
Analysis I

Theorems & Lemmas

Stephan Oseghale

Mathematics Bachelor
ETH Zürich
January 4, 2026

Contents

1	Functions	1
2	The Real Numbers	3
2.1	Groups, Rings, Fields	3
2.2	Order Relation	5
2.3	Ordered Fields	6
2.4	Completeness Axiom	9
2.5	Intervals	9
2.6	Complex Numbers	10
2.7	Maximum and Supremum	13
2.7.1	Existence of the Supremum	13
2.8	Two-Point Compactification	15
2.9	Consequences of Completeness	16
2.9.1	Archimedean Principle	16
2.9.2	Uncountability	19
3	Sequences of Real Numbers	20
3.1	Convergence of Sequences	20
3.2	Convergent Subsequences and Accumulation Points	20
3.3	Addition, Multiplication and Inequalities	22
3.4	Superior and Inferior Limits	24
3.5	Cauchy Sequences	26
3.6	Improper Limits	27
3.7	Sequences of Complex Numbers	28
4	Functions of one Real Variable	28
4.1	Real valued functions	29
4.1.1	Boundedness and Monotonicity	29
4.1.2	Continuity	30
4.1.3	Sequential Continuity	32
4.2	Continuous Functions	33
4.2.1	Intermediate Value Theorem	33
4.3	Continuous Functions on Compact Intervals	34
4.3.1	Boundedness and Extrema	35
4.3.2	Uniform Continuity	36
4.4	Example: Exponential and Logarithmic Functions	37
4.4.1	Definition of the Exponential Function	37
4.4.2	Properties of the Exponential Function	38
4.4.3	The Natural Logarithm	38
4.5	Limits of Functions	39
4.5.1	Limit in the Vicinity of a Point	39
4.5.2	One-Sided Limits	41
4.5.3	Landau Notation	42
4.6	Sequences of Functions	44
4.6.1	Pointwise Convergence	44
4.6.2	Uniform Convergence	45

5	Series and Power Series	46
5.1	Series of Real Numbers	46
5.1.1	Series with Non-negative Elements	48
5.1.2	Conditional Convergence	49
5.1.3	Leibniz Criterion for Alternating Series	50
5.2	Absolute Convergence	51
5.2.1	Criteria for Absolute Convergence	51
5.2.2	Reordering Series	53
5.2.3	Product of Series	54
5.3	Series of Complex Numbers	56
5.4	Power Series	57
5.4.1	Radius of Convergence	57
5.4.2	Complex Power Series	59
5.5	Exponential and Trigonometric Functions	59
5.5.1	The Exponential Map as a Power Series	59
5.5.2	Sine and Cosine	60
5.5.3	The Circle Number	61
5.5.4	Polar Coordinates and Multiplication of Complex Numbers	61
5.5.5	Other Trigonometric and Hyperbolic Functions	62
6	Differential Calculus	62
6.1	The Derivative	62
6.1.1	Definition and Geometrical Interpretation	62
6.1.2	Differentiation Rules	64
6.2	Main Theorems of Differential Calculus	68
6.2.1	Local Extrema	68
6.2.2	The Mean Value Theorem	69
6.2.3	L'Hopital's Rule	71
6.2.4	Monotonicity and Convexity via Differential Calculus	72
6.3	Example: Differentiation of Trigonometric Functions	74
6.3.1	Sine and Arc Sine	74
6.3.2	Cosine and Arc Cosine	74
6.3.3	Tangent and Arc Tangent	74
6.3.4	Hyperbolic Functions	75
7	The Riemann Integral	75
7.1	Step Functions and their Integral	75
7.1.1	Decompositions and Step Functions	75
7.1.2	The Integral of a Step Function	77
7.2	Definition and First Properties of the Riemann Integral	78
7.2.1	Integrability of Real-Valued Functions	78
7.3	Integrability Theorems	82
7.3.1	Integrability of Monotone Functions	82
7.3.2	Integrability of Continuous Functions	83
7.3.3	Integration and Sequences of Functions	84
8	The Derivative and the Riemann Integral	85
8.1	The Fundamental Theorem of Calculus	85

8.1.1	The Fundamental Theorem	85
8.1.2	Integration by Parts and Substitution	87
8.1.3	Improper Integrals	88
8.2	Integration and Differentiation of Power Series	90
8.3	Integration Methods	92
8.3.1	Integration by Parts and by Substitution in Leibniz Notation	92
8.3.2	Integration by Parts: Example	93
8.3.3	Integration by Substitution: Examples	93
8.3.4	The Gamma Function	94
8.4	Taylor Series	95
8.4.1	Analytic Functions	101
9	Ordinary Differential Equations	103
9.1	Ordinary Differential Equations (ODEs)	103
9.1.1	Linear First Order ODEs	104
9.1.2	Autonomous First Order ODEs	107
9.1.3	Homogeneous Linear Second Order ODEs with Constant Coefficients	108
9.1.4	Non-Homogeneous Linear Second Order ODEs with Constant Coefficients	111
9.2	Existence and Uniqueness for ODEs	112
9.2.1	Existence and Uniqueness for First Order ODEs	112
9.2.2	Higher Order ODEs	113

1 Functions

Definition 1.1: Functions/Maps/Transformations

A **function** f from a set X to a set Y is an assignment of an element of Y to each element of X . The element $y \in Y$ to which $x \in X$ is assigned to is denoted $f(x)$. We write $f : X \rightarrow Y$ and sometimes also speak of a **map**, **mapping** or a **transformation**. The set X is the **domain** and the set Y is the **codomain**. We refer to the set X as **domain** or **domain of definition**, and the set Y as **domain of values** or **codomain**. The set

$$\{(x, f(x)) \mid x \in X\} \subseteq X \times Y$$

is called the **graph** of f . In the context of a function $f : X \rightarrow Y$, an element of the domain of definition is also called **argument**, and an element $y = f(x) \in Y$ assumed by the function, is also called **value** of the function. If $f : X \rightarrow Y$ is a function, one also writes

$$\begin{aligned} f : X &\rightarrow Y \\ x &\mapsto f(x), \end{aligned}$$

where $f(x)$ could be a concrete formula. We pronounce ' \mapsto ' as 'is mapped to'. Two functions $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ are said to be equal if $X_1 = X_2$, $Y_1 = Y_2$ and $f_1(x) = f_2(x) \quad \forall x \in X_1$.

Definition 1.2: Injective, Surjective and Bijective Functions

Let $f : X \rightarrow Y$ be a function. We call f :

1. **injective** (or an **injection**) if

$$\forall x_1, x_2 \in X : x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2);$$

2. **surjective** (or a **surjection**) if

$$\forall y \in Y \exists x \in X : f(x) = y;$$

3. **bijective** (or a **bijection**) if f is both injective and surjective.

Thus, a function $f : X \rightarrow Y$ is *not* injective if there exists distinct $x_1 \neq x_2 \in X$ such that $f(x_1) = f(x_2)$, and *not* surjective if there exists $y \in Y$ such that $f(x) \neq y$ for all $x \in X$.

Definition 1.3: Image and Preimage of a Function

For $f : X \rightarrow Y$ and $A \subseteq X$, define the **image** of A under the function f as

$$f(A) := \{y \in Y \mid \exists x \in X : f(x) = y\}.$$

For $B \subseteq Y$, define the **preimage** of B under the function f as

$$f^{-1}(B) := \{x \in X \mid f(x) \in B\}.$$

Remark 1.4. *Saying that $f : X \rightarrow Y$ is surjective is equivalent to $f(X) = Y$. Equivalently, f is surjective if $f^{-1}(\{y\}) \neq \emptyset$ for all $y \in Y$.*

Definition 1.5: Composition of Functions

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. The **composition** is $g \circ f : X \rightarrow Z$, defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.

Associativity: If $f : W \rightarrow X$, $g : X \rightarrow Y$ and $h : Y \rightarrow Z$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Indeed, for all $w \in W$, we have

$$h \circ (g \circ f)(w) = h((g \circ f)(w)) = h(g(f(w))) = (h \circ g)(f(w)) = ((h \circ g) \circ f)(w).$$

Therefore, we may omit parentheses and write $h \circ g \circ f : W \rightarrow Z$.

Definition 1.6: Identity and Inverse Function

Given a set X , the **identity function** $\text{id}_X : X \rightarrow X$ is defined by

$$\text{id}_X(x) = x \quad \forall x \in X.$$

If $f : X \rightarrow Y$ is bijective, then there exists a unique function $g : Y \rightarrow X$ such that, for each $y \in Y$, the value $g(y)$ is the unique element $x \in X$ satisfying $f(x) = y$. With this definition,

$$g \circ f = \text{id}_X \quad \text{and} \quad f \circ g = \text{id}_Y.$$

The function g is called the **inverse function** (or **inverse mapping**) of f , and is denoted by f^{-1} .

Remark 1.7. *A function $f : X \rightarrow Y$ is bijective if and only if there exists a function $g : Y \rightarrow X$ such that $g \circ f = \text{id}_X$ and $f \circ g = \text{id}_Y$.*

Definition 1.8: Affine Functions

An **affine function** is a function of the form $x \mapsto sx + r$, for $s, r \in \mathbb{R}$. The graph of an affine function is a non-vertical line in \mathbb{R}^2 . The parameter s in the equation $y = sx + r$ is called the **slope** of the line.

Definition 1.9: Even and Odd Functions

Let $D \subseteq \mathbb{R}$ be a set satisfying $D = -D$, i.e.,

$$\forall x \in D : x \in D \Rightarrow -x \in D.$$

A function $f : D \rightarrow \mathbb{R}$ is called

- **even** if $f(-x) = f(x)$ for all $x \in D$,
- **odd** if $f(-x) = -f(x)$ for all $x \in D$.

Geometrically, an even function is symmetric with respect to the y -axis, while an odd function is symmetric with respect to the origin.

2 The Real Numbers

2.1 Groups, Rings, Fields

Definition 2.1: Groups

A **group** is a non-empty set G together with a rule (called an *operation*) denoted by $\star : G \times G \rightarrow G$ that combines any two elements of G into another element of G . This operation must satisfy three conditions:

- **Associativity:** No matter how you place parentheses, the result is the same for all $a, b, c \in G$,

$$(a \star b) \star c = a \star (b \star c).$$

- **Neutral element:** There is a special element $e \in G$ such that combining it with any $a \in G$ leaves a unchanged, i.e.,

$$\forall a \in G : a \star e = e \star a = a.$$

- **Inverse element:** Every $a \in G$ has a 'partner' $a^{-1} \in G$ that 'cancels it out', giving the neutral element, i.e.,

$$a \star a^{-1} = a^{-1} \star a = e.$$

Note that, in general, one does not require that $a \star b = b \star a$. If the order of the operation does not matter, i.e., $a \star b = b \star a$ for all $a, b \in G$, the group is called **commutative** or **abelian**.

Lemma 2.2: Basic Properties of Groups

Let (G, \star) be a group. Then:

1. The neutral element is unique.
2. The inverse of an element is unique.
3. The inverse of the inverse of an element is the element itself, namely $(a^{-1})^{-1} = a$ for all $a \in G$.

Proof. 1. Assume that, in addition to $e \in G$, we have a second element e' with the property that $e' \star a = a \star e' = a$ for all elements $a \in G$. Then, we can choose $a = e$ to obtain

$$e \star e' = e.$$

Similarly, since e is a neutral element, we have

$$e \star e' = e'.$$

Combining the two identities, we get

$$e = e \star e' = e'.$$

This proves that $e' = e$, so we speak of *the* neutral element of a group.

2. Assume that for an element $a \in G$, there exists two elements $b, c \in G$ that are both the inverse of a , namely

$$a \star b = b \star a = e, \quad a \star c = c \star a = e.$$

Then, using associativity, we observe that

$$b = b \star e = b \star (a \star c) = (b \star a) \star c = e \star c = c.$$

This proves that the inverse of an element a is unique, so we can speak of *the* inverse element, and the notation a^{-1} makes sense.

3. Since $a \star a^{-1} = e$, we deduce that a is the inverse element of a^{-1} , thus

$$(a^{-1})^{-1} = a. \tag{2.1}$$

□

Groups capture the idea of combining elements with a single operation. But to describe the arithmetic of numbers more faithfully, we also need a second operation (as we do with addition and multiplication). This leads us to the notion of *rings* and *fields*.

Definition 2.3: Rings and Fields

A **ring** is a non-empty set R in which we can both 'add' and 'multiply' elements with two operations '+' and '·'. Also, these two operations are compatible with each other. More precisely:

- $(R, +)$ is a **commutative group**, with neutral element denoted 0.
- Multiplication \cdot is **associative**, has a **neutral element** (usually written as 1), and **distributes over addition**, i.e.,

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (b + c) \cdot a = b \cdot a + c \cdot a \quad \forall a, b, c \in R.$$

If multiplication is also commutative, we call $(R, +, \cdot)$ a **commutative ring**. Note that, unlike addition, we do not require that every element has an inverse for multiplication. A **field** is a special kind of commutative ring, i.e. every non-zero element has an inverse for multiplication. In other words, if $(R, +, \cdot)$ is a commutative ring, then $(R, +, \cdot)$ is a field if $R \setminus \{0\}$ forms a commutative group under multiplication. Traditionally, we use the letter F to denote a field. We also write $F^* = F \setminus \{0\}$ for the set of all invertible elements of F .

Lemma 2.4: Basic Properties of Fields

Let $(F, +, \cdot)$ be a field and let $a, b \in F$. Then:

1. $0 \cdot a = a \cdot 0 = 0$.
2. $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$. In particular $(-1) \cdot a = -a$.
3. $(-a) \cdot (-b) = a \cdot b$. In particular, $(-a)^{-1} = -(a^{-1})$ whenever $a \neq 0$.

Proof. 1. Since 0 is the neutral element for the addition, we have $0+0=0$. Hence, using distributivity, we get

$$0 \cdot a = (0+0) \cdot a = (0 \cdot a) + (0 \cdot a).$$

Adding $-0 \cdot a$ (i.e., the inverse of $0 \cdot a$), we deduce that $0 \cdot a = 0$. The case of $a \cdot 0$ is analogous.

2. By the distributive law,

$$a \cdot b + a \cdot (-b) = a \cdot (b + (-b)) = a \cdot 0 = 0.$$

So $a \cdot (-b)$ is the additive inverse of $a \cdot b$, i.e., $-(a \cdot b) = a \cdot (-b)$. Taking $b = 1$ gives $-a = (-1) \cdot a$. The validity of $(-a) \cdot b = -(a \cdot b)$ follows by exchanging a and b in the argument above.

3. By 2. we know that $-(a \cdot b) = a \cdot (-b)$. Hence, recalling Equation (2.1),

$$a \cdot b = -(a \cdot (-b)).$$

On the other hand, applying 2. with $(-b)$ instead of b , we also have

$$-(a \cdot (-b)) = (-a) \cdot (-b).$$

Combining the two identities above, we conclude that $(-a) \cdot (-b) = a \cdot b$. Finally, taking $b = a^{-1}$ yields $(-a) \cdot (-a^{-1}) = a \cdot a^{-1} = 1$, which gives the second assertion. \square

2.2 Order Relation

Definition 2.5: Cartesian Product

Let X and Y be two sets. The **cartesian product** $X \times Y$ is the set of ordered pairs of elements of X and Y , i.e.,

$$X \times Y := \{(x, y) \mid x \in X, y \in Y\}.$$

Definition 2.6: Subsets

Let P and Q be sets. Then

- P is a **subset** of Q , written $P \subset Q$ (or $P \subseteq Q$), if every element of P also belongs to Q .
- P is a **proper subset** of Q , written $P \subsetneq Q$, if P is a subset of Q but $P \neq Q$.
- We write $P \not\subset Q$ if P is not a subset of Q .

Definition 2.7: Relations

Let X be a set. A **relation** on X is a subset $\mathcal{R} \subseteq X \times X$, that is, a collection of ordered pairs of elements of X . If $(x, y) \in \mathcal{R}$ we write $x\mathcal{R}y$. Common symbols for relations include $<, \leq, \sim, \equiv, \cong$. If \sim is a relation on X , we write $x \not\sim y$ if $x \sim y$ does not hold. A relation \sim may have the following properties:

1. **Reflexive:** if $x \sim x \quad \forall x \in X$.
2. **Transitive:** if $x \sim y$ and $y \sim z$, then $x \sim z$.
3. **Symmetric:** if $x \sim y$, then $y \sim x$.
4. **Antisymmetric:** if $x \sim y$ and $y \sim x$, then $x = y$.

A relation is an **equivalence relation** if it is reflexive, transitive and symmetric. It is an **order relation** if it is reflexive, transitive and antisymmetric.

2.3 Ordered Fields

Definition 2.8: Ordered Field

Let F be a field, and let \leq be an order relation on F . We call (F, \leq) , or simply F , an **ordered field** if the following hold:

1. **Linearity of order:** for all $x, y \in F$, at least one of $x \leq y$ or $y \leq x$ holds.
2. **Compatibility with addition:** for all $x, y, z \in F$,

$$x \leq y \Rightarrow x + z \leq y + z.$$

3. **Compatibility with multiplication:** for all $x, y \in F$,

$$0 \leq x \wedge 0 \leq y \Rightarrow 0 \leq x \cdot y.$$

Lemma 2.9: Ordered Field: Basic Consequences

Let (F, \leq) be an ordered field, and let $x, y, z, w \in F$. Then:

- (a) (Trichotomy) Either $x < y$, or $x = y$, or $x > y$.
- (b) If $x < y$ and $y \leq z$, then $x < z$. (Analogously, $x \leq y$ and $y < z$ imply $x < z$.)
- (c) (Addition of inequalities) If $x \leq y$ and $z \leq w$, then $x + z \leq y + w$. (Analogously, $x < z$ and $z \leq w$ imply $x + z < y + w$.)
- (d) $x \leq y$ if and only if $0 \leq y - x$.
- (e) $x \leq 0$ if and only if $0 \leq -x$.
- (f) $x^2 \geq 0$, and $x^2 > 0$ if $x \neq 0$.
- (g) $0 < 1$.
- (h) If $0 \leq x$ and $y \leq z$, then $xy \leq xz$.
- (i) If $x \leq 0$ and $y \leq z$, then $xy \geq xz$.
- (j) If $0 < x \leq y$, then $0 < y^{-1} \leq x^{-1}$.
- (k) If $0 \leq x \leq y$ and $0 \leq z \leq w$, then $0 \leq xz \leq yw$.
- (l) If $x + y \leq x + z$, then $y \leq z$.
- (m) If $xy \leq xz$ and $x > 0$, then $y \leq z$.

Lemma 2.10: Integers and Rationals Inside an Ordered Field

Let (F, \leq) be an ordered field, and denote by 0 and 1 the neutral elements for addition and multiplication, respectively. Then:

- (i) The elements $\dots, -2, -1, 0, 1, 2, \dots$ defined by

$$2 = 1 + 1, \quad 3 = 2 + 1, \dots, \quad -n = (-1) \cdot n$$

are all distinct and satisfy

$$\dots < -2 < -1 < 0 < 1 < 2 < 3 < \dots$$

We denote this set of elements by \mathbb{Z} , and we call them 'integers'

- (ii) Every fraction pq^{-1} with $p, q \in \mathbb{Z}$, $q \neq 0$, lies in F and the set of all such elements is denoted by \mathbb{Q} . Also,

$$\mathbb{Z} \subsetneq \mathbb{Q} \subseteq F.$$

Proof. (i) By Lemma 2.9(g), we have that $0 < 1$. Then Lemma 2.9(c) yields $0 < 1 < 2 < 3 < \dots$, and taking negatives gives $\dots < -2 < -1 < 0$. Hence all these elements are distinct.

(ii) For $q \neq 0$, q is invertible in F ; define $\frac{p}{q} = pq^{-1}$. The set of such fractions is a field contained in F , which we denote by \mathbb{Q} .

To show that \mathbb{Q} strictly contains \mathbb{Z} , consider $\frac{1}{2}$ (the inverse of 2). Since $2 > 1$, it follows from

Lemma 2.9(j) that $0 < \frac{1}{2} < 1$, so $\frac{1}{2} \notin \mathbb{Z}$. □

Definition 2.11: Absolute Value and Sign

Let (F, \leq) be an ordered field.

- The **absolute value** (or **modulus**) is the function $|\cdot| : F \rightarrow F$ defined by

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

- The **sign** is the function $\text{sgn} : F \rightarrow \{-1, 0, 1\}$ defined by

$$\text{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Lemma 2.12: Absolute Value and Sign: Basic Properties

Let (F, \leq) be an ordered field and let $x, y \in F$. Then:

- (a) $x = \text{sgn}(x)|x|$, $|-x| = |x|$, $\text{sgn}(-x) = -\text{sgn}(x)$.
- (b) $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$ (by Trichotomy Lemma 2.9).
- (c) (Multiplicativity) $\text{sgn}(xy) = \text{sgn}(x)\text{sgn}(y)$ and $|xy| = |x||y|$.
- (d) If $x \neq 0$, then $|x^{-1}| = |x|^{-1}$.
- (e) $|x| \leq y$ iff $-y \leq x \leq y$.
- (f) $|x| < y$ iff $-y < x < y$.
- (g) (Triangle inequality) $|x + y| \leq |x| + |y|$.
- (h) (Inverse triangle inequality) $||x| - |y|| \leq |x - y|$.

Proof. (g) Thanks to (e) we have $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding these two inequalities we get

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

Applying (e) again yields the result.

- (h) From (g) we have $|x| \leq |x - y| + |y|$, therefore

$$|x| - |y| \leq |x - y|.$$

Exchanging the roles of x and y , we also have $|y| - |x| \leq |y - x| = |x - y|$. Combining these two inequalities yields

$$-|x - y| \leq |x| - |y| \leq |x - y|,$$

and the result follows by applying (e) again. □

2.4 Completeness Axiom

Definition 2.13: Completeness Axiom

Let (K, \leq) be an ordered field. We say that (K, \leq) is **complete** (or a **completely ordered field**) if the following statement holds:

Let X, Y be non-empty subsets of K such that $x \leq y$ for all $x \in X$ and $y \in Y$. Then there exists $c \in K$ lying between X and Y , in the sense that $x \leq c \leq y$ for all $x \in X$ and $y \in Y$.

The statement above is called the **completeness axiom**.

Definition 2.14: Real Numbers

We call **the field of real numbers**, any completely ordered field and denote it by \mathbb{R} .

2.5 Intervals

Definition 2.15: Intervals

Let $a, b \in \mathbb{R}$. We define:

- The **closed interval**

$$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\};$$

- The **open interval**

$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\};$$

- The **half-open intervals**

$$[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\} \quad \text{and} \quad (a, b] := \{x \in \mathbb{R} \mid a < x \leq b\};$$

- The **unbounded closed intervals**

$$[a, \infty) := \{x \in \mathbb{R} \mid a \leq x\} \quad \text{and} \quad (-\infty, b] := \{x \in \mathbb{R} \mid x \leq b\};$$

- The **unbounded open intervals**

$$(a, \infty) := \{x \in \mathbb{R} \mid a < x\} \quad \text{and} \quad (-\infty, b) := \{x \in \mathbb{R} \mid x < b\};$$

Definition 2.16: Set Operations

Let P, Q be sets. The **intersection** $P \cap Q$, the **union** $P \cup Q$, the **relative complement** $P \setminus Q$ and the **symmetric difference** $P \triangle Q$ are defined by

$$\begin{aligned} P \cap Q &= \{x \mid x \in P \text{ and } x \in Q\}, \\ P \cup Q &= \{x \mid x \in P \text{ or } x \in Q\}, \\ P \setminus Q &= \{x \mid x \in P \text{ and } x \notin Q\}, \\ P \triangle Q &= (P \setminus Q) \cup (Q \setminus P) = (P \cup Q) \setminus (P \cap Q). \end{aligned}$$

Definition 2.17: Union and Intersection of several Sets

Let \mathcal{A} be a family of sets (i.e., a set whose elements are sets). We define the **union** and **intersection** of the sets in \mathcal{A} as

$$\bigcup_{A \in \mathcal{A}} A = \{x \mid \exists A \in \mathcal{A} : x \in A\}, \quad \bigcap_{A \in \mathcal{A}} A = \{x \mid \forall A \in \mathcal{A} : x \in A\}.$$

If $\mathcal{A} = \{A_1, A_2, \dots\}$, we also write

$$\bigcup_{i=1}^{\infty} A_i = \{x \mid \exists i \geq 1 : x \in A_i\}, \quad \bigcap_{i=1}^{\infty} A_i = \{x \mid \forall i \geq 1 : x \in A_i\}.$$

Definition 2.18: Neighborhoods

Let $x \in \mathbb{R}$. A **neighborhood** of x is a set containing an interval I such that $x \in I$. Given $\delta > 0$, the open interval $(x - \delta, x + \delta)$ is called the δ -**neighborhood** of x .

Definition 2.19: Open and Closed Sets

A subset $U \subseteq \mathbb{R}$ is called **open** in \mathbb{R} if for every $x \in U$ there exists open interval I such that $x \in I$ and $I \subseteq U$. A subset $F \subseteq \mathbb{R}$ is called **closed** in \mathbb{R} if its complement $\mathbb{R} \setminus F$ is open.

Remark 2.20. The sets \emptyset and \mathbb{R} are both open in \mathbb{R} . Hence, they are also closed since $\emptyset^c = \mathbb{R}$ and $\mathbb{R}^c = \emptyset$. We note that $\mathbb{Q} \subseteq \mathbb{R}$ and $[a, b] \subseteq \mathbb{R}$ are neither open nor closed.

Remark 2.21. Let \mathcal{U} be a family of open sets, and \mathcal{F} be a family of closed subsets of \mathbb{R} . Then the union and intersection

$$\bigcup_{U \in \mathcal{U}} U, \quad \bigcap_{F \in \mathcal{F}} F$$

Are open and closed, respectively.

2.6 Complex Numbers

Starting from the field of real numbers \mathbb{R} , we define the set of **complex numbers** as

$$\mathbb{C} = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

We denote the elements $z = (x, y) \in \mathbb{C}$ in the form $z = x + iy$, where i is the **imaginary unit**. Here $x \in \mathbb{R}$ is the **real part** of z , written as $x = \operatorname{Re}(z)$, and $y \in \mathbb{R}$ is the **imaginary part**, written

as $y = \text{Im}(z)$. Elements with $\text{Im}(z) = 0$ are called **real**, while those with $\text{Re}(z) = 0$ are **purely imaginary**. Via the injective map $\mathbb{R} \ni x \mapsto x + i \cdot 0 \in \mathbb{C}$, we identify \mathbb{R} with the subset of real numbers inside \mathbb{C} .

As you may expect from previous knowledge, we want to satisfy $i^2 = -1$. To achieve this, we define addition and multiplication on \mathbb{C} so that it becomes a field. Additionally, we want these operations to coincide with the usual addition and multiplication when considering real numbers.

Since $i^2 = -1$, using commutativity and distributivity we get

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_2x_1 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

This motivates the following definition

Definition 2.22: Addition and Multiplication on \mathbb{C}

On $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ we define **addition** and **multiplication** as follows:

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2), \\ (x_1, y_1) \cdot (x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1). \end{aligned}$$

Proposition 2.23: \mathbb{C} is a Field

With the operation of Definition 2.22, together with the zero element $(0, 0)$ and the unit element $(1, 0)$, the set \mathbb{C} is a field.

Definition 2.24: Complex Conjugation

For $z = x + iy \in \mathbb{C}$ we define its **conjugate** as $\bar{z} = x - iy$. The mapping $\mathbb{C} \ni z \mapsto \bar{z} \in \mathbb{C}$ is called **complex conjugation**.

Lemma 2.25: Properties of Complex Conjugation

For all $z, w \in \mathbb{C}$:

- (i) $z\bar{z} = x^2 + y^2 \in \mathbb{R}_{\geq 0}$. In particular, $z\bar{z} = 0$ if and only if $z = 0$.
- (ii) $\overline{z + w} = \bar{z} + \bar{w}$.
- (iii) $\overline{z\bar{w}} = \bar{z}w$.

Proof. Property (i) follows from the fact that, for $z = x + iy$, $(x + iy)(x - iy) = x^2 + y^2$. Also, $x^2 + y^2 = 0$ if and only if $x + iy = 0$. Properties (ii) and (iii) follow from a direct computation, writing $z = x_1 + iy_1$ and $w = x_2 + iy_2$, which yields

$$\begin{aligned} \overline{z + w} &= \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2) = \bar{z} + \bar{w}, \\ \overline{z \cdot w} &= \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)} = (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) \\ &= (x_1 - iy_1) \cdot (x_2 - iy_2) = \bar{z} \cdot \bar{w}. \quad \square \end{aligned}$$

Definition 2.26: Absolute Value

The **absolute value** (or **norm**) on \mathbb{C} is the map $|\cdot| : \mathbb{C} \rightarrow \mathbb{R}$ given by

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}, \quad z = x + iy \in \mathbb{C}.$$

Lemma 2.27: Cauchy-Schwart Inequality

If $z = x_1 + iy_1$, and $w = x_2 + iy_2$, then

$$x_1x_2 + y_1y_2 \leq |z||w|. \quad (2.2)$$

Proof. We observe that

$$\begin{aligned} |z|^2|w|^2 - (x_1x_2 + y_1y_2)^2 &= (x_1^2 + y_1^2)(x_2^2 + y_2^2) - (x_1x_2 + y_1y_2)^2 \\ &= x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2 - (x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2) \\ &= y_1^2x_2^2 + x_1^2y_2^2 - 2x_1x_2y_1y_2 \\ &= (y_1x_2 - x_1y_2)^2 \geq 0. \end{aligned}$$

This proves that $(x_1x_2 + y_1y_2)^2 \leq |z|^2|w|^2$, so it follows that

$$|x_1x_2 + y_1y_2| \leq |z||w|.$$

Since $x \leq |x|$ for all $x \in \mathbb{R}$, we obtain Equation (2.2). \square

Proposition 2.28: Trianlge Inequality

For all $z, w \in \mathbb{C}$, one has

$$|z + w| \leq |z| + |w|.$$

Proof. For $z = x_1 + iy_1$ and $w = x_2 + iy_2$, using Lemma 2.27, we have

$$\begin{aligned} |z + w|^2 &= (x_1 + x_2)^2 + (y_1 + y_2)^2 \\ &= |z|^2 + |w|^2 + 2(x_1x_2 + y_1y_2) \\ &\leq |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2. \end{aligned}$$

Taking square roots proves the result. \square

Definition 2.29: Cicular Disks

For $z \in \mathbb{C}$ and $r > 0$, we define the **open disk** with radius $r > 0$ around z as

$$B(z, r) := \{w \in \mathbb{C} \mid |z - w| < r\},$$

and the **closed disk** with radius $r > 0$ around z as

$$\overline{B(z, r)} := \{w \in \mathbb{C} \mid |z - w| \leq r\}.$$

In other words, the open disk $B(z, r)$ is the set of points at distance strictly less than r from z . We note that this definition is compatible with the one of neighborhoods in \mathbb{R} : if $x \in \mathbb{R}$ and $r > 0$,

then

$$B(x, r) \cap \mathbb{R} = (x - r, x + r).$$

Definition 2.30: Open and Closed Sets

A set $U \subseteq \mathbb{C}$ is **open** if for every $z \in U$ there exists $r > 0$ such that $B(z, r) \subseteq U$. A set $C \subseteq \mathbb{C}$ is **closed** if its complement $\mathbb{C} \setminus C$ is open.

2.7 Maximum and Supremum

2.7.1 Existence of the Supremum

Definition 2.31: Bounded Sets, Maxima and Minima

Let $X \subseteq \mathbb{R}$ be a subset of real numbers.

- X is **bounded from above** if there exists $s \in \mathbb{R}$ such that $x \leq s$ for all $x \in X$. Such a number s is called an **upper bound** of X . If s is an upper bound and also an element of X , we say that s is the **maximum** of X and write

$$s = \max(X).$$

- Analogously, X is **bounded from below** if there exists $r \in \mathbb{R}$ such that $r \leq x$ for all $x \in X$. Such a number r is called a **lower bound** of X . If r is a lower bound and also an element of X , we say that r is the **minimum** of X and write

$$r = \min(X).$$

- X is called **bounded** if it is both bounded from above and bounded from below.

Remark 2.32. If a set $X \subseteq \mathbb{R}$ has a maximum, then it is unique. Indeed, if $x_1, x_2 \in X$ are both maxima, then $x_1 \leq x_2$ (since x_2 is a maximum) and $x_2 \leq x_1$ (since x_1 is a maximum), so $x_1 = x_2$.

A closed interval $[a, b]$ with $a < b$ has both a minimum and maximum, i.e., $a = \min([a, b])$ and $b = \max([a, b])$. But not all sets have a maximum. For instance, the open interval (a, b) does not have a maximum because the endpoint b , though an upper bound, is not contained in the set. Similarly \mathbb{R} and unbounded intervals such as $[a, \infty)$ or (a, ∞) have no maximum.

Definition 2.33: Supremum

Let $X \subseteq \mathbb{R}$ be a subset and let

$$A := \{a \in \mathbb{R} \mid x \leq a \quad \forall x \in X\}$$

be the set of all upper bounds of X . If A has a minimum, we call this minimum the **supremum** of X and write

$$\sup(X) = \min(A).$$

The **infimum** is defined analogously using the maximum of the set of all lower bounds.

In other words, the supremum of X is the smallest real number that is greater than or equal to

every element of X . Note that we can describe the supremum $s = \sup(X)$ as follows

$$x \leq s \quad \forall x \in X, \quad \text{and} \quad \text{if } t < s, \text{ the } t \text{ is not an upper bound of } X. \quad (2.3)$$

This means that for every $t < s$, there exists some $x \in X$ such that $x > t$. That is,

$$x \leq s \quad \forall x \in X, \quad \text{and} \quad \forall t < s \exists x \in X : x > t. \quad (2.4)$$

The two characterizations 2.3 and 2.4 are equivalent.

Note that not every set has a supremum. If $X = \emptyset$ or if X is unbounded from above, then $\sup(X)$ does not exist. However, for any non-empty and bounded-above subset of \mathbb{R} , the supremum always exists.

Remark 2.34. *If a set X has a maximum, then this element is also the supremum. Indeed, the maximum is an upper bound of X , and since it lies in X , no smaller upper bound can exist.*

Theorem 2.35: Existence of Supremum

Let $X \subseteq \mathbb{R}$ be non-empty and bounded from above. Then $\sup(X)$ exists and is a real number.

Proof. Since X is bounded from above, the set $A := \{a \in \mathbb{R} \mid x \leq a \quad \forall x \in X\}$ of upper bounds is non-empty. Since $x \leq a$ for any $x \in X$ and $a \in A$, we can apply the completeness axiom (Definition 2.13) to find $c \in \mathbb{R}$ such that

$$x \leq c \leq a \quad \forall x \in X, \forall a \in A.$$

The first inequality implies that c is itself an upper bound (so $c \in A$), while the second inequality tells us that c is smaller than or equal to every upper bound. Hence, $c = \min(A) = \sup(X)$. \square

Proposition 2.36: Supremum and Set Operations

Let X and Y be non-empty subsets of \mathbb{R} that are bounded from above. Define

$$X + Y := \{x + y \mid x \in X, y \in Y\} \quad \text{and} \quad X \cdot Y := \{x \cdot y \mid x \in X, y \in Y\}.$$

The sets $X \cup Y$, $X \cap Y$, and $X + Y$ are also bounded from above. Moreover, if $X, Y \subseteq \mathbb{R}_{\geq 0}$ (i.e., $x \geq 0$ and $y \geq 0$ for all $x \in X$ and $y \in Y$), then $X \cdot Y$ is bounded from above as well. In these cases, the following formulas hold:

- (1) $\sup(X \cup Y) = \max\{\sup(X), \sup(Y)\}$,
- (2) If $X \cap Y \neq \emptyset$, then $\sup(X \cap Y) \leq \min\{\sup(X), \sup(Y)\}$,
- (3) $\sup(X + Y) = \sup(X) + \sup(Y)$,
- (4) If $X, Y \subseteq \mathbb{R}_{\geq 0}$, then $\sup(X \cdot Y) = \sup(X) \cdot \sup(Y)$.

Proof. (3) Let $x_0 = \sup(X)$ and $y_0 = \sup(Y)$. For any $z \in X + Y$, there exists $x \in X$ and $y \in Y$ such that $z = x + y$. Since $x \leq x_0$ and $y \leq y_0$, we have

$$z = x + y \leq x_0 + y_0,$$

so $x_0 + y_0$ is an upper bound from $X + Y$. We now want to show that $x_0 + y_0 = \sup(X + Y)$.

Let $z_0 = \sup(X + Y)$ and suppose, by contradiction, that

$$\varepsilon := x_0 + y_0 - z_0 > 0.$$

Since $x_0 = \sup(X)$, by the characterization 2.4 there exists $x \in X$ such that $x > x_0 - \varepsilon/2$. Likewise, there exists $y \in Y$ such that $y > y_0 - \varepsilon/2$. Setting $z = x + y$, we obtain

$$z > x_0 - \frac{\varepsilon}{2} + y_0 - \frac{\varepsilon}{2} = x_0 + y_0 - \varepsilon = z_0,$$

contradicting the assumption that z_0 is an upper bound for $X + Y$. Therefore, $z_0 = x_0 + y_0$.

(4) The proof is analogous. If all elements of X and Y are non-negative, and we set $x_0 = \sup(X)$ and $y_0 = \sup(Y)$, then for any $z = x \cdot y \in X \cdot Y$, we have

$$z = x \cdot y \leq x_0 \cdot y_0,$$

which shows that $x_0 \cdot y_0$ is an upper bound for $X \cdot Y$. Using a similar ' ε -argument' as done above, when proving (3), one shows that this upper bound is sharp, i.e., $x_0 \cdot y_0$ is the least upper bound. \square

2.8 Two-Point Compactification

In this section, we extend the notions of **supremum** and **infimum** to arbitrary subsets of \mathbb{R} . To do so, we introduce two formal symbols

$$+\infty \quad \text{and} \quad -\infty,$$

which are not real numbers. We define the **extended real numbers line** (also called the **two-point compactification** of \mathbb{R}) by

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}.$$

We extend the usual order relation \leq on \mathbb{R} to $\overline{\mathbb{R}}$ by requiring that

$$-\infty < x < +\infty \quad \forall x \in \mathbb{R}.$$

For simplicity, we often write ∞ instead of $+\infty$.

We now introduce some standard (but informal) computation rules involving these symbols. For all $x \in \mathbb{R}$, we adopt the conventions:

$$\infty + x = \infty + \infty = \infty, \quad -\infty + x = -\infty - \infty = -\infty.$$

If $x > 0$, then

$$x \cdot \infty = \infty \cdot \infty = \infty, \quad x \cdot (-\infty) = \infty \cdot (-\infty) = -\infty,$$

while for $x < 0$ we have

$$x \cdot \infty = -\infty \cdot \infty = -\infty, \quad x \cdot (-\infty) = -\infty \cdot (-\infty) = \infty.$$

These rules are widely used as notational shorthand, but one must handle them with care. Expressions like

$$\infty - \infty, \quad 0 \cdot \infty, \quad \text{or similar}$$

are undefined and should be avoided.

Definition 2.37: Supremum and Infimum in the Extended Line

Let $X \subseteq \mathbb{R}$.

- If X is not bounded from above, we define $\sup(X) = \infty$.
- If $X = \emptyset$, we define $\sup(\emptyset) = -\infty$.
- If X is not bounded from below, we define $\inf(X) = -\infty$.
- If $X = \emptyset$, we define $\inf(\emptyset) = \infty$.

In this context, we refer to ∞ and $-\infty$ as **indefinite values**.

In other words:

- Saying $\sup(X) = \infty$ means that X is not bounded from above, i.e.,

$$\forall x_0 \in X \exists x \in X : x > x_0.$$

- Saying $\sup(X) = -\infty$ means that X is empty.
- Similarly, $\inf(X) = -\infty$ means that X is not bounded from below, and $\inf(X) = \infty$ means X is empty.

2.9 Consequences of Completeness

2.9.1 Archimedean Principle

The archimedean principle states that for every real number $x \in \mathbb{R}$ there exists an integer n greater than x . The following theorem, proved using the existence of suprema (and implicitly the completeness axiom), gives a precise formulation of this principle.

Theorem 2.38: Archimedean Principle

For every $x \in \mathbb{R}$ there exists exactly one $n \in \mathbb{Z}$ such that

$$n \leq x < n + 1.$$

Proof. We first treat the case $x \geq 0$. Fix $x \in \mathbb{R}$ with $x \geq 0$ and define

$$E = \{n \in \mathbb{Z} \mid n \leq x\}.$$

Since $0 \in E$ and x is an upper bound, E is a non-empty subset of \mathbb{R} bounded from above. Hence, by Theorem 2.35, the supremum $s_0 = \sup(E)$ exists. From the definition of supremum we deduce:

- (i) $s_0 \leq x$ (because x is an upper bound);
- (ii) there exists $n_0 \in E$ with $s_0 - 1 < n_0$ (otherwise $s_0 - 1$ would also be an upper bound).

From (ii) we obtain $s_0 < n_0 + 1$, which implies

- (iii) $n_0 + 1 \notin E$ (otherwise s_0 would not be an upper bound for E).

Moreover, since $m \leq s_0$ for every $m \in E$, we have $m < n_0 + 1$ for all $m \in E$. As all elements of E are integers,

$$m < n_0 + 1 \Leftrightarrow m - n_0 < 1 \Leftrightarrow m - n_0 \leq 0 \Leftrightarrow m \leq n_0.$$

Thus, every $m \in E$ is less than or equal to n_0 , and since $n_0 \in E$, we conclude that $n_0 = \max(E)$. In particular, by Remark 2.34, the maximum is also the supremum, so $s_0 = n_0$.

Finally, recalling (iii) and the definition of E , we have $n_0 + 1 > x$. Together with (i), this shows

$$n_0 = s_0 \leq x < n_0 + 1,$$

establishing the claim for any $x \geq 0$.

Now, if $x < 0$, apply the previous argument to $-x > 0$. Then there exists $m \in \mathbb{Z}$ such that

$$m \leq -x < m + 1,$$

which is equivalent to

$$-m - 1 < x \leq -m.$$

If $x = -m$, then set $n = -m$. If $x < -m$, set $n = -m - 1$. In both cases, we obtain

$$n \leq x < n + 1.$$

Finally, for uniqueness, assume that $n_1, n_2 \in \mathbb{Z}$ both satisfy $n_i \leq x < n_i + 1$. From $n_1 \leq x < n_2 + 1$ we deduce that $n_1 < n_2 + 1$, and therefore $n_1 \leq n_2$. Reversing the roles of n_1 and n_2 gives $n_2 \leq n_1$. Hence, $n_1 = n_2$. \square

Definition 2.39: Integer and Fractional Parts

The **integer part** $\lfloor x \rfloor$ of $x \in \mathbb{R}$ is the integer $n \in \mathbb{Z}$ uniquely determined by Theorem 2.38 such that $n \leq x < n + 1$. The map $x \mapsto \lfloor x \rfloor$ from \mathbb{R} to \mathbb{Z} is called the **rounding function**. The **fractional part** of x is defined as

$$\{x\} = x - \lfloor x \rfloor \in [0, 1).$$

Corollary 2.40: $\frac{1}{n}$ is Arbitrarily Small

For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$, with $n \geq 1$, such that

$$\frac{1}{n} < \varepsilon.$$

Proof. Applying Theorem 2.38 to $x = \frac{1}{\varepsilon} > 0$, we find $m \in \mathbb{Z}$ such that

$$m \leq \frac{1}{\varepsilon} < m + 1.$$

Set $n := m + 1$. In this way we have $0 < \frac{1}{\varepsilon} < n$, which is equivalent to $n > 0$ (therefore, $n \geq 1$) and $\frac{1}{n} < \varepsilon$. \square

Definition 2.41: Dense Sets

A subset $X \subseteq \mathbb{R}$ is called **dense** in \mathbb{R} if every open non-empty interval contains an element of X .

Corollary 2.42: Density of \mathbb{Q}

For every $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Proof. Set $\varepsilon = b - a$. By Corollary 2.40, there exists $m \in \mathbb{N}$ with $\frac{1}{m} < \varepsilon$. Then, by Theorem 2.38 applied with $x = ma$, there exists $n \in \mathbb{Z}$ with

$$n \leq ma < n + 1,$$

or equivalently,

$$\frac{n}{m} \leq a < \frac{n+1}{m}.$$

Since $\frac{1}{m} < \varepsilon$, by the two inequalities above, we obtain

$$a < \frac{n+1}{m} \leq a + \frac{1}{m} < a + \varepsilon = b.$$

Thus $r = \frac{n+1}{m}$ is a rational number between a and b . □

Corollary 2.43: Density of $\mathbb{R} \setminus \mathbb{Q}$

For every $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < r < b$.

Proof. We want to show that for every $x \in \mathbb{R}$ and $\delta > 0$, there exists an $a \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$a \in (x - \delta, x + \delta).$$

By Corollary 2.42, we find a $q \in \mathbb{Q}$ such that $q \in (x - \delta, x + \delta)$. By Corollary 2.40 we find an $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \frac{(x + \delta) - q}{\sqrt{2}} \quad \Rightarrow \quad \frac{\sqrt{2}}{N} < (x + \delta) - q.$$

This implies that

$$x - \delta < q < \frac{\sqrt{2}}{N} + q < x + \delta.$$

Choosing $r = \frac{\sqrt{2}}{N} + q$ proves the statement. □

2.9.2 Uncountability

Definition 2.44: Cardinality

Let X and Y be sets.

- We say X and Y have the **same cardinality**, written $X \sim Y$, if there is a bijection $f : X \rightarrow Y$.
- We write $X \preceq Y$ if there exists an injection $f : X \rightarrow Y$.
- The empty set has cardinality 0.
- A set X has **finite cardinality** $|X| = n$ if there exists a bijection with $\{1, \dots, n\}$.
- A set is **infinite** if it is not finite.
- A set is **countable** if it has a bijection to \mathbb{N} . Its cardinality is denoted \aleph_0 , pronounced Aleph-0.
- A set is **uncountable** if it is infinite but not countable.

If $X \preceq Y$ and $Y \preceq X$, then $X \sim Y$. In other words, if there exists an injective map $f : X \rightarrow Y$ and an injective map $g : Y \rightarrow X$, then one can find a bijective map $h : X \rightarrow Y$. This non-trivial statement is the **Schröder-Bernstein Theorem**.

We will now list some statements about different sets of numbers from the lecture:

1. \mathbb{N} and the even numbers have the same cardinality.
2. \mathbb{N} and \mathbb{Z} have the same cardinality.
3. \mathbb{Q} is countable, i.e., $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{Z}$.

Proposition 2.45: Uncountability of \mathbb{R}

The set \mathbb{R} is uncountable.

Extra Material

Definition 2.46: Power Set

Let X be a set. The **power set** $\mathcal{P}(X)$ of X is the set of all subsets of X , i.e.,

$$\mathcal{P}(X) := \{A \subseteq X\}.$$

Theorem 2.47: Cantor's Theorem

For any set X , the power set $\mathcal{P}(X)$ has strictly larger cardinality than X .

Proposition 2.48: The Reals have the same cardinality as $\mathcal{P}(\mathbb{N})$

$$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|.$$

3 Sequences of Real Numbers

3.1 Convergence of Sequences

Definition 3.1: Sequences

A **sequence** is a function $a : \mathbb{N} \rightarrow \mathbb{R}$. The image $a(n)$ of $n \in \mathbb{N}$ is also written as a_n and is called the n -th element of a . Instead of $a : \mathbb{N} \rightarrow \mathbb{R}$ one often writes $(a_n)_{n \in \mathbb{N}}, (a_n)_{n=0}^{\infty}, (a_n)_{n \geq 0}$.

Definition 3.2: (Eventually) Constant Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is **constant** if $x_n = x_m \forall n, m \in \mathbb{N}$. It is **eventually constant** if there exists $N \in \mathbb{N}$ such that $x_n = x_m \forall n, m \geq N$.

Definition 3.3: Convergence of Sequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence in \mathbb{R} . We say that $(x_n)_{n=0}^{\infty}$ **converges** (or is **convergent**) if $\exists A \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : |x_n - A| < \varepsilon \quad \forall n \geq N.$$

In this case we write

$$\lim_{n \rightarrow \infty} x_n = A \tag{3.1}$$

and call A the **limit** of $(x_n)_{n=0}^{\infty}$.

Lemma 3.4: Uniqueness of the Limit

A convergent sequence $(x_n)_{n=0}^{\infty}$ has exactly one limit.

Proof. Let $A, B \in \mathbb{R}$ be limits of $(x_n)_{n=0}^{\infty}$. Fix $\varepsilon > 0$. Then there exists $N_A, N_B \in \mathbb{N}$ such that $|x_n - A| < \varepsilon$ for all $n \geq N_A$ and $|x_n - B| < \varepsilon$ for all $n \geq N_B$. We define $N := \max\{N_A, N_B\}$. Then it holds that

$$|A - B| \leq |A - x_N| + |x_N - B| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $A = B$. □

3.2 Convergent Subsequences and Accumulation Points

Definition 3.5: Subsequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence. A **subsequence** is of the form $(x_{n_k})_{k=0}^{\infty}$, where $(n_k)_{k=0}^{\infty}$ is a strictly increasing sequence of non-negative integers, i.e., $n_{k+1} > n_k \forall k \in \mathbb{N}$.

Remark 3.6. Since $n_{k+1} > n_k$ for all $k \in \mathbb{N}$ it follows by induction that $n_k \geq k$ for all $k \in \mathbb{N}$.

Proof. For $k = 0$ we have that $n_0 \geq 0$, because $(n_k)_{k=0}^{\infty}$ is a sequence of non-negative integers. So the condition is fulfilled. For the inductive step we want to show that the condition holds for $k + 1$ under the assumption that the condition is true for k . Because $(n_k)_{k=0}^{\infty}$ is also a strictly increasing sequence, we have that $n_{k+1} > n_k \geq k$. Additionally since $n_k \in \mathbb{N}$, we have that $n_{k+1} \geq n_k + 1$. So it follows that $n_{k+1} \geq n_k + 1 \geq k + 1$, which proves the condition for $k + 1$. □

Lemma 3.7: Subsequences of Convergent Sequences are Convergent

Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Then every subsequence $(x_{n_k})_{k=0}^{\infty}$ also converges to A .

Proof. Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^{\infty}$ converges to A , there exists $N \in \mathbb{N}$ such that $|x_n - A| < \varepsilon \forall n \geq N$. As by Remark 3.6 we know that $n_k \geq k$ for all $k \in \mathbb{N}$. Therefore for all $k \geq N$ it holds that $|x_{n_k} - A| < \varepsilon$. \square

Definition 3.8: Accumulation Points of Sequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence in \mathbb{R} . A point $A \in \mathbb{R}$ is an **accumulation point** of $(x_n)_{n=0}^{\infty}$ if

$$\forall \varepsilon > 0 \forall N \in \mathbb{N} \exists n \geq N : |x_n - A| < \varepsilon.$$

Proposition 3.9: Subsequences and Accumulation Points

Let $(x_n)_{n=0}^{\infty}$ be a sequence in \mathbb{R} . A point $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^{\infty}$ if and only if there exists a convergent subsequence of $(x_n)_{n=0}^{\infty}$ with limit A .

Proof. First assume that $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^{\infty}$. We construct $(n_k)_{k \geq 0}$ recursively:

- first, apply the definition of accumulation point with $N = 1$ and $\varepsilon = 1 = 2^0$ to find $n_0 \geq 1$ with $|x_{n_0} - A| \leq 2^0$,
- second, apply the definition the definition of accumulation point with $N = n_0 + 1$ and $\varepsilon = 2^{-1}$ to find $n_1 \geq n_0 + 1$ with $|x_{n_1} - A| \leq 2^{-1}$,
- more in general given n_{k-1} , we apply the definition of accumulation point with $N = n_{k-1} + 1$ and $\varepsilon = 2^{-k}$ to find $n_k \geq n_{k-1} + 1$ with $|x_{n_k} - A| \leq 2^{-k}$.

Now given $\varepsilon > 0$ choose N such that $2^{-N} < \varepsilon$. Then for all $k \geq N$ we have that

$$|x_{n_k} - A| \leq 2^{-k} \leq 2^{-N} < \varepsilon,$$

so $\lim_{k \rightarrow \infty} x_{n_k} = A$.

Conversely, assume that there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ converging to A . Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. Since $\lim_{k \rightarrow \infty} x_{n_k} = A$, there exists N_0 such that $|x_{n_k} - A| < \varepsilon$ for all $k \geq N_0$. Hence if we choose $k = \max\{N_0, N\}$, because $n_k \geq n$ (recall Remark 3.6) we have that $n_k \geq N$ and $|x_{n_k} - A| < \varepsilon$. Thus A is an accumulation point. \square

Corollary 3.10: Infinitely Many Terms Near an Accumulation Point

If $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^{\infty}$, then for every $\varepsilon > 0$ there are infinitely many n with $x_n \in (A - \varepsilon, A + \varepsilon)$.

Proof. By Proposition 3.9, there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ with $\lim_{k \rightarrow \infty} x_{n_k} = A$. Hence for every $\varepsilon > 0$ there exists K such that $x_{n_k} \in (A - \varepsilon, A + \varepsilon)$ for all $k \geq K$, providing infinitely many elements of the sequence inside the interval $(A - \varepsilon, A + \varepsilon)$. \square

Corollary 3.11: Accumulation Points of Convergent Sequences

convergent sequence has exactly one accumulation point, namely its limit.

3.3 Addition, Multiplication and Inequalities

Proposition 3.12: Limits and Operations

Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be sequences converging to $A, B \in \mathbb{R}$ respectively. Then:

1. The sequence $(x_n + y_n)_{n=0}^{\infty}$ converges to $A + B$.
2. The sequence $(x_n y_n)_{n=0}^{\infty}$ converges to AB .
3. Given $\alpha \in \mathbb{R}$, the sequence $(\alpha x_n)_{n=0}^{\infty}$ converges to αA .
4. Suppose $x_n \neq 0$ for all $n \in \mathbb{N}$ and $A \neq 0$. Then the sequence $(x_n^{-1})_{n=0}^{\infty}$ converges to A^{-1} .

Proposition 3.13: Limits and Inequalities

Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be sequences converging to $A, B \in \mathbb{R}$ respectively.

1. If $A < B$, then there exists $N \in \mathbb{N}$ such that $x_n < y_n$ for all $n \geq N$.
2. If there exists $N \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \geq N$, then $A \leq B$.

Remark 3.14. In Proposition 3.13 even if we assume that $x_n < y_n$ for all $n \in \mathbb{N}$, we cannot conclude that $A < B$. For example take

$$x_n = \frac{1}{n}, \quad y_n = \frac{1}{n}.$$

Then we have that $x_n < y_n$ for all $n \in \mathbb{N}$ but $A = B = 0$.

Lemma 3.15: Sandwich Lemma

Let $(x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}, (z_n)_{n=0}^{\infty}$ be sequences such that for some $N \in \mathbb{N}$, we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N.$$

Suppose that both $(x_n)_{n=0}^{\infty}$ and $(z_n)_{n=0}^{\infty}$ converge to the same limit. Then $(y_n)_{n=0}^{\infty}$ also converges, and we have that

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n.$$

Proof. Let $(x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}, (z_n)_{n=0}^{\infty}$ be sequences such that for some $N_0 \in \mathbb{N}$, we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N_0.$$

Additionally suppose that $(x_n)_{n=0}^{\infty}$ and $(z_n)_{n=0}^{\infty}$ converge to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^{\infty}, (z_n)_{n=0}^{\infty}$ converge to A there exists $N_x, N_z \in \mathbb{N}$ such that

$$\begin{aligned} A - \varepsilon &< x_n < A + \varepsilon \quad \forall n \geq N_x \\ A - \varepsilon &< z_n < A + \varepsilon \quad \forall n \geq N_z. \end{aligned}$$

So we choose $N := \max\{N_0, N_x, N_z\}$. Then we have that

$$A - \varepsilon < x_n \leq y_n \leq z_n < A + \varepsilon \quad \forall n \geq N,$$

which shows that $\lim_{n \rightarrow \infty} y_n = A$. □

Definition 3.16: Bounded Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called **bounded** if there exists a real number $M \geq 0$ such that

$$|x_n| \leq M \quad \forall n \in \mathbb{N}.$$

Lemma 3.17: Convergent Sequences are Bounded

Every convergent sequence is bounded.

Proof. Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Let $\varepsilon = 1$. Then, by convergence of $(x_n)_{n=0}^{\infty}$, there exists N such that $|x_n - A| \leq 1$ for all $n \geq N$. So we have that

$$|x_n| = |x_n - A + A| \leq |x_n - A| + |A| \leq 1 + |A| \quad \forall n \geq N.$$

We choose

$$M = \max(|x_0|, |x_1|, \dots, |x_{N-1}|, 1 + |A|).$$

Then $|x_n| \leq M$ for all $n \in \mathbb{N}$ as desired. □

Definition 3.18: Monotone Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called:

- **(monotonically) increasing** if $m > n \Rightarrow x_m \geq x_n$,
- **strictly (monotonically) increasing** if $m > n \Rightarrow x_m > x_n$,
- **(monotonically) decreasing** if $m > n \Rightarrow x_m \leq x_n$,
- **strictly (monotonically) decreasing** if $m > n \Rightarrow x_m < x_n$.

If a sequence is decreasing or increasing we call it monotone. If a sequence is strictly increasing or strictly decreasing then we call it strictly monotone.

Remark 3.19. *An equivalent formulation of monotone sequences can be given using only successive terms:*

- $(x_n)_{n=0}^{\infty}$ is increasing if $x_{n+1} \geq x_n$ for all n ,
- $(x_n)_{n=0}^{\infty}$ is strictly increasing if $x_{n+1} > x_n$ for all n ,
- $(x_n)_{n=0}^{\infty}$ is decreasing if $x_{n+1} \leq x_n$ for all n ,
- $(x_n)_{n=0}^{\infty}$ is strictly decreasing if $x_{n+1} < x_n$ for all n .

Theorem 3.20: Convergence of Monotone Sequences

A monotone sequence $(x_n)_{n=0}^{\infty}$ converges if and only if it is bounded. More precisely, let $X = \{x_n \mid n \in \mathbb{N}\}$ denote the set of points in the sequence.

- If $(x_n)_{n=0}^{\infty}$ is increasing, then $\lim_{n \rightarrow \infty} x_n = \sup(X)$,
- if $(x_n)_{n=0}^{\infty}$ decreasing, then $\lim_{n \rightarrow \infty} x_n = \inf(X)$.

Proof. If $(x_n)_{n=0}^{\infty}$ converges Lemma 3.17 says that it is bounded.

Conversely, let $(x_n)_{n=0}^{\infty}$ be a bounded monotone sequence. Wlog assume that $(x_n)_{n=0}^{\infty}$ is increasing (otherwise consider $(-x_n)_{n=0}^{\infty}$). Since $(x_n)_{n=0}^{\infty}$ is bounded from above, the set $X = \{x_n \mid n \in \mathbb{N}\}$ has a supremum, that we'll call $A = \sup(X)$.

By definition of A :

- (i) $x_n \leq A \quad \forall n \in \mathbb{N}$,
- (ii) $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_N > A - \varepsilon$.

Then, for all $n \geq N$ using (ii) and monotonicity, we have that $x_n \geq x_N > A - \varepsilon$. Then using (i), we conclude that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N.$$

□

3.4 Superior and Inferior Limits

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence. To study its behavior for large n it is useful to look at its tails

$$X_{\geq n} = \{x_k \mid k \geq n\} \subseteq \mathbb{R}.$$

The concept of limits can be restated using the tails of a sequence, i.e., the sequence $(x_n)_{n=0}^{\infty}$ converges to $A \in \mathbb{R}$ if and only if, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $X_{\geq N} \subseteq (A - \varepsilon, A + \varepsilon)$.

However, since not every sequence has a limit we now introduce a related notion (the **superior** and **inferior limits**), which always exist for bounded sequences.

For each $n \in \mathbb{N}$, define

$$s_n = \sup(X_{\geq n}) = \sup_{k \geq n} x_k, \quad i_n = \inf(X_{\geq n}) = \inf_{k \geq n} x_k.$$

Since $X_{\geq m} \subset X_{\geq n}$, whenever $m > n$, we have that

$$i_n \leq i_m \leq s_m \leq s_n \quad \forall m > n.$$

Thus, $(s_n)_{n=0}^{\infty}$ is a monotonically decreasing sequence, while $(i_n)_{n=0}^{\infty}$ is a monotonically increasing sequence. Moreover, since $(x_n)_{n=0}^{\infty}$ is bounded both $(s_n)_{n=0}^{\infty}$ and $(i_n)_{n=0}^{\infty}$ are bounded as well. Hence by Theorem 3.20, both sequences converge. Their limits will be called the *superior* and the *inferior limit* of $(x_n)_{n=0}^{\infty}$ respectively.

Note that, since $x_n \in X_{\geq n}$, we have that

$$i_n \leq x_n \leq s_n \quad \forall n \in \mathbb{N}. \tag{3.2}$$

Definition 3.21: Superior and Inferior Limits

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence in \mathbb{R} . The numbers

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\sup_{k \geq n} x_k \right), \quad \liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(\inf_{k \geq n} x_k \right)$$

are called the **superior** and **inferior limit** of $(x_n)_{n=0}^{\infty}$ respectively. From Equation (3.2) and Proposition 3.13, we have

$$\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n.$$

Lemma 3.22: Convergence and Superior/Inferior Limits

A bounded sequence $(x_n)_{n=0}^{\infty}$ in \mathbb{R} converges if and only if

$$\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n.$$

Proof. For every $n \in \mathbb{N}$, define

$$i_n = \inf_{k \geq n} x_k, \quad s_n = \sup_{k \geq n} x_k,$$

and set

$$I = \lim_{n \rightarrow \infty} i_n = \liminf_{n \rightarrow \infty} x_n, \quad S = \lim_{n \rightarrow \infty} s_n = \limsup_{n \rightarrow \infty} x_n.$$

First suppose that $I = S$. Since $i_n \leq x_n \leq s_n$ (see Equation (3.2)), the Sandwich Lemma 3.15 implies that the sequence $(x_n)_{n=0}^{\infty}$ converges, and its limit equals $I = S$.

Conversely, assume that $(x_n)_{n=0}^{\infty}$ converges to $A \in \mathbb{R}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N.$$

Then for all $n \geq N$, the same inequalities holds for i_n and s_n , i.e.,

$$A - \varepsilon \leq i_n \leq s_n \leq A + \varepsilon.$$

Taking limits and using Proposition 3.13, we obtain

$$A - \varepsilon \leq I \leq S \leq A + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $A = I = S$, which proves the result. \square

Theorem 3.23: Superior and Inferior Limits are Accumulation Points

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence and let $A = \limsup_{n \rightarrow \infty} x_n$. Then A is an accumulation point of $(x_n)_{n=0}^{\infty}$, and for every $\varepsilon > 0$ the following hold:

1. only finitely many elements satisfy $x_n \geq A + \varepsilon$;
2. infinitely many elements satisfy $A - \varepsilon < x_n < A + \varepsilon$.

An analogous statement holds for the inferior limit.

Proof. Since the sequence $(s_n)_{n=0}^{\infty}$ is monotonically decreasing and converges to A , given $\varepsilon > 0$, there

exists $N_0 \in \mathbb{N}$ such that

$$A \leq s_n < A + \varepsilon \quad \forall n \geq N_0. \quad (3.3)$$

We first prove that A is an accumulation point.

Fix $N \in \mathbb{N}$ and set $N_1 = \max\{N, N_0\}$. Since $s_{N_1} = \sup_{k \geq N_1} x_k$, there exists $n_1 \geq N_1 \geq N_0$ such that

$$s_{N_1} - \varepsilon < x_{n_1} \leq s_{N_1}.$$

Thus, combining this bound with Equation (3.3) we obtain

$$A - \varepsilon < s_{N_1} - \varepsilon < x_{n_1} \leq s_{N_1} < A + \varepsilon.$$

This construct shows that for any $\varepsilon > 0$ and any $N \in \mathbb{N}$, there exists $n_1 \geq N$ such that $A - \varepsilon < x_{n_1} < A + \varepsilon$. Thus A is an accumulation point for $(x_n)_{n=0}^\infty$.

We now prove 1. and 2.. From Equation (3.3) we have $x_n < A + \varepsilon$ for all $n \geq N_0$, so only finitely many terms satisfy $x_n \geq A + \varepsilon$. This shows 1..

Also since A is an accumulation point, it follows from Corollary 3.10 that infinitely many terms of the sequence lie within any interval $(A - \varepsilon, A + \varepsilon)$. \square

Corollary 3.24: Bounded Sequences have Convergent Subsequences

Every bounded sequence has at least one accumulation point and therefore possesses a convergent subsequence.

Proof. By Theorem 3.23, the number

$$A = \limsup_{n \rightarrow \infty} x_n$$

is always an accumulation point of $(x_n)_{n=0}^\infty$. Moreover, by Proposition 3.9, every accumulation point is the limit of a convergent subsequence. Hence every bounded sequence admits at least one convergent subsequence. \square

3.5 Cauchy Sequences

Definition 3.25: Cauchy Sequences

A sequence $(x_n)_{n=0}^\infty$ is called a **Cauchy sequence** if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \quad \forall n, m \geq N.$$

Lemma 3.26: Cauchy Sequences are Bounded

Every Cauchy sequence is bounded.

Proof. By definition, there exists $N \in \mathbb{N}$ such that

$$|x_n - x_N| \leq 1 \quad \forall n \geq N.$$

Hence, for $n \geq N$, we have $|x_n| \leq 1 + |x_N|$. Now, define

$$M = \max\{|x_0|, |x_1|, \dots, |x_{N-1}|, 1 + |x_N|\}.$$

Then, $|x_n| \leq M$ for all $n \in \mathbb{N}$, so $(x_n)_{n=0}^\infty$ is bounded. \square

Theorem 3.27: Convergence and Cauchy Sequences

A sequence $(x_n)_{n=0}^\infty$ of real numbers converges if and only if it is a Cauchy sequence.

Proof. Suppose first that $(x_n)_{n=0}^\infty$ converges to some $A \in \mathbb{R}$, and let us prove that $(x_n)_{n=0}^\infty$ is a Cauchy sequence.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|x_n - A| < \frac{\varepsilon}{2} \quad \forall n \geq N.$$

Then for all $n, m \geq N$, we have that

$$|x_n - x_m| \leq |x_n - A| + |x_m - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence $(x_n)_{n=0}^\infty$ is a Cauchy sequence.

Viceversa, let $(x_n)_{n=0}^\infty$ be a Cauchy sequence. Since it is bounded (by Lemma 3.26), Corollary 3.24 implies that there exists a subsequence $(x_{n_k})_{k=0}^\infty$ converging to some $A \in \mathbb{R}$. Given $\varepsilon > 0$, choose $N_0 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \quad \forall n, m \geq N_0,$$

and choose $N_1 \in \mathbb{N}$ such that

$$|x_{n_k} - A| < \frac{\varepsilon}{2} \quad \forall k \geq N_1.$$

Let $N = \max\{N_0, N_1\}$. Since $n_N \geq N$ (see Remark 3.6), for all $n \geq N$ we have

$$|x_n - A| \leq |x_n - x_{n_N}| + |x_{n_N} - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $(x_n)_{n=0}^\infty$ converges to A . \square

3.6 Improper Limits

We now extend the notion of limit to allow the **improper limit values** $+\infty$ (often abbreviated as ∞) and $-\infty$.

Definition 3.28: Improper Limits

Let $(x_n)_{n=0}^\infty$ be a sequence in \mathbb{R} . We say $(x_n)_{n=0}^\infty$ **diverges to** $+\infty$, and we write

$$\lim_{n \rightarrow \infty} x_n = +\infty,$$

if for every $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n > M$ for all $n \geq N$.

Similarly, $(x_n)_{n=0}^\infty$ **diverges to** $-\infty$ if for every $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n < -M$ for all $n \geq N$. In both cases, we say that $(x_n)_{n=0}^\infty$ has an **improper limit**.

An unbounded sequence doesn't need to diverge to $+\infty$ or $-\infty$. For instance, the sequence $x_n = (-1)^n n$, is unbounded but neither diverges to $+\infty$ nor to $-\infty$.

The notion of improper limit allows us to extend the definitions of superior and inferior limits to

unbounded sequences. If $(x_n)_{n=0}^{\infty}$ is not bounded from above, then

$$\sup_{k \geq n} x_k = +\infty \quad \forall n \in \mathbb{N},$$

and we write

$$\limsup_{n \rightarrow \infty} x_n = +\infty.$$

If $(x_n)_{n=0}^{\infty}$ is bounded from above but not from below, then we define

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (\sup_{k \geq n} x_k),$$

where the right-hand side is a real limit if the decreasing sequence $\sup_{k \geq n} x_k$ is bounded, and the improper limit $-\infty$ otherwise. The definition of the inferior limit extends analogously.

3.7 Sequences of Complex Numbers

Informally, a **sequence of complex numbers** is just like a sequence of real numbers, except that each term is a complex number instead of a real one. Thus, we study ordered lists (z_0, z_1, \dots) , where $z_n : \mathbb{N} \rightarrow \mathbb{C}$. As in the real case, we are mainly interested in their convergence, divergence and limit behavior.

To analyze sequences in \mathbb{C} , it is often sufficient to consider separately the corresponding sequences of real and imaginary parts in \mathbb{R} .

Definition 3.29: Sequences of Complex Numbers

A sequence of complex numbers $(z_n)_{n=0}^{\infty}$, where

$$z_n = x_n + iy_n,$$

is said to **converge** to a limit $A + iB \in \mathbb{C}$ if the two sequences of real numbers $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ converge to A and B , respectively. In this case, we write

$$\lim_{n \rightarrow \infty} z_n = A + iB.$$

We say that $(z_n)_{n=0}^{\infty}$ **diverges to ∞** if the sequence of moduli $(|z_n|)_{n=0}^{\infty}$ diverges to $+\infty$, i.e.,

$$\lim_{n \rightarrow \infty} |z_n| = \lim_{n \rightarrow \infty} \sqrt{x_n^2 + y_n^2} = +\infty.$$

Remark 3.30. As for sequences of real numbers, one can consider subsequences of sequences \mathbb{C} . Given a strictly increasing sequence of non-negative integers $(n_k)_{k=0}^{\infty}$, the corresponding subsequence is

$$(z_{n_k})_{k=0}^{\infty} = (x_{n_k} + iy_{n_k})_{k=0}^{\infty}.$$

4 Functions of one Real Variable

In this chapter we study real-valued functions defined on subsets of \mathbb{R} , typically intervals. The central concept is *continuity*.

4.1 Real valued functions

4.1.1 Boundedness and Monotonicity

For a non-empty set $D \subseteq \mathbb{R}$, the set of **real-valued** functions on D is

$$\mathcal{F}(D) = \{f \mid f : D \rightarrow \mathbb{R}\}.$$

For $f_1, f_2 \in \mathcal{F}(D)$, $\alpha \in \mathbb{R}$, and $x \in D$ we define

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \quad (\alpha f_1)(x) = \alpha f_1(x), \quad (f_1 f_2)(x) = f_1(x) f_2(x).$$

Given $\alpha \in \mathbb{R}$, we write $f \equiv \alpha$ for the constant function $x \mapsto \alpha$ on D .

Remark 4.1. *With the operations above, $\mathcal{F}(D)$ is a commutative ring (the additive identity is $f \equiv 0$ and the multiplicative identity is $f \equiv 1$).*

A point $x \in D$ is a **zero** of $f \in \mathcal{F}(D)$ if $f(x) = 0$. The **zero set** of f is $\{x \in D \mid f(x) = 0\}$. We order $\mathcal{F}(D)$ pointwise: for $f_1, f_2 \in \mathcal{F}(D)$,

$$\begin{aligned} f_1 \leq f_2 &\Leftrightarrow f_1(x) \leq f_2(x) \quad \forall x \in D, \\ f_1 < f_2 &\Leftrightarrow f_1(x) < f_2(x) \quad \forall x \in D. \end{aligned}$$

We say that $f \in \mathcal{F}(D)$ is **non-negative** if $f \geq 0$, and **positive** if $f > 0$.

Definition 4.2: Bounded Functions

Let $D \neq \emptyset$ and $f : D \rightarrow \mathbb{R}$. We say that f is **bounded from above** if there exists $M > 0$ such that

$$f(x) \leq M \quad \forall x \in D.$$

We say that f is **bounded from below** if there exists $M > 0$ such that

$$f(x) \geq -M \quad \forall x \in D.$$

We say that f is **bounded** if it is both bounded from above and from below. Equivalently, f is bounded if there exists $M > 0$ such that

$$|f(x)| \leq M \quad \forall x \in D.$$

Definition 4.3: Monotone Functions

Let $D \subseteq \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$. The function f is:

1. **increasing** if $x < y \Rightarrow f(x) \leq f(y) \quad \forall x, y \in D$;
2. **strictly increasing** if $x < y \Rightarrow f(x) < f(y) \quad \forall x, y \in D$;
3. **decreasing** if $x < y \Rightarrow f(x) \geq f(y) \quad \forall x, y \in D$;
4. **strictly decreasing** if $x < y \Rightarrow f(x) > f(y) \quad \forall x, y \in D$.

We call f **monotone** if it is increasing or decreasing, and **strictly monotone** if it is strictly increasing or strictly decreasing.

4.1.2 Continuity

Definition 4.4: Continuous Functions

Let $D \subseteq \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$. We say that f is **continuous at** $x_0 \in D$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

We say that f is **continuous on** D if it is continuous at every point of D .

Remark 4.5. It suffices to verify the implication above for small ε . Precisely, assume there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ there is a $\delta > 0$ such that

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

Then f is continuous at x_0 .

Indeed, for $\varepsilon_0 > \varepsilon$ we can choose the number $\delta > 0$ corresponding to ε to get

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon < \varepsilon_0.$$

In other words, if δ works for ε , then it works for all $\varepsilon_0 > \varepsilon$.

Definition 4.6: Restriction

Let $D \subseteq \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$. For any $D' \subseteq D$ the **restriction** of f to D' is the function $f|_{D'} : D' \rightarrow \mathbb{R}$ defined by

$$f|_{D'}(x) = f(x) \quad \forall x \in D'.$$

We regard $f|_{D'}$ and f as different functions unless $D' = D$.

Proposition 4.7: Combination of Continuous Functions

Let $D \subseteq \mathbb{R}$, and let $f_1, f_2 : D \rightarrow \mathbb{R}$ be continuous at $x_0 \in D$. Then $f_1 + f_2$, $f_1 f_2$, and αf_1 (for any $\alpha \in \mathbb{R}$) are continuous at x_0 .

Proof. We first prove the result for the sum. Let $\varepsilon > 0$. Since f_1 and f_2 are continuous at x_0 , there exists $\delta_1, \delta_2 > 0$ such that for all $x \in D$,

$$|x - x_0| < \delta_1 \Rightarrow |f_1(x) - f_1(x_0)| < \frac{\varepsilon}{2}, \quad |x - x_0| < \delta_2 \Rightarrow |f_2(x) - f_2(x_0)| < \frac{\varepsilon}{2}.$$

So, choosing $\delta = \min \delta_1, \delta_2$, for $|x - x_0| < \delta$ we get

$$|(f_1 + f_2)(x) - (f_1 + f_2)(x_0)| \leq |f_1(x) - f_1(x_0)| + |f_2(x) - f_2(x_0)| < \varepsilon,$$

which shows that $f_1 + f_2$ is continuous at x_0 .

For the product, note that

$$\begin{aligned} |f_1(x)f_2(x) - f_1(x_0)f_2(x_0)| &= |f_1(x)f_2(x) - f_1(x_0)f_2(x) + f_1(x_0)f_2(x) - f_1(x_0)f_2(x_0)| \\ &\leq |f_1(x)f_2(x) - f_1(x_0)f_2(x)| + |f_1(x_0)f_2(x) - f_1(x_0)f_2(x_0)| \\ &= |f_2(x)||f_1(x) - f_1(x_0)| + |f_1(x_0)||f_2(x) - f_2(x_0)|. \end{aligned}$$

Now, first choose $\delta_0 > 0$ such that $|x - x_0| < \delta_0$ implies $|f_2(x) - f_2(x_0)| < 1$, so that

$$|x - x_0| < \delta_0 \quad \Rightarrow \quad |f_2(x)| < 1 + |f_2(x_0)|.$$

Then choose $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} |x - x_0| < \delta_1 &\Rightarrow |f_1(x) - f_1(x_0)| < \frac{\varepsilon}{2(1 + |f_2(x_0)|)}, \\ |x - x_0| < \delta_2 &\Rightarrow |f_2(x) - f_2(x_0)| < \frac{\varepsilon}{2(1 + |f_1(x_0)|)}. \end{aligned}$$

So choosing $\delta = \min \delta_0, \delta_1, \delta_2$, for $|x - x_0| < \delta$ we get

$$\begin{aligned} |f_1(x)f_2(x) - f_1(x_0)f_2(x_0)| &< |f_2(x)| \frac{\varepsilon}{2(1 + |f_2(x_0)|)} + |f_1(x_0)| \frac{\varepsilon}{2(1 + |f_1(x_0)|)} \\ &< (1 + |f_2(x_0)|) \frac{\varepsilon}{2(1 + |f_2(x_0)|)} + |f_1(x_0)| \frac{\varepsilon}{2(1 + |f_1(x_0)|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

thus f_1f_2 is continuous at x_0 .

Finally, the statement about αf_1 follows by choosing $f_2 \equiv \alpha$ (a constant function) and using the product case proved above: since f_1 and f_2 are continuous at x_0 , their product $f_1f_2 = \alpha f_1$ is continuous at x_0 . \square

Definition 4.8: Sum and Product Notation

Let $n \in \mathbb{N}$ and $a_0, a_1, \dots, a_n \in \mathbb{R}$. We use the notation

$$\sum_{j=0}^n a_j = a_0 + a_1 + \dots + a_n, \quad \prod_{j=0}^n a_j = a_0 \cdot a_1 \cdot \dots \cdot a_n.$$

Here a_j is a **summand** in the sum and a **factor** in the product; j is the **index** (or **running variable**). If J is a finite set and numbers $(a_j)_{j \in J}$ are given, we write

$$\sum_{j \in J} a_j, \quad \prod_{j \in J} a_j.$$

By convention, for the empty index set \emptyset ,

$$\sum_{j \in \emptyset} a_j = 0, \quad \prod_{j \in \emptyset} a_j = 1.$$

Proposition 4.9: Composition of Continuous Functions

Let $D_1, D_2 \subseteq \mathbb{R}, x_0 \in D_1$ and $f : D_1 \rightarrow D_2$ be continuous at x_0 . If $g : D_2 \rightarrow \mathbb{R}$ is continuous at $f(x_0)$, then $g \circ f : D_1 \rightarrow \mathbb{R}$ is continuous at x_0 . In particular, the composition of continuous functions is continuous.

Proof. Let $\varepsilon > 0$. By continuity of g at $f(x_0)$, there exists $\eta > 0$ such that

$$\forall y \in D_2, \quad |y - f(x_0)| < \eta \quad \Rightarrow \quad |g(y) - g(f(x_0))| < \varepsilon.$$

By continuity of f at x_0 , there exists $\delta > 0$ such that

$$\forall x \in D_1, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \eta.$$

Combining the implications gives, for any $x \in D_1$,

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \eta \quad \Rightarrow \quad |g(f(x)) - g(f(x_0))| < \varepsilon. \quad \square$$

Remark 4.10. Applying Proposition 4.9 with $g(y) = |y|$, we see that if $f : D \rightarrow \mathbb{R}$ is continuous, then $x \mapsto |f(x)|$ is continuous.

4.1.3 Sequential Continuity

Definition 4.11: Notation for Limits of Sequences

Let $(x_n)_{n=0}^\infty \subseteq \mathbb{R}$ and $\bar{x} \in \mathbb{R}$. We write

$$x_n \rightarrow \bar{x} \quad \text{or} \quad x_n \xrightarrow{n \rightarrow \infty} \bar{x}$$

to mean

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

Theorem 4.12: Continuity = Sequential Continuity

Let $D \subseteq \mathbb{R}$, $f : D \rightarrow \mathbb{R}$, and $\bar{x} \in D$. Then f is continuous at \bar{x} if and only if for every sequence $(x_n)_{n=0}^\infty \subseteq D$ with $x_n \rightarrow \bar{x}$ we have $f(x_n) \rightarrow f(\bar{x})$.

Proof. ' \Rightarrow ': First Assume that f is continuous at \bar{x} . Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall x \in D, \quad |x - \bar{x}| < \delta \quad \Rightarrow \quad |f(x) - f(\bar{x})| < \varepsilon.$$

Also, since $x_n \rightarrow \bar{x}$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad |x_n - \bar{x}| < \delta.$$

Thus,

$$n \geq N \quad \Rightarrow \quad |f(x_n) - f(\bar{x})| < \varepsilon,$$

which implies that the sequence $(f(x_n))_{n=0}^\infty$ converges to $f(\bar{x})$.

' \Leftarrow ': To prove the converse, assume that f is not continuous at \bar{x} . This means that there exists $\varepsilon > 0$ such that, for every $\delta > 0$, there is $x \in D$ with

$$|x - \bar{x}| < \delta \quad \text{and} \quad |f(x) - f(\bar{x})| \geq \varepsilon.$$

Now, for every $n \in \mathbb{N}$, we apply this property with $\delta = 2^{-n}$ to find a point $x_n \in D$ such that

$$|x_n - \bar{x}| < 2^{-n} \quad \text{and} \quad |f(x_n) - f(\bar{x})| \geq \varepsilon$$

Then the sequence constructed in this way satisfies $x_n \rightarrow \bar{x}$ but $f(x_n) \not\rightarrow f(\bar{x})$. \square

Remark 4.13. *The proof above shows that if $f : D \rightarrow \mathbb{R}$ is not continuous at \bar{x} , then there exists $\varepsilon > 0$ and a sequence $(x_n)_{n=0}^\infty \subseteq D$ with $x_n \rightarrow \bar{x}$ such that $|f(x_n) - f(\bar{x})| \geq \varepsilon$ for all $n \in \mathbb{N}$. This is useful to show that a function f is not continuous at \bar{x} .*

4.2 Continuous Functions

4.2.1 Intermediate Value Theorem

In this section we prove a fundamental theorem that formalizes the idea that the graph of a continuous function on an interval is a continuous curve, and thus cannot make any jumps.

Theorem 4.14: Intermediate Value Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function with $f(a) \leq f(b)$. Then, for every real number c with $f(a) \leq c \leq f(b)$, there exists $\bar{x} \in [a, b]$ such that $f(\bar{x}) = c$.

Proof. Fix $c \in [f(a), f(b)]$. Then define

$$X = \{x \in [a, b] \mid f(x) \leq c\}.$$

Since $a \in X$ and $X \subseteq [a, b]$, the set is non-empty and bounded from above. By Theorem 2.35, its supremum

$$\bar{x} = \sup(X) \in [a, b]$$

exists. We now use the continuity of f at \bar{x} to show that $f(\bar{x}) = c$.

Since \bar{x} is the supremum of X , for each $n \geq 0$, we can find a point $x_n \in X \cap [\bar{x} - 2^{-n}, \bar{x}]$. Then $|x_n - \bar{x}| \leq 2^{-n}$, hence $x_n \rightarrow \bar{x}$. Also, by the definition of X , we have $f(x_n) \leq c$. Thus, by Theorem 4.12 (continuity of f along sequences),

$$\lim_{n \rightarrow \infty} f(x_n) = f(\bar{x}).$$

And Proposition 3.13 yields $\lim_{n \rightarrow \infty} f(x_n) \leq c$. Therefore, $f(\bar{x}) \leq c$.

Suppose, by contradiction, $f(\bar{x}) < c$ and set $\varepsilon := c - f(\bar{x}) > 0$. By continuity at \bar{x} , there exists $\delta > 0$ such that for all $x \in [a, b]$

$$|x - \bar{x}| < \delta \quad \Rightarrow \quad |f(x) - f(\bar{x})| < \varepsilon,$$

hence $f(x) < f(\bar{x}) + \varepsilon = c$. Therefore, by the definition of X ,

$$(\bar{x} - \delta, \bar{x} + \delta) \cap [a, b] \subseteq X.$$

Moreover, since $f(\bar{x}) < c \leq f(b)$, we cannot have $\bar{x} = b$; hence $\bar{x} < b$. Because $\bar{x} < b$, the interval $(\bar{x}, \bar{x} + \delta) \cap [a, b] \subseteq X$ is non-empty. Pick

$$y \in (\bar{x}, \bar{x} + \delta) \cap [a, b] \subseteq X.$$

Then $y \in X$ and $y > \bar{x}$, which contradicts the defining property of the supremum: \bar{x} is an upper bound of X , and X cannot contain elements larger than \bar{x} . This contradiction shows that $f(\bar{x}) \geq c$. Together with $f(\bar{x}) \leq c$ proved above, we conclude that $f(\bar{x}) = c$, as desired. \square

Theorem 4.15: Inverse Function Theorem

Let I be an interval and $f : I \rightarrow \mathbb{R}$ a continuous strictly monotone function. Then $f(I)$ is an interval, and the mapping $f : I \rightarrow f(I)$ has a continuous strictly monotone inverse function $f^{-1} : f(I) \rightarrow I$.

Proof. We may assume that I is non-empty and not a single point. Also, w.l.o.g, suppose f is strictly increasing (otherwise replace f with $-f$).

Let $J = f(I)$. Since f is strictly monotone it is injective. Also, since by definition $J = f(I)$, it is surjective, hence bijective. Therefore there exists a unique inverse $g = f^{-1} : J \rightarrow I$.

Because f is strictly increasing, we have

$$x_1 < x_2 \quad \Leftrightarrow \quad f(x_1) < f(x_2) \quad \forall x_1, x_2 \in I. \quad (4.1)$$

(Note: here we have equivalence in the statements because f is both injective and strictly increasing) Defining $y_1 = f(x_1)$ and $y_2 = f(x_2)$, this is equivalent to

$$y_1 < y_2 \quad \Leftrightarrow \quad g(y_1) < g(y_2) \quad \forall y_1, y_2 \in J$$

Thus, g is strictly increasing.

To show that J is an interval, $y_1, y_2 \in J$, and assume w.l.o.g that $y_1 < y_2$. Since, $J = f(I)$, Equation (4.1) implies that $y_1 = f(x_1), y_2 = f(x_2)$ for some $x_1, x_2 \in I$ with $x_1 < x_2$. Now by the Intermediate Value Theorem 4.14 applied to $f : [x_1, x_2] \rightarrow \mathbb{R}$, we have that all values $c \in [y_1, y_2]$ are in the image of $f : [x_1, x_2] \rightarrow \mathbb{R}$, i.e.,

$$[y_1, y_2] \subseteq f([x_1, x_2]) \subseteq J.$$

Since, y_1, y_2 were two arbitrary points in J , this proves that J is an interval.

It remains to show that $g = f^{-1}$ is continuous. Fix $\bar{y} \in J$ and suppose, by contradiction, that g is not continuous at \bar{y} . Then by Remark 4.13, there exists $\varepsilon > 0$ and a sequence $(y_n)_{n=0}^{\infty} \subseteq J$ such that

$$y_n \longrightarrow \bar{y} \quad \text{but} \quad |g(y_n) - g(\bar{y})| \geq \varepsilon \quad \forall n \in \mathbb{N}. \quad (4.2)$$

Set $x_n = g(y_n) \in I$ and $\bar{x} = g(\bar{y}) \in I$. Then for every $n \in \mathbb{N}$, either $x_n \leq \bar{x} - \varepsilon$ or $x_n \geq \bar{x} + \varepsilon$. In particular, at least one of these cases must occur infinitely often. W.l.o.g, assume $x_n \leq \bar{x} - \varepsilon$ for infinitely many n , and extract a subsequence $(x_{n_k})_{k=0}^{\infty}$ with $x_{n_k} \leq \bar{x} - \varepsilon$ for all k . Since, I is an interval, $\bar{x} - \varepsilon \in I$, and by strict monotonicity of f we obtain

$$y_{n_k} = f(x_{n_k}) \leq f(\bar{x} - \varepsilon) < f(\bar{x}) = \bar{y}.$$

Then Proposition 3.13 gives (recall $y_n \longrightarrow \bar{y}$, see (4.2))

$$\bar{y} = \lim_{k \rightarrow \infty} y_{n_k} \leq f(\bar{x} - \varepsilon) < f(\bar{x}) = \bar{y},$$

a contradiction. Hence, g is continuous. □

4.3 Continuous Functions on Compact Intervals

In this section we show that continuous functions on **bounded closed** intervals, called **compact intervals**, enjoy special properties.

4.3.1 Boundedness and Extrema

Lemma 4.16: Compactness

Let $[a, b]$ be a compact interval, and let $(x_n)_{n=0}^\infty$ be a sequence contained in $[a, b]$. Then there exists a subsequence $(x_{n_k})_{k=0}^\infty$ such that

$$\lim_{k \rightarrow \infty} x_{n_k} = \bar{x} \quad \text{for some } \bar{x} \in [a, b].$$

Proof. Since $(x_n)_{n=0}^\infty$ is bounded (as it lies in $[a, b]$), Corollary 3.24 ensures the existence of a convergent subsequence $(x_{n_k})_{k=0}^\infty$. Let \bar{x} denote its limit. Because $a \leq x_{n_k} \leq b$ for all k , Proposition 3.13 yields $a \leq \bar{x} \leq b$. \square

Theorem 4.17: Boundedness

Let $[a, b]$ be compact interval, and let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Then f is bounded.

Proof. Assume by contradiction that f is unbounded. Then, for every $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|f(x_n)| \geq n$. By Lemma 4.16, there is a subsequence $(x_{n_k})_{k=0}^\infty$ converging to some $\bar{x} \in [a, b]$.

Since f is continuous, so is $|f|$ (recall Remark 4.10), therefore $|f(x_{n_k})| \rightarrow |f(\bar{x})| \in \mathbb{R}$. This contradicts $|f(x_{n_k})| \geq n_k \rightarrow \infty$, so f must be bounded. \square

Exercise 4.18. Find examples of:

1. a continuous but unbounded function on a bounded open interval.

$$f : (0, 1) \rightarrow \mathbb{R}, \quad x \mapsto \frac{1}{x}.$$

2. a continuous but unbounded function on an unbounded closed interval.

$$f : [0, \infty) \rightarrow \mathbb{R}, \quad x \mapsto x.$$

3. an unbounded function on a compact interval but discontinuous at only one point.

$$f : [0, 1] \rightarrow \mathbb{R}, \quad x \mapsto \begin{cases} \frac{1}{x}, & \text{for } x \neq 0 \\ a \in \mathbb{R}, & \text{for } x = 0. \end{cases}$$

Definition 4.19: Extreme Values

Let $D \subseteq \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$.

- We say that f takes its **maximum value** at $x_0 \in D$ if $f(x) \leq f(x_0)$ for all $x \in D$. Then $f(x_0)$ is the **maximum** of f .
- We say that f takes its **minimum value** at $x_0 \in D$ if $f(x) \geq f(x_0)$ for all $x \in D$. Then $f(x_0)$ is the **minimum** of f .

Maxima and minima are called **extreme values** or **extrema**.

Theorem 4.20: Extreme Value Theorem

Let $[a, b]$ be a compact interval, and let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Then f attains both its minimum and its maximum.

Proof. Theorem 4.17 guarantees that f is bounded, or equivalently, that $f([a, b]) \subseteq \mathbb{R}$ is a bounded subset of \mathbb{R} . Thus, Theorem 2.35 implies that

$$S := \sup f([a, b])$$

exists. By definition of the supremum, for each $n \in \mathbb{N}$ there exists $y_n \in f([a, b])$ such that $S - 2^{-n} \leq y_n \leq S$. Hence, $y_n \rightarrow S$. Also, since $y_n \in f([a, b])$, there exists $x_n \in [a, b]$ such that $f(x_n) = y_n$.

Now, by Lemma 4.16, we can find a subsequence $(x_{n_k})_{k=0}^{\infty}$ such that $x_{n_k} \rightarrow \bar{x} \in [a, b]$. By continuity of f , we have that

$$f(\bar{x}) = \lim_{k \rightarrow \infty} f(x_{n_k}) = \lim_{k \rightarrow \infty} y_{n_k} = S,$$

so f attains its maximum at \bar{x} .

Applying the same reasoning to $-f$ shows that f also attains its minimum. \square

4.3.2 Uniform Continuity

Definition 4.21: Uniform Continuity

Let $D \subseteq \mathbb{R}$. A function $f : D \rightarrow \mathbb{R}$ is **uniformly continuous** if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon \quad \forall x, y \in D.$$

Remark 4.22. The difference between the usual definition of continuity and the one of uniform continuity lies in how the choice of δ depends on the points considered.

For a function that is continuous at each $x_0 \in D$, the δ in the definition may depend on both ε and x_0 : for every $\varepsilon > 0$ and each x_0 , we can find a $\delta = \delta(\varepsilon, x_0)$ that works near x_0 .

Uniform continuity is stronger: there exists a single $\delta = \delta(\varepsilon)$ that works **simultaneously** for all $x, y \in D$. In other words, the control on the variation of f does not deteriorate as we move along the domain. This property is automatically satisfied on compact intervals for continuous functions, as we will prove below.

Theorem 4.23: Uniform Continuity on Compact Intervals

Let $[a, b]$ be a compact interval, and $f : [a, b] \rightarrow \mathbb{R}$ continuous on $[a, b]$. Then f is uniformly continuous.

Proof. Assume, by contradiction, that f is not uniformly continuous on $[a, b]$. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$ one can find $x, y \in [a, b]$ with

$$|x - y| < \delta \quad \text{and} \quad |f(x) - f(y)| \geq \varepsilon.$$

Taking $\delta = 2^{-n}$ for each $n \in \mathbb{N}$, we obtain sequences $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ in $[a, b]$ with

$$|x_n - y_n| < 2^{-n} \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \varepsilon. \quad (4.3)$$

By Lemma 4.16, the sequence $(x_n)_{n=0}^\infty$ has a subsequence $(x_{n_k})_{k=0}^\infty$ converging to some $\bar{x} \in [a, b]$. Then

$$|y_{n_k} - \bar{x}| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - \bar{x}| < 2^{-n_k} + |x_{n_k} - \bar{x}| \xrightarrow{k \rightarrow \infty} 0,$$

so $y_{n_k} \rightarrow \bar{x}$ as well. Thus, by continuity of f and Theorem 4.12, we have that

$$\lim_{k \rightarrow \infty} f(x_{n_k}) = \lim_{k \rightarrow \infty} f(y_{n_k}) = f(\bar{x}),$$

therefore,

$$|f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(\bar{x})| + |f(\bar{x}) - f(y_{n_k})| \xrightarrow{k \rightarrow \infty} 0,$$

which contradicts Equation (4.3). Hence, f is uniformly continuous on $[a, b]$. \square

Definition 4.24: Lipschitz Continuity

Let $D \subseteq \mathbb{R}$, and $f : D \rightarrow \mathbb{R}$. We say that f is **Lipschitz continuous** if there exists $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in D.$$

Lemma 4.25: Lipschitz Continuity \Rightarrow Uniform Continuity

Let $D \subseteq \mathbb{R}$, and $f : D \rightarrow \mathbb{R}$ be a Lipschitz continuous function. Then f is uniformly continuous.

Proof. Let $D \subseteq \mathbb{R}$ and assume that $f : D \rightarrow \mathbb{R}$ is a Lipschitz continuous function. Then there exists $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in D.$$

Now, fix $\varepsilon > 0$. We assume that $L \neq 0$ (otherwise the result follows immediately) and choose $\delta = \frac{\varepsilon}{L}$. Because of the Lipschitz continuity of f , we have that for all $x, y \in D$ it holds that

$$\begin{aligned} |x - y| < \delta = \frac{\varepsilon}{L} &\Leftrightarrow L|x - y| < \varepsilon \\ \Rightarrow |f(x) - f(y)| &\leq L|x - y| < \varepsilon, \end{aligned}$$

which shows that f is also uniformly continuous. \square

4.4 Example: Exponential and Logarithmic Functions

4.4.1 Definition of the Exponential Function

Lemma 4.26: Bernoulli's Inequality

For all $a \in \mathbb{R}$ with $a \geq -1$ and all $n \in \mathbb{N}$ with $n \geq 1$, it holds that

$$(1 + a)^n \geq 1 + na.$$

Proof. We proceed by induction. For $n = 1$ we have $(1 + a)^1 = 1 + a = 1 + 1 \cdot a$.

Now assume that the inequality holds for some $n \geq 1$. Since $1 + a \geq 0$ by assumption, we find

$$(1 + a)^{n+1} = (1 + a)^n(1 + a) \geq (1 + na)(1 + a) = 1 + na + a + na^2 \geq 1 + (n + 1)a,$$

which establishes the induction step and completes the proof. \square

Proposition 4.27: Existence of the Exponential

Let $x \in \mathbb{R}$. The sequence $(a_n)_{n=1}^{\infty}$ defined by

$$a_n = \left(1 + \frac{x}{n}\right)^n$$

is convergent, and its limit is a positive real number.

Lemma 4.28: Monotonicity

Given $x \in \mathbb{R}$, let $n_0 \in \mathbb{N}$ satisfy $n_0 \geq 1$ and $n_0 > -x$. Then the sequence $(a_n)_{n=n_0}^{\infty}$ defined in Proposition 4.27 is increasing.

Definition 4.29: Exponential Function

The **exponential function** $\exp : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ is defined by

$$\exp(x) = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n \quad \forall x \in \mathbb{R}.$$

Corollary 4.30: Growth of the Exponential

Given $n \in \mathbb{N}$ with $n \geq 1$, the exponential function satisfies

$$\exp(x) \geq \left(1 + \frac{x}{n}\right)^n \quad \forall x > -n.$$

Proof. By Lemma 4.28 and Definition 4.29, for $x > -n$ we have

$$a_n \leq a_{n+1} \leq \dots \leq \exp(x).$$

□

4.4.2 Properties of the Exponential Function**Theorem 4.31: Properties of the Exponential Function**

The exponential function $\exp : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ is bijective, strictly increasing, and continuous. Moreover,

$$\begin{aligned} \exp(0) &= 1, \\ \exp(-x) &= \exp(x)^{-1}, \\ \exp(x+y) &= \exp(x)\exp(y), \end{aligned}$$

for all $x, y \in \mathbb{R}$.

4.4.3 The Natural Logarithm**Definition 4.32: Logarithm**

The unique inverse function

$$\log : \mathbb{R}_{>0} \rightarrow \mathbb{R}$$

of the bijective map $\exp : \mathbb{R} \rightarrow \mathbb{R}_{>0}$ is called the **logarithm**.

Corollary 4.33: Properties of the Logarithm

The logarithm $\log : \mathbb{R}_{>0} \rightarrow \mathbb{R}$ is strictly increasing, continuous, and bijective. Moreover,

$$\begin{aligned}\log(1) &= 0, \\ \log(a^{-1}) &= -\log(a), \\ \log(ab) &= \log(a) + \log(b),\end{aligned}$$

for all $a, b \in \mathbb{R}_{>0}$.

The logarithm defined here is also called the **natural logarithm** to distinguish it from logarithms with another **base** $a > 1$ (for instance $a = 10$ or $a = 2$). For any $a > 1$, we define

$$\log_a(x) = \frac{\log(x)}{\log(a)} \quad \forall x > 0.$$

Unless stated otherwise, $\log(x)$ always denotes the natural logarithm, i.e., the logarithm to base e .

We can now define powers with arbitrary real exponents. For $a > 0$ and $x \in \mathbb{R}$ we set

$$a^x = \exp(x \log(a)).$$

4.5 Limits of Functions

We consider functions $f : D \rightarrow \mathbb{R}$ defined on a subset $D \subseteq \mathbb{R}$, and we wish to define the limit of $f(x)$ as $x \in D$ approaches a point $x_0 \in \mathbb{R}$. Typical examples include $D = \mathbb{R}$, $D = [0, 1]$ or $D = (0, 1)$, with $x_0 = 0$ in each case.

4.5.1 Limit in the Vicinity of a Point

Let $D \subseteq \mathbb{R}$ be non-empty, and let $x_0 \in \mathbb{R}$ be such that

$$D \cap (x_0 - \delta, x_0 + \delta) \neq \emptyset \quad (4.4)$$

for all $\delta > 0$. Whenever this holds, we say that x_0 is an **accumulation point** of D . Note that if $x_0 \in D$, then Equation (4.4) is automatically satisfied.

Condition 4.4 ensures that there exists a sequence of points in D converging to x_0 .

Definition 4.34: Limit of a Function

Let $f : D \rightarrow \mathbb{R}$, and x_0 be an accumulation point of D . A number $L \in \mathbb{R}$ is called the **limit of $f(x)$ as $x \rightarrow x_0$** if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon \quad \forall x \in D.$$

In general, the limit of $f(x)$ as $x \rightarrow x_0$ may not exist. However, if it exists, it is uniquely determined. Hence we speak of *the* limit and write

$$\lim_{x \rightarrow x_0} f(x) = L$$

to indicate the limit exists and is equal to L . Informally, this means that the function values $f(x)$ are arbitrarily close to L whenever $x \in D$ is sufficiently close to x_0 .

The limit of a function satisfies properties analogous to those of Proposition 3.13. More precisely, if f, g are functions on D such that

$$\lim_{x \rightarrow x_0} f(x) = L_1 \quad \text{and} \quad \lim_{x \rightarrow x_0} g(x) = L_2,$$

then

$$\lim_{x \rightarrow x_0} (f + g)(x) = L_1 + L_2, \quad \lim_{x \rightarrow x_0} (f \cdot g)(x) = L_1 \cdot L_2.$$

Moreover, $f \leq g$ implies $L_1 \leq L_2$, and the sandwich lemma holds: if $f \leq h \leq g$ and $L_1 = L_2$ then $\lim_{x \rightarrow x_0} h(x) = L_1 = L_2$.

Remark 4.35. Let $f : D \rightarrow \mathbb{R}$ be a function. If $x_0 \in D$, then f is continuous at x_0 if and only if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

Suppose that $x_0 \in D$ is an accumulation point of $D \setminus \{x_0\}$. Let $f : D \rightarrow \mathbb{R}$, and consider the restriction $f|_{D \setminus \{x_0\}}$. It may happen that f is discontinuous at x_0 , but the limit

$$L = \lim_{x \rightarrow x_0} f|_{D \setminus \{x_0\}}(x) \tag{4.5}$$

nevertheless exists. In this case, the point x_0 is called a **removable discontinuity** of f , and one also writes

$$L = \lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} f(x). \tag{4.6}$$

If we now define

$$\tilde{f}(x) = \begin{cases} f(x), & x \in D \setminus \{x_0\}, \\ L, & x = x_0, \end{cases} \tag{4.7}$$

then \tilde{f} is continuous at x_0 . In other words, we can remove the discontinuity of f by redefining its value at x_0 to be L .

If instead $x_0 \notin D$ but the limit in Equation (4.6) exists, we call the function \tilde{f} defined in Equation (4.7) the **continuous extension** of f to $D \cup \{x_0\}$.

Arguing as in the proof of Theorem 4.12, we obtain the following result.

Lemma 4.36: Limit and Sequences

Let $f : D \rightarrow \mathbb{R}$. Then $L = \lim_{x \rightarrow \bar{x}} f(x)$ if and only if, for every sequence $(x_n)_{n=0}^{\infty} \subseteq D$ converging to \bar{x} , one has $\lim_{n \rightarrow \infty} f(x_n) = L$.

We now state a result describing the behaviour of limits under composition with a continuous function.

Proposition 4.37: Limit and Composition

Let $E \subseteq \mathbb{R}$, and let $f : D \rightarrow E$ be such that the limit $L = \lim_{x \rightarrow \bar{x}} f(x)$ exists and belongs to E . If $g : E \rightarrow \mathbb{R}$ is continuous at L , then

$$\lim_{x \rightarrow \bar{x}} g(f(x)) = g(L).$$

Proof. Let $(x_n)_{n=0}^{\infty} \subseteq D$ be a sequence converging to \bar{x} . By Lemma 4.36, we have $\lim_{n \rightarrow \infty} f(x_n) = L$. Since g is continuous at L , Theorem 4.12 gives $\lim_{n \rightarrow \infty} g(f(x_n)) = g(L)$. Because $(x_n)_{n=0}^{\infty}$ was arbitrary, using Lemma 4.36 again, we conclude that $\lim_{x \rightarrow \bar{x}} g(f(x)) = g(L)$. \square

We now introduce conventions for improper limits of functions, in analogy with improper limits for sequences.

Definition 4.38: Improper Limits

Let $f : D \rightarrow \mathbb{R}$, and let x_0 be an accumulation point of D . We say that f **diverges to $+\infty$ as $x \rightarrow x_0$** , and write

$$\lim_{x \rightarrow x_0} f(x) = +\infty,$$

if for every $M > 0$, there exists $\delta > 0$ such that

$$\forall x \in D : |x - x_0| < \delta \Rightarrow f(x) \geq M.$$

Analogously, f **diverges to $-\infty$ as $x \rightarrow x_0$** and we write $\lim_{x \rightarrow x_0} f(x) = -\infty$, if for every $M > 0$, there exists $\delta > 0$ such that

$$\forall x \in D : |x - x_0| < \delta \Rightarrow f(x) \leq -M.$$

4.5.2 One-Sided Limits

It is often useful to consider limits taken from one side only and to allow x_0 to be $\pm\infty$ as well. To this end, let $x_0 \in \mathbb{R}$ be such that

$$D \cap (x_0, x_0 + \delta) \neq \emptyset \tag{4.8}$$

for every $\delta > 0$. In this case, we say that x_0 is a **right-hand accumulation point** of D . Analogously, if

$$D \cap (x_0 - \delta, x_0) \neq \emptyset \tag{4.9}$$

for every $\delta > 0$, we say that x_0 is a **left-hand accumulation point** of D .

Definition 4.39: One-Sided Limits

Let $f : D \rightarrow \mathbb{R}$, and let $x_0 \in \mathbb{R}$ be a right-hand accumulation point of D . A number $L \in \mathbb{R}$ is called the **right-hand limit** of f at x_0 if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in D \cap (x_0, x_0 + \delta) \Rightarrow |f(x) - L| < \varepsilon.$$

In this case we write $L = \lim_{x \rightarrow x_0^+} f(x)$. We also allow improper one-sided limits. We say that

$$\lim_{x \rightarrow x_0^+} f(x) = +\infty$$

if for every $M > 0$, there exists $\delta > 0$ such that

$$x \in D \cap (x_0, x_0 + \delta) \Rightarrow f(x) \geq M.$$

Similarly, $\lim_{x \rightarrow x_0^+} f(x) = -\infty$ means that, for every $M > 0$, there exists $\delta > 0$ such that

$$x \in D \cap (x_0, x_0 + \delta) \Rightarrow f(x) \leq -M.$$

The **left-hand limit** is defined analogously, considering a left-hand accumulation point of D and writing $\lim_{x \rightarrow x_0^-} f(x)$.

Next, we define the notion of limit at infinity.

Definition 4.40: Limits at Infinity

Let $f : D \rightarrow \mathbb{R}$, and assume that $D \cap (R, \infty) \neq \emptyset$ for every $R > 0$. A number $L \in \mathbb{R}$ is called the **limit of f as $x \rightarrow +\infty$** if, for every $\varepsilon > 0$, there exists $R > 0$ such that

$$x \in D \cap (R, \infty) \Rightarrow |f(x) - L| < \varepsilon.$$

We say that f **diverges to $+\infty$ as $x \rightarrow +\infty$** if, for every $M > 0$, there exists $R > 0$ such that

$$x \in D \cap (R, \infty) \Rightarrow f(x) \geq M.$$

The corresponding definition for $x \rightarrow -\infty$ and diverges to $-\infty$ are analogous.

Limits at $+\infty$ can be converted into right-hand limits at 0 via inversion. Given $f : D \rightarrow \mathbb{R}$ as above, define

$$E = \{x > 0 \mid x^{-1} \in D\}, \quad g : E \rightarrow \mathbb{R}, \quad g(x) = f(x^{-1}).$$

Then

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow 0^+} g(x),$$

so one limit exists if and only if the other does.

Definition 4.41: One-Sided Continuity and Jumps

Let $f : D \rightarrow \mathbb{R}$ and $x_0 \in D$. If $\lim_{x \rightarrow x_0^+} f(x)$ exists and equals $f(x_0)$, then f is **continuous from the right** at x_0 . **Continuity from the left** is defined similarly. We call x_0 a **jump point** if both one-sided limits exist but are different, i.e.,

$$L_- := \lim_{x \rightarrow x_0^-} f(x) \in \mathbb{R}, \quad L_+ := \lim_{x \rightarrow x_0^+} f(x) \in \mathbb{R}, \quad L_- \neq L_+.$$

4.5.3 Landau Notation

We introduce two standard notations that compare the asymptotic behaviour of a function to that of another function. (often called *relative asymptotics*).

Definition 4.42: Big-O at a Point

Let $f, g : D \rightarrow \mathbb{R}$, and let x_0 be an accumulation point of D . We write

$$f(x) = O(g(x)) \quad \text{as } x \rightarrow x_0$$

if there exists $M > 0$ and $\delta > 0$ such that

$$x \in D \cap (x_0 - \delta, x_0 + \delta) \Rightarrow |f(x)| \leq M|g(x)|.$$

We then say that f is a **Big-O** of g as $x \rightarrow x_0$.

If $g(x) \neq 0$ for all x sufficiently close to x_0 (with $x \in D$), then

$$f(x) = O(g(x)) \quad \text{as } x \rightarrow x_0 \quad \Leftrightarrow \quad \frac{f(x)}{g(x)} \text{ is bounded near } x_0.$$

Definition 4.43: Big-O at Infinity

Let $f, g : D \rightarrow \mathbb{R}$, and assume $D \cap (R, \infty) \neq \emptyset$ for every $R > 0$. We write

$$f(x) = O(g(x)) \quad \text{as } x \rightarrow +\infty$$

if there exists $M > 0$ and $R > 0$ such that

$$x \in D \cap (R, \infty) \quad \Rightarrow \quad |f(x)| \leq M|g(x)|.$$

The definition for $x \rightarrow -\infty$ is analogous.

The big-O notation hides the precise bound by an *implicit constant* M , which is often irrelevant for the argument one is interested in.

Example

- if f and g are bounded and continuous near x_0 with $g(x_0) \neq 0$, then $f(x) = O(g(x))$ as $x \rightarrow x_0$.
- As $x \rightarrow 0$, one has $x^2 = O(x)$, but $x \neq O(x^2)$ (since x/x^2 is unbounded near 0).
- As $x \rightarrow +\infty$, $\frac{3x^3}{x^3+3} = O(1)$, but $\frac{3x^3}{x^3+3} \neq O(x^\alpha)$ for $\alpha < 0$.

As discussed above, the big-O means that f is bounded by a multiple of g . One may also consider a stronger condition, namely that f is asymptotically negligible with respect to g . This leads to the following definition.

Definition 4.44: Little-O at a Point

Let $f, g : D \rightarrow \mathbb{R}$, and let x_0 be an accumulation point of D . We write

$$f(x) = o(g(x)) \quad \text{as } x \rightarrow x_0$$

if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x \in D \cap (x_0 - \delta, x_0 + \delta) \quad \Rightarrow \quad |f(x)| \leq \varepsilon|g(x)|.$$

We then say that f is a **little-o** of g as $x \rightarrow x_0$.

If $g(x) \neq 0$ for all x near x_0 (with $x \in D$), then

$$f(x) = o(g(x)) \quad \text{as } x \rightarrow x_0 \quad \Leftrightarrow \quad \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0.$$

Moreover, $f(x) = o(g(x)) \Rightarrow f(x) = O(g(x))$.

Definition 4.45: Little-o at Infinity

Let $f, g : D \rightarrow \mathbb{R}$, and assume that $D \cap (R, \infty) \neq \emptyset$ for every $R > 0$. We write

$$f(x) = o(g(x)) \quad \text{as } x \rightarrow +\infty$$

if, for every $\varepsilon > 0$ there exists $R > 0$ such that

$$x \in D \cap (R, \infty) \quad \Rightarrow \quad |f(x)| \leq \varepsilon |g(x)|.$$

The definition for $x \rightarrow -\infty$ is analogous.

Example

- $x = o(x^2)$ as $x \rightarrow +\infty$, and $x^2 = o(x)$ as $x \rightarrow 0$

- For any $\alpha < 1$,

$$\frac{3x^3}{2x^2 + x^{10}} = o(|x|^\alpha) \quad \text{as } x \rightarrow 0,$$

but not for $\alpha \geq 1$. Indeed,

$$\left| \frac{3x^3}{|x|^\alpha 2x^2 + x^{10}} \right| = |x|^{1-\alpha} \frac{3}{2 + x^8} \rightarrow 0 \quad \text{as } x \rightarrow 0,$$

whenever $\alpha < 1$.

In computations, one often uses Landau symbols as placeholders. Writing

$$f(x) = o(g(x)) \quad \text{as } x \rightarrow x_0$$

means there is a function $h : D \rightarrow \mathbb{R}$ with $h(x) = o(g(x))$ as $x \rightarrow x_0$. Similarly for big-O.

Example

Polynomial division gives, as $x \rightarrow +\infty$,

$$\frac{x^3 - 7x^2 + 6x + 2}{x^2} = x - 7 + O\left(\frac{1}{x}\right) = x - 7 + o(1) = x + O(1) = x + o(x).$$

4.6 Sequences of Functions**4.6.1 Pointwise Convergence****Definition 4.46: Sequences of Functions**

A **sequence** of real-valued on a subset $D \subseteq \mathbb{R}$ is a family of functions $f_n : D \rightarrow \mathbb{R}$ indexed by \mathbb{N} . The function f_n is called the **n-th element** of the sequence. One often writes $(f_n)_{n \in \mathbb{N}}$, $(f_n)_{n=0}^\infty$, or $(f_n)_{n \geq 0}$ for a sequence of functions.

Definition 4.47: Pointwise Convergence

Let $D \subseteq \mathbb{R}$, and let $(f_n)_{n=0}^\infty$ be a sequence of functions $f_n : D \rightarrow \mathbb{R}$. Let $f : D \rightarrow \mathbb{R}$ be another function. We say that $(f_n)_{n=0}^\infty$ **converges pointwise** to f , if for every $x \in D$, the sequence of real numbers $(f_n(x))_{n=0}^\infty$ converges to $f(x)$, i.e., for every $x \in D$ and for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N.$$

In this case, f is called the **pointwise limit** of the sequence $(f_n)_{n=0}^\infty$.

Remark 4.48. Note that, in the definition of pointwise convergence the index N may depend on both x and ε . In other words, for each point $x \in D$ we examine the convergence of $(f_n)_{n=0}^\infty$ to f separately.

4.6.2 Uniform Convergence**Definition 4.49: Uniform Convergence**

Let $D \subseteq \mathbb{R}$, and let $(f_n)_{n=0}^\infty$ be a sequence of functions $f_n : D \rightarrow \mathbb{R}$. Let $f : D \rightarrow \mathbb{R}$ be another function. We say that $(f_n)_{n=0}^\infty$ **converges uniformly** to f on D if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N, \forall x \in D.$$

Remark 4.50. Note that, in the definition of uniform convergence the index N may only depend on ε , and therefore the condition has to hold for all $x \in D$ for the sequence of functions $(f_n)_{n=0}^\infty$ to converge uniformly to f on D .

Remark 4.51. Let $D \subseteq \mathbb{R}$, and $(f_n)_{n=0}^\infty$ be a sequence of functions $f_n : D \rightarrow \mathbb{R}$ converging uniformly to $f : D \rightarrow \mathbb{R}$ on D . Then the sequence of functions $(f_n)_{n=0}^\infty$ also converges pointwise to f .

Theorem 4.52: Continuity under Uniform Convergence

Let $D \subseteq \mathbb{R}$, and let $(f_n)_{n=0}^\infty$ be a sequence of continuous functions converging uniformly to $f : D \rightarrow \mathbb{R}$. Then f is continuous.

Proof. To prove that f is continuous, we fix $\bar{x} \in D$ and show that f is continuous at \bar{x} . Given $\varepsilon > 0$, the uniform convergence of f_N to f provides $N \in \mathbb{N}$ such that

$$|f_N(y) - f(y)| < \frac{\varepsilon}{3} \quad \forall y \in D.$$

Also, since f_N is continuous at \bar{x} , there exists $\delta > 0$ such that

$$|x - \bar{x}| < \delta \quad \Rightarrow \quad |f_N(x) - f_N(\bar{x})| < \frac{\varepsilon}{3}.$$

Then, for $|x - \bar{x}| < \delta$, we have

$$|f(x) - f(\bar{x})| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(\bar{x})| + |f_N(\bar{x}) - f(\bar{x})| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Which shows that f is continuous at \bar{x} . Since, \bar{x} is arbitrary, f is continuous on D . \square

Intuitively, uniform convergence allows us to *exchange* the order of taking limits. More precisely, assume $(f_n)_{n=0}^\infty$ is a sequence of continuous functions converging pointwise to f . Then, by the pointwise

convergence and the continuity of the functions f_n we have

$$f(\bar{x}) = \lim_{n \rightarrow \infty} f_n(\bar{x}), \quad f_n(\bar{x}) = \lim_{x \rightarrow \bar{x}} f_n(x), \quad f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \forall x \in D.$$

Hence,

$$f(\bar{x}) = \lim_{n \rightarrow \infty} f_n(\bar{x}) = \lim_{n \rightarrow \infty} \left(\lim_{x \rightarrow \bar{x}} f_n(x) \right), \quad \lim_{x \rightarrow \bar{x}} f(x) = \lim_{x \rightarrow \bar{x}} \left(\lim_{n \rightarrow \infty} f_n(x) \right).$$

Note that the function f is continuous at \bar{x} if and only if $f(\bar{x}) = \lim_{x \rightarrow \bar{x}} f(x)$, which by the identities above is equivalent to

$$\lim_{x \rightarrow \bar{x}} \left(\lim_{n \rightarrow \infty} f_n(x) \right) = \lim_{n \rightarrow \infty} \left(\lim_{x \rightarrow \bar{x}} f_n(x) \right).$$

As we have seen, for pointwise convergent this interchange may fail because f need not be continuous. However, Theorem 4.52 ensures that this equality holds under uniform convergence.

5 Series and Power Series

In this chapter we study series (infinite sums). They provide a framework to define many classical functions; in particular, we will use series to define trigonometric functions.

5.1 Series of Real Numbers

Definition 5.1: Convergent and Divergent Series

Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let $A \in \mathbb{R}$. We say that the series $\sum_{k=0}^{\infty} a_k$ **converges** to A if

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n a_k = A.$$

In other words, computing the infinite sum $\sum_{k=0}^{\infty} a_k$ means finding (if it exists) the limit of the **partial sums**

$$s_n = \sum_{k=0}^n a_k, \quad n \in \mathbb{N}.$$

We call a_n the **n -th term** (or **n -th summand**) of the series. If the limit exists, its value A is the **sum of the series**. If the limit does not exist, the series is said to be **not convergent**. In particular, if the sequence of partial sums $(s_n)_{n=0}^{\infty}$ diverges to $+\infty$ (respectively, to $-\infty$), we say that the series **diverges to** $+\infty$ (respectively, **to** $-\infty$). This situation is therefore a specific case of a series that does not converge.

Proposition 5.2: Necessary Condition for Convergence

If the series $\sum_{k=0}^{\infty} a_k$ converges, then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Proof. By assumption the partial sums $s_n = \sum_{k=0}^n a_k$ satisfy $s_n \rightarrow A \in \mathbb{R}$. Then for $n \geq 1$, we have

$$a_n = s_n - s_{n-1} \xrightarrow{n \rightarrow \infty} A - A = 0. \quad \square$$

Geometric Series

For $q \in \mathbb{R}$, the geometric series $\sum_{k=0}^{\infty} q^k$ converges if and only if $|q| < 1$, and in this case

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}.$$

Indeed, if the series converges, then by Proposition 5.2 we must have $q^n \rightarrow 0$ as $n \rightarrow \infty$, hence $|q| < 1$. Conversely, for $|q| < 1$ one provides by induction that

$$s_n = \sum_{k=0}^n q^k = \frac{1 - q^{n+1}}{1 - q} \quad \forall n \in \mathbb{N}, q \neq 1.$$

Also since $|q| < 1$, $q^{n+1} \rightarrow 0$ as $n \rightarrow \infty$. Thus,

$$s_n = \frac{1 - q^{n+1}}{1 - q} \xrightarrow{n \rightarrow \infty} \frac{1}{1 - q}.$$

Harmonic Series

The converse of Proposition 5.2 fails: the **harmonic series** $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge. To see this, consider $n = 2^\ell$ with $\ell \in \mathbb{N}$. Grouping terms gives

$$\begin{aligned} \sum_{k=1}^{2^\ell} \frac{1}{k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{\ell-1}+1} + \dots + \frac{1}{2^\ell}\right) \\ &\geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{=\frac{1}{2}} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{=\frac{1}{2}} + \dots + \underbrace{\frac{1}{2^\ell} + \frac{1}{2^\ell}}_{=\frac{1}{2}} \\ &= 1 + \underbrace{\frac{1}{2} + \dots + \frac{1}{2}}_{\ell \text{ - times}} = 1 + \frac{\ell}{2}, \end{aligned}$$

which is unbounded as $\ell \rightarrow \infty$.

Lemma 5.3: Convergence of the Tail

Let $\sum_{k=0}^{\infty} a_k$ be a series and fix $N \in \mathbb{N}$. Then $\sum_{k=0}^{\infty} a_k$ is convergent if and only if $\sum_{k=N}^{\infty} a_k$ is convergent, and in that case

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{N-1} a_k + \sum_{k=N}^{\infty} a_k.$$

The same equivalence holds for divergence to $+\infty$ or $-\infty$.

Proof. For every $n \geq N$,

$$\sum_{k=0}^n a_k = \sum_{k=0}^{N-1} a_k + \sum_{k=N}^n a_k.$$

Thus, the partial sums of $\sum_{k=0}^{\infty} a_k$ converge if and only if those of $\sum_{k=N}^{\infty} a_k$ do, and the identity in the statement follows by letting $n \rightarrow \infty$. The divergence case is analogous. \square

5.1.1 Series with Non-negative Elements

Proposition 5.4: Non-negative Series: Convergence vs. Divergence

Let $\sum_{k=0}^{\infty} a_k$ be a series with non-negative terms $a_k \geq 0$ for all $k \in \mathbb{N}$. Then the partial sums $s_n = \sum_{k=0}^n a_k$ form an increasing sequence. If $(s_n)_{n=0}^{\infty}$ is bounded, the series $\sum_{k=0}^{\infty} a_k$ converges; otherwise it diverges to $+\infty$.

Proof. Since $a_{n+1} \geq 0$, we have $s_{n+1} = s_n + a_{n+1} \geq s_n$ for all $n \in \mathbb{N}$, so $(s_n)_{n=0}^{\infty}$ is increasing.

If the sequence $(s_n)_{n=0}^{\infty}$ is bounded, then it converges by Theorem 3.20. If the partial sums are not bounded, then they diverge to $+\infty$. \square

Remark 5.5. If $\sum_{k=0}^{\infty} a_k$ has non-negative terms, then $(s_n)_{n=0}^{\infty}$ is bounded if and only if it has a bounded subsequence $(s_{n_k})_{k=0}^{\infty}$.

Corollary 5.6: Comparison Test (Majorant/Minorant)

Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be series with $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Then

$$0 \leq \sum_{k=0}^{\infty} a_k \leq \sum_{k=0}^{\infty} b_k,$$

and in particular

$$\begin{aligned} \sum_{k=0}^{\infty} b_k \text{ convergent} &\Rightarrow \sum_{k=0}^{\infty} a_k \text{ convergent}, \\ \sum_{k=0}^{\infty} a_k \text{ divergent to } +\infty &\Rightarrow \sum_{k=0}^{\infty} b_k \text{ divergent to } +\infty. \end{aligned}$$

These implications remain true if the inequalities $0 \leq a_n \leq b_n$ hold only for all $n \geq N$, for some $N \in \mathbb{N}$.

Proof. From $a_k \leq b_k$ we get $\sum_{k=0}^n a_k \leq \sum_{k=0}^n b_k$ for all $n \in \mathbb{N}$. Therefore,

$$\sum_{k=0}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k \leq \lim_{n \rightarrow \infty} \sum_{k=0}^n b_k = \sum_{k=0}^{\infty} b_k.$$

The last part of the statement follows from Lemma 5.3. \square

Under the assumptions of the corollary, $\sum_{k=0}^{\infty} b_k$ is called a *majorant* of $\sum_{k=0}^{\infty} a_k$, and $\sum_{k=0}^{\infty} a_k$ a *minorant* of $\sum_{k=0}^{\infty} b_k$. Hence the names **majorant** and **minorant criterion**.

Proposition 5.7: Cauchy Condensation Test

Let $(a_k)_{k=0}^{\infty}$ be a decreasing sequence of non-negative numbers. Then

$$\sum_{k=0}^{\infty} a_k \text{ converges} \Leftrightarrow \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ converges}.$$

Proof. Consider the partial sums of the series $\sum_{k=0}^{\infty} a_k$ starting from $k = 2$ up to an index that is a power of 2. Since the terms a_k are decreasing, the following inequalities hold:

$$\begin{aligned} \sum_{k=2}^{2^{n+1}} a_k &= a_2 + (a_3 + a_4) + (a_5 + \dots + a_8) + \dots + (a_{2^n+1} + \dots + a_{2^{n+1}}) \\ &\leq \underbrace{a_1}_{=1 \cdot a_1} + \underbrace{(a_2 + a_2)}_{=2 \cdot a_2} + \underbrace{(a_4 + \dots + a_4)}_{=4 \cdot a_4} + \dots + \underbrace{(a_{2^n} + \dots + a_{2^n})}_{=2^n \cdot a_{2^n}} \\ &= a_1 + 2a_2 + 4a_4 + \dots + 2^n a_{2^n} = \sum_{k=0}^n 2^k a_{2^k}, \end{aligned}$$

and similarly,

$$\begin{aligned} \sum_{k=2}^{2^{n+1}} a_k &= a_2 + (a_3 + a_4) + (a_5 + \dots + a_8) + \dots + (a_{2^n+1} + \dots + a_{2^{n+1}}) \\ &\geq \underbrace{a_2}_{=1 \cdot a_2} + \underbrace{(a_4 + a_4)}_{=2 \cdot a_4} + \underbrace{(a_8 + \dots + a_8)}_{=4 \cdot a_8} + \dots + \underbrace{(a_{2^{n+1}} + \dots + a_{2^{n+1}})}_{=2^{n+1} \cdot a_{2^{n+1}}} \\ &= \frac{1}{2}(2a_2 + 4a_4 + \dots + 2^{n+1} a_{2^{n+1}}) = \frac{1}{2} \sum_{k=1}^{n+1} 2^k a_{2^k}. \end{aligned}$$

In other words,

$$\sum_{k=0}^n 2^k a_{2^k} \geq \sum_{j=2}^{2^{n+1}} a_j \geq \frac{1}{2} \sum_{k=1}^{n+1} 2^k a_{2^k}.$$

By Remark 5.5 and Corollary 5.6, the partial sums of one series are bounded if and only if those of the other are. Hence, the two series converge or diverge together. \square

5.1.2 Conditional Convergence

Definition 5.8: Absolute and Conditional Convergence

A series $\sum_{k=0}^{\infty} a_k$ is **absolutely convergent** if $\sum_{k=0}^{\infty} |a_k|$ converges. It is **conditionally convergent** if $\sum_{k=0}^{\infty} a_k$ converges but $\sum_{k=0}^{\infty} |a_k|$ diverges.

A striking feature of conditionally convergent series is that their terms can be rearranged to obtain any prescribed limit.

Theorem 5.9: Riemann Rearrangement Theorem

Let $\sum_{n=0}^{\infty} a_n$ be a conditionally convergent series and let $A \in \mathbb{R}$. Then there exists a bijection $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$A = \sum_{n=0}^{\infty} a_{\varphi(n)}.$$

The proof of this Theorem is extra material.

5.1.3 Leibniz Criterion for Alternating Series

Definition 5.10: Alternating Series

If $(a_k)_{k=0}^{\infty}$ is a sequence of non-negative numbers, the series

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

is called the **alternating series** associated with the sequence $(a_k)_{k=0}^{\infty}$.

Proposition 5.11: Leibniz Criterion

Let $(a_k)_{k=0}^{\infty}$ be a monotonically decreasing sequence of non-negative numbers with $a_k \rightarrow 0$. Then the alternating series $\sum_{k=0}^{\infty} (-1)^k a_k$ converges, and for all $n \in \mathbb{N}$,

$$\sum_{k=0}^{2n+1} (-1)^k a_k \leq \sum_{k=0}^{\infty} (-1)^k a_k \leq \sum_{k=0}^{2n} (-1)^k a_k. \quad (5.1)$$

Proof. Let $s_n = \sum_{k=0}^n (-1)^k a_k$. Since the sequence $(a_n)_{n=0}^{\infty}$ is decreasing and non-negative, we have

$$\begin{aligned} s_{2n+2} &= s_{2n} - \underbrace{a_{2n+1} + a_{2n+2}}_{\leq 0} \leq s_{2n}, \\ s_{2n+1} &= s_{2n-1} + \underbrace{a_{2n} - a_{2n+1}}_{\geq 0} \geq s_{2n-1}, \\ s_{2n+2} &= s_{2n+1} + \underbrace{a_{2n+2}}_{\geq 0} \geq s_{2n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. In other words,

$$s_1 \leq s_3 \leq \dots \leq s_{2n-1} \leq s_{2n+1} \leq \dots \leq s_{2n+2} \leq s_{2n} \leq \dots \leq s_2 \leq s_0.$$

This implies that the sequence $(s_{2n})_{n=0}^{\infty}$ is decreasing and bounded below, while the sequence $(s_{2n+1})_{n=0}^{\infty}$ is increasing and bounded from above. Thus, both limits $A = \lim_{n \rightarrow \infty} s_{n+1}$ and $B = \lim_{n \rightarrow \infty} s_{2n}$ exist and satisfy

$$s_1 \leq s_3 \leq \dots \leq s_{2n-1} \leq s_{2n+1} \leq A \leq B \leq s_{2n+2} \leq s_{2n} \leq \dots \leq s_2 \leq s_0. \quad (5.2)$$

In particular,

$$0 \leq B - A \leq s_{2n+2} - s_{2n-1} \quad \forall n \in \mathbb{N},$$

and because $a_{2n+2} \rightarrow 0$, we deduce that $A = B$.

Also, Equation (5.2) yields that $s_{2n+1} \leq A = B \leq s_{2n}$, which corresponds exactly to Equation (5.1). \square

Example (Alternating Harmonic Series)

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

converges by Proposition 5.11, whereas $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges. Hence, the alternating harmonic series is only conditionally convergent.

5.2 Absolute Convergence

In this section we will look at absolutely convergent series and prove some convergence criteria. As before, unless otherwise specified, all sequences consist of real numbers.

5.2.1 Criteria for Absolute Convergence

We begin by restating the concept of a Cauchy sequence in the context of convergent series.

Theorem 5.12: Cauchy Criterion for Series

The series $\sum_{k=0}^{\infty} a_k$ converges if and only if, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > m \geq N$,

$$\left| \sum_{k=m+1}^n a_k \right| < \varepsilon.$$

Proof. By definition, the series $\sum_{k=0}^{\infty} a_k$ converges if and only if the sequence of partial sums

$$s_n = \sum_{k=0}^n a_k$$

converges. By Theorem 3.27, this occurs if and only if $(s_n)_{n=0}^{\infty}$ is a Cauchy sequence, i.e., $|s_n - s_m| < \varepsilon$ for all $n, m \geq N$. Since $s_n - s_m = 0$ when $n = m$, and the expression is symmetric in n and m , it suffices to consider the case $n > m$. In this case,

$$s_n - s_m = \sum_{k=m+1}^n a_k,$$

which proves the claim. □

We can now prove that absolutely convergent series do indeed converge.

Proposition 5.13: Absolute Convergence Implies Convergence

If a series $\sum_{n=0}^{\infty} a_n$ converges absolutely, then it converges and satisfies the generalized triangle inequality

$$\left| \sum_{n=0}^{\infty} a_n \right| \leq \sum_{n=0}^{\infty} |a_n|.$$

Proof. Since $\sum_{n=0}^{\infty} |a_n|$ converges, by the Cauchy criterion (Theorem 5.12) there exists $N \in \mathbb{N}$ such that, for all $n > m \geq N$,

$$\sum_{k=m+1}^n |a_k| < \varepsilon.$$

By the triangle inequality,

$$\left| \sum_{k=m+1}^n a_k \right| \leq \sum_{k=m+1}^n |a_k| < \varepsilon,$$

so $\sum_{n=0}^{\infty} a_n$ also satisfies the Cauchy criterion and therefore converges.

Moreover, again by the triangle inequality,

$$\left| \sum_{k=0}^n a_k \right| \leq \sum_{k=0}^n |a_k| \leq \sum_{k=0}^{\infty} a_k \quad \forall n \in \mathbb{N},$$

and taking the limit as $n \rightarrow \infty$ gives the desired inequality. \square

We now establish two classical criteria guaranteeing absolute convergence. In their proof, we repeatedly use the following fact:

Remark 5.14. *If a sequence $(x_n)_{n=0}^{\infty}$ converges to $\alpha \in \mathbb{R}$, then Proposition 3.13 implies the following facts:*

(i) *for any $q > \alpha$ there exists $N \in \mathbb{N}$ such that $x_n < q$ for all $n \geq N$;*

(ii) *for any $r < \alpha$ there exists $N \in \mathbb{N}$ such that $x_n > r$ for all $n \geq N$.*

Proposition 5.15: Cauchy Root Criterion

Given a sequence $(a_n)_{n=0}^{\infty}$, define

$$\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} \in \mathbb{R} \cup \{\infty\}.$$

Then,

$$\alpha < 1 \Rightarrow \sum_{n=0}^{\infty} a_n \text{ converges absolutely,} \quad \alpha > 1 \Rightarrow \sum_{n=0}^{\infty} a_n \text{ does not converge.}$$

Proof. Suppose $\alpha < 1$ and set $q = \frac{1+\alpha}{2}$, so that $q \in (\alpha, 1)$. By definition,

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \sup_{k \geq n} \sqrt[k]{|a_k|}.$$

Thus, $x_n = \sup_{k \geq n} \sqrt[k]{|a_k|} \rightarrow \alpha$. Since $\alpha < q$, Remark 5.14(i) implies the existence of $N \in \mathbb{N}$ such that

$$x_N = \sup_{k \geq N} \sqrt[k]{|a_k|} < q \quad \forall k \geq N,$$

therefore,

$$|a_k| < q^k \quad \forall k \geq N.$$

Since $q < 1$, $\sum_{k=N}^{\infty} |a_k|$ converges by comparison with the geometric series, so $\sum_{n=0}^{\infty} a_n$ converges absolutely.

If $\alpha > 1$, since the limsup is an accumulation point (Theorem 3.23), Proposition 3.9 implies the existence of a subsequence $(a_{n_k})_{k=0}^{\infty}$ such that $\lim_{k \rightarrow \infty} \sqrt[n_k]{|a_{n_k}|} = \alpha$. Hence, thanks to Remark 5.14(ii) with $r = 1$, $\sqrt[n_k]{|a_{n_k}|} > 1$ for all k large, or equivalently, $|a_{n_k}| > 1$ for large k . In particular the sequence $(a_n)_{n=0}^{\infty}$ does not converge to 0. Recalling Proposition 5.2, this implies that the series $\sum_{n=0}^{\infty} a_n$ does not converge. \square

Remark 5.16. *If $\alpha = 1$, the root criterion is inconclusive.*

Proposition 5.17: D'Alembert's Quotient Criterion

Given a sequence $(a_n)_{n=0}^{\infty}$ with $a_n \neq 0$ for all n , assume that

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \alpha \in [0, \infty).$$

Then

$$\alpha < 1 \Rightarrow \sum_{n=0}^{\infty} a_n \text{ converges absolutely,} \quad \alpha > 1 \Rightarrow \sum_{n=0}^{\infty} a_n \text{ does not converge.}$$

Proof. The proof parallels that of the root criterion.

If $\alpha < 1$, set $q = \frac{1+\alpha}{2} \in (\alpha, 1)$. Since $\frac{|a_{n+1}|}{|a_n|} \rightarrow \alpha$ and $\alpha < q$, by Remark 5.14(i) there exists $N \in \mathbb{N}$ such that

$$\frac{|a_{k+1}|}{|a_k|} < q \quad \forall k \geq N.$$

This gives

$$|a_k| = \frac{|a_k|}{|a_{k-1}|} \cdot \frac{|a_{k-1}|}{|a_{k-2}|} \cdot \dots \cdot \frac{|a_{N+1}|}{|a_N|} \cdot |a_N| < q^{k-N} |a_N| = \frac{|a_N|}{q^N} q^k \quad \forall k \geq N.$$

Since $q < 1$, the geometric comparison test shows that $\sum_{n=0}^{\infty} a_n$ converges absolutely.

If $\alpha > 1$, then Remark 5.14(ii) with $r = 1$ implies the existence of $N \in \mathbb{N}$ such that

$$\frac{|a_{k+1}|}{|a_k|} > 1 \quad \forall k \geq N.$$

In particular,

$$|a_k| = \frac{|a_k|}{|a_{k-1}|} \cdot \frac{|a_{k-1}|}{|a_{k-2}|} \cdot \dots \cdot \frac{|a_{N+1}|}{|a_N|} \cdot |a_N| > |a_N| \quad \forall k \geq N.$$

Hence, $(a_n)_{n=0}^{\infty}$ does not tend to 0, and by Proposition 5.2 the series does not converge. \square

5.2.2 Reordering Series**Theorem 5.18: Rearrangement of Absolutely Convergent Series**

Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series, and let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection. Then $\sum_{n=0}^{\infty} a_{\varphi(n)}$ is absolutely convergent, and

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_{\varphi(n)}. \quad (5.3)$$

Proof. Fix $\varepsilon > 0$. Since $\sum_{n=0}^{\infty} |a_n|$ converges, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} |a_k| < \frac{\varepsilon}{2}.$$

Let

$$M = \max\{\varphi^{-1}(0), \dots, \varphi^{-1}(N)\}.$$

Equivalently, $M \in \mathbb{N}$ is the smallest number such that

$$\{a_0, \dots, a_N\} \subseteq \{a_{\varphi(0)}, \dots, a_{\varphi(M)}\}$$

Then

$$\{a_0, \dots, a_n\} \subseteq \{a_{\varphi(0)}, \dots, a_{\varphi(n)}\} \quad \forall n \geq M.$$

Therefore,

$$\sum_{\ell=0}^n a_{\varphi(\ell)} - \sum_{k=0}^N a_k = \sum_{\substack{0 \leq \ell \leq n \\ \varphi(\ell) > N}} a_{\varphi(\ell)}.$$

Moreover, since all indices $\varphi(\ell) > N$ with $0 \leq \ell \leq n$ correspond to terms among $\{|a_k| \mid k \geq N+1\}$, we have

$$\sum_{\substack{0 \leq \ell \leq n \\ \varphi(\ell) > N}} |a_{\varphi(\ell)}| \leq \sum_{k=N+1}^{\infty} |a_k|.$$

This implies that, for $n \geq M$, we can estimate

$$\begin{aligned} \left| \sum_{\ell=0}^n a_{\varphi(\ell)} - \sum_{k=0}^{\infty} a_k \right| &= \left| \sum_{\ell=0}^n a_{\varphi(\ell)} - \sum_{k=0}^N a_k - \sum_{k=N+1}^{\infty} a_k \right| = \left| \sum_{\substack{0 \leq \ell \leq n \\ \varphi(\ell) > N}} a_{\varphi(\ell)} - \sum_{k=N+1}^{\infty} a_k \right| \\ &\leq \sum_{\substack{0 \leq \ell \leq n \\ \varphi(\ell) > N}} |a_{\varphi(\ell)}| + \sum_{k=N+1}^{\infty} |a_k| \leq 2 \sum_{k=N+1}^{\infty} |a_k| < \varepsilon. \end{aligned}$$

This shows that

$$\sum_{\substack{0 \leq \ell \leq n \\ \varphi(\ell) > N}} a_{\varphi(\ell)} \longrightarrow \sum_{k=0}^{\infty} a_k \quad \text{as } n \rightarrow \infty,$$

which proves the identity (5.3). Applying the same reasoning to $\sum_{n=0}^{\infty} |a_n|$ shows that $\sum_{\ell=0}^{\infty} |a_{\varphi(\ell)}| = \sum_{n=0}^{\infty} |a_n| < \infty$, hence $\sum_{\ell=0}^{\infty} a_{\varphi(\ell)}$ is absolutely convergent. \square

5.2.3 Product of Series

Theorem 5.19: Product Theorem

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolutely convergent series, and let $\alpha : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ be a bijection.

Writing $\alpha(n) = (\alpha_1(n), \alpha_2(n))$, one has

$$\left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} a_{\alpha_1(n)} b_{\alpha_2(n)}, \quad (5.4)$$

and the series on the right converges absolutely.

Proof. Consider first a bijection $\alpha : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$, written as $\alpha(n) = (\alpha_1(n), \alpha_2(n))$, such that

$$\{a(k) \mid 0 \leq k < n^2\} = \{0, 1, \dots, n-1\} \quad \forall n \in \mathbb{N}.$$

For example, $(\alpha(n))_{n=0}^{\infty}$ could traverse the grid as shown in the lecture. Then, for every $n \in \mathbb{N}$,

$$\sum_{k=0}^{n^2-1} |a_{\alpha_1(k)}| |b_{\alpha_2(k)}| = \left(\sum_{\ell=0}^{n-1} |a_{\ell}| \right) \left(\sum_{m=0}^{n-1} |b_m| \right).$$

Since the right-hand side is bounded by

$$\left(\sum_{\ell=0}^{\infty} |a_{\ell}| \right) \left(\sum_{m=0}^{\infty} |b_m| \right),$$

we have

$$\sup_{n \in \mathbb{N}} \sum_{k=0}^{n^2-1} |a_{\alpha_1(k)}| |b_{\alpha_2(k)}| \leq \left(\sum_{\ell=0}^{\infty} |a_{\ell}| \right) \left(\sum_{m=0}^{\infty} |b_m| \right) < \infty.$$

This implies that the series $\sum_{k=0}^{\infty} a_{\alpha_1(k)} b_{\alpha_2(k)}$ converges. In particular, since it converges, its value can be computed along every subsequence, therefore

$$\sum_{k=0}^{\infty} a_{\alpha_1(k)} b_{\alpha_2(k)} = \lim_{n \rightarrow \infty} \sum_{k=0}^{n^2-1} a_{\alpha_1(k)} b_{\alpha_2(k)}$$

Now writing the identity

$$\sum_{k=0}^{n^2-1} a_{\alpha_1(k)} b_{\alpha_2(k)} = \left(\sum_{\ell=0}^{n-1} a_{\ell} \right) \left(\sum_{m=0}^{n-1} b_m \right),$$

and taking the limit as $n \rightarrow \infty$, Proposition 3.12(2.) gives

$$\begin{aligned} \sum_{k=0}^{\infty} a_{\alpha_1(k)} b_{\alpha_2(k)} &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n^2-1} a_{\alpha_1(k)} b_{\alpha_2(k)} = \\ &= \left(\lim_{n \rightarrow \infty} \sum_{\ell=0}^{n-1} a_{\ell} \right) \left(\lim_{n \rightarrow \infty} \sum_{m=0}^{n-1} b_m \right) = \left(\sum_{\ell=0}^{\infty} a_{\ell} \right) \left(\sum_{m=0}^{\infty} b_m \right), \end{aligned}$$

which proves Equation (5.4) for this specific bijection α .

For an arbitrary bijection $\beta : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$, define $\varphi = \alpha^{-1} \circ \beta : \mathbb{N} \rightarrow \mathbb{N}$. Then $\beta = \alpha \circ \varphi$ and writing $\beta(n) = (\beta_1(n), \beta_2(n)) = (\alpha_1(\varphi(n)), \alpha_2(\varphi(n)))$, the rearrangement Theorem 5.18 yields

$$\sum_{n=0}^{\infty} a_{\beta_1(n)} b_{\beta_2(n)} = \sum_{n=0}^{\infty} a_{\alpha_1(\varphi(n))} b_{\alpha_2(\varphi(n))} = \sum_{n=0}^{\infty} a_{\alpha_1(n)} b_{\alpha_2(n)} = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right). \quad \square$$

Corollary 5.20: Cauchy Product

If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent, then

$$\left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_{n-k} b_k \right),$$

and the series on the right converges absolutely.

Proof. Consider the bijection $\alpha : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ defined as

$$\begin{aligned} \alpha(0) &= (0, 0), & \alpha(1) &= (1, 0), & \alpha(2) &= (0, 1), & \alpha(3) &= (2, 0), & \alpha(4) &= (1, 1), \dots, \\ \alpha(20) &= (0, 5), \dots, & \alpha(31) &= (4, 3), \dots, & \alpha(49) &= (5, 4), \dots \text{ etc.} \end{aligned}$$

By Theorem 5.19,

$$\left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right) = \sum_{n=0}^{\infty} a_{\alpha_1(n)} b_{\alpha_2(n)}.$$

Listing the terms explicitly and grouping them by diagonals, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} a_{\alpha_1(n)} b_{\alpha_2(n)} &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_2 b_0 + a_1 b_1 + a_0 b_2) \\ &\quad + (a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3) + \dots \\ &= \sum_{n=0}^{\infty} \left(\sum_{\substack{j,k \geq 0 \\ j+k=n}} a_j b_k \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_{n-k} b_k \right). \end{aligned}$$

Finally, absolute convergence follows from the triangle inequality and Theorem 5.19, i.e.,

$$\sum_{n=0}^{\infty} \left| \sum_{k=0}^n a_{n-k} b_k \right| \leq \sum_{n=0}^{\infty} \sum_{k=0}^n |a_{n-k} b_k| = \sum_{n=0}^{\infty} |a_{\alpha_1(n)}| |b_{\alpha_2(n)}| < \infty. \quad \square$$

5.3 Series of Complex Numbers

To define the notion of a convergent series in \mathbb{C} , it is sufficient to consider separately the corresponding series of its real and imaginary parts in \mathbb{R} .

Definition 5.21: Series of Complex Numbers

Let $(z_n)_{n=0}^{\infty} = (x_n + iy_n)_{n=0}^{\infty}$ be a sequence of complex numbers, and let $Z = A + iB \in \mathbb{C}$. The series $\sum_{n=0}^{\infty} z_n$ is said to **converge** to Z if both real series $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ converge, with limits A and B , respectively, i.e.,

$$\sum_{n=0}^{\infty} x_n = A, \quad \sum_{n=0}^{\infty} y_n = B.$$

We say that $\sum_{n=0}^{\infty} z_n$ **converges absolutely** if the series of moduli $\sum_{n=0}^{\infty} |z_n|$ converges.

Whenever the series $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=0}^{\infty} w_n$ converge absolutely, their sum and product are given (exactly as in the real case) by

$$\sum_{n=0}^{\infty} z_n + \sum_{n=0}^{\infty} w_n = \sum_{n=0}^{\infty} (z_n + w_n) \quad (5.5a)$$

$$\left(\sum_{n=0}^{\infty} z_n \right) \left(\sum_{n=0}^{\infty} w_n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n z_{n-k} w_k \right). \quad (5.5b)$$

Remark 5.22. Let $(z_n)_{n=0}^{\infty} = (x_n + iy_n)_{n=0}^{\infty}$ be a sequence of complex numbers, and assume that the series $\sum_{n=0}^{\infty} |z_n|$ converges. Since

$$0 \leq |x_n| \leq |z_n|, \quad 0 \leq |y_n| \leq |z_n|, \quad \forall n \in \mathbb{N},$$

the Majorant Criterion (Corollary 5.6) implies that both $\sum_{n=0}^{\infty} |x_n|$ and $\sum_{n=0}^{\infty} |y_n|$ converge. Hence, the series of real and imaginary parts are absolutely convergent.

Conversely since $|z_n| \leq |x_n| + |y_n|$, the absolute convergence of $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ also implies the absolute convergence of $\sum_{n=0}^{\infty} z_n$. Therefore, absolute convergence in \mathbb{C} is equivalent to absolute convergence of the real and imaginary parts.

5.4 Power Series

Our next goal is to investigate power series. These are series where the terms are powers of the variable $x \in \mathbb{R}$ (or $z \in \mathbb{C}$, if one considers complex power series) multiplied by coefficients.

5.4.1 Radius of Convergence

Definition 5.23: Power Series

A **power series** with real coefficients is a series of the form

$$\sum_{n=0}^{\infty} a_n x^n,$$

where $(a_n)_{n=0}^{\infty}$ is a sequence in \mathbb{R} and $x \in \mathbb{R}$. Here x is the **variable** and a_n is the **coefficient** of x^n . By convention, we set $x^0 = 1$ for all $x \in \mathbb{R}$, including $x = 0$. In other words, the first term of the power series is always a_0 .

Addition and Multiplication of power series are given by

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n &= \sum_{n=0}^{\infty} (a_n + b_n) x^n, \\ \left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} b_n x^n \right) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_{n-k} x^{n-k} b_k x^k \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_{n-k} b_k \right) x^n, \end{aligned}$$

where the product formula follows from Corollary 5.20.

A power series is a polynomial whenever only finitely many coefficients are *non-zero*.

Definition 5.24: Radius of Convergence

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series, and define

$$\rho := \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}.$$

The **radius of convergence** is defined as

$$R := \begin{cases} 0, & \rho = \infty, \\ \rho^{-1}, & 0 < \rho < \infty, \\ \infty, & \rho = 0. \end{cases}$$

In the following, when we write $R \in [0, \infty]$, we mean that R is either a non-negative real number or $R = \infty$.

Theorem 5.25: Convergence of Power Series

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R \in (0, \infty]$. Then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all $x \in \mathbb{R}$ with $|x| < R$, and does not converge for all $x \in \mathbb{R}$ with $|x| > R$. In particular, for $x \in (-R, R)$, we can define the function $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Proof. Let $x \in \mathbb{R}$, and write $\rho = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ as in Definition 5.24. Then

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n x^n|} = \left(\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} \right) |x| = \rho |x|.$$

By the root criterion (see Proposition 5.15), the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $\rho |x| < 1$, and does not converge if $\rho |x| > 1$ (in particular if $\rho = 0$ it converges for all $x \in \mathbb{R}$). Since $R = \frac{1}{\rho}$, the result follows. \square

Theorem 5.26: Continuity of Power Series

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R \in (0, \infty]$, and define polynomials $f_n(x) = \sum_{k=0}^n a_k x^k$. For any $r \in (0, R)$, the sequence $(f_n)_{n=0}^{\infty}$ converges uniformly to f on $[-r, r]$. In particular, the power series defines a continuous function $f : (-R, R) \rightarrow \mathbb{R}$.

Proof. By Theorem 5.25 with $x = r$, the series $\sum_{n=0}^{\infty} |a_n| r^n$ converges, where $r < R$. Hence, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} |a_k| r^k < \varepsilon.$$

Thus, for all $x \in [-r, r]$ and all $n \geq N$,

$$|f_n(x) - f(x)| = \left| \sum_{k=n+1}^{\infty} a_k x^k \right| \leq \sum_{k=n+1}^{\infty} |a_k| |x|^k \leq \sum_{k=N+1}^{\infty} |a_k| r^k < \varepsilon.$$

This shows that $(f_n)_{n=0}^{\infty}$ converges uniformly to f on $[-r, r]$. Since each f_n is continuous (being a polynomial), Theorem 4.52 implies that f is continuous on $[-r, r]$. As $r < R$ is arbitrary, f is continuous on $(-R, R)$. \square

Example

In general, the partial sums $f_n(x) = \sum_{k=0}^n a_k x^k$ do *not* converge uniformly to $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on the whole interval $(-R, R)$.

To see this, consider the geometric series $\sum_{n=0}^{\infty} x^n$. Its radius of Convergence is $R = 1$, and on $(-1, 1)$ we have $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If the convergence on $(-1, 1)$ were uniformly, then applying the notion of uniform convergence with $\varepsilon = 1$ would give $N \in \mathbb{N}$ such that, for all $n \geq N$ and $x \in (-1, 1)$,

$$\left| \sum_{k=0}^n x^k - \frac{1}{1-x} \right| < 1.$$

Taking $n = N$ and using the triangle inequality, we would get

$$\left| \frac{1}{1-x} \right| < 1 + \left| \sum_{k=0}^N x^k \right| \leq 1 + \sum_{k=0}^N |x|^k \leq 1 + (N+1) = N+2 \quad \forall x \in (-1, 1),$$

a contradiction since $\lim_{x \rightarrow 1^-} \frac{1}{1-x} = \infty$.

Proposition 5.27: Radius of Convergence of Sum and Product

Let $R \geq 0$, and let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be power series with radius of convergence of at least R . Then their sum and their Cauchy product also have radius of convergence of at least R .

Proof. By linearity and Corollary 5.20, the absolute convergence of $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ for $|x| < R$ implies that

$$\sum_{n=0}^{\infty} (a_n + b_n) x^n, \quad \left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_{n-k} b_k \right) x^n$$

both converge absolutely for $|x| < R$. Since a power series cannot converge for $|x| > R$, each has radius of convergence of at least R . \square

5.4.2 Complex Power Series**Definition 5.28: Complex Power Series**

complex power series with complex coefficients is a series of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $(a_n)_{n=0}^{\infty}$ is a sequence in \mathbb{C} and $z \in \mathbb{C}$. Again by convention, we set $z^0 = 1$ for all $z \in \mathbb{C}$, including $z = 0$.

Addition and multiplication are defined as in the real case.

Theorem 5.29: Convergence of Complex Power Series

Let $\sum_{n=0}^{\infty} a_n z^n$ be a complex power series with radius of convergence $R \in (0, \infty]$. Then the series $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for all $z \in \mathbb{C}$ with $|z| < R$, and diverges for all $z \in \mathbb{C}$ with $|z| > R$. In particular, for $|z| < R$ one can define the (complex-valued) function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

5.5 Exponential and Trigonometric Functions**5.5.1 The Exponential Map as a Power Series**

We now show that the exponential map can also be defined via the **exponential series**

$$\exp(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \quad (5.6)$$

Since $\frac{n!}{(n+1)!} = \frac{1}{n+1} \rightarrow 0$ as $n \rightarrow \infty$, it follows directly from the quotient criterion that this series has infinite radius of convergence. Hence, Theorem 5.26 implies that the right-hand side of Equation (5.6) defines a continuous function on \mathbb{R} .

Proposition 5.30: Exponential Map as a Power Series

For every $x \in \mathbb{R}$,

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n.$$

Definition 5.31: The Complex Exponential Map

The **complex exponential map** is the function $\exp : \mathbb{C} \rightarrow \mathbb{C}$ defined by

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad z \in \mathbb{C}.$$

Theorem 5.32: Properties of the Complex Exponential

The complex exponential map $\exp : \mathbb{C} \rightarrow \mathbb{C}$ is continuous, and for all $z, w \in \mathbb{C}$,

$$e^{z+w} = e^z e^w, \quad |e^z| = e^{\operatorname{Re}(z)}.$$

In particular, $|e^{ix}| = 1$ for all $x \in \mathbb{R}$.

5.5.2 Sine and Cosine

Given $x \in \mathbb{R}$, we split the power series of e^{ix} into its even and odd terms, i.e.,

$$e^{ix} = \sum_{n=0}^{\infty} \frac{i^n}{n!} x^n = \sum_{n=0}^{\infty} \frac{i^{2n}}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{i^{2n+1}}{(2n+1)!} x^{2n+1}.$$

Since $i^{2n} = (-1)^n$ and $i^{2n+1} = i(-1)^n$, we obtain

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

This motivates the following definition of the **sine** and **cosine functions**, i.e.,

$$\sin(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \cos(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad (5.7)$$

so that the identity

$$e^{ix} = \cos(x) + i \sin(x)$$

holds for all $x \in \mathbb{R}$.

As for the exponential series, radius of convergence of the power series in Equation (5.7) is infinite. Therefore, by Theorems 5.25 and 5.26, \sin and \cos are continuous on \mathbb{R} .

Since $(-x)^{2n+1} = -x^{2n+1}$ and $(-x)^{2n} = x^{2n}$ for all $n \in \mathbb{N}$, it follows directly from Equation (5.7) that

$$\begin{aligned} \sin(-x) &= -\sin(x) \quad \sin \text{ is an } \mathbf{odd} \text{ function,} \\ \cos(-x) &= \cos(x) \quad \cos \text{ is an } \mathbf{even} \text{ function.} \end{aligned}$$

Theorem 5.33: From the Complex Exponential to Sine and Cosine

For all $x \in \mathbb{R}$, the relations

$$e^{ix} = \cos(x) + i \sin(x), \quad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

hold. For all $x, y \in \mathbb{R}$ the trigonometric addition formulas are

$$\begin{aligned} \sin(x + y) &= \sin(x) \cos(y) + \cos(x) \sin(y), \\ \cos(x + y) &= \cos(x) \cos(y) - \sin(x) \sin(y). \end{aligned}$$

5.5.3 The Circle Number**Theorem 5.34: Existence of π as the First Zero of Sine**

There exists exactly one number $\pi \in (0, 4)$ such that $\sin(\pi) = 0$. For this number it holds that

$$e^{i\frac{\pi}{2}} = i, \quad e^{i\pi} = -1, \quad e^{i2\pi} = 1.$$

Corollary 5.35: Periodicity of Sine and Cosine

$$\begin{aligned} \sin\left(x + \frac{\pi}{2}\right) &= \cos(x), & \cos\left(x + \frac{\pi}{2}\right) &= -\sin(x) \\ \sin(x + \pi) &= -\sin(x), & \cos(x + \pi) &= -\cos(x), \\ \sin(x + 2\pi) &= \sin(x), & \cos(x + 2\pi) &= \cos(x). \end{aligned}$$

5.5.4 Polar Coordinates and Multiplication of Complex Numbers

Using the complex exponential function, we can express complex numbers in **polar coordinates**, i.e., in the form

$$z = re^{i\theta} = r \cos(\theta) + ir \sin(\theta),$$

where $r = |z|$ is the distance of z from the origin, and θ is the angle between the positive real axis $\mathbb{R}_{\geq 0}$ and the segment from 0 to z . In other words, if $z = x + iy$, then

$$x = r \cos(\theta), \quad y = r \sin(\theta), \quad r = \sqrt{x^2 + y^2}.$$

If $z \neq 0$, the angle θ is uniquely determined and is called the **argument** of z , denoted $\theta = \arg(z)$. The set of all complex numbers with absolute value 1 is

$$\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{i\theta} \mid \theta \in [0, 2\pi)\},$$

and is called the **unit circle** in \mathbb{C} .

Proposition 5.36: Existence of Polar Coordinates

For every $z \in \mathbb{C} \setminus \{0\}$, there exists uniquely determined real numbers $r > 0$ and $\theta \in [0, 2\pi)$ such that $z = re^{i\theta}$.

In polar coordinates, multiplication of complex numbers takes a simple geometric form: if $z = re^{i\varphi}$ and $w = se^{i\psi}$, then

$$zw = rse^{i(\varphi+\psi)}.$$

Thus, when multiplying two complex numbers, their magnitudes multiply and their arguments add.

5.5.5 Other Trigonometric and Hyperbolic Functions

In addition to sine and cosine functions, several related **trigonometric functions** are defined.

The **tangent** and **cotangent** functions are defined by

$$\tan(x) = \frac{\sin(x)}{\cos(x)}, \quad \cot(x) = \frac{\cos(x)}{\sin(x)},$$

for all $x \in \mathbb{R}$ such that the denominator are non-zero.

The **hyperbolic sine** and **hyperbolic cosine** are defined by the power series

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}, \quad \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Equivalently,

$$\sinh(x) = \frac{e^{ix} - e^{-ix}}{2}, \quad \cosh(x) = \frac{e^{ix} + e^{-ix}}{2},$$

and hence $e^x = \sinh(x) + \cosh(x)$ for all $x \in \mathbb{R}$.

The **hyperbolic tangent** and **hyperbolic cotangent** are defined by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}, \quad \coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}},$$

where $\coth(x)$ is defined for all $x \in \mathbb{R} \setminus \{0\}$ (since $\sinh(x) \neq 0$ for all $x \neq 0$)

The functions \sinh and \tanh are odd, while \cosh is even. Also, they satisfy the addition formulas

$$\begin{aligned} \sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y), \\ \cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y), \end{aligned}$$

and the **hyperbolic identity**

$$\cosh^2(x) - \sinh^2(x) = 1 \quad \forall x \in \mathbb{R}.$$

6 Differential Calculus

In this chapter we deal with differential calculus in one variable. This is of fundamental importance for understanding functions on \mathbb{R} .

6.1 The Derivative

6.1.1 Definition and Geometrical Interpretation

In this section $D \subseteq \mathbb{R}$ denotes a non-empty set with no isolated points, i.e., every $x \in D$ is an accumulation point of $D \setminus \{x\}$. A typical example is a non-empty interval containing more than one point.

Definition 6.1: Derivative

Let $f : D \rightarrow \mathbb{R}$ be a function and $x_0 \in D$. We say that f is **differentiable** at x_0 if the limit

$$f'(x_0) = \lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\substack{h \rightarrow 0 \\ h \neq 0}} \frac{f(x_0 + h) - f(x_0)}{h} \quad (6.1)$$

exists. In this case we call $f'(x_0)$ the **derivative** of f at x_0 . If f is differentiable at every point of D , we say that f is **differentiable** on D , and we call the resulting function $f' : D \rightarrow \mathbb{R}$ the **derivative** of f .

To simplify notation, we will often write

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

without explicitly mentioning that $x \neq x_0$ and $h \neq 0$. Note that the condition

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

can be rewritten as

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0,$$

or equivalently, using the little- o notation from Definition 4.44,

$$f(x) - f(x_0) - f'(x_0)(x - x_0) = o(x - x_0). \quad (6.2)$$

Remark 6.2. If $f : D \rightarrow \mathbb{R}$ is differentiable at x_0 , then f is also continuous at x_0 . Indeed, using Equation 6.2,

$$\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} (f(x_0) + f'(x_0)(x - x_0) + o(x - x_0)) = f(x_0),$$

hence f is continuous at x_0 .

An alternative notation for the derivative of f is $\frac{df}{dx}$. If $x_0 \in D$ is a right accumulation point of D , then f is **differentiable from the right** at x_0 if the **right derivative**

$$f'_+(x_0) = \lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0^+} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. **Differentiability from the left** and the **left derivative** $f'_-(x_0)$ are defined analogously using the limit $x \rightarrow x_0^-$.

If $f : D \rightarrow \mathbb{R}$ is differentiable at $x_0 \in D$, the function $x \mapsto f(x_0) + f'(x_0)(x - x_0)$ is called the **affine approximation** of f at x_0 .

Definition 6.3: Higher Derivatives

Let $f : D \rightarrow \mathbb{R}$ be a function. We define the **higher derivatives** of f , if they exist, by

$$f^{(0)} = f, \quad f^{(1)} = f', \quad f^{(2)} = f'', \quad \dots, \quad f^{(n+1)} = (f^{(n)})'$$

for all $n \in \mathbb{N}$. If $f^{(n)}$ exists, we say that f is **n -times differentiable**. If the n -th derivative is also continuous, we say that f is **n -times continuously differentiable**. We denote the set of n -times continuously differentiable functions on D by $C^n(D)$.

Equivalently, $C^0(D)$ is the set of real-valued continuous function on D , and $C^1(D)$ is the set of all differentiable functions whose derivative is continuous (these are called **continuously differentiable** or of **class C^1**). Recursively, for $n \geq 1$, we define

$$C^n(D) = \{f : D \rightarrow \mathbb{R} \mid f \text{ is differentiable and } f' \in C^{n-1}(D)\},$$

and we say that $f \in C^n(D)$ is of **class C^n** .

Definition 6.4: Smooth Functions

We define

$$C^\infty(D) = \bigcap_{n=0}^{\infty} C^n(D) = \{f : D \rightarrow \mathbb{R} \mid f \text{ is differentiable infinitely many times}\},$$

and call functions $f \in C^\infty(D)$ **smooth** or of **class C^∞** .

The exponential function $\exp : \mathbb{R} \rightarrow \mathbb{R}$ is smooth.

6.1.2 Differentiation Rules

As with continuous functions, we rarely reprove differentiability from first principles for each new example. Instead, we use general rules that reduce differentiability of compound expressions to that of simpler ones.

Proposition 6.5: Derivative of Sum and Product

Let $D \subseteq \mathbb{R}$ and $x_0 \in D$ be an accumulation point of $D \setminus \{x_0\}$. Let $f, g : D \rightarrow \mathbb{R}$ be differentiable at x_0 . Then $f + g$ and $f \cdot g$ are differentiable at x_0 , and

$$(f + g)'(x_0) = f'(x_0) + g'(x_0), \tag{6.3a}$$

$$(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0). \tag{6.3b}$$

In particular, for any $\alpha \in \mathbb{R}$, the scalar multiple αf is differentiable at x_0 and $(\alpha f)'(x_0) = \alpha f'(x_0)$.

Proof. Using the properties of limit discussed in Section 4.5.1, we have

$$\begin{aligned}\lim_{x \rightarrow x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} &= \lim_{x \rightarrow x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \right) \\ &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \rightarrow x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= f'(x_0) + g'(x_0),\end{aligned}$$

and

$$\begin{aligned}\lim_{x \rightarrow x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} &= \lim_{x \rightarrow x_0} \frac{(f(x) - f(x_0))g(x) + f(x_0)(g(x) - g(x_0))}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} g(x) \right) + f(x_0) \lim_{x \rightarrow x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) \cdot \left(\lim_{x \rightarrow x_0} g(x) \right) + f(x_0) \lim_{x \rightarrow x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= f'(x_0)g(x_0) + f(x_0)g'(x_0),\end{aligned}$$

where we used that g is continuous at x_0 (see Remark 6.2) to conclude $\lim_{x \rightarrow x_0} g(x) = g(x_0)$. \square

Corollary 6.6: Higher Order Derivatives of the Sum and Product

Let $f, g : D \rightarrow \mathbb{R}$ be n -times differentiable. Then $f + g$ and $f \cdot g$ are also n -times differentiable, and

$$\begin{aligned}(f + g)^{(n)} &= f^{(n)} + g^{(n)}, \\ (fg)^{(n)} &= \sum_{k=0}^n \binom{n}{k} f^{(k)} g^{(n-k)}.\end{aligned}$$

In particular, for every $\alpha \in \mathbb{R}$, $(\alpha f)^{(n)} = \alpha f^{(n)}$.

Proof. For $n = 1$ this is Proposition 6.5. The general case for $n \geq 1$ follows by induction. \square

Corollary 6.7: Derivatives of Polynomials

Polynomial functions are differentiable on all of \mathbb{R} . Moreover $(1)' = 0$ and $(x^n)' = nx^{n-1}$ for all $n \geq 1$.

Proof. We argue by induction. For $n = 0$, we have

$$(1)' = \lim_{x \rightarrow x_0} \frac{1 - 1}{x - x_0} = 0.$$

For $n = 1$, we have

$$(x)' = \lim_{x \rightarrow x_0} \frac{x - x_0}{x - x_0} = 1.$$

For $n > 1$, assume $(x^n)' = nx^{n-1}$. Then by Equation (6.3b), since $x^{n+1} = x \cdot x^n$, we have

$$(x^{n+1})' = (x \cdot x^n)' = 1 \cdot x^n + x \cdot nx^{n-1} = (n+1)x^n.$$

This proves the inductive step and establishes the result. Finally, the linearity of the derivative (see Equation (6.3a)) yields the differentiability of any polynomial. \square

Example

With $\alpha = \pm 1$ and $\alpha = \pm i$, we have

$$(e^x)' = e^x, \quad (e^{-x})' = -e^{-x}, \quad (e^{ix})' = ie^{ix}, \quad (e^{-ix})' = -ie^{-ix}.$$

By Theorem 5.33, we have

$$\sin'(x) = \frac{(e^{ix})' - (e^{-ix})'}{2i} = \frac{e^{ix} + e^{ix}}{2} = \cos(x),$$

and analogously, $\cos'(x) = -\sin(x)$. Similarly $\sinh'(x) = \cosh(x)$ and $\cosh'(x) = \sinh(x)$.

Theorem 6.8: Chain Rule

Let $D, E \subseteq \mathbb{R}$ and let $x_0 \in D$ be an accumulation point of $D \setminus \{x_0\}$. Let $f : D \rightarrow E$ be differentiable at x_0 such that $y_0 = f(x_0)$ is an accumulation point of $E \setminus \{y_0\}$, and let $g : E \rightarrow \mathbb{R}$ be differentiable at y_0 . Then $g \circ f : D \rightarrow \mathbb{R}$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0).$$

Proof. Observe that we can write

$$\begin{aligned} g(y) &= g(y_0) + [g(y) - g(y_0)] \\ &= g(y_0) + g'(y_0)(y - y_0) + [g(y) - g(y_0) - g'(y_0)(y - y_0)] \\ &= g(y_0) + g'(y_0)(y - y_0) + \omega(y)(y - y_0), \end{aligned}$$

where $\omega : E \rightarrow \mathbb{R}$ is defined as

$$\omega(y) = \begin{cases} \frac{g(y) - g(y_0)}{y - y_0} - g'(y_0) & \text{for } y \in E \setminus \{y_0\}, \\ 0 & \text{for } y = y_0. \end{cases}$$

Since g is continuous at y_0 , it follows that $\omega(y) \rightarrow 0$ as $y \rightarrow y_0$; hence, the function ω is continuous at y_0 . Substituting $y = f(x)$ and using $y_0 = f(x_0)$, we get

$$g(f(x)) = g(f(x_0)) + g'(f(x_0))[f(x) - f(x_0)] + \omega(f(x))[f(x) - f(x_0)],$$

therefore

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x \rightarrow x_0} \left(g'(f(x_0)) \frac{f(x) - f(x_0)}{x - x_0} + \omega(f(x)) \frac{f(x) - f(x_0)}{x - x_0} \right) \\ &= g'(f(x_0))f'(x_0) + \underbrace{\omega(f(x_0))}_{=0} f'(x_0) = g'(f(x_0))f'(x_0), \end{aligned}$$

where we used the continuity of ω at $y_0 = f(x_0)$ to deduce that $\omega(f(x)) \rightarrow \omega(f(x_0)) = 0$ as $x \rightarrow x_0$. \square

Corollary 6.9: Quotient Rule

Let $D \subseteq \mathbb{R}$. Let x_0 be an accumulation point of $D \setminus \{x_0\}$, and let $f, g : D \rightarrow \mathbb{R}$ be differentiable at x_0 . If $g(x_0) \neq 0$, then $\frac{f}{g}$ is differentiable at x_0 , and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Proof. Consider the function $\psi : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ given by $\psi(y) = \frac{1}{y}$. This function is differentiable, with $\psi'(y) = -\frac{1}{y^2}$. Then, by the chain rule (Theorem 6.8), $\frac{1}{g} = \psi \circ g$ is differentiable at x_0 , with

$$\left(\frac{1}{g}\right)'(x_0) = \psi'(g(x_0))g'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}.$$

Applying now the product rule (Proposition 6.5), $\frac{f}{g} = f \cdot \frac{1}{g}$ is differentiable at x_0 , and

$$\left(\frac{f}{g}\right)'(x_0) = \left(f \cdot \frac{1}{g}\right)'(x_0) = f'(x_0)\frac{1}{g(x_0)} - f(x_0)\frac{g'(x_0)}{g(x_0)^2} = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}. \quad \square$$

Theorem 6.10: Derivative of the Inverse

Let $D, E \subseteq \mathbb{R}$, and let $f : D \rightarrow E$ be a continuous bijection whose inverse $f^{-1} : E \rightarrow D$ is also continuous. Let $\bar{x} \in D$ be an accumulation point of $D \setminus \{\bar{x}\}$, and assume f is differentiable at \bar{x} with $f'(\bar{x}) \neq 0$. Then f^{-1} is differentiable at $\bar{y} = f(\bar{x})$ and

$$(f^{-1})'(\bar{y}) = \frac{1}{f'(\bar{x})} = \frac{1}{f'(f^{-1}(\bar{y}))}.$$

Proof. To compute $(f^{-1})'(\bar{y})$, take a sequence $(y_n)_{n=0}^\infty \subseteq E \setminus \{\bar{y}\}$ with $y_n \rightarrow \bar{y}$ and set $x_n = f^{-1}(y_n)$. Then

$$\frac{f^{-1}(y_n) - f^{-1}(\bar{y})}{y_n - \bar{y}} = \frac{x_n - \bar{x}}{f(x_n) - f(\bar{x})} = \left(\frac{f(x_n) - f(\bar{x})}{x_n - \bar{x}}\right)^{-1}.$$

By the continuity of f^{-1} we have $f^{-1}(y_n) = x_n \rightarrow \bar{x} = f^{-1}(\bar{y})$. Thus, since f is differentiable at \bar{x} with $f'(\bar{x}) \neq 0$, Proposition 3.12(4) implies

$$\left(\frac{f(x_n) - f(\bar{x})}{x_n - \bar{x}}\right)^{-1} \rightarrow \frac{1}{f'(\bar{x})},$$

proving that

$$\lim_{n \rightarrow \infty} \frac{f^{-1}(y_n) - f^{-1}(\bar{y})}{y_n - \bar{y}} = \frac{1}{f'(\bar{x})}.$$

Since the sequence $(y_n)_{n=0}^\infty$ was arbitrary, Lemma 4.36 gives

$$\lim_{y \rightarrow \bar{y}} \frac{f^{-1}(y) - f^{-1}(\bar{y})}{y - \bar{y}} = \frac{1}{f'(\bar{x})},$$

as desired. □

6.2 Main Theorems of Differential Calculus

6.2.1 Local Extrema

Definition 6.11: Local Extrema

Let $D \subseteq \mathbb{R}$ and $x_0 \in D$. We say that a function $f : D \rightarrow \mathbb{R}$ has a **local maximum** at x_0 , if there exists $\delta > 0$ such that

$$f(x) \leq f(x_0) \quad \forall x \in D \cap (x_0 - \delta, x_0 + \delta).$$

If the inequality is strict (i.e. $f(x) < f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$), then f has a **strict local maximum** at x_0 . A **(strict) local minimum** is defined analogously. We call x_0 a **local extremum** if f has either a local maximum or a local minimum at x_0 .

Proposition 6.12: Local Extrema vs. First Derivative

Let $D \subseteq \mathbb{R}$ and $f : D \rightarrow \mathbb{R}$. Suppose $x_0 \in D$ is a local extremum of f , that f is differentiable at x_0 and that x_0 is both a right-hand and a left-hand accumulation point of D . Then

$$f'(x_0) = 0.$$

Proof. W.l.o.g, assume that f has a local maximum at x_0 (otherwise replace f by $-f$). We first note that for x close to x_0 and to the right of it, we have $f(x) - f(x_0) \leq 0$ and $x - x_0 > 0$. Hence,

$$f'_+(x_0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0.$$

Similarly for x close to x_0 and to the left of it, we have $f(x) - f(x_0) \leq 0$ and $x - x_0 < 0$, so

$$f'_-(x_0) = \lim_{x \rightarrow x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

Since f is differentiable at x_0 , the two one sided derivatives coincide, i.e.,

$$f'(x_0) = f'_+(x_0) = f'_-(x_0) = 0,$$

and therefore $f'(x_0) = 0$. □

Corollary 6.13: Local Extrema in an Interval

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$. If $x_0 \in I$ is a local extrema, then at least one of the following statements holds:

1. x_0 is an endpoint of I ,
2. f is not differentiable at x_0 ,
3. f is differentiable at x_0 and $f'(x_0) = 0$.

In particular, all local extrema of a differentiable function on an open interval are zeros of the derivative.

6.2.2 The Mean Value Theorem

Theorem 6.14: Rolle's Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . If $f(a) = f(b)$, then there exists $\xi \in (a, b)$ with $f'(\xi) = 0$.

Proof. By Theorem 4.20 f attains both its maximum and minimum on $[a, b]$ at some points $x_0, x_1 \in [a, b]$. By Proposition 6.12 any interior extremum has zero derivative. Thus, we consider two cases:

- (i) If either x_0 or x_1 lies in (a, b) we are done.
- (ii) If both x_0 and x_1 are endpoints, since $f(a) = f(b)$, then $\min f = \max f = f(a) = f(b)$, therefore f is constant. In particular, $f'(x) = 0$ for all $x \in (a, b)$ and the result follows also in this case. \square

Corollary 6.15: Non-Vanishing Derivative Implies Different Endpoint Values

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . If $f'(x) \neq 0$ for all $x \in (a, b)$, then $f(a) \neq f(b)$.

Proof. Assume, by contradiction, that $f(a) = f(b)$. Then by Rolle's Theorem, there would exist $\xi \in (a, b)$ such that $f'(\xi) = 0$, which would contradict the assumption that f' never vanishes. \square

Theorem 6.16: Mean Value Theorem

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . Then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define $g : [a, b] \rightarrow \mathbb{R}$ by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then g is continuous on $[a, b]$ and differentiable on (a, b) , and satisfies

$$g(a) = f(a), \quad g(b) = f(b) - (f(b) - f(a)) = f(a).$$

By Rolle's Theorem 6.14, there exists $\xi \in (a, b)$ such that

$$0 = g'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a},$$

proving the result. \square

Corollary 6.17: Lipschitz Continuity vs. Bounded Derivative

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . Then f is Lipschitz continuous on $[a, b]$ if and only if f' is bounded on (a, b) .

Proof. Suppose first that f is Lipschitz continuous on $[a, b]$ with constant L . This implies that given $x, x_0 \in (a, b)$ with $x \neq x_0$,

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| \leq L.$$

Taking the limit as $x \rightarrow x_0$ gives $|f'(x_0)| \leq L$, so f' is bounded on (a, b) .

Conversely, suppose that f' is bounded on (a, b) , say $f'(z) \leq M$ for all $z \in (a, b)$. Then given $x, y \in [a, b]$ with $x < y$, the Mean Value Theorem 6.16 applied on the interval $[x, y]$ yields $\xi \in (x, y) \subseteq (a, b)$ such that

$$f(y) - f(x) = f'(\xi)(y - x),$$

therefore

$$|f(y) - f(x)| = |f'(\xi)||y - x| \leq M|y - x|.$$

Since $x, y \in [a, b]$ are arbitrary, this shows that f is Lipschitz continuous on $[a, b]$ with Lipschitz constant M . \square

Theorem 6.18: Cauchy Mean Value Theorem

Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . Then there exists $\xi \in (a, b)$ such that

$$g'(\xi)(f(b) - f(a)) = f'(\xi)(g(b) - g(a)). \quad (6.4)$$

If, in addition $g'(x) \neq 0$ for all $x \in (a, b)$, then $g(a) \neq g(b)$ and

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Define the function $F : [a, b] \rightarrow \mathbb{R}$ as

$$F(x) = g(x)(f(b) - f(a)) - f(x)(g(b) - g(a)).$$

Then

$$F(a) = g(a)(f(b) - f(a)) - f(a)(g(b) - g(a)) = g(a)f(b) - f(a)g(b)$$

$$F(b) = g(b)(f(b) - f(a)) - f(b)(g(b) - g(a)) = g(a)f(b) - f(a)g(b).$$

Thus, by Rolle's Theorem 6.14, there exists $\xi \in (a, b)$ such that

$$F'(\xi) = 0 = g'(\xi)(f(b) - f(a)) - f'(\xi)(g(b) - g(a)),$$

which is Equation 6.4.

If $g'(x) \neq 0$ for all $x \in (a, b)$, then Corollary 6.15 yields $g(a) \neq g(b)$. Dividing Equation 6.4 accordingly gives the second formula. \square

6.2.3 L'Hopital's Rule

Theorem 6.19: L'Hopital's Rule

Let $f, g : (a, b) \rightarrow \mathbb{R}$ be differentiable. Suppose:

1. $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a, b)$,
2. $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} g(x) = 0$,
3. the limit $L = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}$ exists.

Then $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)}$ exists and equals L .

Proof. By (2), we can extend f and g continuously to $[a, b)$ by setting $f(a) = g(a) = 0$. Fix $\varepsilon > 0$. By (3), there exists $\delta > 0$ such that

$$\frac{f'(\xi)}{g'(\xi)} \in (L - \varepsilon, L + \varepsilon) \quad \forall \xi \in (a, a + \delta).$$

Now, for any $x \in (a, a + \delta)$, we can apply Cauchy's Mean Value Theorem 6.18 to f and g on $[a, x]$ to find some $\xi_x \in (a, x)$ with

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi_x)}{g'(\xi_x)}.$$

Since $\xi_x \in (a, x) \subseteq (a, a + \delta)$, it follows that

$$\frac{f(x)}{g(x)} = \frac{f'(\xi_x)}{g'(\xi_x)} \in (L - \varepsilon, L + \varepsilon) \quad \forall x \in (a, a + \delta).$$

Because $\varepsilon > 0$ is arbitrary, this proves that $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$. □

Theorem 6.20: L'Hopital's Rule for Improper Limits

Let $f, g : (a, b) \rightarrow \mathbb{R}$ be differentiable. Suppose:

1. $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a, b)$,
2. $\lim_{x \rightarrow a^+} |f(x)| = \lim_{x \rightarrow a^+} |g(x)| = \infty$,
3. the limit $L = \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}$ exists.

Then $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)}$ exists and equals L .

Theorem 6.21: L'Hopital's Rule at Infinity

Let $R > 0$ and $f, g : (R, \infty) \rightarrow \mathbb{R}$ be differentiable. Suppose:

1. $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (R, \infty)$,
2. either $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = 0$ or $\lim_{x \rightarrow \infty} |f(x)| = \lim_{x \rightarrow \infty} |g(x)| = \infty$,
3. the limit $L = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$ exists.

Then $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$ exists and equals L .

6.2.4 Monotonicity and Convexity via Differential Calculus

Here I always denotes a non-trivial interval (non-empty and not a single point).

Proposition 6.22: Monotonicity vs. First Derivative

Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$ be differentiable. Then

$$f' \geq 0 \quad \Leftrightarrow \quad f \text{ is increasing.}$$

Proof. If f is increasing $f(x+h) - f(x) \geq 0$ for $h > 0$, and $f(x+h) - f(x) \leq 0$ for $h < 0$. Hence, in both cases, $\frac{f(x+h)-f(x)}{h} \geq 0$, therefore

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \geq 0.$$

Conversely, assume f is not increasing. Then there exists $x_1 < x_2$ with $f(x_1) > f(x_2)$. By the Mean Value Theorem 6.16, there exists $\xi \in (x_1, x_2)$ with

$$f'(\xi) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0,$$

so $f' \not\geq 0$ on I . □

Remark 6.23. If $f' > 0$, the same argument shows that f is strictly increasing. However, the converse fails: the function $f(x) = x^3$ is strictly increasing but $f'(0) = 0$.

Corollary 6.24: Constant Functions vs. First Derivative

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$. Then f is constant if and only if f is differentiable and $f'(x) = 0$ for all $x \in I$.

Proof. The derivative of a constant function is 0.

Conversely, if $f' = 0$, then $f' \geq 0$ and $-f' \geq 0$, so by Proposition 6.22 both f and $-f$ are increasing, hence f is constant. □

Definition 6.25: Convex Functions

Let $I \subseteq \mathbb{R}$ and $f : I \rightarrow \mathbb{R}$. We call f **convex**, if for all $a, b \in I$ with $a < b$ and all $t \in (0, 1)$, it holds that

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b). \quad (6.5)$$

We call f **strictly convex** if the inequality in (6.5) is strict. A function $g : I \rightarrow \mathbb{R}$ is **(strictly) concave** if $-g$ is (strictly) convex.

An equivalent definition of a convex function is the following: $f : I \rightarrow \mathbb{R}$ is convex if for all $a, b \in I$ with $a < b$ and all $x \in (a, b)$, we have

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(x)}{b - x}, \quad (6.6)$$

and strictly convex if the inequality is strict.

Proposition 6.26: Convexity vs. Monotonicity of the First Derivative

Let $I \subseteq \mathbb{R}$ and let $f : I \rightarrow \mathbb{R}$ be differentiable. Then f is convex if and only if f' is increasing.

Proof. Assume f' is increasing. Then, for $a < b$ and $x \in (a, b)$, the Mean Value Theorem 6.16 applied on the intervals $[a, x]$ and $[x, b]$ yields $\xi \in (a, x)$ and $\zeta \in (x, b)$ such that

$$f'(\xi) = \frac{f(x) - f(a)}{x - a}, \quad f'(\zeta) = \frac{f(b) - f(x)}{b - x}.$$

Since f' is increasing, we have $f'(\xi) \leq f'(\zeta)$, so Equation (6.6) follows. Since $a < b \in I$ and $x \in (a, b)$ are arbitrary, f is convex.

Conversely, assume f is convex. Given $a < b$, consider $h > 0$ small enough, such that $a + h < b - h$ and apply Equation (6.6) twice: first, applying it on the interval $(a, b - h)$ with $x = a + h$ we get

$$\frac{f(a + h) - f(a)}{h} \leq \frac{f(b - h) - f(a + h)}{(b - h) - (a + h)};$$

then, applying it on the interval $(a + h, b)$ with $x = b - h$ we get

$$\frac{f(b - h) - f(a + h)}{(b - h) - (a + h)} \leq \frac{f(b) - f(b - h)}{h}.$$

Combining these two inequalities, we deduce that, for small $h > 0$,

$$\frac{f(a + h) - f(a)}{h} \leq \frac{f(b) - f(b - h)}{h}.$$

Letting $h \rightarrow 0^+$ gives $f'(a) \leq f'(b)$. Since $a < b$ are arbitrary, f' is increasing. \square

Corollary 6.27: Convexity vs. Second Derivative

Let $I \subseteq \mathbb{R}$ and let $f : I \rightarrow \mathbb{R}$ be twice differentiable. Then f is convex if and only if $f'' \geq 0$.

Proof. By Proposition 6.26 f is convex if and only if f' is increasing. Applying Proposition 6.22 to f' , we have that f' is increasing if and only if $f'' \geq 0$. \square

6.3 Example: Differentiation of Trigonometric Functions

In this section we will list the derivatives of various trigonometric functions that can be looked up, when solving integrals, in order to find a certain primitive.

6.3.1 Sine and Arc Sine

The function $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow [-1, 1]$ satisfies

$$\sin'(x) = \cos(x), \quad \cos'(x) = -\sin(x).$$

Its inverse function $\arcsin : [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ has the derivative

$$\arcsin'(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1 - \sin^2(\arcsin(x))}} = \frac{1}{\sqrt{1 - x^2}},$$

where we used the derivative of the inverse (Theorem 6.10) to calculate the above derivative.

6.3.2 Cosine and Arc Cosine

Similarly the function $\cos : [0, \pi] \rightarrow [-1, 1]$ has the inverse function $\arccos : [-1, 1] \rightarrow [0, \pi]$, which has the derivative

$$\arccos'(x) = \frac{1}{-\sin(\arccos(x))} = -\frac{1}{\sqrt{1 - \cos^2(\arccos(x))}} = -\frac{1}{\sqrt{1 - x^2}}.$$

6.3.3 Tangent and Arc Tangent

The function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$ has the derivative

$$\tan'(x) = \left(\frac{\sin(x)}{\cos(x)} \right)' = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{1}{\cos^2(x)}.$$

Its inverse function $\arctan : \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$ has the derivative

$$\arctan'(x) = \frac{1}{\frac{1}{\cos^2(\arctan(x))}} = \cos^2(\arctan(x)).$$

Since

$$\tan^2(x) = \frac{\sin^2(x)}{\cos^2(x)} = \frac{1 - \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} - 1,$$

it follows that

$$\arctan'(x) = \cos^2(\arctan(x)) = \frac{1}{1 + \tan^2(\arctan(x))} = \frac{1}{1 + x^2}.$$

The cotangent and its inverse behave similarly, i.e.

$$\cot'(x) = \left(\frac{\cos(x)}{\sin(x)} \right)' = \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)},$$

and

$$\operatorname{arccot}'(x) = -\frac{1}{1 + x^2}.$$

6.3.4 Hyperbolic Functions

We now perform the analogous analysis for the hyperbolic trigonometric functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}.$$

The function $\sinh : \mathbb{R} \rightarrow \mathbb{R}$ is bijective and satisfies

$$\sinh'(x) = \cosh(x), \quad \sinh''(x) = \sinh(x).$$

Its inverse $\operatorname{arsinh} : \mathbb{R} \rightarrow \mathbb{R}$ is called the inverse hyperbolic sine. Its derivative is given by

$$\operatorname{arsinh}'(x) = \frac{1}{\cosh(\operatorname{arsinh}(x))} = \frac{1}{\sqrt{1 + \sinh^2(\operatorname{arsinh}(x))}} = \frac{1}{\sqrt{1 + x^2}}.$$

The hyperbolic cosine satisfies

$$\cosh'(x) = \sinh(x), \quad \cosh''(x) = \cosh(x).$$

Its inverse $\operatorname{arcosh} : [1, \infty] \rightarrow [0, \infty]$ has the derivative

$$\operatorname{arcosh}'(x) = \frac{1}{\sinh(\operatorname{arcosh}(x))} = \frac{1}{\sqrt{\cosh^2(\operatorname{arcosh}(x)) - 1}} = \frac{1}{\sqrt{x^2 - 1}}.$$

The inverse hyperbolic tangent $\operatorname{artanh} : (-1, 1) \rightarrow \mathbb{R}$ has the derivative

$$\operatorname{artanh}'(x) = \frac{1}{1 - x^2}.$$

Notice that the derivatives containing the inverse functions of either the tangent or hyperbolic tangent do not have a root in the denominator. One can memorize this by the sentence: "You can't get tan under a roof".

7 The Riemann Integral

7.1 Step Functions and their Integral

7.1.1 Decompositions and Step Functions

Definition 7.1: Partitions

Two sets A, B are called **disjoint** if $A \cap B = \emptyset$. For a collection \mathcal{A} of sets, we say that the sets in \mathcal{A} are pairwise disjoint, if for all $A_1, A_2 \in \mathcal{A}$ with $A_1 \neq A_2$ it holds that $A_1 \cap A_2 = \emptyset$.

Let X be a set. A Partition of X is a family \mathcal{P} of non-empty pairwise disjoint subsets of X such that

$$X = \bigcup_{P \in \mathcal{P}} P.$$

Definition 7.2: Decomposition of an Interval

A **decomposition** of $[a, b]$ is a finite sequence of points

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b,$$

with $n \in \mathbb{N}$. The points x_0, \dots, x_n are called the **division points** of the decomposition.

Formally, a decomposition of $[a, b]$ is a finite subset of $[a, b]$ containing a and b , together with the ordering of its elements. Each decomposition induces a natural partition of $[a, b]$, i.e.,

$$[a, b] = \{x_0\} \cup (x_0, x_1) \cup \{x_1\} \cup \dots \cup (x_{n-1}, x_n) \cup \{x_n\},$$

which we will use implicitly from now on.

A decomposition

$$a = y_0 < y_1 < \dots < y_m = b$$

is called a **refinement** of the decomposition

$$a = x_0 < x_1 < \dots < x_n = b$$

if

$$\{x_0, x_1, \dots, x_n\} \subseteq \{y_0, y_1, \dots, y_m\}.$$

The notion of refinement defines a partial order on the set of all decompositions of $[a, b]$. Note that any two decompositions of $[a, b]$ admit a common refinement given by the union of all division points.

Definition 7.3: Step Functions

A function $f : [a, b] \rightarrow \mathbb{R}$ is called a **step function** if there exists a decomposition

$$a = x_0 < x_1 < \dots < x_n = b$$

such that, for each $k = 1, \dots, n$ the restriction of f to the open interval (x_{k-1}, x_k) is constant. In this case, we say that f is a step function *with respect to* the decomposition $a = x_0 < x_1 < \dots < x_n = b$.

Proposition 7.4: Linearity of the Space of Step Functions

Let $f, g : [a, b] \rightarrow \mathbb{R}$ be step functions, and $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is also a step function.

Proof. Let f be a step function with respect to the decomposition $a = x_0 < x_1 < \dots < x_n = b$, and let g be a step function with respect to the decomposition $a = y_0 < y_1 < \dots < y_m = b$. The union of all division points $\{x_0, x_1, \dots, x_n\} \cup \{y_0, y_1, \dots, y_m\}$ defines a new decomposition

$$a = z_0 < z_1 < \dots < z_N = b$$

that is a common refinement of the two. Since both f and g are constant on each open interval (z_{k-1}, z_k) , so is the function $\alpha f + \beta g$. Thus, $\alpha f + \beta g$ is also a step function with respect to this decomposition. \square

Remark 7.5. *As in the proof of Proposition 7.4, one can show that the product of two step functions is again a step function. Moreover, step functions are bounded, since they take only finitely many values.*

7.1.2 The Integral of a Step Function

Definition 7.6: Integral of a Step Function

Let $f : [a, b] \rightarrow \mathbb{R}$ be a step function with respect to a decomposition

$$a = x_0 < x_1 < \dots < x_n = b.$$

We define the **integral** of f on $[a, b]$ as the real number

$$\int_a^b f(x) dx = \sum_{k=1}^n c_k (x_k - x_{k-1}), \quad (7.1)$$

where c_k denotes the constant value of f on the interval (x_{k-1}, x_k) .

Proposition 7.7: Linearity of the Integral of Step Functions

Let $f, g : [a, b] \rightarrow \mathbb{R}$ be step functions, and let $\alpha, \beta \in \mathbb{R}$. Then

$$\int_a^b (\alpha f + \beta g)(x) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx.$$

Proof. As in the proof of Proposition 7.4, we can find a decomposition $a = x_0 < x_1 < \dots < x_n = b$ such that both f and g (and hence $\alpha f + \beta g$) are constant on the interval (x_{k-1}, x_k) . If f takes the values c_k and g the values d_k on (x_{k-1}, x_k) , then $\alpha f + \beta g$ takes the value $\alpha c_k + \beta d_k$. Thus,

$$\begin{aligned} \int_a^b (\alpha f + \beta g)(x) dx &= \sum_{k=1}^n (\alpha c_k + \beta d_k)(x_k - x_{k-1}) \\ &= \alpha \sum_{k=1}^n c_k (x_k - x_{k-1}) + \beta \sum_{k=1}^n d_k (x_k - x_{k-1}) \\ &= \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx, \end{aligned}$$

as claimed. □

Proposition 7.8: Monotonicity of the Integral of Step Functions

Let $f, g : [a, b] \rightarrow \mathbb{R}$ be step functions such that $f \leq g$. Then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx.$$

Proof. As in the proof of Proposition 7.4, we can find a decomposition $a = x_0 < x_1 < \dots < x_n = b$ such that both f and g are constant on each interval (x_{k-1}, x_k) . Writing c_k and d_k for their respective

values, the assumption $f \leq g$ implies that $c_k \leq d_k$ for all $k = 1, \dots, n$. Hence

$$\int_a^b f(x) dx = \sum_{k=1}^n c_k(x_k - x_{k-1}) \leq \sum_{k=1}^n d_k(x_k - x_{k-1}) = \int_a^b g(x) dx. \quad \square$$

Applying Proposition 7.8 with $g \equiv 0$, we deduce the following corollary.

Corollary 7.9: Positivity of the Integral of Step Functions

If $f : [a, b] \rightarrow \mathbb{R}$ is a step functions such that $f(x) \geq 0$ for all $x \in [a, b]$, then

$$\int_a^b f(x) dx \geq 0.$$

7.2 Definition and First Properties of the Riemann Integral

As in the last section, we consider functions on a compact interval $[a, b] \subseteq \mathbb{R}$. To alleviate notation, we write \mathcal{SF} for the set of step functions on $[a, b]$. Also, we often write $\int_a^b f dx$ in place of $\int_a^b f(x) dx$.

7.2.1 Integrability of Real-Valued Functions

Before defining lower and upper sums, we recall a simple but useful property of the supremum and infimum of two related sets. We use this fact several times in what follows.

Definition 7.10: Relation Between Supremum and Infimum

Let $A, B \subseteq \mathbb{R}$ be non-empty sets such that $s \leq t$ for all $s \in A$ and $t \in B$. Then

$$\sup A \leq \inf B. \quad (7.2)$$

Moreover,

$$\sup A = \inf B \iff \forall \varepsilon > 0 \exists s \in A \exists t \in B \text{ such that } t - s < \varepsilon. \quad (7.3)$$

Definition 7.11: Lower and Upper Sums

Let $f : [a, b] \rightarrow \mathbb{R}$ be a function. Define the sets of **lower sums** $\mathcal{L}(f) \subseteq \mathbb{R}$ and **upper sums** $\mathcal{U}(f) \subseteq \mathbb{R}$ by

$$\mathcal{L}(f) = \left\{ \int_a^b \ell dx \mid \ell \in \mathcal{SF} \text{ and } \ell \leq f \right\}, \quad \mathcal{U}(f) = \left\{ \int_a^b u dx \mid u \in \mathcal{SF} \text{ and } f \leq u \right\}.$$

If f is bounded, then these sets are non-empty. Indeed, if $|f| \leq M$, then the constant step functions

$$\ell(x) = -M \quad \forall x \in [a, b], \quad u(x) = M \quad \forall x \in [a, b],$$

satisfy $\ell \in \mathcal{L}(f)$ and $u \in \mathcal{U}(f)$.

For $\ell, u \in \mathcal{SF}$ with $\ell \leq f \leq u$, Proposition 7.8 gives

$$\int_a^b \ell dx \leq \int_a^b u dx.$$

This implies that $s \leq t$ for all $s \in \mathcal{L}(f)$ and $t \in \mathcal{U}(f)$, so Equation (7.2) yields

$$\sup \mathcal{L}(f) \leq \inf \mathcal{U}(f).$$

Definition 7.12: Riemann Integral

A bounded function $f : [a, b] \rightarrow \mathbb{R}$ is **Riemann integrable** if $\sup \mathcal{L}(f) = \inf \mathcal{U}(f)$. In this case, this common value is called the **Riemann integral** of f , and we write

$$\int_a^b f \, dx = \sup \mathcal{L}(f) = \inf \mathcal{U}(f).$$

We call a the **lower (integration) limit** and b the **upper (integration) limit**, and the function f the **integrand** of the integral $\int_a^b f \, dx$. If $f \geq 0$ is Riemann integrable, we interpret the number $\int_a^b f \, dx$ as the **area** of the set

$$\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, \quad 0 \leq y \leq f(x)\}.$$

Proposition 7.13: Riemann Integrability Condition

Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded. Then f is Riemann integrable if and only if for every $\varepsilon > 0$ there exists step functions $\ell, u \in \mathcal{SF}$ such that

$$\ell \leq f \leq u \quad \text{and} \quad \int_a^b (u - \ell) < \varepsilon.$$

In this case,

$$\left| \int_a^b f \, dx - \int_a^b \ell \, dx \right| < \varepsilon, \quad \left| \int_a^b u \, dx - \int_a^b f \, dx \right| < \varepsilon.$$

Proof. By Equation (7.3) applied with $A = \sup \mathcal{L}(f)$ and $B = \inf \mathcal{U}(f)$ we obtain

$$\begin{aligned} f \text{ is Riemann integrable} &\Leftrightarrow \sup \mathcal{L}(f) = \inf \mathcal{U}(f) \\ &\Leftrightarrow \forall \varepsilon > 0 \exists s \in \mathcal{L}(f) \exists t \in \mathcal{U}(f) : t - s < \varepsilon \\ &\Leftrightarrow \forall \varepsilon > 0 \exists \ell, u \in \mathcal{SF} : \ell \leq f \leq u \text{ and } \int_a^b u \, dx - \int_a^b \ell \, dx < \varepsilon \\ &\Leftrightarrow \forall \varepsilon > 0 \exists \ell, u \in \mathcal{SF} : \ell \leq f \leq u \text{ and } \int_a^b (u - \ell) \, dx < \varepsilon, \end{aligned}$$

where we used Proposition 7.7 to deduce that

$$\int_a^b u \, dx - \int_a^b \ell \, dx = \int_a^b (u - \ell) \, dx$$

Finally, the concluding inequalities follow from

$$\int_a^b \ell \, dx \leq \int_a^b f \, dx \leq \int_a^b u \, dx \quad \text{and} \quad \int_a^b (u - \ell) \, dx < \varepsilon. \quad \square$$

Example

Not all functions are Riemann integrable. Indeed, consider the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

We claim that f is not Riemann integrable.

Let $u \in \mathcal{SF}$ with $f \leq u$, and let $0 = x_0 < \dots < x_1 = 1$ be a decomposition such that u is constant c_k on (x_{k-1}, x_k) . Since \mathbb{Q} is dense in \mathbb{R} , there exists $x \in (x_{k-1}, x_k) \cap \mathbb{Q}$, hence $1 = f(x) \leq u(x) = c_k$, so $c_k \geq 1$. Therefore,

$$\int_0^1 u(x) dx = \sum_{k=1}^n c_k (x_k - x_{k-1}) \geq \sum_{k=1}^n (x_k - x_{k-1}) = x_n - x_0 = 1.$$

Thus $\inf \mathcal{U}(f) = 1$, and taking $u \equiv 1$ gives $\inf \mathcal{U}(f)$. A similar argument with lower sums shows that $\sup \mathcal{L}(f) = 0$. Hence, f is not Riemann integrable.

Theorem 7.14: Linearity of the Riemann Integral

If $f, g : [a, b] \rightarrow \mathbb{R}$ are integrable and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is integrable, and

$$\int_a^b (\alpha f + \beta g) dx = \alpha \int_a^b f dx + \beta \int_a^b g dx$$

Proof. Given $\varepsilon > 0$, Proposition 7.13 yields step functions ℓ_1, ℓ_2, u_1, u_2 with

$$\ell_1 \leq f \leq u_1, \quad \ell_2 \leq g \leq u_2, \quad \int_a^b (u_1 - \ell_1) dx < \varepsilon, \quad \int_a^b (u_2 - \ell_2) dx < \varepsilon,$$

and

$$\left| \int_a^b f dx - \int_a^b \ell_1 dx \right| < \varepsilon, \quad \left| \int_a^b g dx - \int_a^b \ell_2 dx \right| < \varepsilon.$$

Assume first $\alpha, \beta \geq 0$. Then

$$\alpha \ell_1 + \beta \ell_2 \leq \alpha f + \beta g \leq \alpha u_1 + \beta u_2,$$

and

$$\int_a^b [(\alpha u_1 + \beta u_2) - (\alpha \ell_1 + \beta \ell_2)] dx = \alpha \int_a^b (u_1 - \ell_1) dx + \beta \int_a^b (u_2 - \ell_2) dx < (\alpha + \beta)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves that $\alpha f + \beta g$ is integrable. Moreover, by the triangle inequality

and Proposition 7.7, we have that

$$\begin{aligned}
 \left| \int_a^b (\alpha f + \beta g) dx - \alpha \int_a^b f dx - \beta \int_a^b g dx \right| &\leq \left| \int_a^b (\alpha f + \beta g) dx - \int_a^b (\alpha \ell_1 + \beta \ell_2) dx \right| \\
 &\quad + \underbrace{\left| \int_a^b (\alpha \ell_1 + \beta \ell_2) dx - \alpha \int_a^b \ell_1 dx - \beta \int_a^b \ell_2 dx \right|}_{=0} \\
 &\quad + \alpha \left| \int_a^b \ell_1 dx - \int_a^b f dx \right| + \beta \left| \int_a^b \ell_2 dx - \int_a^b g dx \right| \\
 &\leq (\alpha + \beta)\varepsilon + \alpha\varepsilon + \beta\varepsilon = 2(\alpha + \beta)\varepsilon.
 \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the linearity follows.

The case when one of α, β is negative is analogous, but one needs to reverse the corresponding inequalities. For instance, if $\alpha \geq 0$ and $\beta < 0$, then

$$\alpha \ell_1 + \beta u_2 \leq \alpha f + \beta g \leq \alpha u_1 + \beta \ell_2,$$

and

$$\int_a^b [(\alpha u_1 + \beta \ell_2) - (\alpha \ell_1 + \beta u_2)] dx = \alpha \int_a^b (u_1 - \ell_1) dx + |\beta| \int_a^b (u_2 - \ell_2) dx < (\alpha + |\beta|)\varepsilon.$$

This implies again that $\alpha f + \beta g$ is integrable, and the linearity identity holds similarly. \square

Proposition 7.15: Monotonicity of the Riemann Integral

Let $f, g : [a, b] \rightarrow \mathbb{R}$ be integrable. If $f \leq g$, then

$$\int_a^b f dx \leq \int_a^b g dx.$$

Proof. Since $f \leq g$, for any step function ℓ with $\ell \leq f$ we have $\ell \leq g$. This implies that $\mathcal{L}(f) \subseteq \mathcal{L}(g)$, therefore

$$\int_a^b f dx = \sup \mathcal{L}(f) \leq \sup \mathcal{L}(g) = \int_a^b g dx. \quad \square$$

Definition 7.16: Positive and Negative Parts

Given a function $f : D \rightarrow \mathbb{R}$, we define its **positive part** $f^+ : D \rightarrow \mathbb{R}$ and **negative part** $f^- : D \rightarrow \mathbb{R}$ by

$$f^+(x) = \max\{0, f(x)\}, \quad f^-(x) = -\min\{0, f(x)\}.$$

These satisfy

$$f = f^+ - f^-, \quad |f| = f^+ + f^-, \quad f^+ = \frac{|f| + f}{2}, \quad f^- = \frac{|f| - f}{2}.$$

Moreover, for any functions $f, g : D \rightarrow \mathbb{R}$,

$$f \leq g \quad \Rightarrow \quad f^+ \leq g^+, \quad f \leq g \quad \Rightarrow \quad f^- \geq g^-.$$

Remark 7.17. For any real numbers $z_1, z_2 \in \mathbb{R}$, one has

$$(z_1 - z_2)^+ \geq z_1^+ - z_2^+. \quad (7.4)$$

Indeed, since $z^+ \geq z$ and $z^+ \geq 0$ for all $z \in \mathbb{R}$, by applying these inequalities with $z = z_1 - z_2$ and $z = z_2$ we obtain

$$z_1 = (z_1 - z_2) + z_2 \leq (z_1 - z_2)^+ + z_2^+ \quad \text{and} \quad 0 \leq (z_1 - z_2)^+ + z_2^+.$$

Hence $(z_1 - z_2)^+ + z_2^+$ is greater or equal to both z_1 and 0, and therefore

$$z_1^+ = \max\{z_1, 0\} \leq (z_1 - z_2)^+ + z_2^+,$$

which yields Equation (7.4) after rearranging.

Theorem 7.18: Triangle Inequality for the Riemann Integral

Let $f, g : [a, b] \rightarrow \mathbb{R}$ be integrable. Then f^+ , f^- , and $|f|$ are integrable, and

$$\left| \int_a^b f \, dx \right| \leq \int_a^b |f| \, dx.$$

Proof. Fix $\varepsilon > 0$. Since f is integrable, there exists step functions $\ell \leq f \leq u$ with $\int_a^b (u - \ell) \, dx < \varepsilon$. Then ℓ^+ and u^+ are step functions with $\ell^+ \leq f^+ \leq u^+$.

Since $u - \ell \geq 0$, we have $(u - \ell) = (u - \ell)^+$. Moreover, applying Equation (7.4) with $z_1 = u(x)$ and $z_2 = \ell(x)$, we obtain

$$(u(x) - \ell(x))^+ \geq u(x)^+ - \ell(x)^+ \quad \forall x \in [a, b].$$

Hence

$$\int_a^b (u^+ - \ell^+) \, dx \leq \int_a^b (u - \ell)^+ \, dx = \int_a^b (u - \ell) \, dx < \varepsilon,$$

so f^+ is integrable. By Theorem 7.14, also $f^- = f^+ - f$ and $|f| = 2f^+ - f$ are integrable. Finally,

$$\left| \int_a^b f \, dx \right| = \left| \int_a^b f^+ \, dx - \int_a^b f^- \, dx \right| \leq \int_a^b f^+ \, dx + \int_a^b f^- \, dx = \int_a^b |f| \, dx. \quad \square$$

Remark 7.19. Let $a < b < c$. A function $f : [a, c] \rightarrow \mathbb{R}$ is integrable if and only if $f|_{[a, b]}$ and $f|_{[b, c]}$ are integrable, and

$$\int_a^c f \, dx = \int_a^b f|_{[a, b]} \, dx + \int_b^c f|_{[b, c]} \, dx.$$

7.3 Integrability Theorems

7.3.1 Integrability of Monotone Functions

As before we work on a compact interval $[a, b] \subseteq \mathbb{R}$. Note that every monotone function $f : [a, b] \rightarrow \mathbb{R}$ is bounded; for instance, if f is increasing then $f(a)$ is a lower bound and $f(b)$ is an upper bound.

Theorem 7.20: Monotone Functions are Integrable

Every monotone function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable.

Proof. W.l.o.g, f is increasing (otherwise replace f by $-f$ and use Theorem 7.14). We want to apply Proposition 7.13. Given $\varepsilon > 0$, we need to construct step functions $\ell, u \in \mathcal{SF}$ such that $\ell \leq f \leq u$ and $\int_a^b (u - \ell) \, dx < \varepsilon$.

Fix $n \in \mathbb{N}$ (to be chosen later) and the uniform partition

$$a = x_0 < x_1 < \dots < x_n = b, \quad x_k = a + \frac{k}{n}(b - a).$$

Define the step functions $\ell, u : [a, b] \rightarrow \mathbb{R}$ as

$$\begin{aligned} \ell(x) = f(x_{k-1}) \quad \text{and} \quad u(x) = f(x_k) & \quad \text{for } x \in (x_{k-1}, x_k), \quad k = 1, \dots, n, \\ \ell(x) = u(x) = f(x) & \quad \text{for } x \in \{x_0, \dots, x_n\}. \end{aligned}$$

Note that, since f is increasing, $\ell \leq f \leq u$. Moreover, for each k we have $u - \ell = f(x_k) - f(x_{k-1})$ on (x_{k-1}, x_k) . Recalling that $x_k - x_{k-1} = \frac{b-a}{n}$, this yields

$$\begin{aligned} \int_a^b (u - \ell) dx &= \sum_{k=1}^n (f(x_k) - f(x_{k-1}))(x_k - x_{k-1}) = \frac{b-a}{n} \sum_{k=1}^n (f(x_k) - f(x_{k-1})) \\ &= \frac{b-a}{n} (f(x_n) - f(x_0)) = \frac{b-a}{n} (f(b) - f(a)). \end{aligned}$$

Choosing $n \in \mathbb{N}$ so large that $\frac{b-a}{n}(f(b) - f(a)) < \varepsilon$, Proposition 7.13 implies that f is Riemann integrable. \square

Definition 7.21: Piecewise Monotone Functions

A function $f : [a, b] \rightarrow \mathbb{R}$ is **piecewise monotone** if there exists a decomposition

$$a = x_0 < x_1 < \dots < x_n = b$$

such that $f|_{(x_{k-1}, x_k)}$ is monotone for every $k = 1, \dots, n$.

Corollary 7.22: Piecewise Monotone Functions are Integrable

Every bounded piecewise monotone function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable.

7.3.2 Integrability of Continuous Functions

Using boundedness and uniform continuity on compact intervals (Theorems 4.17 and 4.23), we can prove that continuous functions are integrable.

Theorem 7.23: Continuous Functions are Integrable

Every continuous function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable.

Proof. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous and fix $\varepsilon > 0$. By uniform continuity 4.23, there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon \quad \forall x, y \in [a, b]. \quad (7.5)$$

Choose a partition $a = x_0 < x_1 < \dots < x_n = b$ with $x_k - x_{k-1} < \delta$. For each k set

$$c_k = \min\{f(x) \mid x_{k-1} \leq x \leq x_k\}, \quad d_k = \max\{f(x) \mid x_{k-1} \leq x \leq x_k\},$$

which exists by Theorem 4.20, and let $y_k, z_k \in [x_{k-1}, x_k]$ satisfy $f(y_k) = c_k$ and $f(z_k) = d_k$. Then since, $|y_k - z_k| \leq x_k - x_{k-1} < \delta$, Equation (7.5) yields $d_k - c_k < \varepsilon$.

Define now the step functions $\ell, u : [a, b] \rightarrow \mathbb{R}$ as

$$\begin{aligned} \ell(x) = c_k, \quad \text{and} \quad u(x) = d_k & \quad \text{for } x \in (x_{k-1}, x_k), \quad k = 1, \dots, n, \\ \ell(x) = u(x) = f(x) & \quad \text{for } x \in \{x_0, \dots, x_n\}. \end{aligned}$$

Then $\ell \leq f \leq u$ and $u - \ell = d_k - c_k$ on (x_{k-1}, x_k) , hence

$$\int_a^b (u - \ell) dx = \sum_{k=1}^n (d_k - c_k)(x_k - x_{k-1}) < \varepsilon \sum_{k=1}^n (x_k - x_{k-1}) = \varepsilon(b - a).$$

Since $\varepsilon > 0$ is arbitrary, f is integrable. \square

Definition 7.24: Piecewise Continuous Functions

A function $f : [a, b] \rightarrow \mathbb{R}$ is **piecewise continuous** if there exists a decomposition

$$a = x_0 < x_1 < \dots < x_n = b$$

such that $f|_{(x_{k-1}, x_k)}$ is continuous for all k and both one sided limits $\lim_{x \rightarrow x_{k-1}^+} f(x)$ and $\lim_{x \rightarrow x_k^-} f(x)$ exist. Equivalently, each $f|_{(x_{k-1}, x_k)}$ extends to a continuous function on $[x_{k-1}, x_k]$.

Corollary 7.25: Piecewise Continuous Functions are Integrable

Every piecewise continuous function $f : [a, b] \rightarrow \mathbb{R}$ is Riemann integrable.

7.3.3 Integration and Sequences of Functions

Let $(f_n)_{n=0}^\infty$ with $f_n : [a, b] \rightarrow \mathbb{R}$ be a sequence of integrable functions. Assume that f_n converges pointwise or uniformly to $f : [a, b] \rightarrow \mathbb{R}$. Is f integrable? And if so, does

$$\lim_{n \rightarrow \infty} \int_a^b f_n dx = \int_a^b f dx$$

hold?

In general, the pointwise limit of integrable functions need not to be integrable. Also, as the following example shows, even when the pointwise limit of f is integrable, one may have that $\lim_{n \rightarrow \infty} \int f_n \neq \int f$.

Example

Let $D = [0, 1]$ and define $f_n : D \rightarrow \mathbb{R}$ by

$$f_n(x) = \begin{cases} n^2 x & \text{if } 0 \leq x \leq \frac{1}{2n}, \\ n^2(\frac{1}{n} - x) & \text{if } \frac{1}{2n} \leq x \leq \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} \leq x \leq 1. \end{cases}$$

Each f_n is continuous (hence integrable). Also, its graph is a triangle of base $\frac{1}{n}$ and height $\frac{n}{2}$, so

$$\int_0^1 f_n(x) dx = \frac{1}{2} \cdot \frac{1}{n} \cdot \frac{n}{2} = \frac{1}{4}.$$

Moreover, $f_n(0) = 0$ for all n , and for every $x > 0$ we have $f_n(x) = 0$ for all $n > 1/x$, hence $f_n(x) \rightarrow 0$. Thus, f_n converges pointwise to the constant function $f = 0$, but

$$\int_0^1 f_n(x) dx = \frac{1}{4} \neq 0 = \int_0^1 f(x) dx.$$

On the other hand, as the next result shows, uniform convergence is sufficient for both integrability of the limit and interchange of limit and integral.

Theorem 7.26: Uniform Convergence and Riemann Integrals Commute

Let $(f_n)_{n=0}^\infty$, with $f_n : [a, b] \rightarrow \mathbb{R}$, be a sequence of integrable functions converging uniformly to $f : [a, b] \rightarrow \mathbb{R}$. Then f is integrable and

$$\int_a^b f dx = \lim_{n \rightarrow \infty} \int_a^b f_n dx. \quad (7.6)$$

Proof. Fix $\varepsilon > 0$. By uniform convergence, there exists $N \in \mathbb{N}$ such that $|f_n - f| < \varepsilon$ on $[a, b]$ for all $n \geq N$.

Since f_N is integrable, there exists step functions $\ell, u \in \mathcal{SF}$ with $\ell \leq f_N \leq u$ and $\int_a^b (u - \ell) dx < \varepsilon$. Set $\hat{\ell} = \ell - \varepsilon$ and $\hat{u} = u + \varepsilon$. Then $\hat{\ell}, \hat{u} \in \mathcal{SF}$. Also, since $|f_N - f| < \varepsilon$,

$$\hat{\ell} = \ell - \varepsilon \leq f_N - \varepsilon \leq f \leq f_N + \varepsilon \leq u + \varepsilon = \hat{u}$$

and (because $\hat{u} - \hat{\ell} = u - \ell + 2\varepsilon$)

$$\int_a^b (\hat{u} - \hat{\ell}) dx = \int_a^b (u - \ell) dx + 2\varepsilon(b - a) < \varepsilon + 2\varepsilon(b - a).$$

As $\varepsilon > 0$ is arbitrary, Proposition 7.13 yields that f is integrable.

Moreover, using monotonicity (Proposition 7.15) and the triangle inequality for the Riemann integral (Theorem 7.18),

$$\left| \int_a^b f dx - \int_a^b f_n dx \right| = \left| \int_a^b (f - f_n) dx \right| \leq \int_a^b |f - f_n| dx \leq \varepsilon(b - a) \quad \forall n \geq N,$$

proving Equation (7.6). □

8 The Derivative and the Riemann Integral

8.1 The Fundamental Theorem of Calculus

Throughout this section we fix a compact interval $I \subseteq \mathbb{R}$ that is non-empty and contains more than one point. For brevity, we write *integrable* for *Riemann integrable*.

8.1.1 The Fundamental Theorem

Definition 8.1: Primitive Function

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$ a function. Any differentiable function $F : I \rightarrow \mathbb{R}$ such that $F' = f$ is called a **primitive** (or **antiderivative**) of f .

Remark 8.2. *A primitive may not always exist.*

The next result is known as the **Fundamental Theorem of (Integral and Differential) Calculus**, going back to Leibniz, Newton and Barrow.

Theorem 8.3: Fundamental Theorem of Calculus

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Then:

(i) For every $C \in \mathbb{R}$, the function $F : [a, b] \rightarrow \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t) dt + C \quad (8.1)$$

is a primitive of f .

(ii) Every primitive $F : [a, b] \rightarrow \mathbb{R}$ of f has the form (8.1) for some constant C .

Proof. By Theorem 7.23, f is integrable.

Let F be defined as in Equation (8.1). To prove (i), we fix $x_0 \in [a, b]$ and we want to show that $F'(x_0) = f(x_0)$. To this aim, fix $\varepsilon > 0$. By continuity, there exists $\delta > 0$ such that

$$z \in [a, b], \quad |z - x_0| < \delta \quad \Rightarrow \quad |f(z) - f(x_0)| < \varepsilon. \quad (8.2)$$

Now, given $x \in (x_0, x_0 + \delta) \cap [a, b]$, it follows from Remark 7.19 that

$$\begin{aligned} \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| &= \left| \frac{1}{x - x_0} \left(\int_a^x f(t) dt - \int_a^{x_0} f(t) dt \right) - f(x_0) \right| \\ &= \left| \frac{1}{x - x_0} \int_{x_0}^x f(t) dt - f(x_0) \right|. \end{aligned}$$

Also,

$$f(x_0) = f(x_0) \frac{1}{x - x_0} \int_{x_0}^x dt = \frac{1}{x - x_0} \int_{x_0}^x f(x_0) dt.$$

Combining these two equations and using Theorem 7.18, we get

$$\begin{aligned} \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| &= \left| \frac{1}{x - x_0} \int_{x_0}^x f(t) dt - \frac{1}{x - x_0} \int_{x_0}^x f(x_0) dt \right| \\ &= \left| \frac{1}{x - x_0} \int_{x_0}^x (f(t) - f(x_0)) dt \right| \\ &\leq \frac{1}{x - x_0} \int_{x_0}^x |f(t) - f(x_0)| dt. \end{aligned}$$

Note now that, in the last integral, $t \in [x_0, x] \subseteq [x_0, x_0 + \delta) \cap [a, b]$. Hence, it follows from Equation (8.2) that $|f(t) - f(x_0)| < \varepsilon$, therefore

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \frac{1}{x - x_0} \int_{x_0}^x \varepsilon dt = \varepsilon.$$

Similarly, if $x \in (x_0 - \delta, x_0) \cap [a, b]$, then

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| = \left| \frac{1}{x_0 - x} \int_x^{x_0} (f(t) - f(x_0)) dt \right| \leq \frac{1}{x_0 - x} \int_x^{x_0} |f(t) - f(x_0)| dt < \varepsilon.$$

In summary, we proved that

$$x \in [a, b], \quad |x - x_0| < \delta \quad \Rightarrow \quad \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon,$$

therefore

$$F'(x_0) = \lim_{x \rightarrow x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0),$$

as desired.

We now prove (ii). Let F be a primitive of f . Then, since $(\int_a^x f(t) dt)' = f(x)$ (by (i)), we have that

$$\left(F(x) - \int_a^x f(t) dt \right)' = F'(x) - f(x) = f(x) - f(x) = 0 \quad \forall x \in (a, b).$$

By Corollary 6.24, this implies that $F(x) - \int_a^x f(t) dt$ is constant on $[a, b]$, concluding the proof of (ii). \square

Corollary 8.4: Integral vs. Derivative

If $F : [a, b] \rightarrow \mathbb{R}$ is continuously differentiable, then for all $x \in [a, b]$,

$$F(x) = F(a) + \int_a^x F'(t) dt.$$

Proof. Since F is a primitive of F' , Theorem 8.3 yields $F(x) = \int_a^x F'(t) dt + C$. Evaluating at $x = a$ gives $C = F(a)$. \square

Corollary 8.5: Riemann Integral and Primitives

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and F is a primitive of f , then

$$\int_a^b f(t) dt = F(b) - F(a).$$

Proof. Apply Corollary 8.4 with $F' = f$ and $x = b$. \square

8.1.2 Integration by Parts and Substitution

Given a Function $h : [a, b] \rightarrow \mathbb{R}$, we use the notation $[h(x)]_a^b := h(b) - h(a)$.

Theorem 8.6: Integration by Parts

If $f, g : [a, b] \rightarrow \mathbb{R}$ are continuously differentiable, then

$$\int_a^b f(x)g'(x) dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x) dx.$$

Proof. By Proposition 6.5, $(fg)' = f'g + fg'$. Rearranging and integrating, thanks to Corollary 8.5, we get

$$\int_a^b fg' dx = \int_a^b (fg)' dx - \int_a^b fg' dx = [fg]_a^b - \int_a^b fg' dx. \quad \square$$

As a convention, for any $h : [a, b] \rightarrow \mathbb{R}$,

$$\int_b^a h(x) dx = - \int_a^b h(x) dx. \quad (8.3)$$

Theorem 8.7: Integration by Substitution, 1st Form

Let $I, J \subseteq \mathbb{R}$ be intervals, $f : I \rightarrow J$ be continuously differentiable, and $g : J \rightarrow \mathbb{R}$ be continuous. For any $[a, b] \subseteq I$,

$$\int_a^b g(f(x))f'(x) dx = \int_{f(a)}^{f(b)} g(y) dy.$$

Proof. Fix $y_0 \in J$ and set $G(y) = \int_{y_0}^y g(t) dt$. Since $G' = g$, by the chain rule (Theorem 6.8) we get $(G \circ f)' = G'(f)f' = g(f)f'$. Integrating this identity and using Corollary 8.5 yields

$$\begin{aligned} \int_a^b g(f(x))f'(x) dx &= \int_a^b (G \circ f)'(x) dx = G(f(b)) - G(f(a)) \\ &= \int_{y_0}^{f(b)} g(t) dt - \int_{y_0}^{f(a)} g(t) dt = \int_{f(a)}^{f(b)} g(t) dt. \end{aligned} \quad \square$$

Before stating the next result, we note the following: If $h : [a, b] \rightarrow \mathbb{R}$ is continuously differentiable with $h' \neq 0$, then h' has constant sign on $[a, b]$, so h is strictly monotone and invertible; h^{-1} is continuous by Theorem 4.15 and differentiable on $(h(a), h(b))$ by Theorem 6.10. Furthermore, since $(h^{-1})' = \frac{1}{h' \circ h^{-1}}$, also $(h^{-1})'$ is continuous.

Theorem 8.8: Integration by Substitution, 2nd Form

Let $I, J \subseteq \mathbb{R}$ be intervals, $f : I \rightarrow J$ be C^1 , and $g : J \rightarrow \mathbb{R}$ be continuous. Let $[a, b] \subseteq I$ and assume $f'(x) \neq 0$ on $[a, b]$. If $f^{-1} : [f(a), f(b)] \rightarrow \mathbb{R}$ denotes the inverse of $f|_{[a, b]}$, then

$$\int_a^b g(f(x)) dx = \int_{f(a)}^{f(b)} g(y)(f^{-1})'(y) dy.$$

Proof. In order to apply Theorem 8.7, we first observe that

$$\int_a^b g(f(x)) dx = \int_a^b \frac{g(f(x))}{f'(x)} f'(x) dx = \int_a^b \frac{g(f(x))}{f' \circ f^{-1}(f(x))} f'(x) dx.$$

So we can apply Theorem 8.7 with $\frac{g}{f' \circ f^{-1}}$ in place of g to get

$$\int_a^b g(f(x)) dx = \int_{f(a)}^{f(b)} \frac{g(y)}{f'(f^{-1}(y))} dy.$$

Since $\frac{1}{f' \circ f^{-1}} = (f^{-1})'$, the result follows. \square

8.1.3 Improper Integrals

A function $f : I \rightarrow \mathbb{R}$ is **locally integrable** if $f|_{[a, b]}$ is integrable for every compact interval $[a, b] \subseteq I$.

Definition 8.9: Improper Integrals

Let $I \subseteq \mathbb{R}$ be a non-empty interval and $f : I \rightarrow \mathbb{R}$ be locally integrable. Set $c = \inf I \in \mathbb{R} \cup \{-\infty\}$ and $d = \sup I \in \mathbb{R} \cup \{\infty\}$, and fix $x_0 \in I$. We define the **improper integral** of f on I by

$$\int_c^d f(x) dx := \lim_{a \rightarrow c^+} \int_a^{x_0} f(x) dx + \lim_{b \rightarrow d^-} \int_{x_0}^b f(x) dx,$$

whenever both limits exist and the sum is well-defined (we do not allow the indeterminate form $\infty - \infty$). Here the first limit is taken over $a \in I$ with $c < a < x_0$ (interpreting $a \rightarrow -\infty$ if $c = -\infty$) and the second over $b \in I$ with $x_0 < b < d$ (interpreting $b \rightarrow +\infty$ if $d = \infty$). If the value is finite we say the integral *converges*; if it is $\pm\infty$ we say it *diverges to $\pm\infty$* ; otherwise, it *does not converge*. When defined, the value is independent of the choice of x_0 .

Lemma 8.10: Improper Integrals of Non-negative Functions

Let $I \subseteq \mathbb{R}$ be a non-empty interval with $c = \inf I$ and $d = \sup I$, and let $f : I \rightarrow [0, \infty)$ be locally integrable. Fix any $x_0 \in I$. Then the one-sided limits

$$L_- := \lim_{a \rightarrow c^+} \int_a^{x_0} f(x) dx, \quad L_+ := \lim_{b \rightarrow d^-} \int_{x_0}^b f(x) dx$$

exist in $[0, \infty]$, and

$$\int_c^d f(x) dx = L_- + L_+ = \sup_{c < \alpha < \beta < d} \int_\alpha^\beta f(x) dx \in [0, \infty].$$

In particular, the improper integral over I always exists in the extended sense and equals $+\infty$ whenever the supremum is $+\infty$.

Proof. Fix $x_0 \in I$. Since $f \geq 0$, the maps $a \mapsto \int_a^{x_0} f dx$ (for $a < x_0$) and $b \mapsto \int_{x_0}^b f dx$ (for $b > x_0$) are monotone, hence the limits L_-, L_+ exist in $[0, \infty]$. Also, for any $c < a < x_0 < b < d$,

$$\int_a^b f dx = \int_a^{x_0} f dx + \int_{x_0}^b f dx.$$

(this is different from the skript. Idk if theres a typo in the skript.) Taking suprema gives

$$\sup_{c < \alpha < \beta < d} \int_\alpha^\beta f dx = \sup_{a < x_0} \int_a^{x_0} f dx + \sup_{b > x_0} \int_{x_0}^b f dx = L_- + L_+,$$

which equals the definition of $\int_c^d f dx$ above. □

Theorem 8.11: Integral Test for Series

Let $f : [0, \infty) \rightarrow [0, \infty)$ be monotone decreasing. Then for every $n \in \mathbb{N}$,

$$\sum_{n=1}^{N+1} f(n) \leq \int_0^{N+1} f(x) dx \leq \sum_{n=0}^N f(n).$$

In particular,

$$\sum_{n=0}^{\infty} f(n) \text{ converges} \quad \Leftrightarrow \quad \int_0^{\infty} f(x) dx \text{ converges}.$$

Proof. By monotonicity, f is locally integrable. Define step functions on $[0, \infty)$ by

$$u(x) = f(\lfloor x \rfloor), \quad \ell(x) = f(\lceil x \rceil),$$

where $\lfloor x \rfloor$ is the rounding function (i.e., the largest integer $\leq x$), while $\lceil x \rceil$ denotes the smallest integer $\geq x$. Then $\ell \leq f \leq u$, and for $N \geq 1$,

$$\sum_{n=1}^{N+1} f(n) = \int_0^{N+1} \ell(x) dx \leq \int_0^{N+1} f(x) dx \leq \int_0^{N+1} u(x) dx = \sum_{n=0}^N f(n).$$

The result follows by taking the limit as $N \rightarrow \infty$. □

8.2 Integration and Differentiation of Power Series

We recall the limit

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n}} = 1. \tag{8.4}$$

Also we shall use the following fact.

Remark 8.12. Let $(a_n)_{n=0}^{\infty}$ be a sequence of non-negative numbers with

$$L = \limsup_{n \rightarrow \infty} a_n < \infty,$$

and let $(b_n)_{n=0}^{\infty}$ and $(\gamma_n)_{n=0}^{\infty}$ satisfy $b_n \rightarrow 1$ and $\gamma_n \rightarrow 1$. Then

$$\limsup_{n \rightarrow \infty} a_n^{\gamma_n} b_n = L.$$

In other words, multiplying by a factor that tends to 1, or raising to an exponent that tends to 1, does not change the value of the lim sup.

Theorem 8.13: Integration of Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

has the same radius of convergence R and is a primitive of f on $(-R, R)$.

Proof. Set $\rho = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$, so that $R = \rho^{-1}$. Define $c_0 = 0$ and $c_n = \frac{a_{n-1}}{n}$ for $n \geq 1$, so

$F(x) = \sum_{n=0}^{\infty} c_n x^n$. Noticing that

$$\sqrt[n]{|c_n|} = \sqrt[n]{\frac{1}{n}} \left(\sqrt[n-1]{|a_{n-1}|} \right)^{\frac{n-1}{n}},$$

it follows from Equation (8.4) and Remark 8.12 (applied with $b_n = \sqrt[n]{\frac{1}{n}}$ and $\gamma_n = \frac{n-1}{n}$) that

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} = \limsup_{n \rightarrow \infty} \sqrt[n-1]{|a_{n-1}|} = \rho,$$

hence also F has radius of convergence R .

We now want to prove that $F' = f$. Fix $[-r, r] \subseteq (-R, R)$ and define the polynomials $f_n(x) = \sum_{k=0}^n a_k x^k$. Then

$$\int_0^x f_n(t) dt = \sum_{k=0}^n \frac{a_k}{k+1} x^{k+1}.$$

By Theorem 5.26, the sequence of functions $(f_n)_{n=0}^{\infty}$ converge uniformly to f on $[-r, r]$, so Theorem 7.26 yields

$$\int_0^x f(t) dt = \lim_{n \rightarrow \infty} \int_0^x f_n(t) dt \quad \forall x \in [-r, r].$$

On the other hand, again by Theorem 5.26,

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{a_k}{k+1} x^{k+1} = F(x) \quad \forall x \in [-r, r].$$

This proves that $F(x) = \int_0^x f(t) dt$ on $[-r, r]$, so Theorem 8.3 implies that $F'(x) = f(x)$ on $[-r, r]$. Since $[-r, r] \subseteq (-R, R)$ is arbitrary, we proved that $F' = f$ on $(-R, R)$. \square

Corollary 8.14: Differentiation of Power Series

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ with

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \forall x \in (-R, R),$$

and the series on the right has radius of convergence R .

Proof. Let $c_n = (n+1)a_{n+1}$ and $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} c_n x^n$. Let \bar{R} be the radius of convergence of g . Then Theorem 8.13 implies that the power series

$$G(x) = \sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$$

has radius \bar{R} and is a primitive of g . But

$$G(x) = \sum_{n=0}^{\infty} \frac{(n+1)a_{n+1}}{n+1} x^{n+1} = \sum_{n=0}^{\infty} a_{n+1} x^{n+1} = \sum_{n=1}^{\infty} a_n x^n = f(x) - a_0.$$

This implies that G and f have the same radius of convergence (so $\bar{R} = R$) and that $g = G' = (f - a_0)' = f'$. \square

8.3 Integration Methods

Let $I \subseteq \mathbb{R}$ be an interval, and $f : I \rightarrow \mathbb{R}$ a function. The notation

$$\int f(x) dx = F(x) + C$$

means that F is a primitive function (antiderivative) of f . In the expression $F(x) + C$, C is read as an indefinite constant, usually called **integration constant**. Since the domain I of f is an interval, two primitive functions of f differ by a constant, which makes the notation meaningful. One calls $F(x) + C$ the **indefinite integral** of f . In this section we summarize general methods to determine indefinite integrals.

Throughout this section, $I \subseteq \mathbb{R}$ denotes a non-empty interval that is not a single point. Also, all functions in this section are real-valued functions with domain I that are integrable on any compact interval $[a, b] \subseteq I$.

8.3.1 Integration by Parts and by Substitution in Leibniz Notation

We recall the derivative of a function h is denoted by h' or by $\frac{dh}{dx}$.

Integration by Parts

Let f and g be functions with primitives F and G , respectively. Recall the Theorem 8.6 (Integration by Parts), which is in the Leibniz notation

$$\int F(x)g(x) dx = F(x)G(x) - \int f(x)G(x) dx + C.$$

In Leibniz notation, $f = \frac{dF}{dx}$ and $g = \frac{dG}{dx}$. This leads to the notation $f dx = dF$ and $g dx = dG$, and integration by parts is sometimes written as

$$\int F dG = FG - \int G dF + C.$$

Integration by Substitution

Let J be an interval and let $f : I \rightarrow J$ be a differentiable function. If $G : J \rightarrow \mathbb{R}$ is a primitive of g , then by the chain rule in Theorem 6.8, $[G(f(x))] = g(f(x))f'(x)$ for all $x \in I$. From this it follows that

$$\int g(f(x))f'(x) dx = G(f(x)) + C = \int g(u) du + C,$$

where we used the change of variables $u = f(x)$. The substitution rule is also called **change of variable**, as one has replaced the variable u in $\int g(u) du$ by $u = f(x)$. In Leibniz notation this is very natural: if $u = f(x)$, then $du = f'(x) dx$.

We also recall the second form of the substitution rule (Theorem 8.8): if $f' \neq 0$ we can set $x = f^{-1}(u)$ so that $\frac{dx}{du} = (f^{-1})'(u)$, and obtain

$$\int g(f(x)) dx = \int g(u) \frac{dx}{du} du + C.$$

8.3.2 Integration by Parts: Example

- We calculate the integral $\int \log(x) dx$:

$$\begin{aligned}
 \int \log(x) dx &= \int \log(x) \cdot 1 dx \\
 &= \log(x) \cdot x - \int (\log(x))' \cdot x dx + C \\
 &= \log(x) \cdot x - \int \frac{1}{x} \cdot x dx + C \\
 &= \log(x) \cdot x - \int 1 dx + C = \log(x) \cdot x - x + C.
 \end{aligned}$$

Suggestion: To ensure that the final result is correct, one can always differentiate the result and check whether you obtain the original function. For instance, in this case, one can easily check that

$$(x \log(x) - x + C)' = \log(x).$$

- We calculate the integral $\int \cos^2(x) dx$:

$$\begin{aligned}
 \int \cos^2(x) dx &= \cos(x) \sin(x) - \int (-\sin(x)) \sin(x) dx + C \\
 &= \cos(x) \sin(x) + \int \underbrace{\sin^2(x)}_{= 1 - \cos^2(x)} dx + C \\
 &= \cos(x) \sin(x) + \int 1 dx - \int \cos^2(x) dx + C \\
 &= \cos(x) \sin(x) + x - \int \cos^2(x) dx + C \\
 \Rightarrow \quad 2 \int \cos^2(x) dx &= \cos(x) \sin(x) + x + C \\
 \Rightarrow \quad \int \cos^2(x) &= \frac{1}{2}(\cos(x) \sin(x) + x) + C.
 \end{aligned}$$

8.3.3 Integration by Substitution: Examples

Example

We want to compute $\int \frac{x}{1+x^2} dx$. Let $u = 1 + x^2$, so that $du = 2x dx$. Then we find

$$\int \frac{x}{1+x^2} dx = \int \frac{x}{1+x^2} \cdot \frac{du}{2x} = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \log |u| + C = \frac{1}{2} \log(1+x^2) + C.$$

Whenever we see the a scalar multiple of the derivative of the denominator in the numerator, we can substitute for the denominator in order to obtain a primitive of the form as above. In the above example, the numerator x is a multiple of the derivative of the denominator $1+x^2$, so we substitute for the denominator.

More Examples

Now, we will list various examples, that require different substitutions. Note that the listed substitutions do not cover all the examples from the skript:

- Let $r > 0$. For the integral $\int \sqrt{r^2 - x^2} dx$, we use the substitution $x = r \sin(\theta)$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

With this choice, we have $dx = r \cos(\theta) d\theta$, therefore

$$\int \sqrt{r^2 - x^2} dx = \int \sqrt{r^2 - r^2 \sin^2(\theta)} \cos(\theta) d\theta = r^2 \int \cos^2(\theta) d\theta.$$

We can compute this integral by using integration by parts.

- In general, for integrals of the form $\int (a^2 - x^2)^{\frac{n}{2}} dx$ for $a > 0$ and $n \in \mathbb{Z}$, one uses the substitution $x = a \sin(\theta)$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, giving $dx = a \cos(\theta) d\theta$ and $(a^2 - x^2)^{\frac{1}{2}} = a \cos(\theta)$.
- For integrals of the form $\int (a^2 + x^2)^{\frac{n}{2}} dx$ for $a > 0$ and $n \in \mathbb{Z}$, the substitution $x = a \tan(\theta)$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ yields $dx = \frac{a}{\cos^2(\theta)} d\theta$ and $(a^2 + x^2)^{\frac{n}{2}} = \left(\frac{a}{\cos(\theta)}\right)^n$.
- For integrals of the form $\int x(a^2 - x^2)^{\frac{n}{2}} dx$ or $\int x(a^2 + x^2)^{\frac{n}{2}} dx$, the substitutions $u = a^2 - x^2$ or $u = a^2 + x^2$, respectively, allow us to compute the indefinite integrals.
- For integrals of the form $\int \sqrt{x^2 - 1} dx$, the substitution $x = \cosh(u)$ yields $dx = \sinh(u) du$ and

$$\int \sqrt{x^2 - 1} dx = \int \sqrt{\cosh^2(u) - 1} \sinh(u) du = \int \sinh^2(u) du.$$

This indefinite integral can again be computed using integration by parts.

- For the integral $\int \frac{1}{\sin(x)} dx$, we consider the substitution $u = \tan(\frac{x}{2})$. This yields, by the doubling angle formulas for sine and cosine, that $\sin(x) = \frac{2u}{1+u^2}$ and $dx = \frac{2}{1+u^2} du$.
- For integrals of the form $\int \frac{1}{a^2 + x^2} dx$, for $a \in \mathbb{R}$, one uses the substitution $u = \frac{x}{a}$. This yields

$$\int \frac{1}{a^2 + x^2} dx = \int \frac{1}{a^2} \cdot \frac{1}{1 + u^2} \cdot a du = \frac{1}{a} \int \frac{1}{1 + u^2} du = \frac{1}{a} \arctan(u) + C = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C.$$

8.3.4 The Gamma Function

The **Gamma-function** Γ is defined, for $s \in (0, \infty)$, by the improper integral

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx. \quad (8.5)$$

To verify that this improper integral indeed converges, we examine the integration limits 0 and ∞ separately. For $0 < a < b$ we find, using integration by parts,

$$\int_a^b x^{s-1} e^{-x} dx = \frac{1}{s} [x^s e^{-x}]_a^b + \frac{1}{s} \int_a^b x^s e^{-x} dx. \quad (8.6)$$

We obtain

$$\begin{aligned} \int_0^b x^{s-1} e^{-x} dx &= \lim_{a \rightarrow 0} \left(\frac{1}{s} [x^s e^{-x}]_a^b + \frac{1}{s} \int_a^b x^s e^{-x} dx \right) \\ &= \frac{1}{s} b^s e^{-b} + \frac{1}{s} \int_0^b x^s e^{-x} dx, \end{aligned}$$

where the integral on the right is an actual Riemann integral since the function $x^s e^{-x}$ is continuous on $[0, b]$. To investigate the upper limit of integration, we note that there exists $R > 0$ such that $e^x > x^{s+2}$ holds for all $x > R$. Thus

$$\int_0^\infty x^s e^{-x} dx \leq \int_0^R x^s e^{-x} dx + \int_R^\infty x^{-2} dx < \infty,$$

which shows that (8.6) converges as $b \rightarrow \infty$. Specifically, we obtain

$$\int_0^\infty x^{s-1} e^{-x} dx = \lim_{b \rightarrow \infty} \left(\frac{1}{s} b^s e^{-b} + \frac{1}{s} \int_0^b x^s e^{-x} dx \right) = \frac{1}{s} \int_0^\infty x^s e^{-x} dx.$$

This shows that the gamma function satisfies the relation

$$\Gamma(s+1) = s\Gamma(s) \quad (8.7)$$

for all $s \in (0, \infty)$, from which one can deduce that the Gamma function extends the factorial function from \mathbb{N} to $(0, \infty)$. In fact

$$\Gamma(1) = \int_0^\infty x^0 e^{-x} dx = \int_0^\infty e^{-x} dx = [e^{-x}]_0^\infty = e^0 = 1,$$

therefore (8.7) implies that

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \dots = n!\Gamma(1) = n! \quad \forall n \in \mathbb{N}.$$

8.4 Taylor Series

As we have seen in Chapter 6, given a differentiable function $f : D \rightarrow \mathbb{R}$, the derivative $f'(x_0)$ gives the slope of the tangent to the graph of f at x_0 . Moreover, the corresponding affine function

$$x \mapsto f(x_0) + f'(x_0)(x - x_0)$$

approximates the function f within an error $o(|x - x_0|)$ as $x \rightarrow x_0$, see Equation (6.2). The idea behind Taylor's theorem is that the "quality" of the approximation can be increased by considering higher-order polynomials instead of affine approximations.

In this section, it will be convenient to use the following abuse of notation: given $a, b \in \mathbb{R}$, irrespective of the order between a and b , $[a, b]$ denotes the interval between them. In other words, for all $a, b \in \mathbb{R}$, $[a, b]$ and $[b, a]$ denote the same interval.

We also recall that, if $a < b$, then

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx,$$

see Theorem 7.18. If instead $b < a$, then a minus sign appears (recall Equation (8.3)) and we get

$$\left| \int_a^b f(x) dx \right| = \left| \int_b^a f(x) dx \right| \leq \int_b^a |f(x)| dx = - \int_a^b |f(x)| dx = \left| \int_a^b |f(x)| dx \right|.$$

In conclusion, independently of the order of a and b , we always have

$$\left| \int_a^b f(x) dx \right| \leq \left| \int_a^b |f(x)| dx \right|.$$

Let $I \subseteq \mathbb{R}$ be an open interval, and $f : I \rightarrow \mathbb{R}$ be an n -times differentiable function. The n -th **Taylor approximation** of f around a point $x_0 \in I$ is the polynomial function

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k. \quad (8.8)$$

Note that, with this choice of the coefficients, $P^{(k)}(x_0) = f^{(k)}(x_0)$ for $k \in \{0, \dots, n\}$.

We will state and prove different versions of Taylor's Theorem. We begin with this first version:

Theorem 8.15: Taylor Expansion to Order n with Integral Remainder

Let $n \geq 1$, $f : [a, b] \rightarrow \mathbb{R}$ be an n -times continuously differentiable function, and fix $x_0 \in [a, b]$. Then, for all $x \in [a, b]$,

$$f(x) = P_{n-1}(x) + \int_{x_0}^x f^{(n)}(t) \frac{(x-t)^{n-1}}{(n-1)!} dt, \quad (8.9)$$

where P_{n-1} is the $(n-1)$ -th Taylor approximation of f defined in Equation (8.8).

Remark 8.16. In the above theorem, the assumption that f is an n -times continuously differentiable function guarantees that the integral of the continuous function $t \mapsto f^{(n)}(t) \frac{(x-t)^{n-1}}{(n-1)!}$ exists.

Proof. The proof follows by induction on n and integration by parts.

If $n = 1$ then $f : [a, b] \rightarrow \mathbb{R}$ is continuously differentiable and, by Corollary 8.4, we get

$$f(x) = f(x_0) + \int_{x_0}^x f'(t) dt = P_0(x) + \int_{x_0}^x f^{(1)}(t) \frac{(x-t)^0}{0!} dt,$$

as desired.

To explain the idea behind the inductive step, assume first $n = 2$ (so f is twice continuously differentiable). Then, in the integral above, we can apply integration by parts to the functions $f'(t)$ and $g(t) = t - x$. Indeed, since $g' = 1$ and $g(x) = 0$, we get

$$\begin{aligned} f(x) &= f(x_0) + \int_{x_0}^x f'(t)g'(t) dt \\ &= f(x_0) + [f'(t)g(t)]_{x_0}^x - \int_{x_0}^x f''(t)g(t) dt \\ &= f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^x f''(t)(x - t) dt \\ &= P_1(x) + \int_{x_0}^x f^{(2)}(t) \frac{(x-t)^1}{1!} dt. \end{aligned}$$

This proves the case $n = 2$.

More generally, assume that the statement of the theorem is true for some $n \geq 1$ and that $f : [a, b] \rightarrow \mathbb{R}$ is an $(n+1)$ -times continuously differentiable function. Then, by induction hypothesis,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \int_{x_0}^x f^{(n)}(t) \frac{(x-t)^{n-1}}{(n-1)!} dt$$

for all $x \in [a, b]$. If we set $g(t) = -\frac{(x-t)^n}{n!}$, then $g'(t) = \frac{(x-t)^{n-1}}{(n-1)!}$ and it follows from integration by

parts that

$$\begin{aligned}
 f(x) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \left[f^{(n)}(t)g(t) \right]_{x_0}^x - \int_{x_0}^x f^{(n+1)}(t)g(t) dt \\
 &= \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + f^{(n)}(x_0) \frac{(x - x_0)^n}{n!} + \int_{x_0}^x f^{(n+1)}(t) \frac{(x - t)^n}{n!} dt \\
 &= \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \int_{x_0}^x f^{(n+1)}(t) \frac{(x - t)^n}{n!} dt.
 \end{aligned}$$

This proves the induction step, and hence the result. \square

Next, we prove the following alternative version of Taylor's Theorem. Here, we do not require $f^{(n)}$ to be continuous, but only to exist. Note that in the case $n = 1$, this result corresponds to the Mean Value Theorem 6.16.

Theorem 8.17: Taylor Expansion to Order n with Lagrange Remainder

Let $n \geq 1$, $f : [a, b] \rightarrow \mathbb{R}$ be n -times differentiable function, and fix $x_0 \in [a, b]$. Then, for all $x \in [a, b]$ with $x \neq x_0$ there exists $\xi_L \in (x_0, x)$ such that

$$f(x) = P_{n-1}(x) + \frac{f^{(n)}(\xi_L)}{n!} (x - x_0)^n. \quad (8.10)$$

Remark 8.18. If $x = x_0$, the formula holds trivially: one may take $\xi_L = x_0$, since

$$f(x_0) = P_{n-1}(x_0) \quad \text{and} \quad \frac{f^{(n)}(\xi_L)}{n!} (x - x_0)^n = 0.$$

Proof. Fix $x \in [a, b]$ (in the skript it's the open interval (a, b) , while in the lecture we used the closed interval $[a, b]$). W.l.o.g, assume $x > x_0$ (the case $x < x_0$ is analogous) and consider the function $g : (a, b) \rightarrow \mathbb{R}$ defined as

$$g(t) = f(t) + f^{(1)}(t)(x - t) + \dots + \frac{f^{(n-1)}(t)}{(n-1)!} (x - t)^{n-1} = \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{k!} (x - t)^k. \quad (8.11)$$

Then $g(x) = f(x)$ and $g(x_0) = P_{n-1}(x_0)$. Also, its derivative is given by

$$\begin{aligned}
 g'(t) &= \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!} (x - t)^k - \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{k!} k(x - t)^{k-1} \\
 &= \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!} (x - t)^k - \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{(k-1)!} (x - t)^{k-1} \\
 &= \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!} (x - t)^k - \sum_{k=0}^{n-2} \frac{f^{(k+1)}(t)}{k!} (x - t)^k = \frac{f^{(n)}(t)}{(n-1)!} (x - t)^{n-1}.
 \end{aligned}$$

Hence, applying the Cauchy Mean Value Theorem 6.18 in the interval $[x_0, x]$ to the functions $g(t)$ and $h(t) = -(x - t)^n$, we deduce the existence of a point $\xi_L \in (x_0, x)$ such that

$$\frac{f(x) - P_{n-1}(x)}{(x - x_0)^n} = \frac{g(x) - g(x_0)}{h(x) - h(x_0)} = \frac{g'(\xi_L)}{h'(\xi_L)} = \frac{\frac{f^{(n)}(\xi_L)}{(n-1)!} (x - \xi_L)^{n-1}}{n(x - \xi_L)^{n-1}} = \frac{f^{(n)}(\xi_L)}{n!}.$$

This implies Equation (8.10) and conclude the proof. \square

We can now state out two versions of Taylor's approximation, using the little-o and the big-O notation.

Corollary 8.19: Taylor Approximation with Little-o

Let $n \geq 1$, $f : [a, b] \rightarrow \mathbb{R}$ be an n -times continuously differentiable function, and fix $x_0 \in [a, b]$. Then, for all $x \in [a, b]$,

$$f(x) = P_n(x) + o(|x - x_0|^n) \quad \text{as } x \rightarrow x_0. \quad (8.12)$$

Proof. Thanks to Theorem 8.15, we have that

$$\begin{aligned} f(x) &= P_{n-1}(x) + \int_{x_0}^x f^{(n)}(t) \frac{(x-t)^{n-1}}{(n-1)!} dt \\ &= P_{n-1}(x) + \int_{x_0}^x f^{(n)}(x_0) \frac{(x-t)^{n-1}}{(n-1)!} dt + \int_{x_0}^x (f^{(n)}(t) - f^{(n)}(x_0)) \frac{(x-t)^{n-1}}{(n-1)!} dt. \end{aligned}$$

Also, using the change of variable $s = x - t$, we note that

$$\int_{x_0}^x \frac{(x-t)^{n-1}}{(n-1)!} dt = \frac{1}{(n-1)!} \int_0^{x-x_0} s^{n-1} ds = \frac{1}{(n-1)!} \frac{(x-x_0)^n}{n} = \frac{(x-x_0)^n}{n!}, \quad (8.13)$$

therefore

$$\int_{x_0}^x f^{(n)}(x_0) \frac{(x-t)^{n-1}}{(n-1)!} dt = f^{(n)}(x_0) \frac{(x-x_0)^n}{n!}.$$

Hence, we can write

$$\begin{aligned} f(x) &= P_{n-1}(x) + f^{(n)}(x_0) \frac{(x-x_0)^n}{n!} + \int_{x_0}^x (f^{(n)}(t) - f^{(n)}(x_0)) \frac{(x-t)^{n-1}}{(n-1)!} dt \\ &= P_n(x) + \int_{x_0}^x (f^{(n)}(t) - f^{(n)}(x_0)) \frac{(x-t)^{n-1}}{(n-1)!} dt. \end{aligned} \quad (8.14)$$

Now, given $\varepsilon > 0$, it follows from the continuity of $f^{(n)}$ at x_0 that there exists $\delta > 0$ such that $|f^{(n)}(x) - f^{(n)}(x_0)| < \varepsilon$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$. Therefore, if $x \in (x_0 - \delta, x_0 + \delta) \cap [a, b]$, we can bound the integrand in the last integral by

$$\begin{aligned} \left| (f^{(n)}(t) - f^{(n)}(x_0)) \frac{(x-t)^{n-1}}{(n-1)!} \right| &\leq |f^{(n)}(t) - f^{(n)}(x_0)| \frac{|x-t|^{n-1}}{(n-1)!} \\ &< \varepsilon \frac{|x-t|^{n-1}}{(n-1)!} \quad \forall t \in [x_0, x]. \end{aligned}$$

Hence, using Equation (8.13) again, we get

$$\begin{aligned} |f(x) - P_n(x)| &\leq \left| \int_{x_0}^x (f^{(n)}(t) - f^{(n)}(x_0)) \frac{(x-t)^{n-1}}{(n-1)!} dt \right| \\ &< \varepsilon \left| \int_{x_0}^x \frac{|x-t|^{n-1}}{(n-1)!} dt \right| = \varepsilon \frac{|x-x_0|^n}{n!} \leq \varepsilon |x-x_0|^n, \end{aligned}$$

which shows that $f(x) - P_n(x) = o(|x - x_0|^n)$ as $x \rightarrow x_0$. \square

Corollary 8.20: Taylor Approximation with Big-O

Let $n \geq 1$, $f : [a, b] \rightarrow \mathbb{R}$ be an n -times differentiable function, and fix $x_0 \in [a, b]$. Assume that there exists $M > 0$ such that $|f^{(n)}(x)| \leq M$ for all $x \in [a, b]$. Then

$$f(x) = P_{n-1}(x) + O(|x - x_0|^n) \quad \text{as } x \rightarrow x_0. \quad (8.15)$$

Proof. Given $x \in [a, b]$, we apply Equation (8.10) to find a point $\xi_L \in [x_0, x]$ such that

$$f(x) - P_{n-1}(x) = \frac{f^{(n)}(\xi_L)}{n!}(x - x_0)^n.$$

Since $|f^{(n)}(\xi_L)| \leq M$, this implies that

$$|f(x) - P_{n-1}(x)| \leq \frac{M}{n!}|x - x_0|^n \leq M|x - x_0|^n,$$

therefore $f(x) - P_{n-1}(x) = O(|x - x_0|^n)$ as $x \rightarrow x_0$, as desired. \square

Example

As mentioned earlier, if f is differentiable at x_0 , then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(|x - x_0|) \quad \text{as } x \rightarrow x_0.$$

Taylor's Theorem allows us to obtain more accurate approximations when f has higher regularity.

1. If f is twice differentiable and f'' is bounded in a neighborhood of x_0 , then Corollary 8.20 (with $n = 2$) gives

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + O(|x - x_0|^2) \quad \text{as } x \rightarrow x_0.$$

If in addition f'' is continuous (equivalently, $f \in C^2$), then Corollary 8.19 yields

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + o(|x - x_0|^2) \quad \text{as } x \rightarrow x_0.$$

2. If f is smooth, then Corollary 8.19 yields

$$f(x) = P_n(x) + o(|x - x_0|^n) \quad \text{as } x \rightarrow x_0,$$

while Corollary 8.20 applied with $n + 1$ in place of n gives

$$f(x) = P_n(x) + O(|x - x_0|^{n+1}) \quad \text{as } x \rightarrow x_0.$$

Hence, in the case when f is smooth, the bound on $f - P_n$ provided by Corollary 8.20 is often more convenient.

Example

We illustrate how Taylor expansions can be used to compute limits of more complicated functions

To compute the limit $\lim_{x \rightarrow 0} \frac{\sin(x) - x}{x^3}$, we consider the Taylor polynomial of \sin at 0 of degree 3. Since

$$\sin(0) = 0, \quad \sin'(0) = 1, \quad \sin''(0) = 0, \quad \sin^{(3)}(0) = -1,$$

the Taylor polynomial of degree 3 at 0 is

$$P_3(x) = 0 + 1 \cdot x + \frac{0}{2} \cdot x^2 + \frac{-1}{6} \cdot x^3 = x - \frac{x^3}{6}.$$

Taylor's theorem gives

$$\sin(x) = P_3(x) + O(x^4) = x - \frac{x^3}{6} + O(x^4) \quad \text{as } x \rightarrow 0.$$

We stop the expansion at order 3 because the limit involves x^3 in the denominator; any term of order ≥ 4 will disappear in the limit. Then

$$\sin(x) - x = \left(x - \frac{x^3}{6} + O(x^4)\right) - x = -\frac{x^3}{6} + O(x^4) \quad \text{as } x \rightarrow 0,$$

and thus

$$\frac{\sin(x) - x}{x^3} = \frac{-\frac{x^3}{6} + O(x^4)}{x^3} = -\frac{1}{6} + O(x^4) \quad \text{as } x \rightarrow 0.$$

Hence,

$$\lim_{x \rightarrow 0} \frac{\sin(x) - x}{x^3} = -\frac{1}{6}.$$

Remark 8.21. *In the previous example we showed that*

$$\sin(x) = x - \frac{x^3}{6} + O(x^4) \quad \text{as } x \rightarrow 0.$$

These coefficients agree with those appearing in the power series expansion of \sin . In Section 8.4.1 below, we will see that \sin is an analytic function, meaning that its Taylor series around 0 actually converges to $\sin(x)$. For such functions, the Taylor polynomial of degree n is simply the truncation of an infinite series that represents the function itself.

Example

We can use the Taylor approximation to refine the discussion in Section 6.2.1. Let $f : (a, b) \rightarrow \mathbb{R}$ be an n -times continuously differentiable function. Suppose $x_0 \in (a, b)$ satisfies

$$f'(x_0) = \dots = f^{(n-1)}(x_0) = 0.$$

Then the following implications hold:

- if $f^{(n)}(x_0) < 0$ and n is even, then f has an isolated local maximum at x_0 .
- if $f^{(n)}(x_0) > 0$ and n is even, then f has an isolated local minimum at x_0 .
- if $f^{(n)}(x_0) \neq 0$ and n is odd, then x_0 is not a local extremum of f .

All three statements follow from Equation (8.10), which, in this case, takes the form

$$f(x) = f(x_0) + \frac{f^{(n)}(\xi_L)}{n!}(x - x_0)^n, \quad \xi_L \in (x_0, x).$$

Indeed, if $f^{(n)}(x_0) > 0$, by continuity there exists $\delta > 0$ such that $f^{(n)}(\xi_L) > 0$ for $\xi_L \in (x_0, x) \subseteq (x_0 - \delta, x_0 + \delta)$. If n is even, then $(x - x_0)^n > 0$ for $x \neq x_0$ and we deduce that $f(x) > f(x_0)$ for $x \in (x_0 - \delta, x_0 + \delta)$ with $x \neq x_0$. When n is odd, then $(x - x_0)^n$ changes sign when considering $x > x_0$ and $x < x_0$, so x_0 is not a local extremum of f .

On the other hand, if $f^{(n)}(x_0) < 0$ and n is even, the same argument as above shows that $f(x) < f(x_0)$ for $x \in (x_0 - \delta, x_0 + \delta)$ with $x \neq x_0$, while the case n odd yields that x_0 is not a local extremum of f .

8.4.1 Analytic Functions

Motivated by Taylor's theorem, one might expect that if f is smooth and we replace the finite Taylor polynomial P_n by the full **Taylor series**

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k,$$

then this infinite series should converge to $f(x)$. Unfortunately, this is not true in general. Only rather special smooth functions enjoy this property.

Note that the Taylor series is centered at x_0 instead of 0 (i.e., x^n is replaced with $(x - x_0)^n$). Hence, all theorems about power series from Section 5.4 still hold, but taking into account that now x_0 plays the role of the center. In particular, if the series has radius of convergence $R > 0$, then it converges for all $x \in (x_0 - R, x_0 + R)$, while it diverges for $|x - x_0| > R$.

Definition 8.22: Analytic Functions

Let $I \subseteq \mathbb{R}$ be an interval and $x_0 \in I$. A smooth function $f : I \rightarrow \mathbb{R}$ is called **analytic** at x_0 if the Taylor series of f around x_0 has radius of convergence $R > 0$ and there exists $\delta \in (0, R)$ such that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \quad \forall x \in (x_0 - \delta, x_0 + \delta) \cap I.$$

We say f is analytic in I if f is analytic at all points in I .

In other words, analytic functions $f : I \rightarrow \mathbb{R}$ are characterized by the fact that, for every point $x_0 \in I$, there exists a power series that converges to f in a neighborhood of x_0 .

As the next example shows, there are smooth functions f whose Taylor series converges to a function different from f .

Example

Consider the function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\psi : x \in \mathbb{R} \mapsto \begin{cases} \exp\left(-\frac{1}{x}\right) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0 \end{cases}.$$

We note that ψ is smooth on \mathbb{R} and satisfies $\psi^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Hence, the Taylor series of the function ψ at the point $x_0 = 0$ is the zero series, i.e.,

$$\sum_{n=0}^{\infty} \frac{\psi^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0.$$

The series has an infinite radius of convergence and converges to the function 0. Since $\psi(x) > 0$ holds for all $x > 0$, the Taylor series does not converge to ψ in a neighborhood of $x_0 = 0$, and so ψ is not analytic at the point $x_0 = 0$.

The next result provides a criterion that guarantees that the Taylor series of f converges to f in a neighborhood of x_0 .

Theorem 8.23: A Criterion for Analyticity at x_0

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$ a smooth function. Given $x_0 \in I$, assume that there exist constants $r, C_0, A > 0$ such that

$$|f^{(n)}(x)| \leq C_0 A^n n! \quad \forall x \in (x_0 - r, x_0 + r) \cap I, \quad \forall n \in \mathbb{N}.$$

Then f is analytic at x_0 .

Proof. Let $I \subseteq \mathbb{R}$ and fix $x_0 \in I$. We first estimate the radius of convergence R of the Taylor series. If we define $a_n = \frac{f^{(n)}(x_0)}{n!}$, then the Taylor series is equal to $\sum_{n=0}^{\infty} a_n (x - x_0)^n$. Thus, thanks to our assumption on the size of $|f^{(n)}|$, it follows that

$$|a_n| \leq \frac{C_0 A^n n!}{n!} = C_0 A^n.$$

This implies that

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} \leq \limsup_{n \rightarrow \infty} \sqrt[n]{C_0 A^n} = \limsup_{n \rightarrow \infty} \underbrace{\sqrt[n]{C_0}}_{=1} A = A,$$

therefore, by the definition of radius of convergence (see Definition 5.24), $R \geq \frac{1}{A}$.

Now, we want to show that there exists a $\delta \in (0, R)$ such that every Taylor series of $f(x)$ at x_0 converges to $f(x)$ for every $x \in (x_0 - \delta, x_0 + \delta) \cap I$. So, fix $\delta < \min\{r, \frac{1}{A}\}$. Given $x \in (x_0 - \delta, x_0 + \delta) \cap I$, we apply Equation (8.10) to find $\xi_L \in (x_0, x)$, and our assumption on the size of $|f^{(n)}|$ to deduce that

$$|f(x) - P_{n-1}(x)| \leq \frac{|f^{(n)}(\xi_L)|}{n!} |x - x_0|^n \leq C_0 A^n |x - x_0|^n \leq C_0 (A\delta)^n.$$

Since $A\delta < 1$ (because $\delta < \frac{1}{A}$), letting $n \rightarrow \infty$ we conclude that

$$f(x) = \lim_{n \rightarrow \infty} P_{n-1}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \quad \forall x \in (x_0 - \delta, x_0 + \delta) \cap I,$$

as desired. □

As a direct consequence of Theorem 8.23, we immediately deduce the following:

Corollary 8.24: A Criterion for Analyticity

Let $f : [a, b] \rightarrow \mathbb{R}$ be a smooth function, and assume there exists constants $C_0, A > 0$ such that

$$|f^{(n)}(x)| \leq C_0 A^n n! \quad \forall x \in [a, b], \quad \forall n \in \mathbb{N}.$$

Then f is analytic on $[a, b]$.

9 Ordinary Differential Equations

9.1 Ordinary Differential Equations (ODEs)

In this section we study *ordinary differential equations* (ODEs). They describe how an unknown quantity depends on one real variable (often denoted by x or t) and how this dependence is constrained by relations involving derivatives.

It is convenient to fix some notation from the beginning:

- we usually denote the *unknown function* by u ;
- the independent variable is denoted by x (or by t when it represents time);
- letters like f, g, a_0, a_1 will typically denote given functions or constants appearing in the equation (data of the problem).

Although derivatives are usually denoted using $'$ (so u', u'' , etc.), it is common to use a dot to denote derivatives with respect to time (so \dot{u}, \ddot{u} , etc.).

Definition 9.1: ODEs

An **ordinary differential equation (ODE)** is a relation involving a function $u : \mathbb{R} \rightarrow \mathbb{R}$ of a real variable $x \in \mathbb{R}$ and its derivatives. The general form of an n -th order ODE is

$$G(x, u(x), u'(x), u''(x), \dots, u^{(n)}(x)) = 0, \quad (9.1)$$

where $G : \mathbb{R}^{n+2} \rightarrow \mathbb{R}$ is a given function.

In many examples the independent variable is time t and the equation describes the evolution of a system, but we keep the generic notation x unless we want to stress the time interpretation.

ODEs can be classified according to several criteria:

1. *Order*: An ODE is of order n if $u^{(n)}$ is the highest derivative appearing in the equation. For instance:
 - (a) $u'' + u = 0 \rightsquigarrow$ second order.
 - (b) $u^{(3)} = x^2u + x \rightsquigarrow$ third order.
 - (c) $(u')^2 + u - x^3 = 0 \rightsquigarrow$ first order.
2. *Linearity*: An ODE is *linear* if it is linear in u and its derivatives. Otherwise, it is *nonlinear*. Here "linear" means that u, u', u'', \dots appears only to the first power and are not multiplied with each other.
 - (a) $u'' + u = 0 \rightsquigarrow$ linear.
 - (b) $u'' + u^2 = 0 \rightsquigarrow$ nonlinear (because of u^2).
 - (c) $u'' + u'u = 0 \rightsquigarrow$ nonlinear (because of the product $u'u$).
 - (d) $u^{(3)} = x^2u + x \rightsquigarrow$ linear.
 - (e) $(u')^2 + u - x^3 = 0 \rightsquigarrow$ nonlinear (because of $(u')^2$).
3. *Homogeneity (for linear ODEs)*: For a linear ODE, we say it is *homogeneous* if all terms involve the function or its derivatives. Equivalently, if u is a solution then Au is a solution for all $A \in \mathbb{R}$. If there is an additional term that does not depend on u (a "forcing term"), the equation is *non-homogeneous*.

- (a) $u'' + u = 0 \rightsquigarrow$ homogeneous.
- (b) $u^{(3)} = x^2u + x \rightsquigarrow$ non-homogeneous (because of the term $+x$).
- (c) $u^{(3)} = x^2u \rightsquigarrow$ homogeneous.

So far, we have only considered single equations, but one can also study *systems* of ODEs with several unknown functions u_1, \dots, u_n . We will not go into this now, but many ideas are similar.

In addition, solutions are often required to satisfy extra conditions such as $u(0) = 0$ (prescribed position at time 0) and/or $u'(0) = 1$ (prescribed velocity at time 0). When these conditions are imposed at a single time (typically $t = 0$), they are called **initial conditions**. More general conditions (for instance at two different points, such as $u(0) = 0$ and $u(1) = 1$) are called **boundary conditions**.

Later we shall see that, under suitable assumptions on the data (for example on a function f appearing in the equation), prescribing initial conditions often leads to a unique solution. This is the content of the Cauchy-Lipschitz (or Picard-Lindelöf) theorem.

9.1.1 Linear First Order ODEs

We now consider linear first-order ODEs. Throughout this subsection we fix a non-empty interval $I \subseteq \mathbb{R}$ that is not a single point, and we study equations of the form

$$u'(x) + f(x)u(x) = g(x),$$

where f and g are given continuous functions on I , and u is the unknown.

We start with the homogeneous case $g \equiv 0$.

Theorem 9.2: Homogeneous Linear 1st Order ODEs

Let $f : I \rightarrow \mathbb{R}$ be continuous and consider the homogeneous first-order linear ODE

$$u'(x) + f(x)u(x) = 0 \quad \forall x \in I. \quad (9.2)$$

Let $F : I \rightarrow \mathbb{R}$ be a primitive of f . Then all C^1 solutions $u : I \rightarrow \mathbb{R}$ of Equation (9.2) are of the form

$$u(x) = Ae^{-F(x)}, \quad A \in \mathbb{R}.$$

In other words, the set of solutions of Equation (9.2) form a one-dimensional linear subspace of $C^1(I)$.

Proof. Given $A \in \mathbb{R}$, define $u(x) = Ae^{-F(x)}$. Then

$$u'(x) = -F'(x)Ae^{-F(x)} = -f(x)Ae^{-F(x)} = -f(x)u(x) \quad \forall x \in I,$$

so u solves the ODE.

Conversely, let $u \in C^1(I)$ solve Equation (9.2) and set $v(x) = e^{F(x)}u(x)$. Then

$$v'(x) = (e^{F(x)})'u(x) = e^{F(x)}u'(x) = e^{F(x)}(f(x)u(x) - f(x)u(x)) = 0 \quad \forall x \in I.$$

By Corollary 6.24, we deduce that $v(x) = A$ for some $A \in \mathbb{R}$, hence $u(x) = Ae^{-F(x)}$. □

Remark 9.3. In the previous result, solutions are written using a primitive F of f . Since primitives are defined up to an additive constant, we can replace F by $F + C$ for any $C \in \mathbb{R}$. This amounts

to replacing $Ae^{-F(x)}$ by $Ae^{-C}e^{-F(x)}$, and since $A \in \mathbb{R}$ is arbitrary, this does not change the set of solutions.

Next, we consider the non-homogeneous linear first-order ODE

$$u'(x) + f(x)u(x) = g(x) \quad \forall x \in I, \quad (9.3)$$

where $f, g : I \rightarrow \mathbb{R}$ are given continuous functions.

To motivate the solution formula, we use the method of **variation of constants**. For the homogeneous equation we know that every solution has the form $Ae^{-F(x)}$ with $A \in \mathbb{R}$. The idea is to *replace this constant by a function* and look for a solution of the non-homogeneous equation of the form

$$u(x) = H(x)e^{-F(x)},$$

for some function $H \in C^1(I)$. With this choice,

$$u'(x) = H'(x)e^{-F(x)} - F'(x)H(x)e^{-F(x)} = H'(x)e^{-F(x)} - f(x)u(x).$$

Thus, u solves Equation (9.3) if and only if

$$H'(x)e^{-F(x)} = g(x),$$

which means that H must be a primitive of the function $g(x)e^{F(x)}$.

This motivates the general solution formula:

Theorem 9.4: Non-Homogeneous Linear 1st Order ODEs

Let $f, g : I \rightarrow \mathbb{R}$ be continuous, and consider the non-homogeneous first-order linear ODE (9.3). Let $F : I \rightarrow \mathbb{R}$ be a primitive of f , and let $H : I \rightarrow \mathbb{R}$ be a primitive of ge^F . Then every C^1 solution $u : I \rightarrow \mathbb{R}$ of Equation (9.3) is of the form

$$u(x) = H(x)e^{-F(x)} + Ae^{-F(x)}, \quad A \in \mathbb{R}.$$

In particular, the set of solutions of Equation (9.3) is a one-dimensional affine subspace of $C^1(I)$.

Proof. If H is a primitive of ge^F , then $H + A$ is also a primitive for any constant A . Hence, by the same computation as the one performed above, it follows that

$$u(x) = (H(x) + A)e^{-F(x)}$$

solves Equation (9.3). Indeed

$$\begin{aligned} u'(x) &= (H(x) + A)'e^{-F(x)} - F'(x)(H(x) + A)e^{-F(x)} \\ &= H'(x)e^{-F(x)} - f(x)u(x) = g(x) - f(x)u(x). \end{aligned}$$

Conversely, let u be any solution of Equation (9.3), and set $v(x) = u(x) - H(x)e^{-F(x)}$. Then,

$$\begin{aligned} v'(x) &= u'(x) - H'(x)e^{-F(x)} + F'(x)H(x)e^{-F(x)} \\ &= -f(x)u(x) + g(x) - g(x)e^{F(x)}e^{-F(x)} + f(x)H(x)e^{-F(x)} \\ &= -f(x)u(x) + g(x) - g(x) + f(x)(u(x) - v(x)) = -f(x)v(x). \end{aligned}$$

Thus v solves the homogeneous Equation (9.2). By Theorem 9.2, we have $v(x) = Ae^{-F(x)}$ for some constant A . Therefore,

$$u(x) = v(x) + H(x)e^{-F(x)} = Ae^{-F(x)} + H(x)e^{-F(x)},$$

which proves the result. \square

The previous results give explicit formulas to solve every linear first-order ODE of the form $u' + fu = g$. In concrete situations, the difficulty lies in computing a primitive F of f and the a primitive of $g(x)e^{F(x)}$.

As we have seen, solutions depend on a free parameter $A \in \mathbb{R}$. This allows us to impose an initial condition of the form $u(x_0) = u_0$, which uniquely determines A .

Example

We solve the ODE

$$u'(x) - 2xu(x) = e^{x^2}, \quad u(0) = 1, \quad (9.4)$$

on \mathbb{R} . Here u is the unknown, and $f(x) = -2x$, $g(x) = e^{x^2}$ are given. According to Theorem 9.4, we first find a primitive of f , i.e.,

$$F(x) = -x^2.$$

Then we consider

$$g(x)e^{F(x)} = e^{x^2}e^{-x^2} = 1,$$

whose primitive is $H(x) = x$. Thus u must be of the form

$$u(x) = (x + A)e^{x^2}.$$

Imposing the condition $u(0) = 1$ gives $A = 1$, hence

$$u(x) = (x + 1)e^{x^2}. \quad (9.5)$$

Remark 9.5. *If one forgets the formula form Theorem 9.4, it is enough to remember the following procedure for solving (9.3). We start from*

$$u'(x) + f(x)u(x) = g(x),$$

and multiply both sides by a function $e^{w(x)}$, i.e.,

$$u'(x)e^{w(x)} + f(x)u(x)e^{w(x)} = g(x)e^{w(x)}.$$

We look for w such that the left-hand side is the derivative of $u(x)e^{w(x)}$, i.e.,

$$(u(x)e^{w(x)})' = u'(x)e^{w(x)} + w(x)u(x)e^{w(x)}.$$

So we require

$$w'(x) = f(x).$$

If we choose $w = F$ to be any primitive of f (the additive constant does not matter), then

$$(u(x)e^{F(x)})' = g(x)e^{F(x)},$$

and therefore

$$u(x)e^{F(x)} = \int g e^F + A,$$

for some $A \in \mathbb{R}$. Thus, if H is a primitive of ge^F , this reads

$$u(x)e^{F(x)} = H(x) + A \quad \Rightarrow \quad u(x) = H(x)e^{-F(x)} + Ae^{-F(x)}.$$

9.1.2 Autonomous First Order ODEs

We next study *autonomous* first-order ODEs, where the rate of change of the unknown depends only on its current value, and not explicitly on x , i.e.,

$$u'(x) = f(u(x)), \tag{9.6}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a given continuous function, and u is the unknown. The function f tells us how u should change depending on its current value.

A standard way to solve such equations is the method of **separation of variables**. If $u(x) = C \in \mathbb{R}$ for some C such that $f(C) = 0$, then the constant function $u = C$ is a solution of the ODE. Otherwise, if $f(u(x)) \neq 0$, we can divide both sides by $f(u(x))$ and get

$$\frac{u'(x)}{f(u(x))} = 1.$$

Integrating and using the substitution formula, we obtain

$$\int \frac{1}{f(u)} du = \int \frac{1}{f(u(x))} u'(x) dx = \int 1 dx = x + A, \tag{9.7}$$

where A is a constant of integration.

If H is a primitive of $1/f$, this reads

$$H(u(x)) = x + A \quad \Rightarrow \quad u(x) = H^{-1}(x + A).$$

Since we assumed $f(u(x)) \neq 0$ on the interval under consideration, we have $H' = \frac{1}{f} \neq 0$ there, so H is strictly monotone and therefore invertible on that interval.

The method of separation of variables applies also to equations of the form

$$u'(x) = f(u(x))g(x),$$

where $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are continuous. Assuming $f(u(x)) \neq 0$ on a suitable interval, we divide and get

$$\frac{u'(x)}{f(u(x))} = g(x).$$

Integrating gives

$$\int \frac{1}{f(u)} du = \int g(x) dx + A.$$

If H is a primitive of $1/f$ and G is a primitive of g , then

$$H(u(x)) = G(x) + A \quad \Rightarrow \quad u(x) = H^{-1}(G(x) + A).$$

We summarize this discussion in the next theorem:

Theorem 9.6: Separable Equations

Let $I, J \subseteq \mathbb{R}$ be intervals, let $f : J \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ be continuous, and assume $f(y) \neq 0$ for all $y \in J$. Let H be a primitive of $1/f$ on J , and let G be a primitive of g on I . Then every C^1 solution $u : I \rightarrow J$ of

$$u'(x) = f(u(x))g(x) \tag{9.8}$$

is of the form

$$u(x) = H^{-1}(G(x) + A), \quad A \in \mathbb{R}.$$

Proof. Since f never vanishes, $H' = 1/f$ has constant sign. Hence, H is strictly monotone and therefore invertible.

If $u(x) = H^{-1}(G(x) + A)$, then by the chain rule (Theorem 6.8) and the formula for the derivative of the inverse (Theorem 6.10), we get

$$u'(x) = (H^{-1})'(G(x) + A)G'(x) = \frac{1}{H' \circ H^{-1}(G(x) + A)}g(x) = \frac{1}{H'(u(x))}g(x) = f(u(x))g(x),$$

so u solves Equation (9.8).

Conversely, suppose u solves Equation (9.8). Then

$$(H \circ u(x))' = H'(u(x))u'(x) = \frac{1}{f(u(x))}f(u(x))g(x) = g(x) = G'(x).$$

Thus, $H(u(x)) - G(x)$ has derivative zero, and is therefore constant, i.e.,

$$H(u(x)) = G(x) + A, \quad A \in \mathbb{R}.$$

Applying H^{-1} gives

$$u(x) = H^{-1}(G(x) + A),$$

which completes the proof. □

9.1.3 Homogeneous Linear Second Order ODEs with Constant Coefficients

We now move to second-order linear ODEs. These are considerably more difficult to solve than first-order ones in general, so we start with the simplest case: homogeneous equations with constant coefficients,

$$u''(x) + a_1 u'(x) + a_0 u(x) = 0, \tag{9.9}$$

where $a_0, a_1 \in \mathbb{R}$ are given constants and u is the unknown. Such equations already cover many important applications (for instance, oscillations and damped vibrations).

We choose the *Ansatz* that the solutions of Equation (9.9) is of the form

$$u(x) = e^{\alpha x}, \quad \alpha \in \mathbb{C}.$$

With this choice,

$$u''(x) + a_1 u'(x) + a_0 u(x) = (\alpha^2 + a_1 \alpha + a_0)u(x) = 0,$$

so u is a solution if and only if

$$\alpha^2 + a_1 \alpha + a_0 = 0.$$

The quadratic polynomial

$$p(t) = t^2 + a_1 t + a_0$$

is called the **characteristic polynomial**. Its roots determine the shape of the solutions. We distinguish three cases according to the discriminant $\Delta = a_1^2 - 4a_0$.

- Case 1: $\Delta > 0$. The polynomial $p(t)$ has two distinct roots

$$\alpha = \frac{-a_1 + \sqrt{\Delta}}{2}, \quad \beta = \frac{-a_1 - \sqrt{\Delta}}{2}. \quad (9.10)$$

Then $x \mapsto e^{\alpha x}$ and $x \mapsto e^{\beta x}$ are two linearly independent solutions, and therefore

$$u(x) = Ae^{\alpha x} + Be^{\beta x}, \quad A, B \in \mathbb{R},$$

is a solutions of Equation (9.9)

- Case 2: $\Delta < 0$. Then $p(t)$ has two complex-conjugate roots

$$\alpha + i\beta = -\frac{a_1}{2} + i\frac{\sqrt{-\Delta}}{2}, \quad \alpha - i\beta = -\frac{a_1}{2} - i\frac{\sqrt{-\Delta}}{2}, \quad (9.11)$$

with $\beta > 0$. The complex-valued functions $x \mapsto e^{(\alpha \pm i\beta)x}$ solve Equation (9.9), and hence their real and imaginary parts are solutions. This gives

$$u(x) = Ae^{\alpha x} \sin(\beta x) + Be^{\alpha x} \cos(\beta x), \quad A, B \in \mathbb{R}.$$

- Case 3: $\Delta = 0$. Then $p(t)$ has a double root

$$\alpha = -\frac{a_1}{2}, \quad (9.12)$$

so $x \mapsto e^{\alpha x}$ is a solution of Equation (9.9). To find another independent solution, recall the special case $u'' = 0$, where two linearly independent solutions are 1 and x , which can be written as $e^{\gamma x}$ and $xe^{\gamma x}$ with $\gamma = 0$. This suggests that $x \mapsto xe^{\alpha x}$ might be a solution. Indeed,

$$(xe^{\alpha x})'' + a_1(xe^{\alpha x})' + a_0xe^{\alpha x} = \underbrace{(\alpha^2 + a_1\alpha + a_0)}_{=0}xe^{\alpha x} + \underbrace{(2\alpha + a_1)}_{=0}e^{\alpha x} = 0,$$

where the first term vanishes because α is a root of p , and the second vanishes by (9.12). Hence

$$u(x) = Ae^{\alpha x} + Bxe^{\alpha x}, \quad A, B \in \mathbb{R},$$

solves Equation (9.9).

For second-order ODEs it is customary to prescribe both the value of u and the value of its derivative at some point (for instance, $u(0) = 1$ and $u'(0) = 0$). The two constants A, B in the formulas above are precisely what we need in order to satisfy two such conditions.

Theorem 9.7: Existence and Uniqueness: The Homogeneous Case

Given $a_0, a_1 \in \mathbb{R}$. let $\Delta = a_1^2 - 4a_0$ and consider the following solutions of Equation (9.9):

$$\underline{\Delta > 0}: \quad u_1(x) = e^{\alpha x}, \quad u_2(x) = e^{\beta x}, \quad \alpha, \beta \text{ as in (9.10)}$$

$$\underline{\Delta < 0}: \quad u_1(x) = e^{\alpha x} \sin(\beta x), \quad u_2(x) = e^{\alpha x} \cos(\beta x), \quad \alpha, \beta \text{ as in (9.11)}$$

$$\underline{\Delta = 0}: \quad u_1(x) = e^{\alpha x}, \quad u_2(x) = x e^{\alpha x}, \quad \alpha, \beta \text{ as in (9.12)}.$$

If $u \in C^2(I)$ solves Equation (9.9), then there exists $A, B \in \mathbb{R}$ such that

$$u = Au_1 + Bu_2.$$

In other words, the set of solutions of Equation (9.9) forms a two-dimensional linear subspace of $C^2(I)$.

We will skip the proof of this theorem. (It's in the skript)

Corollary 9.8: Wronskian and Linear Dependence

Let $v_1, v_2 \in C^2(I)$ be solutions of Equation (9.9) on an interval $I \subseteq \mathbb{R}$, and let

$$W(x) = v_1(x)v_2'(x) - v_1'(x)v_2(x)$$

be their Wronskian. If $W(x_0) = 0$ for some $x_0 \in I$, then v_1 and v_2 are linearly dependent on I .

Proof. Consider the 2×2 matrix

$$M(x_0) = \begin{pmatrix} v_1(x_0) & v_2(x_0) \\ v_1'(x_0) & v_2'(x_0) \end{pmatrix}.$$

The condition $W(x_0) = 0$ means precisely that $\det M(x_0) = 0$, so there exists a non-zero vector $(A, B) \in \mathbb{R}^2$ such that

$$Av_1(x_0) + Bv_2(x_0) = 0, \quad Av_1'(x_0) + Bv_2'(x_0) = 0.$$

Define

$$w(x) = Av_1(x) + Bv_2(x).$$

Then w solves Equation (9.9) and satisfies

$$w(x_0) = 0, \quad w'(x_0) = 0.$$

The only solution with these initial data is the trivial one, so $w = 0$ on I . This proves that $Av_1 + Bv_2 = 0$, thus v_1 and v_2 are linearly dependent on I . \square

9.1.4 Non-Homogeneous Linear Second Order ODEs with Constant Coefficients

We now add a forcing term and consider the non-homogeneous linear second-order ODE with constant coefficients

$$u''(x) + a_1 u'(x) + a_0 u(x) = g(x), \quad (9.13)$$

where $a_0, a_1 \in \mathbb{R}$ are constants and $g \in C^0(I)$ is a given function. The unknown is again u .

With the notation of Paragraph 9.1.3, let u_1, u_2 be two linearly independent solutions of the homogeneous Equation (9.9):

$$\underline{\Delta} > 0: \quad u_1(x) = e^{\alpha x}, \quad u_2(x) = e^{\beta x}, \quad \alpha, \beta \text{ as in (9.10)}$$

$$\underline{\Delta} < 0: \quad u_1(x) = e^{\alpha x} \sin(\beta x), \quad u_2(x) = e^{\alpha x} \cos(\beta x), \quad \alpha, \beta \text{ as in (9.11)}$$

$$\underline{\Delta} = 0: \quad u_1(x) = e^{\alpha x}, \quad u_2(x) = x e^{\alpha x}, \quad \alpha, \beta \text{ as in (9.12)}.$$

We look for a solution u of Equation (9.13) of the form

$$u(x) = H_1(x)u_1(x) + H_2(x)u_2(x),$$

where H_1, H_2 are unknown functions. This is the method of **variation of constants** in the second-order setting. We will spare you the details of the computation and state the theorem directly:

Theorem 9.9: Existence and Uniqueness: The Non-Homogeneous Case

Given $a_0, a_1 \in \mathbb{R}$, let $\Delta = a_1^2 - 4a_0$ and consider the following solutions of Equation (9.9):

$$\underline{\Delta} > 0: \quad u_1(x) = e^{\alpha x}, \quad u_2(x) = e^{\beta x}, \quad \alpha, \beta \text{ as in (9.10)}$$

$$\underline{\Delta} < 0: \quad u_1(x) = e^{\alpha x} \sin(\beta x), \quad u_2(x) = e^{\alpha x} \cos(\beta x), \quad \alpha, \beta \text{ as in (9.11)}$$

$$\underline{\Delta} = 0: \quad u_1(x) = e^{\alpha x}, \quad u_2(x) = x e^{\alpha x}, \quad \alpha, \beta \text{ as in (9.12)}.$$

Let H_1 and H_2 be primitives of $\frac{u_2 g}{u_1' u_2 - u_2' u_1}$ and $\frac{u_1 g}{u_2' u_1 - u_1' u_2}$, respectively. If $u \in C^2(I)$ solves Equation (9.13), then there exists $A, B \in \mathbb{R}$ such that

$$u = Au_1 + Bu_2 + H_1 u_1 + H_2 u_2.$$

In other words, the set of solutions of Equation (9.13) forms a two-dimensional affine subspace of $C^2(I)$.

Proof. First we show existence. By the computation above, that was skipped (but is in the skript), if H_1 and H_2 are primitives of

$$\frac{u_2 g}{u_1' u_2 - u_2' u_1} \quad \text{and} \quad \frac{u_1 g}{u_2' u_1 - u_1' u_2},$$

then the function

$$u_p = H_1 u_1 + H_2 u_2$$

satisfies

$$u_p'' + a_1 u_p' + a_0 u_p = g,$$

so u_p is a particular solution of Equation (9.13). Since the equation is linear, for any $A, B \in \mathbb{R}$ the function

$$u = Au_1 + Bu_2 + u_p$$

also solves Equation (9.13). This proves the existence of solutions of the stated form.

For uniqueness, let $u \in C^2(I)$ be any solution of Equation (9.9), and define

$$v = u - u_p = u - (H_1 u_1 + H_2 u_2).$$

Then v solves the homogeneous Equation (9.9) and therefore, by Theorem 9.7, there exist $A, B \in \mathbb{R}$ such that

$$v = Au_1 + Bu_2.$$

Hence, $u = v + u_p = Au_1 + Bu_2 + H_1 u_1 + H_2 u_2$, as desired. \square

9.2 Existence and Uniqueness for ODEs

9.2.1 Existence and Uniqueness for First Order ODEs

Our goal now is to present the general theory of first-order ODEs for real-valued functions on the real line. In general, a first-order ODE is an equation relating $x, u(x), u'(x)$, i.e.,

$$G(x, u(x), u'(x)) = 0.$$

In this section we restrict to equations for which one can "isolate" u' , so that one can write the ODE in *normal form*, i.e.,

$$u'(x) = f(x, u(x)).$$

Definition 9.10: First-Order ODEs in Normal Form

first-order ODE in **normal form** is an equation of the type

$$u'(x) = f(x, u(x)),$$

where $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a given function and $u : \mathbb{R} \rightarrow \mathbb{R}$ is the unknown.

The Cauchy-Lipschitz Theorem, also known as Picard-Lindelöf Theorem, is a fundamental result in the theory of ODEs. It ensures the existence and uniqueness of solutions under suitable conditions on f .

In the next theorem we need to assume that f is continuous as a function of the two variables x and y . This means that, for any point (x_0, y_0) in the domain and for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \quad \text{and} \quad |y - y_0| < \delta \quad \Rightarrow \quad |f(x, y) - f(x_0, y_0)| < \varepsilon.$$

Theorem 9.11: Cauchy-Lipschitz: Global Version

Let $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following conditions:

1. f is continuous in $\mathbb{R} \times \mathbb{R}$;
2. f is Lipschitz continuous with respect to the second variable, i.e., there exists a constant $L > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \quad \forall x, y_1, y_2 \in \mathbb{R}.$$

Then, for any point $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ there exists a unique C^1 function $u : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\begin{cases} u'(x) = f(x, u(x)) & \forall x \in \mathbb{R}, \\ u(x_0) = y_0. \end{cases} \quad (9.14)$$

Theorem 9.12: Cauchy-Lipschitz: Local Version

Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ satisfy the following conditions:

1. f is continuous in $I \times \mathbb{R}$;
2. f is locally Lipschitz continuous with respect to the second variable, i.e., for every pair of compact intervals $[a, b] \subseteq I$ and $[c, d] \subseteq \mathbb{R}$ there exists a constant such that

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \quad \forall x \in [a, b], \forall y_1, y_2 \in [c, d].$$

Then, for any point $(x_0, y_0) \in I \times \mathbb{R}$ there exists an interval $I' \subseteq I$ containing x_0 and a unique function $u : I' \rightarrow \mathbb{R}$ such that

$$\begin{cases} u'(x) = f(x, u(x)) & \forall x \in I', \\ u(x_0) = y_0. \end{cases} \quad (9.15)$$

In other words, under a local Lipschitz assumption, one can only guarantee the existence and uniqueness of a solution on some interval around x_0 . Moreover, as long as the solution $u(x)$ remains bounded within I' , one can continue applying Theorem 9.12 to extend the interval I' as much as possible.

9.2.2 Higher Order ODEs

Suppose we are given an n -th order ODE of the form

$$G(x, u'(x), u''(x), \dots, u^{(n)}(x)) = 0,$$

and assume that the highest derivative can be isolated and written as

$$u^{(n)}(x) = f(x, u(x), u'(x), \dots, u^{(n-1)}(x)), \quad (9.16)$$

where $f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a given function

We introduce the variables

$$U_1 = u, \quad U_2 = u', \quad U_3 = u'', \dots, U_n = u^{(n-1)}.$$

By definition, we have

$$U'_1 = U_2, \quad U'_2 = U_3, \dots, U'_{n-1} = U_n,$$

and Equation (9.16) becomes

$$U'_n = u^{(n)} = f(x, U_1, U_2, \dots, U_n).$$

Therefore, the n -th order Equation (9.16) is equivalent to the first-order system

$$\begin{cases} U'_1 &= U_2, \\ U'_2 &= U_3, \\ &\vdots \\ U'_{n-1} &= U_n, \\ U'_n &= f(x, U_1, U_2, \dots, U_n). \end{cases}$$

The Cauchy-Lipschitz Theorem (both its global and local versions) extends to systems of first-order ODEs and ensures existence and uniqueness of solutions whenever the right-hand side is continuous and (locally) Lipschitz continuous with respect to the variable (U_1, \dots, U_n) . In particular, once the initial conditions

$$U_1(x_0) = u(x_0), \quad U_2(x_0) = u'(x_0), \dots, U_n(x_0) = u^{(n-1)}(x_0)$$

are prescribed at some $x_0 \in I$, there exists a unique (local) solution to the system, and hence a unique solution to the original n -th order ODE.

Theorem 9.13: Existence and Uniqueness For Linear Second-Order ODEs

Let $a_0, a_1 : I \rightarrow \mathbb{R}$ be continuous and bounded functions. Then the set of solutions of the linear homogeneous equation

$$u''(x) + a_1(x)u'(x) + a_0(x)u(x) = 0 \quad \forall x \in I$$

is a two-dimensional linear subspace of $C^2(I)$.

proof can be found in the skript.