Analysis I
Theorems & Lemmas

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Contents

1 Functions

Definition 1.1: Functions/Maps/Transformations

A **function** f from a set X to a set Y is an assignment of an element of Y to each element of X. The element $y \in Y$ to which $x \in X$ is assigned to is denoted $f(x)$. We write $f : X \to Y$ and sometimes also speak of a **map**, **mapping** or a **transformation**. The set X is the **domain** and the set Y is the **codomain**. We refer to the set X as **domain** or **domain of definition**, and the set Y as **domain of values** or **codomain**. The set

$$\{(x, f(x)) \mid x \in X\} \subseteq X \times Y$$

is called the **graph** of f. In the context of a function $f : X \to Y$, an element of the domain of definition is also called **argument**, and an element $y = f(x) \in Y$ assumed by the function, is also called **value** of the function. If $f : X \to Y$ is a function, one also writes

$$\begin{aligned} f : X &\to Y \\ x &\mapsto f(x), \end{aligned}$$

where $f(X)$ could be a concrete formula. We pronounce ´$\mapsto$´ as ´is mapped to'. Two functions $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are said to be equal if $X_1 = X_2$, $Y_1 = Y_2$ and $f_1(x) = f_2(x) \quad \forall x \in X_1$.

Definition 1.2: Injective, Surjective and Bijective Functions

Let $f : X \to Y$ be a function. We call f:

1. **injective** (or an **injection**) if

$$\forall x_1, x_2 \in X \; : \; x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2);$$

2. **surjective** (or a **sujection**) if

$$\forall y \in Y \; \exists x \in X \; : \; f(x) = y;$$

3. **bijective** (or a **bijection**) if f is both injective and surjective.

Thus, a function $f : X \to Y$ is *not* injective if there exists distinct $x_1 \neq x_2 \in X$ such that $f(x_1) = f(x_2)$, and *not* surjective if there exists $y \in Y$ such that $f(x) \neq y$ for all $x \in X$.

Definition 1.3: Image and Preimage of a Function

For $f : X \to Y$ and $A \subseteq X$, define the **image** of A under the function f as

$$f(A) := \{y \in Y \mid \exists x \in X : f(x) = y\}.$$

For $B \subseteq Y$, define the **preimage** of B under the function f as

$$f^{-1}(B) := \{x \in X \mid f(x) \in B\}.$$

Remark 1.4. *Saying that $f : X \to Y$ is surjective is equivalent to $f(X) = Y$. Equivalently, f is surjective if $f^{-1}(\{y\}) \neq \emptyset$ for all $y \in Y$.*

Definition 1.5: Composition of Functions

Let $f : X \to Y$ and $g : Y \to Z$. The **composition** is $g \circ f : X \to Z$, defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$.
Associativity: If $f : W \to X, g : X \to Y$ and $h : Y \to Z$, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Indeed, for all $w \in W$, we have

$$h \circ (g \circ f)(w) = h((g \circ f)(w)) = h(g(f(w))) = (h \circ g)(f(w)) = ((h \circ g) \circ f)(w).$$

Therefore, we may omit parentheses and write $h \circ g \circ f : W \to Z$.

Definition 1.6: Identity and Inverse Function

Given a set X, the **identity function** $\mathrm{id}_X : X \to X$ is defined by

$$\mathrm{id}_X(x) = x \qquad \forall x \in X.$$

If $f : X \to Y$ is bijective, then there exists a unique function $g : Y \to X$ such that, for each $y \in Y$, the value $g(y)$ is the unique element $x \in X$ satisfying $f(x) = y$. With this definition,

$$g \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ g = \mathrm{id}_Y .$$

The function g is called the **inverse function** (or **inverse mapping**) of f, and is denoted by f^{-1}.

2 The Real Numbers

2.1 Groups, Rings, Fields

Definition 2.1: Groups

A **group** is a non-empty set G together with a rule (called an *operation*) denoted by $\star : G \times G \to G$ that combines any two elements of G into another element of G. This operation must satisfy three conditions:

- **Associativity:** No matter how you place parentheses, the result is the same for all $a, b, c \in G$,
$$(a \star b) \star c = a \star (b \star c).$$
- **Neutral element:** There is a special element $e \in G$ such that combining it with any $a \in G$ leaves a unchanged, i.e.,
$$\forall a \in G : a \star e = e \star a = a.$$
- **Inverse element:** Every $a \in G$ has a 'partner' $a^{-1} \in G$ that 'cancels it out', giving the neutral element, i.e.,
$$a \star a^{-1} = a^{-1} \star a = e.$$

Note that, in general, one does not require that $a \star b = b \star a$. If the order of the operation does not matter, i.e., $a \star b = b \star a$ for all $a, b \in G$, the group is called **commutative** or **abelian**.

Lemma 2.2: Basic Properties of Groups

Let $(G, \star)$ be a group. Then:

1. *The neutral element is unique.*
2. *The inverse of an element is unique.*
3. *The inverse of the inverse of an element is the element itself, namely $(a^{-1})^{-1} = a$ for all $a \in G$.*

Proof. 1. Assume that, in addition to $e \in G$, we have a second element e' with the property that $e' \star a = a \star e' = a$ for all elements $a \in G$. Then, we can choose $a = e$ to obtain
$$e \star e' = e.$$
Similarly, since e is a neutral element, we have
$$e \star e' = e'.$$
Combining the two identities, we get
$$e = e \star e' = e'.$$
This proves that $e' = e$, so we speak of *the* neutral element of a group.

2. Assume that for an element $a \in G$, there exists two elements $b, c \in G$ that are both the inverse

of a, namely

$$a \star b = b \star a = e, \qquad a \star c = c \star a = e.$$

Then, using associativity, we observe that

$$b = b \star e = b \star (a \star c) = (b \star a) \star c = e \star c = c.$$

This proves that the inverse of an element a is unique, so we can speak of *the* inverse element, and the notation a^{-1} makes sense.

3. Since $a \star a^{-1} = e$, we deduce that a is the inverse element of a^{-1}, thus

$$(a^{-1})^{-1} = a. \tag{2.1}$$

□

Groups capture the idea of combining elements with a single operation. But to describe the arithmetic of numbers more faithfully, we also need a second operation (as we do with addition and multiplication). This leads us to the notion of *rings* and *fields*.

Definition 2.3: Rings and Fields

A **ring** is a non-empty set R in which we can both 'add' and 'multiply' elements with two operations '+' and '·'. Also, these two operations are compatible with each other. More precisely:

- $(R, +)$ is a **commutative group**, with neutral element denoted 0.
- Multiplication $\cdot$ is **associative**, has a **neutral element** (usually written as 1), and **distributes over addition**, i.e.,

$$a \cdot (b + c) = a \cdot b + a \cdot c, \qquad (b + c) \cdot a = b \cdot a + c \cdot a \qquad \forall a, b, c \in R.$$

If multiplication is also commutative, we call $(R, +, \cdot)$ a **commutative ring**. Note that, unlike addition, we do not require that every element has an inverse for multiplication. A **field** is a special kind of commutative ring, i.e. every non-zero element has an inverse for multiplication. In other words, if $(R, +, \cdot)$ is a commutative ring, then $(R, +, \cdot)$ is a field if $R \setminus \{0\}$ forms a commutative group under multiplication. Traditionally, we use the letter F to denote a field. We also write $F^* = F \setminus \{0\}$ for the set of all invertible elements of F.

Lemma 2.4: Basic Properties of Fields

Let $(F, +, \cdot)$ be a field and let $a, b \in F$. Then:

1. $0 \cdot a = a \cdot 0 = 0$.
2. $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$. *In particular* $(-1) \cdot a = -a$.
3. $(-a) \cdot (-b) = a \cdot b$. *In particular,* $(-a)^{-1} = -(a^{-1})$ *whenever* $a \neq 0$.

Proof. 1. Since 0 is the neutral element for the addition, we have $0+0 = 0$. Hence, using distributivity, we get

$$0 \cdot a = (0 + 0) \cdot a = (0 \cdot a) + (0 \cdot a).$$

Adding $-0 \cdot a$ (i.e., the inverse of $0 \cdot a$), we deduce that $0 \cdot a = 0$. The case of $a \cdot 0$ is analogous.

2. By the distributive law,

$$a \cdot b + a\dot{(} - b) = a \cdot (b + (-b)) = a \cdot 0 = 0.$$

So $a \cdot (-b)$ is the additive inverse of $a \cdot b$, i.e., $-(a \cdot b) = a \cdot (-b)$. Taking $b = 1$ gives $-a = (-1) \cdot a$. The validity of $(-a) \cdot b = -(a \cdot b)$ follows by exchanging a and b in the argument above.

3. By 2. we know that $-(a \cdot b) = a \cdot (-b)$. Hence, recalling Equation 2.1,

$$a \cdot b = -(a \cdot (-b)).$$

On the other hand, applying 2. with $(-b)$ instead of b, we also have

$$-(a \cdot (-b)) = (-a) \cdot (-b).$$

Combining the two identities above, we conclude that $(-a) \cdot (-b) = a \cdot b$. Finally, taking $b = a^{-1}$ yields $(-a) \cdot (-(a^{-1})) = a \cdot a^{-1} = 1$, which gives the second assertion. □

2.2 Order Relation

Definition 2.5: Cartesian Product

Let X and Y be two sets. The **cartesian product** $X \times Y$ is the set of ordered pairs of elements of X and Y, i.e.,

$$X \times Y := \{(x, y) \mid x \in X, y \in Y\}.$$

Definition 2.6: Subsets

Let P and Q be sets. Then

- P is a **subset** of Q, written $P \subset Q$ (or $P \subseteq Q$), if every element of P also belongs to Q.
- P is a **proper subset** of Q, written $P \subsetneq Q$, if P is a subset of Q but $P \neq Q$.
- We write $P \nsubseteq Q$ if P is not a subset of Q.

Definition 2.7: Relations

Let X be a set. A **relation** on X is a subset $\mathcal{R} \subseteq X \times X$, that is, a collection of ordered pairs of elements of X. If $(x, y) \in \mathcal{R}$ we write $x\mathcal{R}y$. Common symbols for relations include $<, \leq, \sim, \equiv, \cong$. If $\sim$ is a relation on X, we write $x \nsim y$ if $x \sim y$ does not hold. A realtion $\sim$ may have the following properties:

1. **Reflexive:** if $x \sim x \qquad \forall x \in X$.
2. **Transitive:** if $x \sim y$ and $y \sim z$, then $x \sim z$.
3. **Symmetric:** if $x \sim y$, then $y \sim x$.
4. **Antisymmetric:** if $x \sim y$ and $y \sim x$, then $x = y$.

A relation is an **equivalence relation** if it is reflexive, transitive and symmetric. It is an **order relation** if it is reflexive, transitive and antisymmetric.

2.3 Ordered Fields

Definition 2.8: Ordered Field

Let F be a field, and let $\leq$ be an order relation on F. We call $(F, \leq)$, or simply F, an **ordered field** if the following hold:

1. **Linearity of order:** for all $x, y \in F$, at least one of $x \leq y$ or $y \leq x$ holds.
2. **Compatibility with addition:** for all $x, y, z \in F$,
$$x \leq y \Rightarrow x + z \leq y + z.$$
3. **Compatibility with multiplication:** for all $x, y \in F$,
$$0 \leq x \ \wedge \ 0 \leq y \Rightarrow 0 \leq x \cdot y.$$

Lemma 2.9: Ordered Field: Basic Consequences

Let $(F, \leq)$ be and ordered field, and let $x, y, z, w \in F$. Then:

(a) (Trichotomy) Either $x < y$, or $x = y$, or $x > y$.

(b) If $x < y$ and $y \leq z$, then $x < z$. (Analogously, $x \leq yandy < zimplyx < z$.)

(c) (Addition of inequalities) If $x \leq y$ and $z \leq w$, then $x + z \leq y + w$. (Analogously, $x < z$ and $z \leq w$ imply $x + z < y + w$.)

(d) $x \leq y$ if and only if $0 \leq y - x$.

(e) $x \leq 0$ if and only if $0 \leq -x$.

(f) $x^2 \geq 0$, and $x^2 > 0$ if $x \neq 0$.

(g) 0 < 1.

(h) If $0 \leq x$ and $y \leq z$, then $xy \leq xz$.

(i) If $x \leq 0$ and $y \leq z$, then $xy \geq xz$.

(j) If $0 < x \leq y$, then $0 < y^{-1} \leq x^{-1}$.

(k) If $0 \leq x \leq y$ and $0 \leq z \leq w$, then $0 \leq xz \leq yw$.

(l) If $x + y \leq x + z$, then $y \leq z$.

(m) If $xy \leq xz$ and $x > 0$, then $y \leq z$.

Lemma 2.10: Integers and Rationals Inside an Ordered Field

Let $(F, \leq)$ be and ordered field, and denote by 0 and 1 the neutral elements for addition and multiplication, respectively. Then:

(i) The elements $\ldots, -2, -1, 0, 1, 2, \ldots$ defined by

$$2 = 1 + 1, \quad 3 = 2 + 1, \ldots, \quad -n = (-1) \cdot n$$

are all distinct and satisfy

$$\ldots < -2 < -1 < 0 < 1 < 2 < 3 < \ldots.$$

We denote this set of elements by $\mathbb{Z}$, and we call them 'integers'

(ii) Every fraction pq^{-1} with $p, q \in \mathbb{Z}$, $q \neq 0$, lies in F and the set of all such elements is denoted by $\mathbb{Q}$. Also,

$$\mathbb{Z} \subsetneq \mathbb{Q} \subseteq F.$$

Proof. (i) By Lemma 2.9(g), we have that $0 < 1$. Then Lemma 2.9(c) yields $0 < 1 < 2 < 3 < \ldots$, and taking negatives gives $\ldots < -2 < -1 < 0$. Hence all these elements are distinct.

(ii) For $q \neq 0$, q is invertible in F; define $\frac{p}{q} = pq^{-1}$. The set of such fractions is a field contained in F, which we denote by $\mathbb{Q}$.

To show that $\mathbb{Q}$ strictly contains $\mathbb{Z}$, consider $\frac{1}{2}$ (the inverse of 2). Since $2 > 1$, it follows from Lemma 2.9(j) that $0 < \frac{1}{2} < 1$, so $\frac{1}{2} \notin \mathbb{Z}$. □

Definition 2.11: Absolute Value and Sign

Let $(F, \leq)$ be and ordered field.

- The **absolute value** (or **modulus**) is the function $|\cdot| : F \to F$ defined by

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

- The **sign** is the function $\operatorname{sgn} : F \to \{-1, 0, 1\}$ defined by

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Lemma 2.12: Absolute Value and Sign: Basic Properties

Let $(F, \leq)$ be an ordered field and let $x, y \in F$. Then:

(a) $x = \operatorname{sgn}(x)|x|, \quad |-x| = |x|, \quad \operatorname{sgn}(-x) = -\operatorname{sgn}(x)$.

(b) $|x| \geq 0$, *and* $|x| = 0$ *if and only if* $x = 0$ *(by Trichotomy Lemma **??**).*

(c) (Multiplicativity) $\operatorname{sgn}(xy) = \operatorname{sgn}(x)\operatorname{sgn}(y)$ *and* $|xy| = |x||y|$.

(d) If $x \neq 0$, *then* $|x^{-1}| = |x|^{-1}$.

(e) $|x| \leq y$ *iff* $-y \leq x \leq y$.

(f) $|x| < y$ *iff* $-y < x < y$.

(g) (Triangle inequality) $|x + y| \leq |x| + |y|$.

(h) (Inverse triangle inequality) $||x| - |y|| \leq |x - y|$.

Proof. (g) Thanks to (e) we have $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding these two inequalities we get

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

Applying (e) again yields the result.

(h) From (g) we have $|x| \leq |x - y| + |y|$, therefore

$$|x| - |y| \leq |x - y|.$$

Exchanging the roles of x and y, we also have $|y| - |x| leq |y - x| = |x - y|$. Combining these two inequalities yields

$$-|x - y| \leq |x| - |y| \leq |x - y|,$$

and the result follows by applying (e) again. □

2.4 Completeness Axiom

Definition 2.13: Completeness Axiom

Let $(K, \leq)$ be an ordered field. We say that $(K, \leq)$ is **complete** (or a **completely ordered field**) if the following statement holds:

> Let X, Y be non-empty subsets of K such that $x \leq y$ for all $x \in X$ and $y \in Y$. Then there exists $c \in K$ lying between X and Y, in the sense that $x \leq c \leq y$ for all $x \in X$ and $y \in Y$.

The statement above is called the **completeness axiom**.

Definition 2.14: Real Numbers

We call **the field of real numbers**, any completely ordered field and denote it by $\mathbb{R}$.

2.5 Intervals

Definition 2.15: Intervals

Let $a, b \in \mathbb{R}$. We define:

- The **closed interval**
$$[a, b] := \{x \in R \mid a \leq x \leq b\};$$
- The **open interval**
$$(a, b) := \{x \in R \mid a < x < b\};$$
- The **half-open intervals**
$$[a, b) := \{x \in R \mid a \leq x < b\} \quad \text{and} \quad (a, b] := \{x \in R \mid a < x \leq b\};$$
- The **unbounded closed intervals**
$$[a, \infty) := \{x \in R \mid a \leq x\} \quad \text{and} \quad (-\infty, b] := \{x \in R \mid x \leq b\};$$
- The **unbounded open intervals**
$$(a, \infty) := \{x \in R \mid a < x\} \quad \text{and} \quad (-\infty, b) := \{x \in R \mid x < b\};$$

Definition 2.16: Set Operations

Let P, Q be sets. The **intersection** $P \cap Q$, the **union** $P \cup Q$, the **relative complement** $P \setminus Q$ and the **symmetric difference** $P \triangle Q$ are defined by

$$\begin{aligned}
P \cap Q &= \{x \mid x \in P \text{and} x \in Q\}, \\
P \cup Q &= \{x \mid x \in P \text{or} x \in Q\}, \\
P \setminus Q &= \{x \mid x \in P \text{and} x \notin Q\}, \\
P \triangle Q &= (P \setminus Q) \cup (Q \setminus P) = (P \cup Q) \setminus (P \cap Q).
\end{aligned}$$

Definition 2.17: Union and Intersection of several Sets

Let $\mathcal{A}$ be a family of sets (i.e., a set whose elements are sets). We define the **union** and **intersection** of the sets in $\mathcal{A}$ as

$$\bigcup_{A \in \mathcal{A}} A = \{x \mid \exists A \in A : x \in A\}, \quad \bigcap_{A \in \mathcal{A}} A = \{x \mid \forall A \in \mathcal{A} : x \in A\}.$$

If $\mathcal{A} = \{A_1, A_2, \ldots\}$, we also write

$$\bigcup_{i=1}^{\infty} A_i = \{x \mid \exists i \geq 1 : x \in A_i\}, \quad \bigcap_{i=1}^{\infty} A_i = \{x \mid \forall i \geq 1 : x \in A_i\}.$$

Definition 2.18: Neighborhoods

Let $x \in \mathbb{R}$. A **neighborhood** of x is a set containing an interval I such that $x \in I$. Given $\delta > 0$, the open interval $(x-\delta, x+\delta)$ is called the δ**-neighborhood** of x.

Definition 2.19: Open and Closed Sets

A subset $U \subseteq \mathbb{R}$ is called **open** in $\mathbb{R}$ if for every $x \in U$ there exists open interval I such that $x \in I$ and $I \subseteq U$. A subset $F \subseteq \mathbb{R}$ is called **closed** in $\mathbb{R}$ if its complement $\mathbb{R} \setminus F$ is open.

Remark 2.20. *The sets $\emptyset$ and $\mathbb{R}$ are both open in $\mathbb{R}$. Hence, they are also closed since $\emptyset^c = \mathbb{R}$ and $\mathbb{R}^c = \emptyset$. We note that $\mathbb{Q} \subseteq \mathbb{R}$ and $[a, b) \subseteq \mathbb{R}$ are neither open nor closed.*

Remark 2.21. *Let $\mathcal{U}$ be a family of open sets, and $\mathcal{F}$ be a family of closed subsets of $\mathbb{R}$. Then the union and intersection*

$$\bigcup_{U \in \mathcal{U}} U, \qquad \bigcap_{F \in \mathcal{F}} F$$

Are open and closed, respectively.

2.6 Complex Numbers

Starting from the field of real numbers $\mathbb{R}$, we define the set of **complex numbers** as

$$\mathbb{C} = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

We denote the elements $z = (x, y) \in \mathbb{C}$ in the form $z = x + iy$, where i is the **imaginary unit**. Here $x \in \mathbb{R}$ is the **real part** of z, written as $x = \mathrm{Re}(z)$, and $y \in \mathbb{R}$ is the **imaginary part**, written as $y = \mathrm{Im}(z)$. Elements with $\mathrm{Im}(z) = 0$ are called **real**, while those with $\mathrm{Re}(z) = 0$ are **purely imaginary**. Via the injective map $\mathbb{R} \ni x \mapsto x + i \cdot 0 \in \mathbb{C}$, we identify $\mathbb{R}$ with the subset of real numbers inside $\mathbb{C}$.

As you may expect from previous knowledge, we want to satisfy $i^2 = -1$. To achieve this, we define addition and multiplication on $\mathbb{C}$ so that it becomes a field. Additionally, we want these operations to coincide with the usual addition and multiplication when considering real numbers.

Since $i^2 = -1$, using commutativity and distributivity we get

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_2x_1 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

This motivates the following definition

Definition 2.22: Addition and Multiplication on $\mathbb{C}$

On $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ we define **addition** and **multiplication** as follows:

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2,\ y_1 + y_2),\\ (x_1, y_1) \cdot (x_2, y_2) &= (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1).\end{aligned}$$

Proposition 2.23: $\mathbb{C}$ is a Field

With the operation of Definition 2.22, together with the zero element $(0, 0)$ and the unit element $(1, 0)$, the set $\mathbb{C}$ is a field.

Definition 2.24: Complex Conjugation

For $z = x + iy \in \mathbb{C}$ we define its **conjugate** as $\bar{z} = x - iy$. The mapping $\mathbb{C} \ni z \mapsto \bar{z} \in \mathbb{C}$ is called **complex conjugation**.

Lemma 2.25: Properties of Complex Conjugation

For all $z, w \in \mathbb{C}$:

(i) $z\bar{z} = x^2 + y^2 \in \mathbb{R}_{\geq 0}$. *In particular,* $z\bar{z} = 0$ *if and only if* $z = 0$.

(ii) $\overline{z + w} = \bar{z} + \bar{w}$.

(iii) $\overline{zw} = \bar{z}\bar{w}$.

Proof. Property (i) follows from the fact that, for $z = x + iy$, $(x + iy)(x - iy) = x^2 + y^2$. Also, $x^2 + y^2 = 0$ if and only if $x + iy = 0$. Properties (ii) and (iii) follow from a direct computation, writing $z = x_+ + iy_1$ and $w = x_2 + iy_2$, which yields

$$\begin{aligned}
\overline{z + w} &= \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_2) + (x_2 - iy_2) = \bar{z} + \bar{w}, \\
\overline{z \cdot w} &= \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)} = (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2) \\
&= (x_1 - iy_1) \cdot (x_2 - iy_2) = \bar{z} \cdot \bar{w}.
\end{aligned}$$

□

Definition 2.26: Absolute Value

The **absolute value** (or **norm**) on $\mathbb{C}$ is the map $|\cdot| : \mathbb{C} \to \mathbb{R}$ given by

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}, \qquad z = x + iy \in \mathbb{C}.$$

Lemma 2.27: Cauchy-Schwart Inequality

If $z = x_1 + iy_1$, *and* $w = x_2 + iy_2$, *then*

$$x_1x_2 + y_1y_2 \leq |z||w|. \tag{2.2}$$

Proof. We observe that

$$\begin{aligned}
|z|^2|w|^2 - (x_1x_2 + y_1y_2)^2 &= (x_1^2 + y_1^2)(x_2^2 + y_2^2) - (x_1x_2 + y_1y_2)^2 \\
&= x_1^2x_2^2 + y_1^2y_2^2 + y_1^2x_2^2 + x_+^2y_2^2 - (x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2) \\
&= y_1^2x_2^2 + x_+^2y_2^2 - 2x_1x_2y_1y_2 \\
&= (y_1x_2 - x_1y_2)^2 \geq 0.
\end{aligned}$$

This proves that $(x_1x_2 + y_1y_2)^2 \leq |z|^2|w|^2$, so it follows that

$$|x_1x_2 + y_1y_2| \leq |z||w|.$$

Since $x \leq |x|$ for all $x \in \mathbb{R}$, we obtain Equation 2.2. □

Proposition 2.28: Trianlge Inequality

For all $z, w \in \mathbb{C}$, one has
$$|z+w| \leq |z| + |w|.$$

Proof. For $z = x_1 + iy_1$ and $w = x_2 + iy_2$, using Lemma 2.27, we have

$$\begin{aligned} |z+w|^2 &= (x_1+x_2)^2 + (y_1+y_2)^2 \\ &= |z|^2 + |w|^2 + 2(x_1x_2 + y_1y_2) \\ &\leq |z|^2 + |w|^2 + 2|z||w| = (|z|+|w|)^2. \end{aligned}$$

Taking square roots proves the result. □

Definition 2.29: Cicular Disks

For $z \in \mathbb{C}$ and $r > 0$, we define the **open disk** with radius $r > 0$ around z as

$$B(z,r) := \{w \in \mathbb{C} \mid |z-w| < r\},$$

and the **closed disk** with radius $r > 0$ around z as

$$\overline{B(z,r)} := \{w \in \mathbb{C} \mid |z-w| \leq r\}.$$

In other words, the open disk $B(z,r)$ is the set of points at distance strictly less than r form z. We note that this definition is compatible with the one of neighborhoods in $\mathbb{R}$: if $x \in \mathbb{R}$ and $r > 0$, then
$$B(x,r) \cap \mathbb{R} = (x-r, x+r).$$

Definition 2.30: Open and Closed Sets

A set $U \subseteq \mathbb{C}$ is **open** if for every $z \in U$ there exists $r > 0$ such that $B(z,r) \subseteq U$. A set $C \subseteq \mathbb{C}$ is **closed** if its complement $\mathbb{C} \setminus C$ is open.

2.7 Maximum and Supremum

2.7.1 Existence of the Supremum

Definition 2.31: Bounded Sets, Maxima and Minima

et $X \subseteq \mathbb{R}$ be a subset of real numbers.

- X is **bounded from above** if there exists $s \in \mathbb{R}$ such that $x \leq s$ for all $x \in X$. Such a number s is called an **upper bound** of X. If s is an upper bound and also an element of X, we say that s is the **maximum** of X and write

$$s = \max(X).$$

- Analogously, X is **bounded from below** if there exists $r \in \mathbb{R}$ such that $r \leq x$ for all $x \in X$. Such a number r is called a **lower bound** of X. If r is a lower bound and also an element of X, we say that r is the **minimum** of X and write

$$r = \min(X).$$

- X is called **bounded** if it is both bounded from above and bounded from below.

Remark 2.32. *If a set $X \subseteq \mathbb{R}$ has a maximum, then it is unique. Indeed, if $x_1, x_2 \in X$ are both maxima, then $x_1 \leq x_2$ (since x_2 is a maximum) and $x_2 \leq x_1$ (since x_1 is a maximum), so $x_1 = x_2$.*

A closed interval $[a, b]$ with $a < b$ has both a minimum and maximum, i.e., $a = \min([a, b])$ and $b = \max([a, b])$. But not all sets have a maximum. For instance, the open interval (a, b) does not have a maximum because the endpoint b, though an upper bound, is not contained in the set. Similarly $\mathbb{R}$ and unbounded intervals such as $[a, \infty)$ or (a, ∞) have no maximum.

Definition 2.33: Supremum

Let $X \subseteq \mathbb{R}$ be a subset and let

$$A := \{a \in \mathbb{R} \mid x \leq a \quad \forall x \in X\}$$

be the set of all upper bounds of X. If A has a minimum, we call this minimum the **supremum** of X and write

$$\sup(X) = \min(A).$$

The **infimum** is defined analogously using the maximum of the set of all lower bounds.

In other words, the supremum of X is the smallest real number that is greater than or equal to every element of X. Note that we can describe the supremum $s = \sup(X)$ as follows

$$x \leq s \qquad \forall x \in X, \qquad \text{and} \qquad \text{if } t < s, \text{the } t \text{ is not an upper bound of } X. \tag{2.3}$$

This means that for every $t < s$, there exists some $x \in X$ such that $x > t$. That is,

$$x \leq s \qquad \forall x \in X, \qquad \text{and} \qquad \forall t < s \, \exists x \in X : x > t. \tag{2.4}$$

The two characterizations 2.3 and 2.4 are equivalent.

Note that not every set has a supremum. If $X = \emptyset$ or if X is unbounded form above, then $\sup(X)$ does not exist. However, for any non-empty and bounded-above subset of $\mathbb{R}$, the supremum always exists.

Remark 2.34. *If a set X has a maximum, then this element is also the supremum. Indeed, the maximum is an upper bound of X, and since it lies in X, no smaller upper bound can exist.*

Theorem 2.35: Existence of Supremum

Let $X \subseteq \mathbb{R}$ be non-empty and bounded from above. Then $\sup(X)$ exists and is a real number.

Proof. Since X is bounded from above, the set $A := \{a \in \mathbb{R} \mid x \leq a \quad \forall x \in X\}$ of upper bounds is non-empty. Since $x \leq a$ for any $x \in X$ and $a \in A$, we can apply the completeness axiom (Definition 2.13) to find $c \in \mathbb{R}$ such that

$$x \leq c \leq a \qquad \forall x \in X, \forall a \in A.$$

The first inequality implies that c is itself an upper bound (so $c \in A$), while the second inequality tells us that c is smaller than or equal to every upper bound. Hence, $c = \min(A) = \sup(X)$. $\square$

Proposition 2.36: Supremum and Set Operations

Let X and Y be non-empty subsets of $\mathbb{R}$ that are bounded from above. Define

$$X + Y := \{x + y \mid x \in X, y \in Y\} \quad \text{and} \quad X \cdot Y := \{x \cdot y \mid x \in X, y \in Y\}.$$

The the sets $X \cup Y, X \cap Y$, and $X + Y$ are also bounded from above. Moreover, if $X, Y \subseteq \mathbb{R}_{\geq 0}$ (i.e., $x \geq 0$ and $y \geq 0$ for all $x \in X$ and $y \in Y$), then $X \cdot Y$ is bounded from above as well. In these cases, the following formulas hold:

(1) $\sup(X \cup Y) = \max\{\sup(X), \sup(Y)\}$,

(2) If $X \cap Y \neq \emptyset$, then $\sup(X \cap Y) \leq \min\{\sup(X), \sup(Y)\}$,

(3) $\sup(X + Y) = \sup(X) + \sup(Y)$,

(4) If $X, Y \subseteq \mathbb{R}_{\geq 0}$, then $\sup(X \cdot Y) = \sup(X) \cdot \sup(Y)$.

Proof. (3) Let $x_0 = \sup(X)$ and $y_0 = \sup(Y)$. For any $z \in X + Y$, there exists $x \in X$ and $y \in Y$ such that $z = x + y$. Since $x \leq x_0$ and $y \leq y_0$, we have

$$z = x + y \leq x_0 + y_0,$$

so $x_0 + y_0$ is an upper bound form $X + Y$. We now want to show that $x_0 + y_0 = \sup(X + Y)$.

Let $z_0 = \sup(X + Y)$ and suppose, by contradiction, that

$$\varepsilon := x_0 + y_0 - z_0 > 0.$$

Since $x_0 = \sup(X)$, by the characterization 2.4 there exists $x \in X$ such that $x > x_0 - \varepsilon/2$. Likewise, there exists $y \in Y$ such that $y > y_0 - \varepsilon/2$. Setting $z = x + y$, we obtain

$$z > x_0 - \frac{\varepsilon}{2} + y_0 - \frac{\varepsilon}{2} = x_0 + y_0 - \varepsilon = z_0,$$

contradicting the assumption that z_0 is an upper bound for $X + Y$. Therefore, $z_0 = x_0 + y_0$.

(4) The proof is analogous. If all elements of X and Y are non-negative, and we set $x_0 = \sup(X)$ and $y_0 = \sup(Y)$, then for any $z = x \cdot y \in X \cdot Y$, we have

$$z = x \cdot y \leq x_0 \cdot y_0,$$

which shows that $x_0 \cdot y_0$ is an upper bound for $X \cdot Y$. Using a similar 'ε-argument' as done above, when proving (3), one shows that this upper bound is sharp, i.e., $x_0 \cdot y_0$ is the least upper bound. □

2.8 Two-Point Compactification

In this section, we extend the notions of **supremum** and **infimum** to arbitrary subsets of $\mathbb{R}$. To do so, we introduce two formal symbols

$$+\infty \quad \text{and} \quad -\infty,$$

which are not real numbers. We define the **extended real numbers line** (also called the **two-point compactification** of $\mathbb{R}$) by

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}.$$

We extend the usual order relation $\leq$ on $\mathbb{R}$ to $\overline{\mathbb{R}}$ by requiring that

$$-\infty < x < +\infty \qquad \forall x \in \mathbb{R}.$$

For simplicity, we often write ∞ instead of $+\infty$.

We now introduce some standard (but informal) computation rules involving these symbols. For all $x \in \mathbb{R}$, we adopt the conventions:

$$\infty + x = \infty + \infty = \infty, \qquad -\infty + x = -\infty - \infty = -\infty.$$

If $x > 0$, then

$$x \cdot \infty = \infty \cdot \infty = \infty, \qquad x \cdot (-\infty) = \infty \cdot (-\infty) = -\infty,$$

while for $x < 0$ we have

$$x \cdot \infty = -\infty \cdot \infty = -\infty, \qquad x \cdot (-\infty) = -\infty \cdot (-\infty) = \infty.$$

These rules are widely used as notational shorthand, but one must handle them with care. Expressions like

$$\infty - \infty, \quad 0 \cdot \infty, \quad \text{or similar}$$

are undefined and should be avoided.

Definition 2.37: Supremum and Infimum in the Extended Line

Let $X \subseteq \mathbb{R}$.

- If X is not bounded from above, we define $\sup(X) = \infty$.
- If $X = \emptyset$, we define $\sup(\emptyset) = -\infty$.
- If X is not bounded from below, we define $\inf(X) = -\infty$.
- If $X = \emptyset$, we define $\inf(\emptyset) = \infty$.

In this context, we refer to ∞ and $-\infty$ as **indefinite values**.

In other words:

- Saying $\sup(X) = \infty$ means that X is not bounded from above, i.e.,

$$\forall x_0 \in X \; \exists x \in X : x > x_0.$$

- Saying $\sup(X) = -\infty$ means that X is empty.
- Similarly, $\inf(X) = -\infty$ means that X is not bounded from below, and $\inf(X) = \infty$ means X is empty.

2.9 Consequences of Completeness

2.9.1 Archimedean Principle

The archimedean principle states that for every real number $x \in \mathbb{R}$ there exists and integers n greater than x. The following theorem, proved using the existence of suprema (and implicitly the completeness axiom), gives a precise formulation of this principle.

Theorem 2.38: Archimedean Principle

For every $x \in \mathbb{R}$ there exists exactly one $n \in \mathbb{Z}$ such that

$$n \leq x < n + 1.$$

Proof. We first treat the case $x \geq 0$. Fix $\mathbb{R} \ni x \geq 0$ and define

$$E = \{n \in \mathbb{Z} \mid n \leq x\}.$$

Since $0 \in E$ and x is an upper bound, E is a non-empty subset of $\mathbb{R}$ bounded from above. Hence, by Theorem 2.35, the supremum $s_0 = \sup(E)$ exists. From the definition of supremum we deduce:

(i) $s_0 \leq x$ (because x is an upper bound);

(ii) there exists $n_0 \in E$ with $s_0 - 1 < n_0$ (otherwise $s_0 - 1$ would also be an upper bound).

From (ii) we obtain $s_0 < n_0 + 1$, which implies

(iii) $n_0 + 1 \notin E$ (otherwise s_0 would not be an upper bound for E).

Moreover, since $m \le s_0$ for every $m \in E$, we have $m < n_0 + 1$ for all $m \in E$. As all elements of E are integers,

$$m < n_0 + 1 \quad \Leftrightarrow \quad m - n_0 < 1 \quad \Leftrightarrow \quad m - n_0 \le 0 \quad \Leftrightarrow \quad m \le n_0.$$

Thus, every $m \in E$ is less than or equal to n_0, and since $n_0 \in E$, we conclude that $n_0 = \max(E)$. In particular, by Remark 2.34, the maximum is also the supremum, so $s_0 = n_0$.

Finally, recalling (iii) and the definition of E, we have $n_0 + 1 > x$. Together with (i), this shows

$$n_0 = s_0 \le x < n_0 + 1,$$

establishing the claim for any $x \ge 0$.

Now, if $x < 0$, apply the previous argument to $-x > 0$. Then there exists $m \in \mathbb{Z}$ such that

$$m \le -x < m + 1,$$

which is equivalent to

$$-m - 1 < x \le -m.$$

If $x = -m$, then set $n = -m$. If $x < -m$, set $n = -m - 1$. In both cases, we obtain

$$n \le x < n + 1.$$

Finally, for uniqueness, assume that $n_1, n_2 \in \mathbb{Z}$ both satisfy $n_i \le x < n_i + 1$. From $n_1 \le x < n_2 + 1$ we deduce that $n_1 < n_2 + 1$, and therefore $n_1 \le n_2$. Reversing the roles of n_1 and n_2 gives $n_2 \le n_1$. Hence, $n_1 = n_2$. □

Definition 2.39: Integer and Fractional Parts

The **integer part** $\lfloor x \rfloor$ of $x \in \mathbb{R}$ is the integer $n \in \mathbb{Z}$ uniquely determined by Theorem 2.38 such that $n \le x < n + 1$. The map $x \mapsto \lfloor x \rfloor$ from $\mathbb{R}$ to $\mathbb{Z}$ is called the **rounding function**. The **fractional part** of x is defined as

$$\{x\} = x - \lfloor x \rfloor \in [0, 1).$$

Corollary 2.40: $\frac{1}{n}$ is Arbitrarily Small

For every $\varepsilon > 0$ there exists $n \in \mathbb{N}$, with $n \ge 1$, such that

$$\frac{1}{n} < \varepsilon.$$

Proof. Applying Theorem 2.38 to $x = \frac{1}{\varepsilon} > 0$, we find $m \in \mathbb{Z}$ such that

$$m \le \frac{1}{\varepsilon} < m + 1.$$

Set $n := m + 1$. In this way we have $0 < \frac{1}{\varepsilon} < n$, which is equivalent to $n > 0$ (therefore, $n \ge 1$) and $\frac{1}{n} < \varepsilon$. □

Definition 2.41: Dense Sets

A subset $X \subseteq \mathbb{R}$ is called **dense** in $\mathbb{R}$ if every open non-empty interval contains an element of X.

Corollary 2.42: Density of $\mathbb{Q}$

For every $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < r < b$.

Proof. Set $\varepsilon = b - a$. By Corollary 2.40, there exists $m \in \mathbb{N}$ with $\frac{1}{m} < \varepsilon$. Then, by Theorem 2.38 applied with $x = ma$, there exists $n \in \mathbb{Z}$ with

$$n \leq ma < n + 1,$$

or equivalently,

$$\frac{n}{m} \leq a < \frac{n+1}{m}.$$

Since $\frac{1}{m} < \varepsilon$, by the two inequalities above, we obtain

$$a < \frac{n+1}{m} \leq a + \frac{1}{m} < a + \varepsilon = b.$$

Thus $r = \frac{n+1}{m}$ is a rational number between a and b. $\square$

Corollary 2.43: Density of $\mathbb{R} \setminus \mathbb{Q}$

For every $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < r < b$.

Proof. We want to show that for every $x \in \mathbb{R}$ and $\delta > 0$, there exists an $a \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$a \in (x - \delta, x + \delta).$$

By Corollary 2.42, we find a $q \in \mathbb{Q}$ such that $q \in (x - \delta, x + \delta)$. By Corollary 2.40 we find an $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \frac{(x+\delta) - q}{\sqrt{2}} \quad \Rightarrow \quad \frac{\sqrt{2}}{N} < (x + \delta) - q.$$

This implies that

$$x - \delta < q < \frac{\sqrt{2}}{N} + q < x + \delta.$$

Choosing $r = \frac{\sqrt{2}}{N} + q$ proves the statement. $\square$

2.9.2 Uncountability

Definition 2.44: Cardinality

Let X and Y be sets.

- We say X and Y have the **same cardinality**, written $X \sim Y$, if there is a bijection $f : X \to Y$.
- We write $X \preceq Y$ if there exists an injection $f : X \to Y$.
- The empty set has cardinality 0.
- A set X has **finite cardinality** $|X| = n$ if there exists a bijection with $\{1, \ldots, n\}$.
- A set is **infinite** it it is not finite.
- A set is **countable** if it has a bijection to $\mathbb{N}$. Its cardinality is denoted $\aleph_0$, pronounced Aleph-0.
- A set is **uncountable** if it is infinite but not countable.

If $X \preceq Y$ and $Y \preceq X$, then $X \sim Y$. In other words, if there exists an injective map $f : X \to Y$ and an injective map $g : Y \to X$, then one can find a bijective map $h : X \to Y$. This non-trivial statement is the **Schröder-Bernstein Theorem**.

We will now list some statements about different sets of numbers from the lecture:

1. $\mathbb{N}$ and the even numbers have the same cardinality.
2. $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.
3. $\mathbb{Q}$ is countable, i.e., $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{Z}$.

Proposition 2.45: Uncountability of $\mathbb{R}$

The set $\mathbb{R}$ is uncountable.

Extra Material

Definition 2.46: Power Set

Let X be a set. The **power set** $\mathcal{P}(X)$ of X is the set of all subsets of X, i.e.,

$$\mathcal{P}(X) := \{A \subseteq X\}.$$

Theorem 2.47: Cantor's Theorem

For any set X, the power set $\mathcal{P}(X)$ has strictly larger cardinality than X.

Proposition 2.48: The Reals have the same cardinality as $\mathcal{P}(\mathbb{N})$

$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

3 Sequences of Real Numbers

3.1 Convergence of Sequences

Definition 3.1: Sequences

A **sequence** is a function $a : \mathbb{N} \to \mathbb{R}$. The image $a(n)$ of $n \in \mathbb{N}$ is also written as a_n and is called the n-th element of a. Instead of $a : \mathbb{N} \to \mathbb{R}$ one often writes $(a_n)_{n\in\mathbb{N}}, (a_n)_{n=0}^{\infty}, (a_n)_{n\geq 0}$.

Definition 3.2: (Eventually) Constant Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is **constant** if $x_n = x_m \forall n, m \in \mathbb{N}$. It is **eventually constant** if there exists $N \in \mathbb{N}$ such that $x_n = x_m \forall n, m \geq N$.

Definition 3.3: Convergence of Sequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. We say that $(x_n)_{n=0}^{\infty}$ **converges** (or is **convergent**) if $\exists A \in \mathbb{R}$ such that

$$\forall \varepsilon > 0 \, \exists N \in \mathbb{N} : |x_n - A| < \varepsilon \quad \forall n \geq N.$$

In this case we write

$$\lim_{n\to\infty} x_n = A \tag{3.1}$$

and call A the **limit** of $(x_n)_{n=0}^{\infty}$.

Lemma 3.4: Uniqueness of the Limit

A convergent sequence $(x_n)_{n=0}^{\infty}$ has exactly one limit.

Proof. Let $A, B \in \mathbb{R}$ be limits of $(x_n)_{n=0}^{\infty}$. Fix $\varepsilon > 0$. Then there exists $N_A, N_B \in \mathbb{N}$ such that $|x_n - A| < \varepsilon$ for all $n \geq N_A$ and $|x_n - B| < \varepsilon$ for all $n \geq N_B$. We define $N := \max\{N_A, N_B\}$. Then it holds that

$$|A - B| \leq |A - x_N| + |x_N - B| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $A = B$. □

3.2 Convergent Subsequences and Acculmulation Points

Definition 3.5: Subsequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence. A **subsequence** is of the form $(x_{n_k})_{k=0}^{\infty}$, where $(n_k)_{k=0}^{\infty}$ is a strictly increasing sequence of non-negative integers, i.e., $n_{k+1} > n_k \; \forall k \in \mathbb{N}$.

Remark 3.6. *Since $n_{k+1} > n_k$ for all $k \in \mathbb{N}$ it follows by induction that $n_k \geq k$ for all $k \in \mathbb{N}$.*

Proof. For $k = 0$ we have that $n_0 \geq 0$, because $(n_k)_{k=0}^{\infty}$ is a sequence of non-negative integers. So the condition is fulfilled. For the inductive step we want to show that the condition holds for $k+1$ under the assumption that the condition is true for k. Because $(n_k)_{k=0}^{\infty}$ is also a strictly increasing sequence, we have that $n_{k+1} > n_k \geq k$. Additionally since $n_k \in \mathbb{N}$, we have that $n_{k+1} \geq n_k + 1$. So it follows that $n_{k+1} \geq n_k + 1 \geq k + 1$, which proofs the condition for $k + 1$. □

Lemma 3.7: Subsequences of Convergent Sequences are Convergent

Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Then every subsequence $(x_{n_k})_{k=0}^{\infty}$ also converges to A.

Proof. Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^{\infty}$ converges to A, there exists $N \in \mathbb{N}$ such that $|x_n - A| < \varepsilon \ \forall n \geq N$. As by Remark 3.6 we know that $n_k \geq k$ for all $k \in \mathbb{N}$. Therefore for all $k \geq N$ it holds that $|x_{n_k} - A| < \varepsilon$. $\square$

Definition 3.8: Accumulation Points of Sequences

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. A point $A \in \mathbb{R}$ is an **accumulation point** of $(x_n)_{n=0}^{\infty}$ if

$$\forall \varepsilon > 0 \, \forall N \in \mathbb{N} \, \exists n \geq N : |x_n - A| < \varepsilon.$$

Proposition 3.9: Subsequences and Accumulation Points

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. A point A is and accumulation point of $(x_n)_{n=0}^{\infty}$ if and only if there exists a convergent subsequence of $(x_n)_{n=0}^{\infty}$ with limit A.

Proof. First assume that $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^{\infty}$. We construct $(n_k)_{k\geq 0}$ recursively:

- first, apply the definition of accumulation point with $N = 1$ and $\varepsilon = 1 = 2^0$ to find $n_0 \geq 1$ with $|x_{n_0} - A| \leq 2^0$,
- second, apply the definition the definition of accumulation point with $N = n_0 + 1$ and $\varepsilon = 2^{-1}$ to find $n_1 \geq n_0 + 1$ with $|x_{n_1} - A| \leq 2^{-1}$,
- more in general given n_{k-1}, we apply the definition of accumulation point with $N = n_{k-1} + 1$ and $\varepsilon = 2^{-k}$ to find $n_k \geq n_{k-1} + 1$ with $|x_{n_k} - A| \leq 2^{-k}$.

Now given $\varepsilon > 0$ choose N such that $2^{-N} < \varepsilon$. Then for all $k \geq N$ we have that

$$|x_{n_k} - A| \leq 2^{-k} \leq 2^{-N} < \varepsilon,$$

so $\lim_{k\to\infty} x_{n_k} = A$.

Conversely, assume that there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ converging to A. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$. Since $\lim_{k\to\infty} x_{n_k} = A$, there exists N_0 such that $|x_{n_k} - A| < \varepsilon$ for all $k \geq N_0$. Hence if we choose $k = \max\{N_0, N\}$, because $n_k \geq n$ (recall Remark 3.6) we have that $n_k \geq N$ and $|x_{n_k} - A| < \varepsilon$. Thus A is an accumulation point. $\square$

Corollary 3.10: Infinitely Many Terms Near an Accumulation Point

If $A \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=0}^{\infty}$, then for every $\varepsilon > 0$ there are infinitely many n with $x_n \in (A - \varepsilon, A + \varepsilon)$.

Proof. By Proposition 3.9, there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ with $\lim_{k\to\infty} = A$. Hence for every $\varepsilon > 0$ there exists K such that $x_{n_k} \in (A - \varepsilon, A + \varepsilon)$ for all $k \geq K$, providing infinitely many elements of the sequence inside the interval $(A - \varepsilon, A + \varepsilon)$. $\square$

Corollary 3.11: Accumulation Points of Convergent Sequences

convergent sequence has exactly one accumulation point, namely its limit.

3.3 Addition, Multiplication and Inequalities

Proposition 3.12: Limits and Operations

Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be sequences converging to $A, B \in \mathbb{R}$ respectively. Then:

1. *The sequence $(x_n + y_n)_{n=0}^{\infty}$ converges to $A + B$.*
2. *The sequence $(x_n y_n)_{n=0}^{\infty}$ converges to AB.*
3. *Given $\alpha \in \mathbb{R}$, the sequence $(\alpha x_n)_{n=0}^{\infty}$ converges to αA.*
4. *Suppose $x_n \neq 0$ for all $n \in \mathbb{N}$ and $A \neq 0$. Then the sequence $(x_n^{-1})_{n=0}^{\infty}$ converges to A^{-1}.*

Proposition 3.13: Limits and Inequalities

Let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be sequences converging to $A, B \in \mathbb{R}$ respectively.

1. *If $A < B$, then there exists $N \in \mathbb{N}$ such that $x_n < y_n$ for all $n \geq \mathbb{N}$.*
2. *If there exists $N \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \geq N$, then $A \leq B$.*

Remark 3.14. *In Propostion 3.13 even if we assume that $x_n < y_n$ for all $n \in \mathbb{N}$, we cannot conclude that $A < B$. for example take*

$$x_n = \frac{1}{n}, \quad y_n = \frac{1}{n}.$$

Then we have that $x_n < y_n$ for all $n \in \mathbb{N}$ but $A = B = 0$.

Lemma 3.15: Sandwich Lemma

Let $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty}$, $(z_n)_{n=0}^{\infty}$ be sequences such that for some $N \in \mathbb{N}$, we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N.$$

Suppose that both $(x_n)_{n=0}^{\infty}$ and $(z_n)_{n=0}^{\infty}$ converge to the same limit. Then $(y_n)_{n=0}^{\infty}$ also converges, and we have that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n.$$

Proof. Let $(x_n)_{n=0}^{\infty}$, $(y_n)_{n=0}^{\infty}$, $(z_n)_{n=0}^{\infty}$ be sequences such that for some $N_0 \in \mathbb{N}$, we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N_0.$$

Additionally suppose that $(x_n)_{n=0}^{\infty}$ and $(z_n)_{n=0}^{\infty}$ converge to $A \in \mathbb{R}$. Fix $\varepsilon > 0$. Since $(x_n)_{n=0}^{\infty}$, $(z_n)_{n=0}^{\infty}$ converge to A there exists $N_x, N_z \in \mathbb{N}$ such that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N_x$$
$$A - \varepsilon < z_n < A + \varepsilon \quad \forall n_z \geq N_z.$$

So we choose $N := \max\{N_0, N_x, N_z\}$. Then we have that

$$A - \varepsilon < x_n \le y_n \le z_n < A + \varepsilon \quad \forall n \ge N,$$

which shows that $\lim_{n\to\infty} y_n = A$. □

Definition 3.16: Bounded Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called **bounded** if there exists a real number $M \ge 0$ such that

$$|x_n| \le M \quad \forall n \in \mathbb{N}.$$

Lemma 3.17: Convergent Sequences are Bounded

Every convergent sequence is bounded.

Proof. Let $(x_n)_{n=0}^{\infty}$ be a sequence converging to $A \in \mathbb{R}$. Let $\varepsilon = 1$. Then, by convergence of $(x_n)_{n=0}^{\infty}$, there exists N such that $|x_n - A| \le 1$ for all $n \ge N$. So we have that

$$|x_n| = |x_n - A + A| \le |x_n - A| + |A| \le 1 + |A| \qquad \forall n \ge N.$$

We choose

$$M = \max(|x_0|, |x_1|, \dots, |x_{N-1}|, 1 + |A|).$$

Then $|x_n| \le M$ for all $n \in \mathbb{N}$ as desired. □

Definition 3.18: Monotone Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called:

- **(monotonically) increasing** if $m > n \Rightarrow x_m \ge x_n$,
- **strictly (monotonically) increasing** if $m > n \Rightarrow x_m > x_n$,
- **(monotonically) decreasing** if $m > n \Rightarrow x_m \le x_n$,
- **strictly (monotonically) decreasing** if $m > n \Rightarrow x_m < x_n$.

If a sequence is decreasing or increasing we call it monotone. If a sequence is strictly increasing or strictly decreasing then we call it strictly monotone.

Remark 3.19. *An equivalent formulation of monotone sequences can be given using only successive terms:*

- $(x_n)_{n=0}^{\infty}$ *is increasing if* $x_{n+1} \ge x_n$ *for all* n,
- $(x_n)_{n=0}^{\infty}$ *is strictly increasing if* $x_{n+1} > x_n$ *for all* n,
- $(x_n)_{n=0}^{\infty}$ *is decreasing if* $x_{n+1} \le x_n$ *for all* n,
- $(x_n)_{n=0}^{\infty}$ *is strictly decreasing if* $x_{n+1} < x_n$ *for all* n.

Theorem 3.20: Convergence of Monotone Sequences

A monotone sequence $(x_n)_{n=0}^{\infty}$ converges if and only if it is bounded. More precisely, let $X = \{x_n \mid n \in \mathbb{N}\}$ denote the set of points in the sequence.

- *If $(x_n)_{n=0}^{\infty}$ is increasing, then $\lim_{n\to\infty} x_n = \sup(X)$,*
- *if $(x_n)_{n=0}^{\infty}$ decreasing, then $\lim_{n\to\infty} x_n = \inf(X)$.*

Proof. If $(x_n)_{n=0}^{\infty}$ converges Lemma 3.17 says that its bounded.

Conversely, let $(x_n)_{n=0}^{\infty}$ be a bounded monotone sequence. Wlog assume that $(x_n)_{n=0}^{\infty}$ is increasing (otherwise consider $(-x_n)_{n=0}^{\infty}$). Since $(x_n)_{n=0}^{\infty}$ is bounded from above, the set $X = \{x_n \mid n \in \mathbb{N}\}$ has a supremum, that we'll call $A = \sup(X)$.

By definiton of A:

(i) $x_n \leq A \quad \forall n \in \mathbb{N}$,

(ii) $\forall \varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_N > A - \varepsilon$.

Then, for all $n \geq N$ using (ii) and monotonicity, we have that $x_n \geq x_N > A - \varepsilon$. Then using (i), we conclude that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N.$$

□

3.4 Superior and Inferior Limits

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence. To study its behavior for large n its is useful to look at its tails

$$X_{\geq n} = \{x_k \mid k \geq n\} \subseteq \mathbb{R}.$$

The concept of limits can be restated using the tails of a sequence, i.e., the sequence $(x_n)_{n=0}^{\infty}$ converges to $A \in \mathbb{R}$ if and only if, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $X_N \subseteq (A - \varepsilon, A + \varepsilon)$.

However, since not every sequence has a limit we now introduce a related notion (the **superior** and **inferior limits**), which always exist for bounded sequences.

For each $n \in \mathbb{N}$, define

$$s_n = \sup(X_{\geq n}) = \sup_{k \geq n} x_k, \qquad i_n = \inf(X_{\geq n}) = \inf_{k \geq n} x_k.$$

Since $X_{\geq m} \subset X_{\geq n}$, whenever $m > n$, we have that

$$i_n \leq i_m \leq s_m \leq s_n \qquad \forall m > n.$$

Thus, $(s_n)_{n=0}^{\infty}$ is a monotonically decreasing sequence, while $(i_n)_{n=0}^{\infty}$ is a monotonically increasing sequence. Moreover, since $(x_n)_{n=0}^{\infty}$ is bounded both $(s_n)_{n=0}^{\infty}$ and $(i_n)_{n=0}^{\infty}$ are bounded as well. Hence by Theorem 3.20, both sequences converge. Their limits will be called the *superior* and the *inferior limit* of $(x_n)_{n=0}^{\infty}$ respectively.

Note that, since $x_n \in X_{\geq n}$, we have that

$$i_n \leq x_n \leq s_n \qquad \forall n \in \mathbb{N}. \tag{3.2}$$

Definition 3.21: Superior and Inferior Limits

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence in $\mathbb{R}$. The numbers

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \big(\sup_{k\ge n} x_k\big), \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty} \big(\inf_{k\ge n} x_k\big)$$

are called the **superior** and **inferior limit** of $(x_n)_{n=0}^{\infty}$ respectively. From Equation 3.2 and Proposition 3.13, we have

$$\liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n.$$

Lemma 3.22: Convergence and Superior/Inferior Limits

A bounded sequence $(x_n)_{n=0}^{\infty}$ in $\mathbb{R}$ converges if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n.$$

Proof. For every $n \in \mathbb{N}$, define

$$i_n = \inf_{k\ge n} x_k, \qquad s_n = \sup_{k\ge n} x_k,$$

and set

$$I = \lim_{n\to\infty} i_n = \liminf_{n\to\infty} x_n, \qquad S = \lim_{n\to\infty} s_n = \limsup_{n\to\infty} x_n.$$

First suppose that $I = S$. Since $i_n \le x_n \le s_n$ (see Equation 3.2), the Sandwich Lemma 3.15 implies that the sequence $(x_n)_{n=0}^{\infty}$ converges, and its limit equals $I = S$.

Conversely, assume that $(x_n)_{n=0}^{\infty}$ converges to $A \in \mathbb{R}$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$A - \varepsilon < x_n < A + \varepsilon \qquad \forall n \ge N.$$

Then for all $n \ge N$, the same inequalities holds for i_n and s_n, i.e.,

$$A - \varepsilon \le i_n \le s_n \le A + \varepsilon.$$

Taking limits and using Proposition 3.13, we obtain

$$A - \varepsilon \le I \le S \le A + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $A = I = S$, which proves the result. $\square$

Theorem 3.23: Superior and Inferior Limits are Accumulation Points

Let $(x_n)_{n=0}^{\infty}$ be a bounded sequence and let $A = \limsup_{n\to\infty} x_n$. Then A is an accumulation point of $(x_n)_{n=0}^{\infty}$, and for every $\varepsilon > 0$ the following hold:

1. *only finitely many elements satisfy $x_n \ge A + \varepsilon$;*
2. *infinitely many elements satisfy $A - \varepsilon < x_n < A + \varepsilon$.*

An analogous statement holds for the inferior limit.

Proof. Since the sequence $(s_n)_{n=0}^{\infty}$ is monotonically decreasing and converges to A, given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$A \le s_n < A + \varepsilon \qquad \forall n \ge N_0. \tag{3.3}$$

We first prove that A is an accumulation point.

Fix $N \in \mathbb{N}$ and set $N_1 = \max\{N, N_0\}$. Since $s_{N_1} = \sup_{k \ge N_1} x_k$, there exists $n_1 \ge N_1 \ge N_0$ such that

$$s_{N_1} - \varepsilon < x_{n_1} \le s_{N_1}.$$

Thus, combining this bound with Equation 3.3 we obtain

$$A - \varepsilon < s_{N_1} - \varepsilon < x_{n_1} \le s_{N_1} < A + \varepsilon.$$

This construct shows that for any $\varepsilon > 0$ and any $N \in \mathbb{N}$, there exists $n_1 \ge N$ such that $A - \varepsilon < x_{n_1} < A + \varepsilon$. Thus A is an accumulation point for $(x_n)_{n=0}^{\infty}$.

We now prove1. and 2.. From Equation 3.3 we have $x_n < A + \varepsilon$ for all $n \ge N_0$, so only finitely many terms satisfy $x_n \ge A + \varepsilon$. This shows 1..

Also since A is an accumulation point, it follows from Corollary 3.10 that infinitely many terms of the sequence lie within any interval $(A - \varepsilon, A + \varepsilon)$. □

Corollary 3.24: Bounded Sequences have Convergent Subsequences

Every bounded sequence has at least one accumulation point and therefore possesses a convergent subsequence.

Proof. By Theorem 3.23, the number

$$A = \limsup_{n \to \infty} x_n$$

is always an accumulation point of $(x_n)_{n=0}^{\infty}$. Moreover, by Proposition 3.9, every accumulation point is the limit of a convergent subsequence. Hence every bounded sequence admits at least one convergent subsequence. □

3.5 Cauchy Sequences

Definition 3.25: Cauchy Sequences

A sequence $(x_n)_{n=0}^{\infty}$ is called a **Cauchy sequence** if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \qquad \forall n, m \ge N.$$

Lemma 3.26: Cauchy Sequences are Bounded

Every Cauchy sequence is bounded.

Proof. By definition, there exists $N \in \mathbb{N}$ such that

$$|x_n - x_N| \le 1 \qquad \forall n \ge N.$$

Hence, for $n \ge N$, we have $|x_n| \le 1 + |x_N|$. Now, define

$$M = \max\{|x_0|, |x_1|, \dots, |x_{N-1}|, 1 + |x_N|\}.$$

Then, $|x_n| \leq M$ for all $n \in \mathbb{N}$, so $(x_n)_{n=0}^{\infty}$ is bounded. □

Theorem 3.27: Convergence and Cauchy Sequences

A sequence $(x_n)_{n=0}^{\infty}$ of real numbers converges if and only if it is a Cauchy sequence.

Proof. Suppose first that $(x_n)_{n=0}^{\infty}$ converges to some $A \in \mathbb{R}$, and let us prove that $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that

$$|x_n - A| < \frac{\varepsilon}{2} \qquad \forall n \geq N.$$

Then for all $n, m \geq N$, we have that

$$|x_n - x_m| \leq |x_n - A| + |x_m - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence.

Viceversa, let $(x_n)_{n=0}^{\infty}$ be a Cauchy sequence. Since it is bounded (by Lemma 3.26), Corollary 3.24 implies that there exists a subsequence $(x_{n_k})_{k=0}^{\infty}$ converging to some $A \in \mathbb{R}$. Given $\varepsilon > 0$, choose $N_0 \in \mathbb{N}$ such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \qquad \forall n, m \geq N_0,$$

and choose $N_1 \in \mathbb{N}$ such that

$$|x_{n_k} - A| < \frac{\varepsilon}{2} \qquad \forall k \geq N_1.$$

Let $N = \max\{N_0, N_1\}$. Since $n_N \geq N$ (see Remark 3.6), for all $n \geq N$ we have

$$|x_n - A| \leq |x_n - x_{n_N}| + |x_{n_N} - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $(x_n)_{n=0}^{\infty}$ converges to A. □

3.6 Improper Limits

We now extend the notion of limit to allow the **improper limit values** $+\infty$ (often abbreviated as ∞) and $-\infty$.

Definition 3.28: Improper Limits

Let $(x_n)_{n=0}^{\infty}$ be a sequence in $\mathbb{R}$. We say $(x_n)_{n=0}^{\infty}$ **diverges to** $+\infty$, and we write

$$\lim_{n\to\infty} x_n = +\infty,$$

if for every $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n > M$ for all $n \geq N$.
Similarly, $(x_n)_{n=0}^{\infty}$ **diverges to** $-\infty$ if for every $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n < -M$ for all $n \geq N$. In both cases, we say that $(x_n)_{n=0}^{\infty}$ has an **improper limit**.

An unbounded sequence doesn't need to diverge to $+\infty$ or $-\infty$. For instance, the sequence $x_n = (-1)^n n$, is unbounded but neither diverges to $+\infty$ nor to $-\infty$.

The notion of improper limit allows us to extend the definitions of superior and inferior limits to *unbounded* sequences. If $(x_n)_{n=0}^{\infty}$ is not bounded from above, then

$$\sup_{k\geq n} x_k = +\infty \qquad \forall n \in \mathbb{N},$$

and we write

$$\limsup_{n\to\infty} x_n = +\infty.$$

If $(x_n)_{n=0}^{\infty}$ is bounded from above but not from below, then we define

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} (\sup_{k\geq n} x_k),$$

where the right-hand side is a real limit if the decreasing sequence $\sup_{k\geq n} x_k$ is bounded, and the improper limit $-\infty$ otherwise. The definition of the inferior limit extends analogously.

3.7 Sequences of Complex Numbers

Informally, a **sequence of complex numbers** is just like a sequence of real numbers, except that each term is a complex number instead of a real one. Thus, we study ordered lists $(z_0, z_1, \ldots)$, where $z_n : \mathbb{N} \to \mathbb{C}$. As in the real case, we are mainly interested in their convergence, divergence and limit behavior.

To analyze sequences in $\mathbb{C}$, it is often sufficient to consider separately the corresponding sequences of real and imaginary parts in $\mathbb{R}$.

Definition 3.29: Sequences of Complex Numbers

A sequence of complex numbers $(z_n)_{n=0}^{\infty}$, where

$$z_n = x_n + iy_n,$$

is said to **converge** to a limit $A + iB \in \mathbb{C}$ if the two sequences of real numbers $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ converge to A and B, respectively. In this case, we write

$$\lim_{n\to\infty} z_n = A + iB.$$

We say that $(z_n)_{n=0}^{\infty}$ **diverges to** ∞ if the sequence of moduli $(|z_n|)_{n=0}^{\infty}$ diverges to $+\infty$, i.e.,

$$\lim_{n\to\infty} |z_n| = \lim_{n\to\infty} \sqrt{x_n^2 + y_n^2} = +\infty.$$

Remark 3.30. *As for sequences of real numbers, one can consider subsequences of sequences $\mathbb{C}$. Given a strictly increasing sequence of non-negative integers $(n_k)_{k=0}^{\infty}$, the corresponding subsequence is*

$$(z_{n_k})_{k=0}^{\infty} = (x_{n_k} + iy_{n_k})_{n=0}^{\infty}.$$

4 Functions of one Real Variable

In this chapter we study real-valued functions defined on subsets of $\mathbb{R}$, typically intervals. The central concept is *continuity*.

4.1 Real valued functions

4.1.1 Boundedness and Monotonitcity

For a non-empty set $D \subseteq \mathbb{R}$, the set of **real-valued** functions on D is

$$\mathcal{F}(D) = \{f \mid f : D \to \mathbb{R}\}.$$

For $f_1, f_2 \in \mathcal{F}(D)$, $\alpha \in \mathbb{R}$, and $x \in D$ we define

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \qquad (\alpha f_1)(x) = \alpha f_1(x), \qquad (f_1 f_2)(x) = f_1(x) f_2(x).$$

Given $\alpha \in \mathbb{R}$, we write $f \equiv \alpha$ for the constant function $x \mapsto \alpha$ on D.

Remark 4.1. *With the operations above, $\mathcal{F}(D)$ is a commutative ring (the additive identity is $f \equiv 0$ and the multiplicative identity is $f \equiv 1$).*

A point $x \in D$ is a **zero** of $f \in \mathcal{F}(D)$ if $f(x) = 0$. The **zero set** of f is $\{x \in D \mid f(x) = 0\}$. We order $\mathcal{F}(D)$ pointwise: for $f_1, f_2 \in \mathcal{F}(D)$,

$$\begin{aligned} f_1 \leq f_2 \quad &\Leftrightarrow \quad f_1(x) \leq f_2(x) \quad \forall x \in D, \\ f_1 < f_2 \quad &\Leftrightarrow \quad f_1(x) < f_2(x) \quad \forall x \in D. \end{aligned}$$

We say that $f \in \mathcal{F}(D)$ is **non-negative** if $f \geq 0$, and **positive** if $f > 0$.

Definition 4.2: Bounded Functions

Let $D \neq \emptyset$ and $f : D \to \mathbb{R}$. We say that f is **bounded from above** if there exists $M > 0$ such that

$$f(x) \leq M \qquad \forall x \in D.$$

We say that f is **bounded from below** if there exists $M > 0$ such that

$$f(x) \geq -M \qquad \forall x \in D.$$

We say that f is **bounded** if it is both bounded from above and from below. Equivalently, f if bounded if there exists $M > 0$ such that

$$|f(x)| \leq M \qquad \forall x \in D.$$

Definition 4.3: Monotone Functions

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. The function f is:

1. **increasing** if $x < y \quad \Rightarrow \quad f(x) \leq f(y) \quad \forall x, y \in D$;
2. **strictly increasing** if $x < y \quad \Rightarrow \quad f(x) < f(y) \quad \forall x, y \in D$;
3. **decreasing** if $x < y \quad \Rightarrow \quad f(x) \geq f(y) \quad \forall x, y \in D$;
4. **strictly decreasing** if $x < y \quad \Rightarrow \quad f(x) > f(y) \quad \forall x, y \in D$.

We call f **monotone** if it is increasing or decreasing, and **strictly monotone** if it is strictly increasing or strictly decreasing.

4.1.2 Continuity

Definition 4.4: Continuous Functions

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. We say that f is **continuous at** $x_0 \in D$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

We say that f is **continuous on** D if it is continuous at every point of D.

Remark 4.5. *It suffices to verify the implication above for small ε. Precisely, assume there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ there is a $\delta > 0$ such that*

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

Then f is continuous at x_0.

Indeed, for $\varepsilon_0 > \varepsilon$ we can choose the number $\delta > 0$ corresponding to ε to get

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon < \varepsilon_0.$$

In other words, if δ works for ε, then it works for all $\varepsilon_0 > \varepsilon$.

Definition 4.6: Restriction

Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. For any $D' \subseteq D$ the **restriction** of f to D' is the function $f|_{D'} : D' \to \mathbb{R}$ defined by

$$f|_{D'}(x) = f(x) \qquad \forall x \in D'.$$

We regard $f|_{D'}$ and f as different functions unless $D' = D$.

Proposition 4.7: Combination of Continuous Functions

Let $D \subseteq \mathbb{R}$, and let $f_1, f_2 : D \to \mathbb{R}$ be continuous at $x_0 \in D$. Then $f_1 + f_2$, $f_1 f_2$, and αf_1 (for any $\alpha \in \mathbb{R}$) are continuous at x_0.

Proof. We first prove the result for the sum. Let $\varepsilon > 0$. Since f_1 and f_2 are continuous at x_0, there exists $\delta_1, \delta_2 > 0$ such that for all $x \in D$,

$$|x - x_0| < \delta_1 \ \Rightarrow \ |f_1(x) - f_1(x_0)| < \frac{\varepsilon}{2}, \quad |x - x_0| < \delta_2 \ \Rightarrow \ |f_2(x) - f_2(x_0)| < \frac{\varepsilon}{2}.$$

So, choosing $\delta = \min \delta_1, \delta_2$, for $|x - x_0| < \delta$ we get

$$|(f_1 + f_2)(x) - (f_1 + f_2)(x_0)| \le |f_1(x) - f_1(x_0)| + |f_2(x) - f_2(x_0)| < \varepsilon,$$

which shows that $f_1 + f_2$ is continuous at x_0.

For the product, note that

$$\begin{aligned} |f_1(x)f_2(x) - f_1(x_0)f_2(x_0)| &= |f_1(x)f_2(x) - f_1(x_0)f_2(x) + f_1(x_0)f_2(x) - f_1(x_0)f_2(x_0)| \\ &\le |f_1(x)f_2(x) - f_1(x_0)f_2(x)| + |f_1(x_0)f_2(x) - f_1(x_0)f_2(x_0)| \\ &= |f_2(x)||f_1(x) - f_1(x_0)| + |f_1(x_0)||f_2(x) - f_2(x_0)|. \end{aligned}$$

Now, first choose $\delta_0 > 0$ such that $|x - x_0| < \delta_0$ implies $|f_2(x) - f_2(x_0)| < 1$, so that

$$|x - x_0| < \delta_0 \quad \Rightarrow \quad |f_2(x)| < 1 + |f_2(x_0)|.$$

Then choose $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} |x - x_0| < \delta_1 \quad &\Rightarrow \quad |f_1(x) - f_1(x_0)| < \frac{\varepsilon}{2(1 + |f_2(x_0)|)}, \\ |x - x_0| < \delta_2 \quad &\Rightarrow \quad |f_2(x) - f_2(x_0)| < \frac{\varepsilon}{2(1 + |f_1(x_0)|)}. \end{aligned}$$

So choosing $\delta = \min \delta_0, \delta_1, \delta_2$, for $|x - x_0| < \delta$ we get

$$\begin{aligned} |f_1(x)f_2(x) - f_1(x_0)f_2(x_0)| &< |f_2(X)| \frac{\varepsilon}{2(1 + |f_2(x_0)|)} + |f_1(x_0)| \frac{\varepsilon}{2(1 + |f_1(x_0)|)} \\ &< (1 + |f_2(x_0)|) \frac{\varepsilon}{2(1 + |f_2(x_0)|)} + |f_1(x_0)| \frac{\varepsilon}{2(1 + |f_1(x_0)|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

thus $f_1 f_2$ is continuous at x_0.

Finally, the statement about αf_1 follows by choosing $f_2 \equiv \alpha$ (a constant function) and using the product case proved above: since f_1 and f_2 are continuous at x_0, their product $f_1 f_2 = \alpha f_1$ is continuous at x_0. □

Definition 4.8: Sum and Product Notation

Let $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n \in \mathbb{R}$. We use the notation

$$\sum_{j=0}^{n} a_j = a_0 + a_1 + \ldots + a_n, \qquad \prod_{j=0}^{n} a_0 \cdot a_1 \cdot \ldots \cdot a_n.$$

Here a_j is a **summand** in the sum and a **factor** in the product; j is the **index** (or **running variable**). If J is a finite set and numbers $(a_j)_{j \in J}$ are given, we write

$$\sum_{j \in J} a_j, \qquad \prod_{j \in J} a_j.$$

By convention, for the empty index set $\emptyset$,

$$\sum_{j \in \emptyset} a_j = 0, \qquad \prod_{j \in \emptyset} a_j = 1.$$

Proposition 4.9: Composition of Continuous Functions

Let $D_1, D_2 \subseteq \mathbb{R}, x_0 \in D_1$ and $f : D_1 \to D_2$ be continuous at x_0. If $g : D_2 \to \mathbb{R}$ is continuous at $f(x_0)$, then $g \circ f : D_1 \to \mathbb{R}$ is continuous at x_0. In particular, the composition of continuous functions is continuous.

Proof. Let $\varepsilon > 0$. By continuity of g at $f(x_0)$, there exists $\eta > 0$ such that

$$\forall y \in D_2, \quad |y - f(x_0)| < \eta \quad \Rightarrow \quad |g(y) - g(f(x_0))| < \varepsilon.$$

By continuity of f at x_0, there exists $\delta > 0$ such that

$$\forall x \in D_1, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \eta.$$

Combining the implications gives, for any $x \in D_1$,

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \eta \quad \Rightarrow \quad |g(f(x)) - g(f(x_0))| < \varepsilon.$$ □

Remark 4.10. *Applying Proposition 4.9 with $g(y) = |y|$, we see that if $f : D \to \mathbb{R}$ is continuous, then $x \mapsto |f(x)|$ is continuous.*

4.1.3 Sequential Continuity

Definition 4.11: Notation for Limits of Sequences

Let $(x_n)_{n=0}^{\infty} \subseteq \mathbb{R}$ and $\overline{x} \in \mathbb{R}$. We write

$$x_n \to \bar{x} \qquad \text{or} \qquad x_n \underset{n\to\infty}{\longrightarrow} \bar{x}$$

to mean

$$\lim_{n\to\infty} x_n = \bar{x}.$$

Theorem 4.12: Continuity = Sequential Continuity

Let $D \subseteq \mathbb{R}$, $f : D \to \mathbb{R}$, and $\bar{x} \in D$. Then f is continuous at $\bar{x}$ if and only if for every sequence $(x_n)_{n=0}^{\infty} \subseteq D$ with $x_n \to \bar{x}$ we have $f(x_n) \to f(\bar{x})$.

Proof. '⇒': First Assume that f is continuous at $\bar{x}$. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall x \in D, \quad |x - \bar{x}| < \delta \quad \Rightarrow \quad |f(x) - f(\bar{x})| < \varepsilon.$$

Also, since $x_n \to \bar{x}$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad |x_n - \bar{x}| < \delta.$$

Thus,

$$n \geq N \quad \Rightarrow \quad |f(x_n) - f(\bar{x})| < \varepsilon,$$

which implies that the sequence $(f(x_n))_{n=0}^{\infty}$ converges to $f(\bar{x})$.

'⇐': To prove the converse, assume that f is not continuous at x_0. This means that there exists $\varepsilon > 0$ such that, for every $\delta > 0$, there is $x \in D$ with

$$|x - \bar{x}| < \delta \quad \text{and} \quad |f(x) - f(\bar{x})| \geq \varepsilon.$$

Now, for every $n \in \mathbb{N}$, we apply this property with $\delta = 2^{-n}$ to find a point $x_n \in D$ such that

$$|x_n - \bar{x}| < 2^{-n} \quad \text{and} \quad |f(x_n) - f(\bar{x})| \geq \varepsilon$$

Then the sequence constructed in this way satisfies $x_n \to \bar{x}$ but $f(x_n) \not\to f(\bar{x})$. □

Remark 4.13. *The proof above shows that if $f : D \to \mathbb{R}$ is not continuous at $\bar{x}$, then there exists*

$\varepsilon > 0$ and a sequence $(x_n)_{n=0}^{\infty} \subseteq D$ with $x_n \to \bar{x}$ such that $|f(x_n) - f(\bar{x})| \geq \varepsilon$ for all $n \in \mathbb{N}$. This is useful to show that a function f is not continuous at $\bar{x}$.

4.2 Continuous Functions

4.2.1 Intermediate Value Theorem