leq s_{2n+2} - s_{2n-1} \qquad \forall n \in \mathbb{N},$$

and because $a_{2n+2} \to 0$, we deduce that $A = B$.

Also, Equation 5.2 yields that $s_{2n+1} \leq A = B \leq s_{2n}$, which corresponds exactly to Equation 5.1. □

Example (Alternating Harmonic Series)

The series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

converges by Proposition 5.11, whereas $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges. Hence, the alternating harmonic series is only conditionally convergent.

5.2 Absolute Convergence

In this section we will look at absolutely convergent series and prove some convergence criteria. As before, unless otherwise specified, all sequences consist of real numbers.

5.2.1 Criteria for Absolute Covergence

We begin by restating the concept of a Cauchy sequence in the context of convergent series.

Theorem 5.12: Cauchy Criterion for Series

The series $\sum_{k=0}^{\infty} a_k$ converges if and only if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > m \geq N$,

$$\left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon.$$

Proof. By definition, the series $\sum_{k=0}^{\infty} a_k$ converges if and only if the sequence of partial sums

$$s_n = \sum_{k=0}^{n} a_k$$

converges. By Theorem 3.27, this occurs if and only if $(s_n)_{n=0}^{\infty}$ is a Cauchy sequence, i.e., $|s_n - s_m| < \varepsilon$ for all $n, m \geq N$. Since $s_n - s_m = 0$ when $n = m$, and the expression is symmetric in n and m, it suffices to consider the case $n > m$. In this case,

$$s_n - s_m = \sum_{k=m+1}^{n} a_k,$$

which proves the claim. □

We can now prove that absolutely convergent series do indeed converge.

Proposition 5.13: Absolute Convergence Implies Convergence

If a series $\sum_{n=0}^{\infty} a_n$ converges absolutely, then it converges and satisfies the generalized triangle inequality

$$\left| \sum_{n=0}^{\infty} a_n \right| \leq \sum_{n=0}^{\infty} |a_n|.$$

Since $\sum_{n=0}^{\infty} a_n$ converges, by the Cauchy criterion (Theorem 5.12) there exists $N \in \mathbb{N}$ such that, for all $n > m \geq N$,

$$\sum_{k=m+1}^{n} |a_k| < \varepsilon.$$

By the triangle inequality,

$$\left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k| < \varepsilon,$$

so $\sum_{n=0}^{\infty} a_n$ also satisfies the Cauchy criterion and therefore converges.

Moreover, again by the triangle inequality,

$$\left|\sum_{k=0}^{n} a_k\right| \leq \sum_{k=0}^{n} |a_k| \leq \sum_{k=0}^{\infty} a_k \qquad \forall n \in \mathbb{N},$$

and taking the limit as $n \to \infty$ gives the desired inequality.

We now establish two classical criteria guaranteeing absolute convergence. In their proof, we repeatedly use the following fact:

Remark 5.14. *If a sequence $(x_n)_{n=0}^{\infty}$ converges to $\alpha \in \mathbb{R}$, then Proposition 3.13 implies the following facts:*

(i) for any $q > \alpha$ there exists $N \in \mathbb{N}$ such that $x_n < q$ for all $n \geq N$;

(ii) for any $r < \alpha$ there exists $N \in \mathbb{N}$ such that $x_n > r$ for all $n \geq N$.

Proposition 5.15: Cauchy Root Criterion

Given a sequence $(a_n)_{n=0}^{\infty}$, define

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \in \mathbb{R} \cup \{\infty\}.$$

Then,

$$\alpha < 1 \Rightarrow \sum_{n=0}^{\infty} a_n \text{ converges absolutely}, \qquad \alpha > 1 \Rightarrow \sum_{n=0}^{\infty} a_n \text{ does not converge.}$$

Proof. Suppose $\alpha < 1$ and set $q = \frac{1+\alpha}{2}$, so that $q \in (\alpha, 1)$. By definition,

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sup_{k\geq n} \sqrt[k]{|a_k|}.$$

Thus, $x_n = \sup_{k\geq n} \sqrt[k]{|a_k|} \to \alpha$. Since $\alpha < q$, Remark 5.14(i) implies the existence of $N \in \mathbb{N}$ such that

$$x_N = \sup_{k\geq N} \sqrt[k]{|a_k|} < q \qquad \forall k \geq N,$$

therefore,

$$|a_k| < q^k \qquad \forall k \geq N.$$

Since $q < 1$, $\sum_{k=N}^{\infty} |a_k|$ converges by comparison with the geometric series, so $\sum_{n=0}^{\infty} a_n$ converges absolutely.

If $\alpha > 1$, since the limsup is an accumulation point (Theorem 3.23), Proposition 3.9 implies the existence of a subsequence $(a_{n_k})_{k=0}^{\infty}$ such that $\lim_{k\to\infty} \sqrt[n_k]{|a_{n_k}|} = \alpha$. Hence, thanks to Remark 5.14(ii) with $r = 1$, $\sqrt[n_k]{|a_{n_k}|} > 1$ for all k large, or equivalently, $|a_{n_k}| > 1$ for large k. In particular the sequence $(a_n)_{n=0}^{\infty}$ does not converge to 0. Recalling Proposition 5.2, this implies that the series $\sum_{n=0}^{\infty} a_n$ does not converge. $\square$

Remark 5.16. *If $\alpha = 1$, the root criterion is inconclusive.*

Proposition 5.17: D'Alembert's Quotient Criterion

Given a sequence $(a_n)_{n=0}^\infty$ with $a_n \neq 0$ for all n, assume that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \alpha \in [0, \infty).$$

Then

$$\alpha < 1 \ \Rightarrow \ \sum_{n=0}^\infty a_n \text{ converges absolutely,} \qquad \alpha > 1 \ \Rightarrow \ \sum_{n=0}^\infty a_n \text{ does not converge.}$$

Proof. The proof parallels that of the root criterion.

If $\alpha < 1$. set $q = \frac{1+\alpha}{2} \in (\alpha, 1)$. Since $\frac{|a_{n+1}|}{|a_n|} \to \alpha$ and $\alpha < q$, by Remark 5.14(i) there exists $N \in \mathbb{N}$ such that

$$\frac{|a_{k+1}|}{|a_k|} < q \qquad \forall k \geq N.$$

This gives

$$|a_k| = \frac{|a_k|}{|a_{k-1}|} \cdot \frac{|a_{k-1}|}{|a_{k-2}|} \cdot \ldots \cdot \frac{|a_{N+1}|}{|a_N|} \cdot |a_N| < q^{k-N}|a_N| = \frac{|a_N|}{q^N} q^k \qquad \forall k \geq N.$$

Since $q < 1$, the geometric comparison test shows that $\sum_{n=0}^\infty a_n$ converges absolutely.

If $\alpha > 1$, then Remark 5.14(ii) with $r = 1$ implies the existence of $N \in \mathbb{N}$ such that

$$\frac{|a_{k+1}|}{|a_k|} > 1 \qquad \forall k \geq N.$$

In particular,

$$|a_k| = \frac{|a_k|}{|a_{k-1}|} \cdot \frac{|a_{k-1}|}{|a_{k-2}|} \cdot \ldots \cdot \frac{|a_{N+1}|}{|a_N|} \cdot |a_N| > |a_N| \qquad \forall k \geq N.$$

Hence, $(a_n)_{n=0}^\infty$ does not tend to 0, and by Proposition 5.2 the series does not converge. ☐

5.2.2 Reordering Series

Theorem 5.18: Rearrangement of Absolutely Convergent Series

Let $\sum_{n=0}^\infty a_n$ be an absolutely convergent series, and let $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{n=0}^\infty a_{\varphi(n)}$ is absolutely convergent, and

$$\sum_{n=0}^\infty a_n = \sum_{n=0}^\infty a_{\varphi(n)}. \tag{5.3}$$

Proof. Fix $\varepsilon > 0$. Since $\sum_{n=0}^\infty |a_n|$ converges, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^\infty |a_k| < \frac{\varepsilon}{2}.$$

Let

$$M = \max\{\varphi^{-1}(0), \ldots, \varphi^{-1}(N)\}.$$

Equivalently, $M \in \mathbb{N}$ is the smallest number such that

$$\{a_0, \dots, a_N\} \subseteq \{a_{\varphi(0)}, \dots, a_{\varphi(M)}\}$$

Then

$$\{a_0, \dots, a_N\} \subseteq \{a_{\varphi(0)}, \dots, a_{\varphi(n)}\} \qquad \forall n \geq M.$$

Therefore,

$$\sum_{\ell=0}^{n} a_{\varphi(\ell)} - \sum_{k=0}^{N} a_k = \sum_{\substack{0 \leq \ell \leq n \\ \varphi(\ell) > N}} a_{\varphi(\ell)}.$$

Moreover, since all indices $\varphi(\ell) > N$ with $0 \leq \ell \leq n$ correspond to terms among $\{|a_k| \mid k \geq N+1\}$, we have

$$\sum_{\substack{0 \leq \ell \leq n \\ \varphi(\ell) > N}} |a_{\varphi(\ell)}| \leq \sum_{k=N+1}^{\infty} |a_k|.$$

This implies that, for $n \geq M$, we can estimate

$$\left| \sum_{\ell=0}^{n} a_{\varphi(\ell)} - \sum_{k=0}^{\infty} a_k \right| = \left| \sum_{\ell=0}^{n} a_{\varphi(\ell)} - \sum_{k=0}^{N} a_k - \sum_{k=N+1}^{\infty} a_k \right| = \left| \sum_{\substack{0 \leq \ell \leq n \\ \varphi(\ell) > N}} a_{\varphi(\ell)} - \sum_{k=N+1}^{\infty} a_k \right|$$
$$\leq \sum_{\substack{0 \leq \ell \leq n \\ \varphi(\ell) > N}} |a_{\varphi(\ell)}| + \sum_{k=N+1}^{\infty} |a_k| \leq 2 \sum_{k=N+1}^{\infty} |a_k| < \varepsilon.$$

This shows that

$$\sum_{\substack{0 \leq \ell \leq n \\ \varphi(\ell) > N}} a_{\varphi(\ell)} \longrightarrow \sum_{k=0}^{\infty} a_k \quad \text{as } n \to \infty,$$

which proves the identity 5.3. Applying the same reasoning to $\sum_{n=0}^{\infty} |a_n|$ shows that $\sum_{\ell=0}^{\infty} |a_{\varphi(\ell)}| = \sum_{n=0}^{\infty} |a_n| < \infty$, hence $\sum_{\ell=0}^{\infty} a_{\varphi(\ell)}$ is absolutely convergent. □

5.2.3 Product of Series

Theorem 5.19: Product Theorem

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolutely convergent series, and let $\alpha : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection. Writing $\alpha(n) = (\alpha_1(n), \alpha_2(n))$, one has

$$\left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} a_{\alpha_1(n)} b_{\alpha_2(n)}, \tag{5.4}$$

and the series on the right converges absolutely.

Proof. Consider first a bijection $\alpha : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, written as $\alpha(n) = (\alpha_1(n), \alpha_2(n))$, such that

$$\{a(k) \mid 0 \leq k < n^2\} = \{0, 1, \dots, n-1\} \qquad \forall n \in \mathbb{N}.$$

For example, $(\alpha(n))_{n=0}^{\infty}$ could traverse the grid as shown in the lecture. Then, for every $n \in \mathbb{N}$,

$$\sum_{k=0}^{n^2-1} |a_{\alpha_1(k)}||b_{\alpha_2(k)}| = \left(\sum_{\ell=0}^{n-1} |a_\ell| \right) \left(\sum_{m=0}^{n-1} |b_m| \right).$$

Since the right-hand side is bounded by

$$\left(\sum_{\ell=0}^{\infty}|a_\ell|\right)\left(\sum_{m=0}^{\infty}|b_m|\right),$$

we have

$$\sup_{n\in\mathbb{N}}\sum_{k=0}^{n^2-1}|a_{\alpha_1(k)}||b_{\alpha_2(k)}| \le \left(\sum_{\ell=0}^{\infty}|a_\ell|\right)\left(\sum_{m=0}^{\infty}|b_m|\right) < \infty.$$

This implies that the series $\sum_{k=0}^{\infty} a_{\alpha_1(k)}b_{\alpha_2(k)}$. In particular, since it converges, its value can be computed along every subsequence, therefore

$$\sum_{k=0}^{\infty} a_{\alpha_1(k)}b_{\alpha_2(k)} = \lim_{n\to\infty}\sum_{k=0}^{n^2-1} a_{\alpha_1(k)}b_{\alpha_2(k)}$$

Now writing the identity

$$\sum_{k=0}^{n^2-1} a_{\alpha_1(k)}b_{\alpha_2(k)} = \left(\sum_{\ell=0}^{n-1}a_\ell\right)\left(\sum_{m=0}^{n-1}b_m\right),$$

and taking the limit as $n\to\infty$, Proposition 3.12(2.) gives

$$\begin{aligned}\sum_{k=0}^{\infty} a_{\alpha_1(k)}b_{\alpha_2(k)} &= \lim_{n\to\infty}\sum_{k=0}^{n^2-1} a_{\alpha_1(k)}b_{\alpha_2(k)} = \\ &= \left(\lim_{n\to\infty}\sum_{\ell=0}^{n-1}a_\ell\right)\left(\lim_{n\to\infty}\sum_{m=0}^{n-1}b_m\right) = \left(\sum_{\ell=0}^{\infty}a_\ell\right)\left(\sum_{m=0}^{\infty}b_m\right),\end{aligned}$$

which proves Equation 5.4 for this specific bijection α.

For an arbitrary bijection $\beta:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$, define $\varphi=\alpha^{-1}\circ\beta:\mathbb{N}\to\mathbb{N}$. Then $\beta=\alpha\circ\varphi$ and writing $\beta(n)=(\beta_1(n),\beta_2(n))=(\alpha_1(\varphi(n)),\alpha_2(\varphi(n)))$, the rearrangement Theorem 5.18 yields

$$\sum_{n=0}^{\infty} a_{\beta_1(n)}b_{\beta_2(n)} = \sum_{n=0}^{\infty} a_{\alpha_1(\varphi(n))}b_{\alpha_2(\varphi(n))} = \sum_{n=0}^{\infty} a_{\alpha_1(n)}b_{\alpha_2(n)} = \left(\sum_{n=0}^{\infty}a_n\right)\left(\sum_{n=0}^{\infty}b_n\right). \qquad \square$$

Corollary 5.20: Cauchy Product

If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent, then

$$\left(\sum_{n=0}^{\infty}a_n\right)\left(\sum_{n=0}^{\infty}b_n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_{n-k}b_k\right),$$

and the series on the right converges absolutely.

Proof. Consider the bijection $\alpha:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$ defined as

$$\begin{aligned}&\alpha(0)=(0,0),\quad \alpha(1)=(1,0),\quad \alpha(2)=(0,1),\quad \alpha(3)=(2,0),\quad \alpha(4)=(1,1),\dots,\\ &\alpha(20)=(0,5),\dots,\alpha(31)=(4,3),\dots,\alpha(49)=(5,4),\dots \text{etc.}\end{aligned}$$

By Theorem 5.19,

$$\left(\sum_{n=0}^{\infty}a_n\right)\left(\sum_{n=0}^{\infty}b_n\right) = \sum_{n=0}^{\infty} a_{\alpha_1(n)}b_{\alpha_2(n)}.$$

Listing the terms explicitly and grouping them by diagonals, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} a_{\alpha_1(n)} b_{\alpha_2(n)} &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_2 b_0 + a_1 b_1 + a_0 b_2) \\ &\quad + (a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3) + \dots \\ &= \sum_{n=0}^{\infty} \Bigg(\sum_{\substack{j,k \geq 0 \\ j+k=n}} a_j b_k \Bigg) = \sum_{n=0}^{\infty} \Bigg(\sum_{k=0}^{n} a_{n-k} b_k \Bigg).\end{aligned}$$

Finally, absolute convergence follows from the triangle inequality and Theorem 5.19, i.e.,

$$\sum_{n=0}^{\infty} \left| \sum_{k=0}^{n} a_{n-k} b_k \right| \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |a_{n-k} b_k| = \sum_{n=0}^{\infty} |a_{\alpha_1(n)}| |b_{\alpha_2(n)}| < \infty. \qquad \square$$

5.3 Series of Complex Numbers

To define the notion of a convergent series in $\mathbb{C}$, it is sufficient to consider separately the corresponding series of its real and imaginary parts in $\mathbb{R}$.

Definition 5.21: Series of Complex Numbers

Let $(z_n)_{n=0}^{\infty} = (x_n + iy_n)_{n=0}^{\infty}$ be a sequence of complex numbers, and let $Z = A + iB \in \mathbb{C}$. The series $\sum_{n=0}^{\infty} z_n$ is said to **converge** to Z if both real series $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ converge, with limits A and B, respectively, i.e.,

$$\sum_{n=0}^{\infty} x_n = A, \qquad \sum_{n=0}^{\infty} y_n = B.$$

We say that $\sum_{n=0}^{\infty} z_n$ **converges absolutely** if the series of moduli $\sum_{n=0}^{\infty} |z_n|$ converges.

Whenever the series $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=0}^{\infty} w_n$ converge absolutely, their sum and product are given (exactly as in the real case) by

$$\sum_{n=0}^{\infty} z_n + \sum_{n=0}^{\infty} w_n = \sum_{n=0}^{\infty} z_n + w_n \tag{5.5a}$$

$$\Bigg(\sum_{n=0}^{\infty} z_n \Bigg) \Bigg(\sum_{n=0}^{\infty} w_n \Bigg) = \sum_{n=0}^{\infty} \Bigg(\sum_{k=0}^{n} z_{n-k} w_k \Bigg). \tag{5.5b}$$

Remark 5.22. *Let $(z_n)_{n=0}^{\infty} = (x_n + iy_n)_{n=0}^{\infty}$ be a sequence of complex numbers, and assume that the series $\sum_{n=0}^{\infty} |z_n|$ converges. Since*

$$0 \leq |x_n| \leq |z_n|, \qquad 0 \leq |y_n| \leq |z_n|, \qquad \forall n \in \mathbb{N},$$

the Majorant Criterion (Corollary 5.6) implies that both $\sum_{n=0}^{\infty} |x_n|$ and $\sum_{n=0}^{\infty} |y_n|$ converge. Hence, the series of real and imaginary parts are absolutely convergent.

Conversely since $|z_n| \leq |x_n| + |y_n|$, the absolute convergence of $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ also implies the absolute convergence of $\sum_{n=0}^{\infty} z_n$. Therefore, absolute convergence in $\mathbb{C}$ is equivalent to absolute convergence of the real and imaginary parts.

5.4 Power Series

Our next goal is to investigate power series. These are series where the terms are powers of the variable $x \in \mathbb{R}$ (or $z \in \mathbb{C}$, if one considers complex power series) multiplied by coefficients.

5.4.1 Radius of Convergence

Definition 5.23: Power Series

A **power series** with real coefficients is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n,$$

where $(a_n)_{n=0}^{\infty}$ is a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$. Here x is the **variable** and a_n is the **coefficient** of x^n. By convention, we set $x^0 = 1$ for all $x \in \mathbb{R}$, including $x = 0$. In other words, the first term of the power series is always a_0.

Addition and Multiplication of power series are given by

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n,$$
$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_{n-k} x^{n-k} b_k x^k\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{n-k} b_k\right) x^n,$$

where the product formula follows from Corollary 5.20.

A power series is a polynomial whenever only finitely many coefficients are *non-zero*.

Definition 5.24: Radius of Convergence

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series, and define

$$\rho := \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

The **radius of convergence** is defined as

$$R := \begin{cases} 0, & \rho = \infty, \\ \rho^{-1}, & 0 < \rho < \infty, \\ \infty, & \rho = 0. \end{cases}$$

In the following, when we write $R \in [0, \infty]$, we mean that R is either a non-negative real number or $R = \infty$.

Theorem 5.25: Convergence of Power Series

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R \in (0, \infty]$. Then $\sum_{n=0}^{\infty}$ converges absolutely for all $x \in \mathbb{R}$ with $|x| < R$, and does not converge for all $x \in \mathbb{R}$ with $|x| > R$. In particular, for $x \in (-R, R)$, we can define the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Proof. Let $x \in \mathbb{R}$, and write $\rho = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$ as in Definition 5.24. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n x^n|} = \left(\limsup_{n\to\infty} \sqrt[n]{|a_n|}\right)|x| = \rho|x|.$$

By the root criterion (see Proposition 5.15), the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely if$\rho|x| < 1$, and does not converge if $\rho|x| > 1$ (in particular if $\rho = 0$ it converges for all $x \in \mathbb{R}$). Since $R = \frac{1}{\rho}$, the result follows. $\square$

Theorem 5.26: Continuity of Power Series

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R \in (0, \infty]$, and define polynomials $f_n(x) = \sum_{k=0}^{n} a_k x^k$. For any $r \in (0, R)$, the sequence $(f_n)_{n=0}^{\infty}$ converges uniformly to f on $[-r, r]$. In particular, the power series defines a continuous function $f : (-R, R) \to \mathbb{R}$.

Proof. By Theorem 5.25 with $x = r$, the series $\sum_{n=0}^{\infty} |a_n| r^n$ converges, where $r < R$. Hence, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} |a_k| r^k < \varepsilon.$$

Thus, for all $x \in [-r, r]$ and all $n \geq N$,

$$|f_n(x) - f(x)| = \left|\sum_{k=n+1}^{\infty} a_k x^k\right| \leq \sum_{k=n+1}^{\infty} |a_k||x^k| \leq \sum_{k=N+1}^{\infty} |a_k| r^k < \varepsilon.$$

This shows that $(f_n)_{n=0}^{\infty}$ converges uniformly to f on $[-r, r]$. Since each f_n is continuous (being a polynomial), Theorem 4.52 implies that f is continuous on $[-r, r]$. As $r < R$ is arbitrary, f is continuous on $(-R, R)$. $\square$

Example

In general, the partial sums $f_n(x) = \sum_{k=0}^{n} a_k x^k$ do *not* converge uniformly to $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on the whole interval $(-R, R)$.

To see this, consider the geometric series $\sum_{n=0}^{\infty} x^n$. Its radius of Convergence is $R = 1$, and on $(-1, 1)$ we have $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If the convergence on $(-1, 1)$ were uniformly, then applying the notion of uniform convergence with $\varepsilon = 1$ would give $N \in \mathbb{N}$ such that, for all $n \geq N$ and $x \in (-1, 1)$,

$$\left|\sum_{k=0}^{n} x^k - \frac{1}{1-x}\right| < 1.$$

Taking $n = N$ and using the triangle inequality, we would get

$$\left|\frac{1}{1-x}\right| < 1 + \left|\sum_{k=0}^{N} x^k\right| \leq 1 + \sum_{k=0}^{N} |x^k| \leq 1 + (N+1) = N + 2 \qquad \forall x \in (-1, 1),$$

a contradiction since $\lim_{x\to 1^-} \frac{1}{1-x} = \infty$.

Proposition 5.27: Radius of Convergence of Sum and Product

Let Rgeq0, and let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be power series with radius of convergence of at least R. Then their sum and their Cauchy product also have radius of convergence of at least R.

Proof. By linearity and Corollary 5.20, the absolute convergence of $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ for $|x| < R$ implies that

$$\sum_{n=0}^{\infty}(a_n + b_n)x^n, \qquad \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_{n-k} b_k\right) x^n$$

both converge absolutely for $|x| < R$. Since a power series cannot converge for $|x| > R$, each has radius of convergence of at least R. $\square$

5.4.2 Complex Power Series

Definition 5.28: Complex Power Series

complex power series with complex coefficients is a series of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $(a_n)_{n=0}^{\infty}$ is a sequence in $\mathbb{C}$ and $z \in \mathbb{C}$. Again by convention, we set $z^0 = 1$ for all $z \in \mathbb{C}$, including $z = 0$.

Addition and multiplication are defined as in the real case.

Theorem 5.29: Convergence of Complex Power Series

Let $\sum_{n=0}^{\infty} a_n z^n$ be a complex power series with radius of convergence $R \in (0, \infty]$. Then the series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all $z \in \mathbb{C}$ with $|z| < R$, and diverges for all $z \in \mathbb{C}$ with $|z| > R$. In particular, for $|z| < R$ one can define the (complex-valued) function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

5.5 Exponential and Trigonometric Functions

5.5.1 The Exponential Map as a Power Series

We now show that the exponential map can also be defined via the **exponential series**

$$\exp(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{5.6}$$

Since $\frac{n!}{(n+1)!} = \frac{1}{(n+1)} \to 0$ as $n \to \infty$, it follows directly from the quotient criterion that this series has infinite radius of convergence. Hence, Theorem 5.26 implies that the right-hand side of Equation 5.6 defines a continuous function on $\mathbb{R}$.

Proposition 5.30: Exponential Map as a Power Series

For every $x \in \mathbb{R}$,

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

Definition 5.31: The Complex Exponential Map

The **complex exponential map** is the function $\exp : \mathbb{C} \to \mathbb{C}$ defined by

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad z \in \mathbb{C}.$$

Theorem 5.32: Properties of the Complex Exponential

The complex exponential map $\exp : \mathbb{C} \to \mathbb{C}$ is continuous, and for all $z, w \in \mathbb{C}$,

$$e^{z+w} = e^z e^w, \quad |e^z| = e^{\mathrm{Re}(z)}.$$

In particular, $|e^{ix}| = 1$ for all $x \in \mathbb{R}$.

5.5.2 Sine and Cosine

Given $x \in \mathbb{R}$, we split the power series of e^{ix} into its even and odd terms, i.e.,

$$e^{ix} = \sum_{n=0}^{\infty} \frac{i^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{i^{2n+1}}{(2n+1)!} x^{2n+1}.$$

Since $i^{2n} = (-1)^n$ and $i^{2n+1} = i(-1)^n$, we obtain

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

This motivates the following definition of the **sine** and **cosine functions**, i.e.,

$$\sin(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \qquad \cos(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \tag{5.7}$$

so that the identity

$$e^{ix} = \cos(x) + i \sin(x)$$

holds for all $x \in \mathbb{R}$.

As for the exponential series, radius of convergence of the power series in Equation 5.7 is infinite. Therefore, by Theorems 5.25 and 5.26, sin and cos are continuous on $\mathbb{R}$.

Since $(-x)^{2n+1} = -x^{2n+1}$ and $(-x)^{2n} = x^{2n}$ for all $n \in \mathbb{N}$, it follows directly from Equation 5.7 that

$$\sin(-x) = -\sin(x) \text{ sin is an } \textbf{odd} \text{ function},$$
$$\cos(-x) = \cos(x) \text{ cos is an } \textbf{even} \text{ function}.$$

Theorem 5.33: From the Complex Exponential to Sine and Cosine

For all $x \in \mathbb{R}$, the relations

$$e^{ix} = \cos(x) + i\sin(x), \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

hold. For all $x, y \in \mathbb{R}$ the trigonometric addition formulas are

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

5.5.3 The Circle Number

Theorem 5.34: Existence of π as the First Zero of Sine

There exists exactly one number $\pi \in (0,4)$ such that $\sin(\pi) = 0$. For this number it holds that

$$e^{i\frac{\pi}{2}} = i, \qquad e^{i\pi} = -1, \qquad e^{i2\pi} = 1.$$

Corollary 5.35: Periodicity of Sine and Cosine

$$\sin(x + \frac{\pi}{2}) = \cos(x), \qquad \cos(x + \frac{\pi}{2}) = -\sin(x)$$
$$\sin(x+\pi) = -\sin(x), \qquad \cos(x+\pi) = -\cos(x),$$
$$\sin(x+2\pi) = \sin(x), \qquad \cos(x+2\pi) = \cos(x).$$

5.5.4 Polar Coordinates and Multiplication of Complex Numbers

Using the complex exponential function, we can express complex numbers in **polar coordinates**, i.e., in the form

$$z = re^{i\theta} = r\cos(\theta) + ir\sin(\theta),$$

where $r = |z|$ is the distance of z from the origin, and θ is the angle between the positive real axis $\mathbb{R}_{\geq 0}$ and the segment from 0 to z. In other words, if $z = x + iy$, then

$$x = r\cos(\theta), \qquad y = r\sin(\theta), \qquad r = \sqrt{x^2 + y^2}.$$

If $z \neq 0$, the angle θ is uniquely determined and is called the **argument** of z, denoted $\theta = \arg(z)$. The set of all complex numbers with absolute value 1 is

$$\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{i\theta} \mid \theta \in [0, 2\pi)\},$$

and is called the **unit circle** in $\mathbb{C}$.

Proposition 5.36: Existence of Polar Coordinates

For every $z \in \mathbb{C} \setminus \{0\}$, there exists uniquely determined real numbers $r > 0$ and $\theta \in [0, 2\pi)$ such that $z = re^{i\theta}$.

In polar coordinates, multiplication of complex numbers takes a simple geometric form: if $z = re^{i\varphi}$ and $w = se^{i\psi}$, then

$$zw = rse^{i(\varphi+\psi)}.$$

Thus, when multiplying two complex numbers, their magnitudes multiply and their arguments add.

5.5.5 Other Trigonometric and Hyperbolic Functions

In addition to sine and cosine functions, several related **trigonometric functions** are defined.

The **tangent** and **cotangent** functions are defined by

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \qquad \cot(x) = \frac{\cos(x)}{\sin(x)},$$

for all $x \in \mathbb{R}$ such that the denominator are non-zero.

The **hyperbolic sine** and **hyperbolic cosine** are defined by the power series

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \qquad \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Equivalently,

$$\sinh(x) = \frac{e^{ix} - e^{-ix}}{2}, \qquad \cosh(x) = \frac{e^{ix} + e^{-ix}}{2},$$

and hence $e^x = \sinh(x) + \cosh(x)$ for all $x \in \mathbb{R}$.

The **hyperbolic tangent** and **hyperbolic cotangent** are defined by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}, \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}},$$

where $\coth(x)$ is defined for all $x \in \mathbb{R} \setminus \{0\}$ (since $\sinh(x) \neq 0$ for all $x \neq 0$)

The functions sinh and tanh are odd, while cosh is even. Also, they satisfy the addition formulas

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y),$$
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y),$$

and the **hyperbolic identity**

$$\cosh^2(x) - \sinh^2(x) = 1 \qquad \forall x \in \mathbb{R}.$$

6 Differential Calculus

In this chapter we deal with differential calculus in one variable. This is of fundamental importance for understanding functions on $\mathbb{R}$.

6.1 The Derivative

6.1.1 Definition and Geometrical Interpretation

In this section $D \subseteq \mathbb{R}$ denotes a non-empty set with no isolated points, i.e., every $x \in D$ is an accumulation point of $D \setminus \{x\}$. A typical example is a non-empty interval containing more than one point.

Definition 6.1: Derivative

Let $f : D \to \mathbb{R}$ be a function and $x_0 \in D$. We say that f is **differentiable** at x_0 if the limit

$$f'(x_0) = \lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\substack{h \to 0 \\ h \neq 0}} \frac{f(x_0 + h) - f(x_0)}{h} \tag{6.1}$$

exists. In this case we call $f'(x_0)$ the **derivative** of f at x_0. If f is differentiable at every point of D, we say that f is **differentiable** on D, and we call the resulting function $f' : D \to \mathbb{R}$ the **derivative** of f.

To simplify notation, we will often write

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

without explicitly mentioning that $x \neq x_0$ and $h \neq 0$. Note that the condition

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

can be rewritten as

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0,$$

or equivalently, using the little-o notation from Definition 4.44,

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = o(x - x_0). \tag{6.2}$$

Remark 6.2. *If $f : D \to \mathbb{R}$ is differentiable at x_0, then f is also continuous at x_0. Indeed, using Equation 6.2,*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} \left(f(x_0) + f'(x_0)(x - x_0) + o(x - x_0) \right) = f(x_0),$$

hence f is continuous at x_0.

An alternative notation for the derivative of f is $\frac{df}{dx}$. If $x_0 \in D$ is a right accumulation point of D, then f is **differentiable from the right** at x_0 if the **right derivative**

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. **Differentiablility from the left** and the **left derivative** $f'_-(x_0)$ are defined analogously using the limit $x \to x_0^-$.

If $f : D \to \mathbb{R}$ is differentiable at $x_0 \in D$, the function $x \mapsto f(x_0) + f'(x_0)(x - x_0)$ is called the **affine approximation** of f at x_0.

Definition 6.3: Higher Derivatives

Let $f : D \to \mathbb{R}$ be a function. We define the **higher derivatives** of f, if they exist, by

$$f^{(0)} = f, \qquad f^{(1)} = f', \qquad f^{(2)} = f'', \; \dots \; , f^{(n+1)} = (f^{(n)})'$$

for all $n \in \mathbb{N}$. If $f^{(n)}$ exists, we say that f is **n-times differentiable**. If the n-th derivative is also continuous, we say that f is **n-times continuously differentiable**. We denote the set of n-times continuously differentiable functions on D by $C^n(D)$.

Equivalently, $C^0(D)$ is the set of real-valued continuous function on D, and $C^1(D)$ is the set of all differentiable functions whose derivative is continuous (these are called **continuously differentiable** or of **class C^1**). Recursively, for $n \geq 1$, we define

$$C^n(D) = \{f : D \to \mathbb{R} \mid f \text{ is differentiable and } f' \in C^{n-1}(D)\},$$

and we say that $f \in C^n(D)$ is of **class C^n**.

Definition 6.4: Smooth Functions

We define

$$C^\infty(D) = \bigcap_{n=0}^{\infty} C^n(D) = \{f : D \to \mathbb{R} \mid f \text{ is differentiable infinitely many times}\},$$

and call functions $f \in C^\infty(D)$ **smooth** or of **class C^∞**.

The exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is smooth.

6.1.2 Differentiation Rules

As with continuous functions, we rarely reprove differentiability from first principles for each new example. Instead, we use general rules that reduce differentiability of compound expressions to that of simpler ones.

Proposition 6.5: Derivative of Sum and Product

Let $D \subseteq \mathbb{R}$ and $x_0 \in D$ be an accumulation point of $D \setminus \{x_0\}$. Let $f, g : D \to \mathbb{R}$ be differentiable at x_0. Then $f + g$ and $f \cdot g$ are differentiable at x_0, and

$$(f + g)'(x_0) = f'(x_0) + g'(x_0), \tag{6.3a}$$

$$(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0). \tag{6.3b}$$

In particular, for any $\alpha \in \mathbb{R}$, the scalar multiple αf is differentiable at x_0 and $(\alpha f)'(x_0) = \alpha f'(x_0)$.

Proof. Using the properties of limit discussed in Section 4.5.1, we have

$$\begin{aligned}
\lim_{x\to x_0} \frac{(f+g)(x)-(f+g)(x_0)}{x-x_0} &= \lim_{x\to x_0} \left(\frac{f(x)-f(x_0)}{x-x_0} + \frac{g(x)g(x_0)}{x-x_0}\right) \\
&= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} + \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} \\
&= f'(x_0)+g'(x_0),
\end{aligned}$$

and

$$\begin{aligned}
\lim_{x\to x_0} \frac{(fg)(x)-(fg)(x_0)}{x-x_0} &= \lim_{x\to x_0} \frac{(f(x)-f(x_0))g(x)+f(x_0)(g(x)-g(x_0))}{x-x_0} \\
&= \lim_{x\to x_0} \left(\frac{f(x)-f(x_0)}{x-x_0} g(x)\right) + f(x_0) \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} \\
&= \lim_{x\to x_0} \left(\frac{f(x)-f(x_0)}{x-x_0}\right) \cdot \left(\lim_{x\to x_0} g(x)\right) + f(x_0) \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} \\
&= f'(x_0)g(x_0)+f(x_0)g'(x_0),
\end{aligned}$$

where we used that g is continuous at x_0 (see Remark 6.2) to conclude $\lim_{x\to x_0} g(x) = g(x_0)$. □

Corollary 6.6: Higher Order Derivatives of the Sum and Product

Let $f, g : D \to \mathbb{R}$ be n-times differentiable. Then $f+g$ and $f \cdot g$ are also n-times differentiable, and

$$\begin{aligned}
(f+g)^{(n)} &= f^{(n)} + g^{(n)}, \\
(fg)^{(n)} &= \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.
\end{aligned}$$

In particular, for every $\alpha \in \mathbb{R}$, $(\alpha f)^{(n)} = \alpha f^{(n)}$.

Proof. For $n = 1$ this is Proposition 6.5. The general case for $n \geq 1$ follows by induction. □

Corollary 6.7: Derivatives of Polynomials

Polynomial functions are differentiable on all of $\mathbb{R}$. Moreover $(1)' = 0$ and $(x^n)' = nx^{n-1}$ for all $n \geq 1$.

Proof. We argue by induction. For $n = 0$, we have

$$(1)' = \lim_{x\to x_0} = \frac{1-1}{x-x_0} = 0.$$

For $n = 1$, we have

$$(x)' = \lim_{x\to x_0} \frac{x-x_0}{x-x_0} = 1.$$

For $n > 1$, assume $(x^n)' = nx^{n-1}$. Then by Equation 6.3b, since $x^{n+1} = x \cdot x^n$, we have

$$(x^{n+1})' = (x \cdot x^n)' = 1 \cdot x^n + x \cdot nx^{n-1} = (n+1)x^n.$$

This proves the inductive step and establishes the result. Finally, the linearity of the derivative (see Equation 6.3a) yields the differentiability of any polynomial. □

Example

With $\alpha = \pm 1$ and $\alpha = \pm i$, we have

$$(e^x)' = e^x, \qquad (e^{-x})' = -e^{-x}, \qquad (e^{ix})' = ie^{ix}, \qquad (e^{-ix})' = -ie^{-ix}.$$

By Theorem 5.33, we have

$$\sin'(x) = \frac{(e^{ix})' - (e^{-ix})'}{2i} = \frac{e^{ix} + e^{ix}}{2} = \cos(x),$$

and analogously, $\cos'(x) = -\sin(x)$. Similarly $\sinh'(x) = \cosh(x)$ and $\cosh'(x) = \sinh(x)$.

Theorem 6.8: Chain Rule

Let $D, E \subseteq \mathbb{R}$ and let $x_0 \in D$ be an accumulation point of $D \setminus \{x_0\}$. Let $f : D \to E$ be differentiable at x_0 such that $y_0 = f(x_0)$ is an accumulation point of $E \setminus \{y_0\}$, and let $g : E \to \mathbb{R}$ be differentiable at y_0. Then $g \circ f : D \to \mathbb{R}$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Proof. Observer that we can write

$$\begin{aligned} g(y) &= g(y_0)[g(y) - g(y_0)] \\ &= g(y_0) + g'(y_0)(y - y_0) + [g(y) - g(y_0) - g'(y_0)(y - y_0)] \\ &= g(y_0) + g'(y_0)(y - y_0) + \omega(y)(y - y_0), \end{aligned}$$

where $\omega : E \to \mathbb{R}$ is defined as

$$\omega(y) = \begin{cases} \dfrac{g(y) - g(y_0)}{y - y_0} - g'(y_0) & \text{for } y \in E \setminus \{y_0\}, \\ 0 & \text{for } y = y_0. \end{cases}$$

Since g is continuous at y_0, it follows that $\omega(y) \to 0$ as $y \to y_0$; hence, the function ω is continuous at y_0. Substituting $y = f(x)$ and using $y_0 = f(x_0)$, we get

$$g(f(x)) = g(f(x_0)) + g'(f(x_0))[f(x) - f(x_0)] + \omega(f(x))[f(x) - f(x_0)],$$

therefore

$$\begin{aligned} \lim_{x \to x_0} \frac{g(g(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \to x_0} \left(g'(f(x_0)) \frac{f(x) - f(x_0)}{x - x_0} + \omega(f(x)) \frac{f(x) - f(x_0)}{x - x_0} \right) \\ &= g'(f(x_0))f'(x_0) + \underbrace{\omega(f(x_0))}_{=0} f'(x_0) = g'(f(x_0))f'(x_0), \end{aligned}$$

where we used the continuity of ω at $y_0 = f(x_0)$ to deduce that $\omega(f(x)) \to \omega(f(x_0))$ as $x \to x_0$. $\square$

Corollary 6.9: Quotient Rule

Let $D \subseteq \mathbb{R}$. let x_0 be an accumulation point of $D \setminus \{x_0\}$, and let $f, g : D \to \mathbb{R}$ be differentiable at x_0. If $g(x_0) \neq 0$, then $\frac{f}{g}$ is differentiable at x_0, and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Proof. Consider the function $\psi : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $\psi(y) = \frac{1}{y}$. This function is differentiable, with $\phi'(y) = -\frac{1}{y^2}$. Then, by the chain rule (Theorem 6.8), $\frac{1}{g} = \psi \circ g$ is differentiable at x_0, with

$$\left(\frac{1}{g}\right)'(x_0) = \psi'(g(x_0))g'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}.$$

Applying now the product rule (Proposition 6.5), $\frac{f}{g} = f \cdot \frac{1}{g}$ is differentiable at x_0, and

$$\left(\frac{f}{g}\right)'(x_0) = \left(f \cdot \frac{1}{g}\right)'(x_0) = f'(x_0)\frac{1}{g(x_0)} - f(x_0)\frac{g'(x_0)}{g(x_0)^2} = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0^2)}.$$ □

Theorem 6.10: Derivative of the Inverse

Let $D, E \subseteq \mathbb{R}$, and let $f : D \to E$ be a continuous bijection whose inverse $f^{-1} : E \to D$ is also continuous. Let $\bar{x} \in D$ be an accumulation point of $D \setminus \{\bar{x}\}$, and assume f is differentiable at $\bar{x}$ with $f'(\bar{x}) \neq 0$. Then f^{-1} is differentiable at $\bar{y} = f(\bar{x})$ and

$$(f^{-1})'(\bar{y}) = \frac{1}{f'(\bar{x})} = \frac{1}{f'(f^{-1}(\bar{y}))}.$$

Proof. To compute $(f^{-1})'(\bar{y})$, take a sequence $(y_n)_{n=0}^{\infty} \subseteq E \setminus \{\bar{y}\}$ with $y_n \to \bar{y}$ and set $x_n = f^{-1}(y_n)$. Then

$$\frac{f^{-1}(y_n) - f^{-1}(\bar{y})}{y_n - \bar{y}} = \frac{x_n - \bar{x}}{f(x_n) - f(\bar{x})} = \left(\frac{f(x_n) - f(\bar{x})}{x_n - \bar{x}}\right)^{-1}.$$

By the continuity of f^{-1} we have $f^{-1}(y_n) = x_n \to \bar{x} = f^{-1}(\bar{y})$ as $y_n \to \bar{y}$. Thus, since f is differentiable at $\bar{x}$ with $f'(\bar{x}) \neq 0$, Proposition 3.12(4) implies

$$\left(\frac{f(x_n) - f(\bar{x})}{x_n - \bar{x}}\right)^{-1} \longrightarrow \frac{1}{f'(\bar{x})},$$

proving that

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(\bar{y})}{y_n - \bar{y}} = \frac{1}{f'(\bar{x})}.$$

Since the sequence $(y_n)_{n=0}^{\infty}$ was arbitrary, Lemma 4.36 gives

$$\lim_{y\to\bar{y}} \frac{f^{-1}(y) - f^{-1}(\bar{y})}{y - \bar{y}} = \frac{1}{f'(\bar{x})},$$

as desired. □

6.2 Main Theorems of Differential Calculus

6.2.1 Local Extrema

Definition 6.11: Local Extrema

Let $D \subseteq \mathbb{R}$ and $x_0 \in D$. We say that a function $f : D \to \mathbb{R}$ has a **local maximum** at x_0, if there exists $\delta > 0$ such that

$$f(x) \leq f(x_0) \qquad \forall x \in D \cap (x_0 - \delta, x_0 + \delta).$$

If the inequality is strict (i.e. $f(x) < f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$), then f has a **strict local maximum** at x_0. A (**strict**) **local minimum** is defined analogously. We call x_0 a **local extremum** if f has either a local maximum or a local minimum at x_0.

Proposition 6.12: Local Extrema vs. First Derivative

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. Suppose $x_0 \in D$ is a local extremum of f, that f is differentiable at x_0 and that x_0 is both a right-hand and a left-hand accumulation point of D. Then

$$f'(x_0) = 0.$$

Proof. W.l.o.g, assume that f has a local maximum at x_0 (otherwise replace f by $-f$). We first note that for x close to x_0 and to the right of it, we have $f(x) - f(x_0) \leq 0$ and $x - x_0 > 0$. Hence,

$$f'_+(x_0) = \lim_{x \to 0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

Similarly for x close to x_0 and to the left of it, we have $f(x) - f(x_0) \leq 0$ and $x - x_0 < 0$, so

$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

Since f is differentiable at x_0, the two one sided derivatives coincide, i.e.,

$$f'(x_0) = f'_+(x_0) = f'_-(x_0) = 0,$$

and therefore $f'(x_0) = 0$. □

Corollary 6.13: Local Extrema in an Inteval

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. If $x_0 \in I$ is a local extrema, then at least one of the following statements holds:

1. *x_0 is an endpoint of I,*
2. *f is not differentiable at x_0,*
3. *f is differentiable at x_0 and $f'(x_0) = 0$.*

In particular, all local extrema of a differentiable function on an open interval are zeros of the derivative.

6.2.2 The Mean Value Theorem

Theorem 6.14: Rolle's Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $\xi \in (a, b)$ with $f'(\xi) = 0$.

Proof. By Theorem 4.20 f attains both its maximum and minimum on $[a, b]$ at some points $x_0, x_1 \in [a, b]$. By Proposition 6.12 any interior extremum has zero derivative. Thus, we consider two cases:

(i) If either x_0 or x_1 lies in (a, b) we are done.

(ii) If both x_0 and x_1 are endpoints, since $f(a) = f(b)$, then $\min f = \max f = f(a) = f(b)$, therefore f is constant. In particular, $f'(x) = 0$ for all $x \in (a, b)$ and the result follows also in this case. □

Corollary 6.15: Non-Vanishing Derivative Implies Different Endpoint Values

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) \neq 0$ for all $x \in (a, b)$, then $f(a) \neq f(b)$.

Proof. Assume, by contradiction, that $f(a) = f(b)$. Then by Rolle's Theorem, there would exists $\xi \in (a, b)$ such that $f'(\xi) = 0$, which would contradict the assumption that f' never vanishes. □

Theorem 6.16: Mean Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then g is continuous on $[a, b]$ and differentiable on (a, b), and satisfies

$$g(a) = f(a), \qquad g(b) = f(b) - (f(b) - f(a)) = f(a).$$

By Rolle's Theorem 6.14, there exists $\xi \in (a, b)$ such that

$$0 = g'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a},$$

proving the result. □

Corollary 6.17: Lipschitz Continuity vs. Bounded Derivative

et $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then f is Lipschitz continuous on $[a, b]$ if and only if f' is bounded on (a, b).

Proof. Suppose first that f is Lipschitz continuous on $[a, b]$ with constant L. This implies that given $x, x_0 \in (a, b)$ with $x \neq x_0$,

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq L.$$

Taking the limit as $x \to x_0$ gives $|f'(x_0)| \leq L$, so f' is bounded on (a, b).

Conversely, suppose that f' is bounded on (a, b), say $f'(z) \leq M$ for all $z \in (a, b)$. Then given $x, y \in [a, b]$ with $x < y$, the Mean Value Theorem 6.16 applied on the interval $[x, y]$ yields $\xi \in (x, y) \subseteq (a, b)$ such that

$$f(y) - f(x) = f'(\xi)(y - x),$$

therefore

$$|f(y) - f(x)| = |f'(\xi)||y - x| \leq M|y - x|.$$

Since $x, y \in [a, b]$ are arbitrary, this shows that f is Lipschitz continuous on $[a, b]$ with Lipschitz constant M. $\square$

Theorem 6.18: Cauchy Mean Value Theroem

Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $\xi \in (a, b)$ such that

$$g'(\xi)(f(b) - f(a)) = f'(\xi)(g(b) - g(a)). \tag{6.4}$$

If, in addition $g'(x) \neq 0$ for all $x \in (a, b)$, then $g(a) \neq g(b)$ and

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Define the function $F : [a, b] \to \mathbb{R}$ as

$$F(x) = g(x)(f(b) - f(a)) - f(x)(g(b) - g(a)).$$

Then

$$F(a) = g(a)(f(b) - f(a)) - f(a)(g(b) - g(a)) = g(a)f(b) - f(a)g(b)$$
$$F(b) = g(b)(f(b) - f(a)) - f(b)(g(b) - g(a)) = g(a)f(b) - f(a)g(b).$$

Thus, by Rolle's Theorem 6.14, there exists $\xi \in (a, b)$ such that

$$F'(\xi) = 0 = g'(\xi)(f(b) - f(a)) - f'(\xi)(g(b) - g(a)),$$

which is Equation 6.4.

If $g'(x) \neq 0$ for all $x \in (a, b)$, then Corollary 6.15 yields $g(a) \neq g(b)$. Dividing Equation 6.4 accordingly gives the second formula. $\square$

6.2.3 L'Hopital's Rule

Theorem 6.19: L'Hopital's Rule

Let $f, g : (a, b) \to \mathbb{R}$ be differentiable. Suppose:

1. $g(x) \neq 0$ *and* $g'(x) \neq 0$ *for all* $x \in (a, b)$*,*
2. $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$*,*
3. *the limit* $L = \lim_{x\to a^+} \dfrac{f'(x)}{g'(x)}$ *exists.*

Then $\lim_{x\to a^+} \dfrac{f(x)}{g(x)}$ *exists and equals* L*.*

Proof. By (2), we can extend f and g continuously to $[a, b]$ by setting $f(a) = g(a) = 0$. Fix $\varepsilon > 0$. By (3), there exists $\delta > 0$ such that

$$\frac{f'(\xi)}{g'(\xi)} \in (L - \varepsilon, L + \varepsilon) \qquad \forall x \in (a, a + \delta).$$

Now, for any $x \in (a, a + \delta)$, we can apply Cauchy's Mean Value Theorem 6.18 to f and g on $[a, x]$ to find some $\xi_x \in (a, x)$ with

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi_x)}{g'(\xi_x)}.$$

Since $\xi_x \in (a, x) \subseteq (a, a + \delta)$, it follows that

$$\frac{f(x)}{g(x)} = \frac{f'(\xi_x)}{g'(\xi_x)} \in (L - \varepsilon, L + \varepsilon) \qquad \forall x \in (a, a + \delta).$$

Because $\varepsilon > 0$ is arbitrary, this proves that $\lim_{x\to a^+} \dfrac{f(x)}{g(x)} = L$. □

Theorem 6.20: L'Hopital's Rule for Improper Limits

Let $f, g : (a, b) \to \mathbb{R}$ be differentiable. Suppose:

1. $g(x) \neq 0$ *and* $g'(x) \neq 0$ *for all* $x \in (a, b)$*,*
2. $\lim_{x\to a^+} |f(x)| = \lim_{x\to a^+} |g(x)| = \infty$*,*
3. *the limit* $L = \lim_{x\to a^+} \dfrac{f'(x)}{g'(x)}$ *exists.*

Then $\lim_{x\to a^+} \dfrac{f(x)}{g(x)}$ *exists and equals* L*.*

Theorem 6.21: L'Hopital's Rule at Infinity

Let $R > 0$ and $f, g : (R, \infty) \to \mathbb{R}$ be differentiable. Suppose:

1. *$g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (R, \infty)$,*
2. *either $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ or $\lim_{x\to\infty} |f(x)| = \lim_{x\to\infty} |g(x)| = \infty$,*
3. *the limit $L = \lim_{x\to\infty} \dfrac{f'(x)}{g'(x)}$ exists.*

Then $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$ exists and equals L.

6.2.4 Monotonicity and Convexity via Differential Calculus

Here I always denotes a non-trivial interval (non-empty and not a single point).

Proposition 6.22: Monotonicity vs. First Derivative

Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be differentiable. Then

$$f' \geq 0 \quad \Leftrightarrow \quad f \text{ is increasing.}$$

Proof. If f is increasing $f(x+h) - f(x) \geq 0$ for $h > 0$, and $f(x+h) - f(x) \leq 0$ for $h < 0$. Hence, in both cases, $\frac{f(x+h)-f(x)}{h} \geq 0$, therefore

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \geq 0.$$

Conversely, assume f is not increasing. Then there exists $x_1 < x_2$ with $f(x_1) > f(x_2)$. By the Mean Value Theorem 6.16, there exists $\xi \in (x_1, x_2)$ with

$$f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0,$$

so $f' \not\geq 0$ on I. □

Remark 6.23. *If $f' > 0$, the same argument shows that f is strictly increasing. However, the converse fails: the function $f(x) = x^3$is strictly increasing but $f'(0) = 0$.*

Corollary 6.24: Constant Functions vs. First Derivative

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. Then f is constant if and only if f is differentiable and $f'(x) = 0$ for all $x \in I$.

Proof. The derivative of a constant function is 0.

Conversely, if $f' = 0$, then $f' \geq 0$ and $-f' \geq 0$, so by Proposition 6.22 both f and $-f$ are increasing, hence f is constant. □

Definition 6.25: Convex Functions

Let $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$. We call f **convex**, if for all $a, b \in I$ with $a < b$ and all $t \in (0, 1)$, it holds that

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b). \tag{6.5}$$

We call f **strictly convex** if the inequality in 6.5 is strict. A function $g : I \to \mathbb{R}$ is (**strictly**) **concave** if $-g$ is (strictly) convex.

An equivalent definition of a convex function is the following: $f : I \to \mathbb{R}$ is convex if for all $a, b \in I$ with $a < b$ and all $x \in (a, b)$, we have

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(x)}{b - x}, \tag{6.6}$$

and strictly convex if the inequality is strict.

Proposition 6.26: Convexity vs. Monotonicity of the First Derivative

Let $I \subseteq \mathbb{R}$ and let $f : I \to \mathbb{R}$ be differentiable. Then f is convex if and only if f' is increasing.

Proof. Assume f' is increasing. Then, for $a < b$ and $x \in (a, b)$, the Mean Value Theorem 6.16 applied on the intervals $[a, x]$ and $[x, b]$ yields $\xi \in (a, x)$ and $\zeta \in (x, b)$ such that

$$f'(\xi) = \frac{f(x) - f(a)}{x - a}, \qquad f'(\zeta) = \frac{f(b) - f(x)}{b - x}.$$

Since f' is increasing, we have $f'(\xi) \leq f'(\zeta)$, so Equation 6.6 follows. Since $a < b \in I$ and $x \in (a, b)$ are arbitrary, f is convex.

Conversely, assume f is convex. Given $a < b$, consider $h > 0$ small enough, such that $a + h < b - h$ and apply Equation 6.6 twice: first, applying it on the interval $(a, b - h)$ with $x = a + h$ we get

$$\frac{f(a+h) - f(a)}{h} \leq \frac{f(b-h) - f(a+h)}{(b-h) - (a+h)};$$

then, applying it on the interval $(a + h, b)$ with $x = b - h$ we get

$$\frac{f(b-h) - f(a+h)}{(b-h) - (a+h)} \leq \frac{f(b) - f(b-h)}{h}.$$

Combining these two inequalities, we deduce that, for small $h > 0$,

$$\frac{f(a+h) - f(a)}{h} \leq \frac{f(b) - f(b-h)}{h}.$$

Letting $h \to 0^+$ gives $f'(a) \leq f'(b)$. Since $a < b$ are arbitrary, f' is increasing. $\square$

Corollary 6.27: Convexity vs. Second Derivative

Let $I \subseteq \mathbb{R}$ and let $f : I \to \mathbb{R}$ be twice differentiable. Then f is convex if and only if $f'' \geq 0$.

Proof. By Proposition 6.26 f is convex if and only if f' is increasing. Applying Proposition 6.22 to f', we have that f' is increasing if and only if $f'' \geq 0$. $\square$

7 The Riemann Integral

7.1 Step Functions and their Integral

7.1.1 Decompositions and Step Functions

Definition 7.1: Partitions

Two sets A, B are called **disjoint** if $A \cap B = \emptyset$. For a collection $\mathcal{A}$ of sets, we say that the sets in $\mathcal{A}$ are pairwise disjoint, if for all $A_1, A_2 \in \mathcal{A}$ with $A_1 \neq A_2$ it holds that $A_1 \cap A_2 = \emptyset$.
Let X be a set. A Partition of X is a family $\mathcal{P}$ of non-empty pairwise disjoint subsets of X such that

$$X = \bigcup_{P \in \mathcal{P}} P.$$

Definition 7.2: Decomposition of an Interval

A **decomposition** of $[a, b]$ is a finite sequence of points

$$a = x_0 < x_1 < \ \dots \ < x_{n-1} < x_n = b,$$

with $n \in \mathbb{N}$. The points $x_0, \dots, x_n$ are called the **division points** of the decomposition.

Formally, a decomposition of $[a, b]$ is a finite subset of $[a, b]$ containing a and b, together with the ordering of its elements. Each decomposition induces a natural partition of $[a, b]$, i.e.,

$$[a, b] = \{x_0\} \cup (x_0, x_1) \cup \{x_1\} \cup \ \dots \ \cup (x_{n-1}, x_n) \cup \{x_n\},$$

which we will use implicitly from now on.

A decomposition

$$a = y_0 < y_1 < \ \dots \ < y_m = b$$

is called a **refinement** of the decomposition

$$a = x_0 < x_1 < \ \dots \ < x_n = b$$

if

$$\{x_0, x_1, \dots, x_n\} \subseteq \{y_0, y_1, \dots, y_m\}.$$

The notion of refinement defines a partial order on the set of all decompositions of $[a, b]$. Note that any two decompositions of $[a, b]$ admit a common refinement given by the union of all division points.

Definition 7.3: Step Functions

A function $f : [a, b] \to \mathbb{R}$ is called a **step function** if there exists a decomposition

$$a = x_0 < x_1 < \ \dots \ < x_n = b$$

such that, for each $k = 1, \dots, n$ the restriction of f to the open interval (x_{k-1}, x_k) is constant. In this case, we say that f is a step function *with respect to* the decomposition $a = x_0 < x_1 < \ \dots \ < x_n = b$.

Proposition 7.4: Linearity of the Space of Step Functions

Let $f, g : [a, b] \to \mathbb{R}$ be step functions, and $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is also a step function.

Proof. Let f be a step function with respect to the decomposition $a = x_0 < x_1 < \ldots < x_n = b$, and let g be a step function with respect to the decomposition $a = y_0 < y_1 < \ldots < y_m = b$. The union of all division points $\{x_0, x_1, \ldots, x_n\} \cup \{y_0, y_1, \ldots, y_m\}$ defines a new decomposition

$$a = z_0 < z_1 < \ldots < z_N = b$$

that is a common refinement of the two. Since both f and g are constant on each open interval (z_{k-1}, z_k), so is the function $\alpha f + \beta g$. Thus, $\alpha f + \beta g$ is also a step function with respect to this decomposition. □

Remark 7.5. *As in the proof of Proposition 7.4, one can show that the product of two step functions is again a step function. Moreover, step functions are bounded, since they take only finitely many values.*

7.1.2 The Integral of a Step Function

Definition 7.6: Integral of a Step Function

Let $f : [a, b] \to \mathbb{R}$ be a step function with respect to a decomposition

$$a = x_0 < x_1 < \ldots < x_n = b.$$

We define the **integral** of f on $[a, b]$ as the real number

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} c_k (x_k - x_{k-1}), \tag{7.1}$$

where c_k denotes the constant value of f on the interval (x_{k-1}, x_k).

Proposition 7.7: Linearity of the Integral of Step Functions

Let $f, g : [a, b] \to \mathbb{R}$ be step functions, and let $\alpha, \beta \in \mathbb{R}$. Then

$$\int_a^b (\alpha f + \beta g)(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

Proof. As in the proof of Proposition 7.4, we can find a decomposition $a = x_0 < x_1 < \ldots < x_n = b$ such that both f and g (and hence $\alpha f + \beta g$) are constant on the interval (x_{k-1}, x_k). If f takes the values c_k and g the values d_k on (x_{k-1}, x_k), then $\alpha f + \beta g$ takes the value $\alpha c_k + \beta d_k$. Thus,

$$\begin{aligned}
\int_a^b (\alpha f + \beta g)(x)\,dx &= \sum_{k=1}^{n} (\alpha c_k + \beta d_k)(x_k - x_{k-1}) \\
&= \alpha \sum_{k=1}^{n} c_k (x_k - x_{k-1}) + \beta \sum_{k=1}^{n} d_k (x_k - x_{k-1}) \\
&= \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx,
\end{aligned}$$

as claimed. □

Proposition 7.8: Monotonicity of the Integral of Step Functions

Let $f, g : [a,b] \to \mathbb{R}$ be step functions such that $f \le g$. Then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Proof. As in the proof of Proposition 7.4, we can find a decomposition $a = x_0 < x_1 < \ \ldots \ < x_n = b$ such that both f and g are constant on each interval (x_{k-1}, x_k). Writing c_k and d_k for their respective values, the assumption $f \le g$ implies that $c_k \le d_k$ for all $k = 1, \ldots, n$. Hence

$$\int_a^b f(x)\,dx = \sum_{k=1}^n c_k(x_k - x_{k-1}) \le \sum_{k=1}^n d_k(x_k - x_{k-1}) = \int_a^b g(x)\,dx.$$ □

Applying Proposition 7.8 with $g \equiv 0$, we deduce the following corollary.

Corollary 7.9: Positivity of the Integral of Step Functions

If $f : [a,b] \to \mathbb{R}$ is a step functions such that $f(x) \ge 0$ for all $x \in [a,b]$, then

$$\int_a^b f(x)\,dx \ge 0.$$

7.2 Definition and First Properties of the Riemann Integral

As in the last section, we consider functions on a compact interval $[a,b] \subseteq \mathbb{R}$. To alleviate notation, we write $\mathcal{SF}$ for the set of step functions on $[a,b]$. Also, we often write $\int_a^b f\,dx$ in place of $\int_a^b f(x)\,dx$.

7.2.1 Integrability of Real-Valued Functions

Before defining lower and upper sums, we recall a simple but useful property of the supremum and infimum of two related sets. We use this fact several times in what follows.

Definition 7.10: Relation Between Supremum and Infimum

Let $A, B \subseteq \mathbb{R}$ be non-empty sets such that $s \le t$ for all $s \in A$ and $t \in B$. Then

$$\sup A \le \inf B. \tag{7.2}$$

Moreover,

$$\sup A = \inf B \quad \Leftrightarrow \quad \forall \varepsilon > 0\ \exists s \in A\ \exists t \in B \text{ such that } t - s < \varepsilon. \tag{7.3}$$

Definition 7.11: Lower and Upper Sums

Let $f : [a,b] \to \mathbb{R}$ be a function. Define the sets of **lower sums** $\mathcal{L}(f) \subseteq \mathbb{R}$ and **upper sums** $\mathcal{U}(f) \subseteq \mathbb{R}$ by

$$\mathcal{L}(f) = \left\{ \int_a^b \ell\,dx \,\middle|\, \ell \in \mathcal{SF} \text{ and } \ell \le f \right\}, \qquad \mathcal{U}(f) = \left\{ \int_a^b u\,dx \,\middle|\, u \in \mathcal{SF} \text{ and } f \le u \right\}.$$

If f is bounded, then these sets are non-empty. Indeed, if $|f| \leq M$, then the constant step functions

$$\ell(x) = -M \quad \forall x \in [a,b], \qquad u(x) = M \quad \forall x \in [a,b],$$

satisfy $\ell \in \mathcal{L}(f)$ and $u \in \mathcal{U}(f)$.

For $\ell, u \in \mathcal{SF}$ with $\ell \leq f \leq u$, Proposition **??** gives

$$\int_a^b \ell \, dx \leq \int_a^b u \, dx.$$

This implies that $s \leq t$ for all $s \in \mathcal{L}(f)$ and $t \in \mathcal{U}(f)$, so Equation 7.2 yields

$$\sup \mathcal{L}(f) \leq \inf \mathcal{U}(f).$$

Definition 7.12: Riemann Integral

A bounded function $f : [a,b] \to \mathbb{R}$ is **Riemann integrable** if $\sup \mathcal{L}(f) = \inf \mathcal{U}(f)$. In this case, this common value is called the **Riemann integral** of f, and we write

$$\int_a^b f \, dx = \sup \mathcal{L}(f) = \inf \mathcal{U}(f).$$

We call a the **lower (integration) limit** and b the **upper (integration) limit**, and the function f the **integrand** of the integral $\int_a^b f \, dx$. If $f \geq 0$ is Riemann integrable, we interpret the number $\int_a^b f \, dx$ as the **area** of the set

$$\{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad 0 \leq y \leq f(x)\}.$$

Proposition 7.13: Riemann Integrability Condition

Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable if and only if for every $\varepsilon > 0$ there exists step functions $\ell, u \in \mathcal{SF}$ such that

$$\ell \leq f \leq u \qquad \textit{and} \qquad \int_a^b (u - \ell) < \varepsilon.$$

In this case,

$$\left| \int_a^b f \, dx - \int_a^b \ell \, dx \right| < \varepsilon, \qquad \left| \int_a^b u \, dx - \int_a^b f \, dx \right| < \varepsilon.$$

Proof. By Equation 7.3 applied with $A = \sup \mathcal{L}(f)$ and $B = \inf \mathcal{U}(f)$ we obtain

$$\begin{aligned}
f \text{ is Riemann integrable} \quad &\Leftrightarrow \quad \sup \mathcal{L}(f) = \mathcal{U}(f) \\
&\Leftrightarrow \quad \forall \varepsilon > 0 \; \exists s \in \mathcal{L}(f) \, \exists t \in \mathcal{U}(f) : \; t - s < \varepsilon \\
&\Leftrightarrow \quad \forall \varepsilon > 0 \; \exists \ell, u \in \mathcal{SF} : \; \ell \leq f \leq u \text{ and } \int_a^b u \, dx - \int_a^b \ell \, dx < \varepsilon \\
&\Leftrightarrow \quad \forall \varepsilon > 0 \; \exists \ell, u \in \mathcal{SF} : \; \ell \leq f \leq u \text{ and } \int_a^b (u - \ell) \, dx < \varepsilon,
\end{aligned}$$

where we used Proposition 7.7 to deduce that

$$\int_a^b u\,dx - \int_a^b \ell\,dx = \int_a^b (u-\ell)\,dx$$

Finally, the concluding inequalities follow from

$$\int_a^b \ell\,dx \le \int_a^b f\,dx \le \int_a^b u\,dx \qquad \text{and} \qquad \int_a^b (u-\ell)\,dx < \varepsilon. \qquad \square$$

Example

Not all functions are Riemann integrable. Indeed, consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

We claim that f is not Riemann integrable.

Let $u \in \mathcal{SF}$ with $f \le u$, and let $0 = x_0 < \ \ldots \ < x_1 = 1$ be a decomposition such that u is constant c_k on (x_{k-1}, x_k). Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $x \in (x_{k-1}, x_k) \cap \mathbb{Q}$, hence $1 = f(x) \le u(x) = c_k$, so $c_k \ge 1$. Therefore,

$$\int_0^1 u(x)\,dx = \sum_{k=1}^n c_k(x_k - x_{k-1}) \ge \sum_{k=1}^n (x_k - x_{k-1}) = x_n - x_0 = 1.$$

Thus $\inf \mathcal{U}(f) = 1$, and taking $u \equiv 1$ gives $\inf \mathcal{U}(f)$. A similar argument with lower sums shows that $\sup \mathcal{L}(f) = 0$. Hence, f is not Riemann integrable.

Theorem 7.14: Linearity of the Riemann Integral

If $f, g : [a,b] \to \mathbb{R}$ are integrable and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is integrable, and

$$\int_a^b (\alpha f + \beta g)\,dx = \alpha \int_a^b f\,dx + \beta \int_a^b g\,dx$$

Proof. Given $\varepsilon > 0$, Proposition 7.13 yields step functions ℓ_1, ℓ_2, u_1, u_2 with

$$\ell_1 \le f \le u_1, \quad \ell_2 \le g \le u_2, \quad \int_a^b (u_1 - \ell_1)\,dx < \varepsilon, \quad \int_a^b (u_2 - \ell_2)\,dx < \varepsilon,$$

and

$$\left| \int_a^b f\,dx - \int_a^b \ell_1\,dx \right| < \varepsilon, \qquad \left| \int_a^b g\,dx - \int_a^b \ell_2\,dx \right| < \varepsilon.$$

Assume first $\alpha, \beta \ge 0$. Then

$$\alpha\ell_1 + \beta\ell_2 \le \alpha f + \beta g \le \alpha u_1 + \beta u_2,$$

and

$$\int_a^b [(\alpha u_1 + \beta u_2) - (\alpha\ell_1 + \beta\ell_2)]\,dx = \alpha \int_a^b (u_1 - \ell_1)\,dx + \beta \int_a^b (u_2 - \ell_2)\,dx < (\alpha + \beta)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves that $\alpha f + \beta g$ is integrable. Moreover, by the triangle inequality and Proposition 7.7, we have that

$$\begin{aligned}\left|\int_a^b(\alpha f+\beta g)\,dx-\alpha\int_a^b f\,dx-\beta\int_a^b g\,dx\right| &\le \left|\int_a^b(\alpha f+\beta g)\,dx-\int_a^b(\alpha\ell_1+\beta\ell_2)\,dx\right|\\ &+\underbrace{\left|\int_a^b(\alpha\ell_1+\beta\ell_2)\,dx-\alpha\int_a^b\ell_1\,dx-\beta\int_a^b\ell_2\,dx\right|}_{=0}\\ &+\alpha\left|\int_a^b\ell_1\,dx-\int_a^b f\,dx\right|+\beta\left|\int_a^b\ell_2\,dx-\int_a^b g\,dx\right|\\ &\le(\alpha+\beta)\varepsilon+\alpha\varepsilon+\beta\varepsilon=2(\alpha+\beta)\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, the linearity follows.

The case when one of α,β is negative is analogous, but one needs to reverse the corresponding inequalities. For instance, if $\alpha\ge 0$ and $\beta<0$, then

$$\alpha\ell_1+\beta u_2\le\alpha f+\beta g\le\alpha u_1+\beta\ell_2,$$

and

$$\int_a^b[(\alpha u_1+\beta\ell_2)-(\alpha\ell_1+\beta u_2)]\,dx=\alpha\int_a^b(u_1-\ell_1)\,dx+|\beta|\int_a^b(u_2-\ell_2)\,dx<(\alpha+|\beta|)\varepsilon.$$

This implies again that $\alpha f+\beta g$ is integrabl, and the linearity identity holds similarly. □

Proposition 7.15: Monotonicity of the Riemann Integral

Let $f,g:[a,b]\to\mathbb{R}$ be integrable. If $f\le g$, then

$$\int_a^b f\,dx\le\int_a^b g\,dx.$$

Proof. Since $f\le g$, for any step function ℓ with $\ell\le f$ we have $\ell\le g$. This implies that $\mathcal{L}(f)\subseteq\mathcal{L}(g)$, therefore

$$\int_a^b f\,dx=\sup\mathcal{L}(f)\le\sup\mathcal{L}(g)=\int_a^b g\,dx.$$ □

Definition 7.16: Positive and Negative Parts

Given a function $f:D\to\mathbb{R}$, we define its **positive part** $f^+:D\to\mathbb{R}$ and **negative part** $f^-:D\to\mathbb{R}$ by

$$f^+(x)=\max\{0,f(x)\},\qquad f^-(x)=-\min\{0,f(x)\}.$$

These satisfy

$$f=f^+-f^-,\quad |f|=f^++f^-,\quad f^+=\frac{|f|+f}{2},\quad f^-=\frac{|f|-f}{2}.$$

Moreover, for any functions $f,g:D\to\mathbb{R}$,

$$f\le g\quad\Rightarrow\quad f^+\le g^+,\qquad f\le g\quad\Rightarrow\quad f^-\ge g^-.$$

Remark 7.17. *For any real numbers $z_1,z_2\in\mathbb{R}$, one has*

$$(z_1-z_2)^+\ge z_1^+-z_2^+. \tag{7.4}$$

Indeed, since $z^+ \geq z$ and $z^+ \geq 0$ for all $z \in \mathbb{R}$, by applying these inequalities with $z = z_1 - z_2$ and $z = z_2$ we obtain

$$z_1 = (z_1 - z_2) + z_2 \leq (z_1 - z_2)^+ + z_2^+ \quad \text{and} \quad 0 \leq (z_1 - z_2)^+ + z_2^+.$$

Hence $(z_1 - z_2)^+ + z_2^+$ is greater or equal to both z_1 and 0, and therefore

$$z_1^+ = \max\{z_1, 0\} \leq (z_1 - z_2)^+ + z_2^+,$$

which yields Equation 7.4 after rearranging.

Theorem 7.18: Triangle Inequality for the Riemann Integral

Let $f, g : [a,b] \to \mathbb{R}$ be integrable. Then f^+, f^-, and $|f|$ are integrable, and

$$\left| \int_a^b f\,dx \right| \leq \int_a^b |f|\,dx.$$

Proof. Fix $\varepsilon > 0$. Since f is integrable, there exists step functions $\ell \leq f \leq u$ with $\int_a^b (u - \ell)\,dx < \varepsilon$. Then ℓ^+ and u^+ are step functions with $\ell^+ \leq f^+ \leq u^+$.

Since $u - \ell \geq 0$, we have $(u - \ell) = (u - \ell)^+$. Moreover, applying Equation 7.4 with $z_1 = u(x)$ and $z_2 = \ell(x)$, we obtain

$$(u(x) - \ell(x))^+ \geq u(x)^+ - \ell(x)^+ \qquad \forall x \in [a,b].$$

Hence

$$\int_a^b (u^+ - \ell^+)\,dx \leq \int_a^b (u - \ell)^+\,dx = \int_a^b (u - \ell)\,dx < \varepsilon,$$

so f^+ is integrable. By Theorem 7.14, also $f^- = f^+ - f$ and $|f| = 2f^+ - f$ are integrable. Finally,

$$\left| \int_a^b f\,dx \right| = \left| \int_a^b f^+\,dx - \int_a^b f^-\,dx \right| \leq \int_a^b f^+\,dx + \int_a^b f^-\,dx = \int_a^b |f|\,dx. \qquad \square$$

7.3 Integrability Theorems

7.3.1 Integrability of Monotone Functions

As before we work on a compact interval $[a,b] \subseteq \mathbb{R}$. Note that very monotone function $f : [a,b] \to \mathbb{R}$ is bounded; for instance, if f is increasing then $f(a)$ is a lower bound and $f(b)$ is an upper bound.

Theorem 7.19: Monotone Functions are Integrable

Every monotone function $f : [a,b] \to \mathbb{R}$ is Riemann integrable.

Proof. W.l.o.g, f is increasing (otherwise replace f by $-f$ and use Theorem 7.14). We want to apply Proposition 7.13. Given $\varepsilon > 0$, we need to construct step functions $\ell, u \in \mathcal{SF}$ such that $\ell \leq f \leq u$ and $\int_a^b (u - \ell) <, dx < \varepsilon$.

Fix $n \in \mathbb{N}$ (to be chosen later) and the uniform partition

$$a = x_0 < x_1 < \ \ldots \ < x_n = b, \qquad x_k = a + \frac{k}{n}(b - a).$$

Define the step functions $ell, u : [a, b] \to \mathbb{R}$ as

$$\begin{aligned} \ell(x) = f(x_{k-1}) \quad \text{and} \quad u(x) &= f(x_k) \qquad \text{for } x \in (x_{k-1}, x_k), \quad k = 1, \dots, n, \\ \ell(x) = u(x) &= f(x) \qquad \text{for } x \in \{x_0, \dots, x_n\}. \end{aligned}$$

Note that, since f is increasing, $\ell \le f \le u$. Moreover, for each k we have $u - \ell = f(x_k) - f(x_{k-1})$ on (x_{k-1}, x_k). Recalling that $x_k - x_{k-1} = \frac{b-a}{n}$, this yields

$$\begin{aligned} \int_a^b (u - \ell)\, dx &= \sum_{k=1}^n (f(x_k) - f(x_{k-1}))(x_k - x_{k-1}) = \frac{b-a}{n} \sum_{k=1}^n (f(x_k) - f(x_{k-1})) \\ &= \frac{b-a}{n}(f(x_n) - f(x_0)) = \frac{b-a}{n}(f(b) - f(a)). \end{aligned}$$

Choosing $n \in \mathbb{N}$ so large that $\frac{b-a}{n}(f(b) - f(a)) < \varepsilon$, Proposition 7.13 implies that f is Riemann integrable. □

Definition 7.20: Piecewise Monotone Functions

A function $f : [a, b] \to \mathbb{R}$ is **piecewise monotone** if there exists a decomposition

$$a = x_0 < x_1 < \ \dots \ < x_n = b$$

such that $f|_{(x_{k-1}, x_k)}$ is monotone for every $k = 1, \dots, n$.

Corollary 7.21: Piecewise Monotone Functions are Integrable

Every bounded piecewise monotone function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

7.3.2 Integrability of Continuous Functions

Using boundedness and uniform continuity on compact intervals (Theorems 4.17 and 4.23), we can prove that continuous functions are integrable.

Theorem 7.22: Continuous Functions are Integrable

Every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Proof. Let $f : [a, b] \to \mathbb{R}$ be continuous and fix $\varepsilon > 0$. By uniform continuity **??**, there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon \qquad \forall x, y \in [a, b]. \tag{7.5}$$

Choose a partition $a = x_0 < x_1 < \ \dots \ < x_n = b$ with $x_k - x_{k-1} < \delta$. For each k set

$$c_k = \min\{f(x) \mid x_{k-1} \le x \le x_k\}, \qquad d_k = \max\{f(x) \mid x_{k-1} \le x \le x_k\},$$

which exists by Theorem 4.20, and let $y_k, z_k \in [x_{k-1}, x_k]$ satisfy $f(y_k) = c_k$ and $f(z_k) = d_k$. Then since, $|y_k - z_k| \le x_k - x_{k-1} < \delta$, Equation 7.5 yields $d_k - c_k < \varepsilon$.

Define now the step functions $\ell, u : [a, b] \to \mathbb{R}$ as

$$\begin{aligned} \ell(x) = c_k, \quad \text{and} \quad u(x) &= d_k \qquad \text{for } x \in (x_{k-1}, x_k), \quad k = 1, \dots, n, \\ \ell(x) = u(x) &= f(x) \qquad \text{for } x \in \{x_0, \dots, x_n\}. \end{aligned}$$

Then $\ell \leq f \leq u$ and $u - \ell = d_k - c_k$ on (x_{k-1}, x_k), hence

$$\int_a^b (u - \ell)\, dx = \sum_{k=1}^n (d_k - c_k)(x_k - x_{k-1}) < \varepsilon \sum_{k=1}^n (x_k - x_{k-1}) = \varepsilon(b - a).$$

Since $\varepsilon > 0$ is arbitrary, f is integrable. $\square$

Definition 7.23: Piecewise Continuous Functions

A function $f : [a, b] \to \mathbb{R}$ is **piecewise continuous** if there exists a decomposition

$$a = x_0 < x_1 < \ \ldots \ < x_n = b$$

such that $f|_{(x_{k-1}, x_k)}$ is continuous for all k and both one sided limits $\lim_{x \to x_{k-1}^+} f(x)$ and $\lim_{x \to x_k^-} f(x)$ exist. Equivalently, each $f|_{(x_{k-1}, x_k)}$ extends to a continuous function on $[x_{k-1}, x_k]$.

Corollary 7.24: Piecewise Continuous Functions are Integrable

Every piecewise continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

7.3.3 Integration and Sequences of Functions

Let $(f_n)_{n=0}^\infty$ with $f_n : [a, b] \to \mathbb{R}$ be a sequence of integrable functions. Assume that f_n converges pointwise or uniformly to $f : [a, b] \to \mathbb{R}$. Is f integrable? And if so, does

$$\lim_{n \to \infty} \int_a^b f_n \, dx = \int_a^b f \, dx$$

hold?

In general, the pointwise limit of integrable functions need not to be integrable. Also, as the following example shows, even when the pointwise limit of f is integrable, one may have that $\lim_{n \to \infty} \int f_n \neq \int f$.

Example

Let $D = [0, 1]$ and define $f_n : D \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x \leq \frac{1}{2n}, \\ n^2(\frac{1}{n} - x) & \text{if } \frac{1}{2n} \leq x \leq \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} \leq x \leq 1. \end{cases}$$

Each f_n is continuous (hence integrable). Also, its graph is a triangle of base $\frac{1}{n}$ and height $\frac{n}{2}$, so

$$\int_0^1 f_n(x)\, dx = \frac{1}{2} \cdot \frac{1}{n} \cdot \frac{n}{2} = \frac{1}{4}.$$

Moreover, $f_n(0) = 0$ for all n, and for every $x > 0$ we have $f_n(x) = 0$ for all $n > 1/x$, hence $f_n(x) \to 0$. Thus, f_n converges pointwise to the constant function $f = 0$, but

$$\int_0^1 f_n(x)\, dx = \frac{1}{4} \neq 0 = \int_0^1 f(x)\, dx.$$

On the other hand, as the next result shows, uniform convergence is sufficient for both integrability of the limit and interchange of limit and integral.

Theorem 7.25: Uniform Convergence and Riemann Integral Commute

Let $(f_n)_{n=0}^{\infty}$, with $f_n : [a,b] \to \mathbb{R}$, be a sequence of integrable functions converging uniformly to $f : [a,b] \to \mathbb{R}$. Then f is integrable and

$$\int_a^b f\,dx = \lim_{n\to\infty} \int_a^b f_n\,dx. \tag{7.6}$$

Proof. Fix $\varepsilon > 0$. By uniform convergence, there exists $N \in \mathbb{N}$ such that $|f_n - f| < \varepsilon$ for all $n \geq N$.

Since f_N is integrable, there exists step functions $\ell, u \in \mathcal{SF}$ with $\ell \leq f \leq u$ and $\int_a^b (u-\ell)\,dx < \varepsilon$. Set $\hat{\ell} = \ell - \varepsilon$ and $\hat{u} = u + \varepsilon$. Then $\hat{\ell}, \hat{u} \in \mathcal{SF}$. Also, since $|f_N - f| < \varepsilon$,

$$\hat{\ell} = \ell - \varepsilon \leq f_N - \varepsilon \leq f \leq f_N + \varepsilon \leq u + \varepsilon = \hat{u}$$

and (because $\hat{u} - \hat{\ell} = u - \ell + 2\varepsilon$)

$$\int_a^b (\hat{u} - \hat{\ell})\,dx = \int_a^b (u-\ell)\,dx + 2\varepsilon(b-a) < \varepsilon + 2\varepsilon(b-a).$$

As $\varepsilon > 0$ is arbitrary, Proposition 7.13 yields that f is integrable.

Moreover, using monotonicity (Proposition 7.15) and the triangle inequality for the Riemann integral (Theorem 7.18),

$$\left| \int_a^b f\,dx - \int_a^b f_n\,dx \right| = \left| \int_a^b (f - f_n)\,dx \right| \leq \int_a^b |f - f_n|\,dx \leq \varepsilon(b-a) \qquad \forall n \geq N,$$

proving Equation 7.6. □