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# Analysis I

## Theorems & Lemmas

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# 1 Functions

## Definition 1.1: Functions/Maps/Transformations

A **function**  $f$  from a set  $X$  to a set  $Y$  is an assignment of an element of  $Y$  to each element of  $X$ . The element  $y \in Y$  to which  $x \in X$  is assigned to is denoted  $f(x)$ . We write  $f : X \rightarrow Y$  and sometimes also speak of a **map**, **mapping** or a **transformation**. The set  $X$  is the **domain** and the set  $Y$  is the **codomain**. We refer to the set  $X$  as **domain** or **domain of definition**, and the set  $Y$  as **domain of values** or **codomain**. The set

$$\{(x, f(x)) \mid x \in X\} \subseteq X \times Y$$

is called the **graph** of  $f$ . In the context of a function  $f : X \rightarrow Y$ , an element of the domain of definition is also called **argument**, and an element  $y = f(x) \in Y$  assumed by the function, is also called **value** of the function. If  $f : X \rightarrow Y$  is a function, one also writes

$$\begin{aligned} f : X &\rightarrow Y \\ x &\mapsto f(x), \end{aligned}$$

where  $f(x)$  could be a concrete formula. We pronounce ' $\mapsto$ ' as 'is mapped to'. Two functions  $f_1 : X_1 \rightarrow Y_1$  and  $f_2 : X_2 \rightarrow Y_2$  are said to be equal if  $X_1 = X_2$ ,  $Y_1 = Y_2$  and  $f_1(x) = f_2(x) \quad \forall x \in X_1$ .

## Definition 1.2: Injective, Surjective and Bijective Functions

Let  $f : X \rightarrow Y$  be a function. We call  $f$ :

1. **injective** (or an **injection**) if

$$\forall x_1, x_2 \in X : x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2);$$

2. **surjective** (or a **surjection**) if

$$\forall y \in Y \exists x \in X : f(x) = y;$$

3. **bijective** (or a **bijection**) if  $f$  is both injective and surjective.

Thus, a function  $f : X \rightarrow Y$  is *not* injective if there exists distinct  $x_1 \neq x_2 \in X$  such that  $f(x_1) = f(x_2)$ , and *not* surjective if there exists  $y \in Y$  such that  $f(x) \neq y$  for all  $x \in X$ .

## Definition 1.3: Image and Preimage of a Function

For  $f : X \rightarrow Y$  and  $A \subseteq X$ , define the **image** of  $A$  under the function  $f$  as

$$f(A) := \{y \in Y \mid \exists x \in X : f(x) = y\}.$$

For  $B \subseteq Y$ , define the **preimage** of  $B$  under the function  $f$  as

$$f^{-1}(B) := \{x \in X \mid f(x) \in B\}.$$

**Remark 1.4.** *Saying that  $f : X \rightarrow Y$  is surjective is equivalent to  $f(X) = Y$ . Equivalently,  $f$  is surjective if  $f^{-1}(\{y\}) \neq \emptyset$  for all  $y \in Y$ .*

### Definition 1.5: Composition of Functions

Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ . The **composition** is  $g \circ f : X \rightarrow Z$ , defined by  $(g \circ f)(x) = g(f(x))$  for all  $x \in X$ .

**Associativity:** If  $f : W \rightarrow X$ ,  $g : X \rightarrow Y$  and  $h : Y \rightarrow Z$ , then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Indeed, for all  $w \in W$ , we have

$$h \circ (g \circ f)(w) = h((g \circ f)(w)) = h(g(f(w))) = (h \circ g)(f(w)) = ((h \circ g) \circ f)(w).$$

Therefore, we may omit parentheses and write  $h \circ g \circ f : W \rightarrow Z$ .

### Definition 1.6: Identity and Inverse Function

Given a set  $X$ , the **identity function**  $\text{id}_X : X \rightarrow X$  is defined by

$$\text{id}_X(x) = x \quad \forall x \in X.$$

If  $f : X \rightarrow Y$  is bijective, then there exists a unique function  $g : Y \rightarrow X$  such that, for each  $y \in Y$ , the value  $g(y)$  is the unique element  $x \in X$  satisfying  $f(x) = y$ . With this definition,

$$g \circ f = \text{id}_X \quad \text{and} \quad f \circ g = \text{id}_Y.$$

The function  $g$  is called the **inverse function** (or **inverse mapping**) of  $f$ , and is denoted by  $f^{-1}$ .

**Remark 1.7.** *A function  $f : X \rightarrow Y$  is bijective if and only if there exists a function  $g : Y \rightarrow X$  such that  $g \circ f = \text{id}_X$  and  $f \circ g = \text{id}_Y$ .*

## 2 The Real Numbers

### 2.1 Groups, Rings, Fields

#### Definition 2.1: Groups

A **group** is a non-empty set  $G$  together with a rule (called an *operation*) denoted by  $\star : G \times G \rightarrow G$  that combines any two elements of  $G$  into another element of  $G$ . This operation must satisfy three conditions:

- **Associativity:** No matter how you place parentheses, the result is the same for all  $a, b, c \in G$ ,

$$(a \star b) \star c = a \star (b \star c).$$

- **Neutral element:** There is a special element  $e \in G$  such that combining it with any  $a \in G$  leaves  $a$  unchanged, i.e.,

$$\forall a \in G : a \star e = e \star a = a.$$

- **Inverse element:** Every  $a \in G$  has a 'partner'  $a^{-1} \in G$  that 'cancels it out', giving the neutral element, i.e.,

$$a \star a^{-1} = a^{-1} \star a = e.$$

Note that, in general, one does not require that  $a \star b = b \star a$ . If the order of the operation does not matter, i.e.,  $a \star b = b \star a$  for all  $a, b \in G$ , the group is called **commutative** or **abelian**.

#### Lemma 2.2: Basic Properties of Groups

Let  $(G, \star)$  be a group. Then:

1. The neutral element is unique.
2. The inverse of an element is unique.
3. The inverse of the inverse of an element is the element itself, namely  $(a^{-1})^{-1} = a$  for all  $a \in G$ .

*Proof.* 1. Assume that, in addition to  $e \in G$ , we have a second element  $e'$  with the property that  $e' \star a = a \star e' = a$  for all elements  $a \in G$ . Then, we can choose  $a = e$  to obtain

$$e \star e' = e.$$

Similarly, since  $e$  is a neutral element, we have

$$e \star e' = e'.$$

Combining the two identities, we get

$$e = e \star e' = e'.$$

This proves that  $e' = e$ , so we speak of *the* neutral element of a group.

2. Assume that for an element  $a \in G$ , there exists two elements  $b, c \in G$  that are both the inverse

of  $a$ , namely

$$a \star b = b \star a = e, \quad a \star c = c \star a = e.$$

Then, using associativity, we observe that

$$b = b \star e = b \star (a \star c) = (b \star a) \star c = e \star c = c.$$

This proves that the inverse of an element  $a$  is unique, so we can speak of *the* inverse element, and the notation  $a^{-1}$  makes sense.

3. Since  $a \star a^{-1} = e$ , we deduce that  $a$  is the inverse element of  $a^{-1}$ , thus

$$(a^{-1})^{-1} = a. \quad (2.1)$$

□

Groups capture the idea of combining elements with a single operation. But to describe the arithmetic of numbers more faithfully, we also need a second operation (as we do with addition and multiplication). This leads us to the notion of *rings* and *fields*.

### Definition 2.3: Rings and Fields

A **ring** is a non-empty set  $R$  in which we can both 'add' and 'multiply' elements with two operations '+' and '·'. Also, these two operations are compatible with each other. More precisely:

- $(R, +)$  is a **commutative group**, with neutral element denoted 0.
- Multiplication  $\cdot$  is **associative**, has a **neutral element** (usually written as 1), and **distributes over addition**, i.e.,

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (b + c) \cdot a = b \cdot a + c \cdot a \quad \forall a, b, c \in R.$$

If multiplication is also commutative, we call  $(R, +, \cdot)$  a **commutative ring**. Note that, unlike addition, we do not require that every element has an inverse for multiplication. A **field** is a special kind of commutative ring, i.e. every non-zero element has an inverse for multiplication. In other words, if  $(R, +, \cdot)$  is a commutative ring, then  $(R, +, \cdot)$  is a field if  $R \setminus \{0\}$  forms a commutative group under multiplication. Traditionally, we use the letter  $F$  to denote a field. We also write  $F^* = F \setminus \{0\}$  for the set of all invertible elements of  $F$ .

### Lemma 2.4: Basic Properties of Fields

Let  $(F, +, \cdot)$  be a field and let  $a, b \in F$ . Then:

1.  $0 \cdot a = a \cdot 0 = 0$ .
2.  $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$ . In particular  $(-1) \cdot a = -a$ .
3.  $(-a) \cdot (-b) = a \cdot b$ . In particular,  $(-a)^{-1} = -(a^{-1})$  whenever  $a \neq 0$ .

*Proof.* 1. Since 0 is the neutral element for the addition, we have  $0+0=0$ . Hence, using distributivity, we get

$$0 \cdot a = (0 + 0) \cdot a = (0 \cdot a) + (0 \cdot a).$$

Adding  $-0 \cdot a$  (i.e., the inverse of  $0 \cdot a$ ), we deduce that  $0 \cdot a = 0$ . The case of  $a \cdot 0$  is analogous.

2. By the distributive law,

$$a \cdot b + a \cdot (-b) = a \cdot (b + (-b)) = a \cdot 0 = 0.$$

So  $a \cdot (-b)$  is the additive inverse of  $a \cdot b$ , i.e.,  $-(a \cdot b) = a \cdot (-b)$ . Taking  $b = 1$  gives  $-a = (-1) \cdot a$ . The validity of  $(-a) \cdot b = -(a \cdot b)$  follows by exchanging  $a$  and  $b$  in the argument above.

3. By 2. we know that  $-(a \cdot b) = a \cdot (-b)$ . Hence, recalling Equation 2.1,

$$a \cdot b = -(a \cdot (-b)).$$

On the other hand, applying 2. with  $(-b)$  instead of  $b$ , we also have

$$-(a \cdot (-b)) = (-a) \cdot (-b).$$

Combining the two identities above, we conclude that  $(-a) \cdot (-b) = a \cdot b$ . Finally, taking  $b = a^{-1}$  yields  $(-a) \cdot (-a^{-1}) = a \cdot a^{-1} = 1$ , which gives the second assertion.  $\square$

## 2.2 Order Relation

### Definition 2.5: Cartesian Product

Let  $X$  and  $Y$  be two sets. The **cartesian product**  $X \times Y$  is the set of ordered pairs of elements of  $X$  and  $Y$ , i.e.,

$$X \times Y := \{(x, y) \mid x \in X, y \in Y\}.$$

### Definition 2.6: Subsets

Let  $P$  and  $Q$  be sets. Then

- $P$  is a **subset** of  $Q$ , written  $P \subset Q$  (or  $P \subseteq Q$ ), if every element of  $P$  also belongs to  $Q$ .
- $P$  is a **proper subset** of  $Q$ , written  $P \subsetneq Q$ , if  $P$  is a subset of  $Q$  but  $P \neq Q$ .
- We write  $P \not\subseteq Q$  if  $P$  is not a subset of  $Q$ .

### Definition 2.7: Relations

Let  $X$  be a set. A **relation** on  $X$  is a subset  $\mathcal{R} \subseteq X \times X$ , that is, a collection of ordered pairs of elements of  $X$ . If  $(x, y) \in \mathcal{R}$  we write  $x\mathcal{R}y$ . Common symbols for relations include  $<, \leq, \sim, \equiv, \cong$ . If  $\sim$  is a relation on  $X$ , we write  $x \not\sim y$  if  $x \sim y$  does not hold. A relation  $\sim$  may have the following properties:

1. **Reflexive:** if  $x \sim x \quad \forall x \in X$ .
2. **Transitive:** if  $x \sim y$  and  $y \sim z$ , then  $x \sim z$ .
3. **Symmetric:** if  $x \sim y$ , then  $y \sim x$ .
4. **Antisymmetric:** if  $x \sim y$  and  $y \sim x$ , then  $x = y$ .

A relation is an **equivalence relation** if it is reflexive, transitive and symmetric. It is an **order relation** if it is reflexive, transitive and antisymmetric.

## 2.3 Ordered Fields

### Definition 2.8: Ordered Field

Let  $F$  be a field, and let  $\leq$  be an order relation on  $F$ . We call  $(F, \leq)$ , or simply  $F$ , an **ordered field** if the following hold:

1. **Linearity of order:** for all  $x, y \in F$ , at least one of  $x \leq y$  or  $y \leq x$  holds.
2. **Compatibility with addition:** for all  $x, y, z \in F$ ,

$$x \leq y \Rightarrow x + z \leq y + z.$$

3. **Compatibility with multiplication:** for all  $x, y \in F$ ,

$$0 \leq x \wedge 0 \leq y \Rightarrow 0 \leq x \cdot y.$$

### Lemma 2.9: Ordered Field: Basic Consequences

Let  $(F, \leq)$  be an ordered field, and let  $x, y, z, w \in F$ . Then:

- (a) (Trichotomy) Either  $x < y$ , or  $x = y$ , or  $x > y$ .
- (b) If  $x < y$  and  $y \leq z$ , then  $x < z$ . (Analogously,  $x \leq y$  and  $y < z$  imply  $x < z$ .)
- (c) (Addition of inequalities) If  $x \leq y$  and  $z \leq w$ , then  $x + z \leq y + w$ . (Analogously,  $x < z$  and  $z \leq w$  imply  $x < w$ .)
- (d)  $x \leq y$  if and only if  $0 \leq y - x$ .
- (e)  $x \leq 0$  if and only if  $0 \leq -x$ .
- (f)  $x^2 \geq 0$ , and  $x^2 > 0$  if  $x \neq 0$ .
- (g)  $0 < 1$ .
- (h) If  $0 \leq x$  and  $y \leq z$ , then  $xy \leq xz$ .
- (i) If  $x \leq 0$  and  $y \leq z$ , then  $xy \geq xz$ .
- (j) If  $0 < x \leq y$ , then  $0 < y^{-1} \leq x^{-1}$ .
- (k) If  $0 \leq x \leq y$  and  $0 \leq z \leq w$ , then  $0 \leq xz \leq yw$ .
- (l) If  $x + y \leq x + z$ , then  $y \leq z$ .
- (m) If  $xy \leq xz$  and  $x > 0$ , then  $y \leq z$ .



### Lemma 2.10: Integers and Rationals Inside an Ordered Field

Let  $(F, \leq)$  be an ordered field, and denote by  $0$  and  $1$  the neutral elements for addition and multiplication, respectively. Then:

(i) The elements  $\dots, -2, -1, 0, 1, 2, \dots$  defined by

$$2 = 1 + 1, \quad 3 = 2 + 1, \dots, \quad -n = (-1) \cdot n$$

are all distinct and satisfy

$$\dots < -2 < -1 < 0 < 1 < 2 < 3 < \dots$$

We denote this set of elements by  $\mathbb{Z}$ , and we call them 'integers'

(ii) Every fraction  $pq^{-1}$  with  $p, q \in \mathbb{Z}$ ,  $q \neq 0$ , lies in  $F$  and the set of all such elements is denoted by  $\mathbb{Q}$ . Also,

$$\mathbb{Z} \subsetneq \mathbb{Q} \subseteq F.$$

*Proof.* (i) By Lemma 2.9(g), we have that  $0 < 1$ . Then Lemma 2.9(c) yields  $0 < 1 < 2 < 3 < \dots$ , and taking negatives gives  $\dots < -2 < -1 < 0$ . Hence all these elements are distinct.

(ii) For  $q \neq 0$ ,  $q$  is invertible in  $F$ ; define  $\frac{p}{q} = pq^{-1}$ . The set of such fractions is a field contained in  $F$ , which we denote by  $\mathbb{Q}$ .

To show that  $\mathbb{Q}$  strictly contains  $\mathbb{Z}$ , consider  $\frac{1}{2}$  (the inverse of 2). Since  $2 > 1$ , it follows from Lemma 2.9(j) that  $0 < \frac{1}{2} < 1$ , so  $\frac{1}{2} \notin \mathbb{Z}$ .  $\square$

### Definition 2.11: Absolute Value and Sign

Let  $(F, \leq)$  be an ordered field.

- The **absolute value** (or **modulus**) is the function  $|\cdot| : F \rightarrow F$  defined by

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

- The **sign** is the function  $\text{sgn} : F \rightarrow \{-1, 0, 1\}$  defined by

$$\text{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

### Lemma 2.12: Absolute Value and Sign: Basic Properties

Let  $(F, \leq)$  be an ordered field and let  $x, y \in F$ . Then:

- (a)  $x = \operatorname{sgn}(x)|x|$ ,  $|-x| = |x|$ ,  $\operatorname{sgn}(-x) = -\operatorname{sgn}(x)$ .
- (b)  $|x| \geq 0$ , and  $|x| = 0$  if and only if  $x = 0$  (by Trichotomy Lemma ??).
- (c) (Multiplicativity)  $\operatorname{sgn}(xy) = \operatorname{sgn}(x)\operatorname{sgn}(y)$  and  $|xy| = |x||y|$ .
- (d) If  $x \neq 0$ , then  $|x^{-1}| = |x|^{-1}$ .
- (e)  $|x| \leq y$  iff  $-y \leq x \leq y$ .
- (f)  $|x| < y$  iff  $-y < x < y$ .
- (g) (Triangle inequality)  $|x + y| \leq |x| + |y|$ .
- (h) (Inverse triangle inequality)  $||x| - |y|| \leq |x - y|$ .

*Proof.* (g) Thanks to (e) we have  $-|x| \leq x \leq |x|$  and  $-|y| \leq y \leq |y|$ . Adding these two inequalities we get

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

Applying (e) again yields the result.

(h) From (g) we have  $|x| \leq |x - y| + |y|$ , therefore

$$|x| - |y| \leq |x - y|.$$

Exchanging the roles of  $x$  and  $y$ , we also have  $|y| - |x| \leq |y - x| = |x - y|$ . Combining these two inequalities yields

$$-|x - y| \leq |x| - |y| \leq |x - y|,$$

and the result follows by applying (e) again. □

## 2.4 Completeness Axiom

### Definition 2.13: Completeness Axiom

Let  $(K, \leq)$  be an ordered field. We say that  $(K, \leq)$  is **complete** (or a **completely ordered field**) if the following statement holds:

Let  $X, Y$  be non-empty subsets of  $K$  such that  $x \leq y$  for all  $x \in X$  and  $y \in Y$ . Then there exists  $c \in K$  lying between  $X$  and  $Y$ , in the sense that  $x \leq c \leq y$  for all  $x \in X$  and  $y \in Y$ .

The statement above is called the **completeness axiom**.

### Definition 2.14: Real Numbers

We call **the field of real numbers**, any completely ordered field and denote it by  $\mathbb{R}$ .

## 2.5 Intervals

### Definition 2.15: Intervals

Let  $a, b \in \mathbb{R}$ . We define:

- The **closed interval**

$$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\};$$

- The **open interval**

$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\};$$

- The **half-open intervals**

$$[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\} \quad \text{and} \quad (a, b] := \{x \in \mathbb{R} \mid a < x \leq b\};$$

- The **unbounded closed intervals**

$$[a, \infty) := \{x \in \mathbb{R} \mid a \leq x\} \quad \text{and} \quad (-\infty, b] := \{x \in \mathbb{R} \mid x \leq b\};$$

- The **unbounded open intervals**

$$(a, \infty) := \{x \in \mathbb{R} \mid a < x\} \quad \text{and} \quad (-\infty, b) := \{x \in \mathbb{R} \mid x < b\};$$

### Definition 2.16: Set Operations

Let  $P, Q$  be sets. The **intersection**  $P \cap Q$ , the **union**  $P \cup Q$ , the **relative complement**  $P \setminus Q$  and the **symmetric difference**  $P \triangle Q$  are defined by

$$P \cap Q = \{x \mid x \in P \text{ and } x \in Q\},$$

$$P \cup Q = \{x \mid x \in P \text{ or } x \in Q\},$$

$$P \setminus Q = \{x \mid x \in P \text{ and } x \notin Q\},$$

$$P \triangle Q = (P \setminus Q) \cup (Q \setminus P) = (P \cup Q) \setminus (P \cap Q).$$

### Definition 2.17: Union and Intersection of several Sets

Let  $\mathcal{A}$  be a family of sets (i.e., a set whose elements are sets). We define the **union** and **intersection** of the sets in  $\mathcal{A}$  as

$$\bigcup_{A \in \mathcal{A}} A = \{x \mid \exists A \in \mathcal{A} : x \in A\}, \quad \bigcap_{A \in \mathcal{A}} A = \{x \mid \forall A \in \mathcal{A} : x \in A\}.$$

If  $\mathcal{A} = \{A_1, A_2, \dots\}$ , we also write

$$\bigcup_{i=1}^{\infty} A_i = \{x \mid \exists i \geq 1 : x \in A_i\}, \quad \bigcap_{i=1}^{\infty} A_i = \{x \mid \forall i \geq 1 : x \in A_i\}.$$

**Definition 2.18: Neighborhoods**

Let  $x \in \mathbb{R}$ . A **neighborhood** of  $x$  is a set containing an interval  $I$  such that  $x \in I$ . Given  $\delta > 0$ , the open interval  $(x - \delta, x + \delta)$  is called the  $\delta$ -**neighborhood** of  $x$ .

**Definition 2.19: Open and Closed Sets**

A subset  $U \subseteq \mathbb{R}$  is called **open** in  $\mathbb{R}$  if for every  $x \in U$  there exists open interval  $I$  such that  $x \in I$  and  $I \subseteq U$ . A subset  $F \subseteq \mathbb{R}$  is called **closed** in  $\mathbb{R}$  if its complement  $\mathbb{R} \setminus F$  is open.

**Remark 2.20.** The sets  $\emptyset$  and  $\mathbb{R}$  are both open in  $\mathbb{R}$ . Hence, they are also closed since  $\emptyset^c = \mathbb{R}$  and  $\mathbb{R}^c = \emptyset$ . We note that  $\mathbb{Q} \subseteq \mathbb{R}$  and  $[a, b] \subseteq \mathbb{R}$  are neither open nor closed.

**Remark 2.21.** Let  $\mathcal{U}$  be a family of open sets, and  $\mathcal{F}$  be a family of closed subsets of  $\mathbb{R}$ . Then the union and intersection

$$\bigcup_{U \in \mathcal{U}} U, \quad \bigcap_{F \in \mathcal{F}} F$$

are open and closed, respectively.

**2.6 Complex Numbers**

Starting from the field of real numbers  $\mathbb{R}$ , we define the set of **complex numbers** as

$$\mathbb{C} = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

We denote the elements  $z = (x, y) \in \mathbb{C}$  in the form  $z = x + iy$ , where  $i$  is the **imaginary unit**. Here  $x \in \mathbb{R}$  is the **real part** of  $z$ , written as  $x = \operatorname{Re}(z)$ , and  $y \in \mathbb{R}$  is the **imaginary part**, written as  $y = \operatorname{Im}(z)$ . Elements with  $\operatorname{Im}(z) = 0$  are called **real**, while those with  $\operatorname{Re}(z) = 0$  are **purely imaginary**. Via the injective map  $\mathbb{R} \ni x \mapsto x + i \cdot 0 \in \mathbb{C}$ , we identify  $\mathbb{R}$  with the subset of real numbers inside  $\mathbb{C}$ .

As you may expect from previous knowledge, we want to satisfy  $i^2 = -1$ . To achieve this, we define addition and multiplication on  $\mathbb{C}$  so that it becomes a field. Additionally, we want these operations to coincide with the usual addition and multiplication when considering real numbers.

Since  $i^2 = -1$ , using commutativity and distributivity we get

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + iy_2x_1 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

This motivates the following definition

**Definition 2.22: Addition and Multiplication on  $\mathbb{C}$** 

On  $\mathbb{C} = \mathbb{R} \times \mathbb{R}$  we define **addition** and **multiplication** as follows:

$$\begin{aligned} (x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2), \\ (x_1, y_1) \cdot (x_2, y_2) &= (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1). \end{aligned}$$

**Proposition 2.23:  $\mathbb{C}$  is a Field**

With the operation of Definition 2.22, together with the zero element  $(0, 0)$  and the unit element  $(1, 0)$ , the set  $\mathbb{C}$  is a field.

**Definition 2.24: Complex Conjugation**

For  $z = x + iy \in \mathbb{C}$  we define its **conjugate** as  $\bar{z} = x - iy$ . The mapping  $\mathbb{C} \ni z \mapsto \bar{z} \in \mathbb{C}$  is called **complex conjugation**.

**Lemma 2.25: Properties of Complex Conjugation**

For all  $z, w \in \mathbb{C}$ :

(i)  $z\bar{z} = x^2 + y^2 \in \mathbb{R}_{\geq 0}$ . In particular,  $z\bar{z} = 0$  if and only if  $z = 0$ .

(ii)  $\overline{z + w} = \bar{z} + \bar{w}$ .

(iii)  $\overline{z\bar{w}} = \bar{z}\bar{w}$ .

*Proof.* Property (i) follows from the fact that, for  $z = x + iy$ ,  $(x + iy)(x - iy) = x^2 + y^2$ . Also,  $x^2 + y^2 = 0$  if and only if  $x + iy = 0$ . Properties (ii) and (iii) follow from a direct computation, writing  $z = x_1 + iy_1$  and  $w = x_2 + iy_2$ , which yields

$$\begin{aligned}\overline{z + w} &= \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2) = \bar{z} + \bar{w}, \\ \overline{z \cdot w} &= \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)} = (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) \\ &= (x_1 - iy_1) \cdot (x_2 - iy_2) = \bar{z} \cdot \bar{w}. \quad \square\end{aligned}$$

**Definition 2.26: Absolute Value**

The **absolute value** (or **norm**) on  $\mathbb{C}$  is the map  $|\cdot| : \mathbb{C} \rightarrow \mathbb{R}$  given by

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}, \quad z = x + iy \in \mathbb{C}.$$

**Lemma 2.27: Cauchy-Schwarz Inequality**

If  $z = x_1 + iy_1$ , and  $w = x_2 + iy_2$ , then

$$x_1x_2 + y_1y_2 \leq |z||w|. \quad (2.2)$$

*Proof.* We observe that

$$\begin{aligned}|z|^2|w|^2 - (x_1x_2 + y_1y_2)^2 &= (x_1^2 + y_1^2)(x_2^2 + y_2^2) - (x_1x_2 + y_1y_2)^2 \\ &= x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2 - (x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2) \\ &= x_1^2y_2^2 + y_1^2x_2^2 - 2x_1x_2y_1y_2 \\ &= (y_1x_2 - x_1y_2)^2 \geq 0.\end{aligned}$$

This proves that  $(x_1x_2 + y_1y_2)^2 \leq |z|^2|w|^2$ , so it follows that

$$|x_1x_2 + y_1y_2| \leq |z||w|.$$

Since  $x \leq |x|$  for all  $x \in \mathbb{R}$ , we obtain Equation 2.2. □

### Proposition 2.28: Triangle Inequality

For all  $z, w \in \mathbb{C}$ , one has

$$|z + w| \leq |z| + |w|.$$

*Proof.* For  $z = x_1 + iy_1$  and  $w = x_2 + iy_2$ , using Lemma 2.27, we have

$$\begin{aligned} |z + w|^2 &= (x_1 + x_2)^2 + (y_1 + y_2)^2 \\ &= |z|^2 + |w|^2 + 2(x_1x_2 + y_1y_2) \\ &\leq |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2. \end{aligned}$$

Taking square roots proves the result. □

### Definition 2.29: Circular Disks

For  $z \in \mathbb{C}$  and  $r > 0$ , we define the **open disk** with radius  $r > 0$  around  $z$  as

$$B(z, r) := \{w \in \mathbb{C} \mid |z - w| < r\},$$

and the **closed disk** with radius  $r > 0$  around  $z$  as

$$\overline{B(z, r)} := \{w \in \mathbb{C} \mid |z - w| \leq r\}.$$

In other words, the open disk  $B(z, r)$  is the set of points at distance strictly less than  $r$  from  $z$ . We note that this definition is compatible with the one of neighborhoods in  $\mathbb{R}$ : if  $x \in \mathbb{R}$  and  $r > 0$ , then

$$B(x, r) \cap \mathbb{R} = (x - r, x + r).$$

### Definition 2.30: Open and Closed Sets

A set  $U \subseteq \mathbb{C}$  is **open** if for every  $z \in U$  there exists  $r > 0$  such that  $B(z, r) \subseteq U$ . A set  $C \subseteq \mathbb{C}$  is **closed** if its complement  $\mathbb{C} \setminus C$  is open.

## 2.7 Maximum and Supremum

### 2.7.1 Existence of the Supremum

#### Definition 2.31: Bounded Sets, Maxima and Minima

Let  $X \subseteq \mathbb{R}$  be a subset of real numbers.

- $X$  is **bounded from above** if there exists  $s \in \mathbb{R}$  such that  $x \leq s$  for all  $x \in X$ . Such a number  $s$  is called an **upper bound** of  $X$ . If  $s$  is an upper bound and also an element of  $X$ , we say that  $s$  is the **maximum** of  $X$  and write

$$s = \max(X).$$

- Analogously,  $X$  is **bounded from below** if there exists  $r \in \mathbb{R}$  such that  $r \leq x$  for all  $x \in X$ . Such a number  $r$  is called a **lower bound** of  $X$ . If  $r$  is a lower bound and also an element of  $X$ , we say that  $r$  is the **minimum** of  $X$  and write

$$r = \min(X).$$

- $X$  is called **bounded** if it is both bounded from above and bounded from below.

**Remark 2.32.** If a set  $X \subseteq \mathbb{R}$  has a maximum, then it is unique. Indeed, if  $x_1, x_2 \in X$  are both maxima, then  $x_1 \leq x_2$  (since  $x_2$  is a maximum) and  $x_2 \leq x_1$  (since  $x_1$  is a maximum), so  $x_1 = x_2$ .

A closed interval  $[a, b]$  with  $a < b$  has both a minimum and maximum, i.e.,  $a = \min([a, b])$  and  $b = \max([a, b])$ . But not all sets have a maximum. For instance, the open interval  $(a, b)$  does not have a maximum because the endpoint  $b$ , though an upper bound, is not contained in the set. Similarly  $\mathbb{R}$  and unbounded intervals such as  $[a, \infty)$  or  $(a, \infty)$  have no maximum.

#### Definition 2.33: Supremum

Let  $X \subseteq \mathbb{R}$  be a subset and let

$$A := \{a \in \mathbb{R} \mid x \leq a \quad \forall x \in X\}$$

be the set of all upper bounds of  $X$ . If  $A$  has a minimum, we call this minimum the **supremum** of  $X$  and write

$$\sup(X) = \min(A).$$

The **infimum** is defined analogously using the maximum of the set of all lower bounds.

In other words, the supremum of  $X$  is the smallest real number that is greater than or equal to every element of  $X$ . Note that we can describe the supremum  $s = \sup(X)$  as follows

$$x \leq s \quad \forall x \in X, \quad \text{and} \quad \text{if } t < s, \text{ the } t \text{ is not an upper bound of } X. \quad (2.3)$$

This means that for every  $t < s$ , there exists some  $x \in X$  such that  $x > t$ . That is,

$$x \leq s \quad \forall x \in X, \quad \text{and} \quad \forall t < s \exists x \in X : x > t. \quad (2.4)$$

The two characterizations 2.3 and 2.4 are equivalent.

Note that not every set has a supremum. If  $X = \emptyset$  or if  $X$  is unbounded from above, then  $\sup(X)$  does not exist. However, for any non-empty and bounded-above subset of  $\mathbb{R}$ , the supremum always exists.

**Remark 2.34.** *If a set  $X$  has a maximum, then this element is also the supremum. Indeed, the maximum is an upper bound of  $X$ , and since it lies in  $X$ , no smaller upper bound can exist.*

### Theorem 2.35: Existence of Supremum

*Let  $X \subseteq \mathbb{R}$  be non-empty and bounded from above. Then  $\sup(X)$  exists and is a real number.*

*Proof.* Since  $X$  is bounded from above, the set  $A := \{a \in \mathbb{R} \mid x \leq a \quad \forall x \in X\}$  of upper bounds is non-empty. Since  $x \leq a$  for any  $x \in X$  and  $a \in A$ , we can apply the completeness axiom (Definition 2.13) to find  $c \in \mathbb{R}$  such that

$$x \leq c \leq a \quad \forall x \in X, \forall a \in A.$$

The first inequality implies that  $c$  is itself an upper bound (so  $c \in A$ ), while the second inequality tells us that  $c$  is smaller than or equal to every upper bound. Hence,  $c = \min(A) = \sup(X)$ .  $\square$

### Proposition 2.36: Supremum and Set Operations

*Let  $X$  and  $Y$  be non-empty subsets of  $\mathbb{R}$  that are bounded from above. Define*

$$X + Y := \{x + y \mid x \in X, y \in Y\} \quad \text{and} \quad X \cdot Y := \{x \cdot y \mid x \in X, y \in Y\}.$$

*The sets  $X \cup Y$ ,  $X \cap Y$ , and  $X + Y$  are also bounded from above. Moreover, if  $X, Y \subseteq \mathbb{R}_{\geq 0}$  (i.e.,  $x \geq 0$  and  $y \geq 0$  for all  $x \in X$  and  $y \in Y$ ), then  $X \cdot Y$  is bounded from above as well. In these cases, the following formulas hold:*

- (1)  $\sup(X \cup Y) = \max\{\sup(X), \sup(Y)\}$ ,
- (2) If  $X \cap Y \neq \emptyset$ , then  $\sup(X \cap Y) \leq \min\{\sup(X), \sup(Y)\}$ ,
- (3)  $\sup(X + Y) = \sup(X) + \sup(Y)$ ,
- (4) If  $X, Y \subseteq \mathbb{R}_{\geq 0}$ , then  $\sup(X \cdot Y) = \sup(X) \cdot \sup(Y)$ .

*Proof.* (3) Let  $x_0 = \sup(X)$  and  $y_0 = \sup(Y)$ . For any  $z \in X + Y$ , there exists  $x \in X$  and  $y \in Y$  such that  $z = x + y$ . Since  $x \leq x_0$  and  $y \leq y_0$ , we have

$$z = x + y \leq x_0 + y_0,$$

so  $x_0 + y_0$  is an upper bound for  $X + Y$ . We now want to show that  $x_0 + y_0 = \sup(X + Y)$ .

Let  $z_0 = \sup(X + Y)$  and suppose, by contradiction, that

$$\varepsilon := x_0 + y_0 - z_0 > 0.$$

Since  $x_0 = \sup(X)$ , by the characterization 2.4 there exists  $x \in X$  such that  $x > x_0 - \varepsilon/2$ . Likewise, there exists  $y \in Y$  such that  $y > y_0 - \varepsilon/2$ . Setting  $z = x + y$ , we obtain

$$z > x_0 - \frac{\varepsilon}{2} + y_0 - \frac{\varepsilon}{2} = x_0 + y_0 - \varepsilon = z_0,$$

contradicting the assumption that  $z_0$  is an upper bound for  $X + Y$ . Therefore,  $z_0 = x_0 + y_0$ .



(4) The proof is analogous. If all elements of  $X$  and  $Y$  are non-negative, and we set  $x_0 = \sup(X)$  and  $y_0 = \sup(Y)$ , then for any  $z = x \cdot y \in X \cdot Y$ , we have

$$z = x \cdot y \leq x_0 \cdot y_0,$$

which shows that  $x_0 \cdot y_0$  is an upper bound for  $X \cdot Y$ . Using a similar ' $\varepsilon$ -argument' as done above, when proving (3), one shows that this upper bound is sharp, i.e.,  $x_0 \cdot y_0$  is the least upper bound.  $\square$

## 2.8 Two-Point Compactification

In this section, we extend the notions of **supremum** and **infimum** to arbitrary subsets of  $\mathbb{R}$ . To do so, we introduce two formal symbols

$$+\infty \quad \text{and} \quad -\infty,$$

which are not real numbers. We define the **extended real numbers line** (also called the **two-point compactification** of  $\mathbb{R}$ ) by

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}.$$

We extend the usual order relation  $\leq$  on  $\mathbb{R}$  to  $\overline{\mathbb{R}}$  by requiring that

$$-\infty < x < +\infty \quad \forall x \in \mathbb{R}.$$

For simplicity, we often write  $\infty$  instead of  $+\infty$ .

We now introduce some standard (but informal) computation rules involving these symbols. For all  $x \in \mathbb{R}$ , we adopt the conventions:

$$\infty + x = \infty + \infty = \infty, \quad -\infty + x = -\infty - \infty = -\infty.$$

If  $x > 0$ , then

$$x \cdot \infty = \infty \cdot \infty = \infty, \quad x \cdot (-\infty) = \infty \cdot (-\infty) = -\infty,$$

while for  $x < 0$  we have

$$x \cdot \infty = -\infty \cdot \infty = -\infty, \quad x \cdot (-\infty) = -\infty \cdot (-\infty) = \infty.$$

These rules are widely used as notational shorthand, but one must handle them with care. Expressions like

$$\infty - \infty, \quad 0 \cdot \infty, \quad \text{or similar}$$

are undefined and should be avoided.

### Definition 2.37: Supremum and Infimum in the Extended Line

Let  $X \subseteq \mathbb{R}$ .

- If  $X$  is not bounded from above, we define  $\sup(X) = \infty$ .
- If  $X = \emptyset$ , we define  $\sup(\emptyset) = -\infty$ .
- If  $X$  is not bounded from below, we define  $\inf(X) = -\infty$ .
- If  $X = \emptyset$ , we define  $\inf(\emptyset) = \infty$ .

In this context, we refer to  $\infty$  and  $-\infty$  as **indefinite values**.

In other words:

- Saying  $\sup(X) = \infty$  means that  $X$  is not bounded from above, i.e.,

$$\forall x_0 \in X \exists x \in X : x > x_0.$$

- Saying  $\sup(X) = -\infty$  means that  $X$  is empty.
- Similarly,  $\inf(X) = -\infty$  means that  $X$  is not bounded from below, and  $\inf(X) = \infty$  means  $X$  is empty.

## 2.9 Consequences of Completeness

### 2.9.1 Archimedean Principle

The archimedean principle states that for every real number  $x \in \mathbb{R}$  there exists an integer  $n$  greater than  $x$ . The following theorem, proved using the existence of suprema (and implicitly the completeness axiom), gives a precise formulation of this principle.

#### Theorem 2.38: Archimedean Principle

*For every  $x \in \mathbb{R}$  there exists exactly one  $n \in \mathbb{Z}$  such that*

$$n \leq x < n + 1.$$

*Proof.* We first treat the case  $x \geq 0$ . Fix  $x \geq 0$  and define

$$E = \{n \in \mathbb{Z} \mid n \leq x\}.$$

Since  $0 \in E$  and  $x$  is an upper bound,  $E$  is a non-empty subset of  $\mathbb{R}$  bounded from above. Hence, by Theorem 2.35, the supremum  $s_0 = \sup(E)$  exists. From the definition of supremum we deduce:

- $s_0 \leq x$  (because  $x$  is an upper bound);
- there exists  $n_0 \in E$  with  $s_0 - 1 < n_0$  (otherwise  $s_0 - 1$  would also be an upper bound).

From (ii) we obtain  $s_0 < n_0 + 1$ , which implies

- $n_0 + 1 \notin E$  (otherwise  $s_0$  would not be an upper bound for  $E$ ).

Moreover, since  $m \leq s_0$  for every  $m \in E$ , we have  $m < n_0 + 1$  for all  $m \in E$ . As all elements of  $E$  are integers,

$$m < n_0 + 1 \quad \Leftrightarrow \quad m - n_0 < 1 \quad \Leftrightarrow \quad m - n_0 \leq 0 \quad \Leftrightarrow \quad m \leq n_0.$$

Thus, every  $m \in E$  is less than or equal to  $n_0$ , and since  $n_0 \in E$ , we conclude that  $n_0 = \max(E)$ . In particular, by Remark 2.34, the maximum is also the supremum, so  $s_0 = n_0$ .

Finally, recalling (iii) and the definition of  $E$ , we have  $n_0 + 1 > x$ . Together with (i), this shows

$$n_0 = s_0 \leq x < n_0 + 1,$$

establishing the claim for any  $x \geq 0$ .

Now, if  $x < 0$ , apply the previous argument to  $-x > 0$ . Then there exists  $m \in \mathbb{Z}$  such that

$$m \leq -x < m + 1,$$

which is equivalent to

$$-m - 1 < x \leq -m.$$

If  $x = -m$ , then set  $n = -m$ . If  $x < -m$ , set  $n = -m - 1$ . In both cases, we obtain

$$n \leq x < n + 1.$$

Finally, for uniqueness, assume that  $n_1, n_2 \in \mathbb{Z}$  both satisfy  $n_i \leq x < n_i + 1$ . From  $n_1 \leq x < n_2 + 1$  we deduce that  $n_1 < n_2 + 1$ , and therefore  $n_1 \leq n_2$ . Reversing the roles of  $n_1$  and  $n_2$  gives  $n_2 \leq n_1$ . Hence,  $n_1 = n_2$ .  $\square$

### Definition 2.39: Integer and Fractional Parts

The **integer part**  $\lfloor x \rfloor$  of  $x \in \mathbb{R}$  is the integer  $n \in \mathbb{Z}$  uniquely determined by Theorem 2.38 such that  $n \leq x < n + 1$ . The map  $x \mapsto \lfloor x \rfloor$  from  $\mathbb{R}$  to  $\mathbb{Z}$  is called the **rounding function**. The **fractional part** of  $x$  is defined as

$$\{x\} = x - \lfloor x \rfloor \in [0, 1).$$

### Corollary 2.40: $\frac{1}{n}$ is Arbitrarily Small

For every  $\varepsilon > 0$  there exists  $n \in \mathbb{N}$ , with  $n \geq 1$ , such that

$$\frac{1}{n} < \varepsilon.$$

*Proof.* Applying Theorem 2.38 to  $x = \frac{1}{\varepsilon} > 0$ , we find  $m \in \mathbb{Z}$  such that

$$m \leq \frac{1}{\varepsilon} < m + 1.$$

Set  $n := m + 1$ . In this way we have  $0 < \frac{1}{\varepsilon} < n$ , which is equivalent to  $n > 0$  (therefore,  $n \geq 1$ ) and  $\frac{1}{n} < \varepsilon$ .  $\square$

**Definition 2.41: Dense Sets**

A subset  $X \subseteq \mathbb{R}$  is called **dense** in  $\mathbb{R}$  if every open non-empty interval contains an element of  $X$ .

**Corollary 2.42: Density of  $\mathbb{Q}$** 

*For every  $a, b \in \mathbb{R}$  with  $a < b$ , there exists  $r \in \mathbb{Q}$  such that  $a < r < b$ .*

*Proof.* Set  $\varepsilon = b - a$ . By Corollary 2.40, there exists  $m \in \mathbb{N}$  with  $\frac{1}{m} < \varepsilon$ . Then, by Theorem 2.38 applied with  $x = ma$ , there exists  $n \in \mathbb{Z}$  with

$$n \leq ma < n + 1,$$

or equivalently,

$$\frac{n}{m} \leq a < \frac{n+1}{m}.$$

Since  $\frac{1}{m} < \varepsilon$ , by the two inequalities above, we obtain

$$a < \frac{n+1}{m} \leq a + \frac{1}{m} < a + \varepsilon = b.$$

Thus  $r = \frac{n+1}{m}$  is a rational number between  $a$  and  $b$ . □

**Corollary 2.43: Density of  $\mathbb{R} \setminus \mathbb{Q}$** 

*For every  $a, b \in \mathbb{R}$  with  $a < b$ , there exists  $r \in \mathbb{R} \setminus \mathbb{Q}$  such that  $a < r < b$ .*

*Proof.* We want to show that for every  $x \in \mathbb{R}$  and  $\delta > 0$ , there exists an  $a \in \mathbb{R} \setminus \mathbb{Q}$  such that

$$a \in (x - \delta, x + \delta).$$

By Corollary 2.42, we find a  $q \in \mathbb{Q}$  such that  $q \in (x - \delta, x + \delta)$ . By Corollary 2.40 we find an  $N \in \mathbb{N}$  such that

$$\frac{1}{N} < \frac{(x + \delta) - q}{\sqrt{2}} \quad \Rightarrow \quad \frac{\sqrt{2}}{N} < (x + \delta) - q.$$

This implies that

$$x - \delta < q < \frac{\sqrt{2}}{N} + q < x + \delta.$$

Choosing  $r = \frac{\sqrt{2}}{N} + q$  proves the statement. □

## 2.9.2 Uncountability

## Definition 2.44: Cardinality

Let  $X$  and  $Y$  be sets.

- We say  $X$  and  $Y$  have the **same cardinality**, written  $X \sim Y$ , if there is a bijection  $f : X \rightarrow Y$ .
- We write  $X \preceq Y$  if there exists an injection  $f : X \rightarrow Y$ .
- The empty set has cardinality 0.
- A set  $X$  has **finite cardinality**  $|X| = n$  if there exists a bijection with  $\{1, \dots, n\}$ .
- A set is **infinite** if it is not finite.
- A set is **countable** if it has a bijection to  $\mathbb{N}$ . Its cardinality is denoted  $\aleph_0$ , pronounced Aleph-0.
- A set is **uncountable** if it is infinite but not countable.

If  $X \preceq Y$  and  $Y \preceq X$ , then  $X \sim Y$ . In other words, if there exists an injective map  $f : X \rightarrow Y$  and an injective map  $g : Y \rightarrow X$ , then one can find a bijective map  $h : X \rightarrow Y$ . This non-trivial statement is the **Schröder-Bernstein Theorem**.

We will now list some statements about different sets of numbers from the lecture:

1.  $\mathbb{N}$  and the even numbers have the same cardinality.
2.  $\mathbb{N}$  and  $\mathbb{Z}$  have the same cardinality.
3.  $\mathbb{Q}$  is countable, i.e.,  $\mathbb{N} \sim \mathbb{Q} \sim \mathbb{Z}$ .

Proposition 2.45: Uncountability of  $\mathbb{R}$ 

*The set  $\mathbb{R}$  is uncountable.*

## Extra Material

## Definition 2.46: Power Set

Let  $X$  be a set. The **power set**  $\mathcal{P}(X)$  of  $X$  is the set of all subsets of  $X$ , i.e.,

$$\mathcal{P}(X) := \{A \subseteq X\}.$$

## Theorem 2.47: Cantor's Theorem

*For any set  $X$ , the power set  $\mathcal{P}(X)$  has strictly larger cardinality than  $X$ .*

Proposition 2.48: The Reals have the same cardinality as  $\mathcal{P}(\mathbb{N})$ 

$$|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|.$$

### 3 Sequences of Real Numbers

#### 3.1 Convergence of Sequences

##### Definition 3.1: Sequences

A **sequence** is a function  $a : \mathbb{N} \rightarrow \mathbb{R}$ . The image  $a(n)$  of  $n \in \mathbb{N}$  is also written as  $a_n$  and is called the  $n$ -th element of  $a$ . Instead of  $a : \mathbb{N} \rightarrow \mathbb{R}$  one often writes  $(a_n)_{n \in \mathbb{N}}, (a_n)_{n=0}^{\infty}, (a_n)_{n \geq 0}$ .

##### Definition 3.2: (Eventually) Constant Sequences

A sequence  $(x_n)_{n=0}^{\infty}$  is **constant** if  $x_n = x_m \forall n, m \in \mathbb{N}$ . It is **eventually constant** if there exists  $N \in \mathbb{N}$  such that  $x_n = x_m \forall n, m \geq N$ .

##### Definition 3.3: Convergence of Sequences

Let  $(x_n)_{n=0}^{\infty}$  be a sequence in  $\mathbb{R}$ . We say that  $(x_n)_{n=0}^{\infty}$  **converges** (or is **convergent**) if  $\exists A \in \mathbb{R}$  such that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} : |x_n - A| < \varepsilon \quad \forall n \geq N.$$

In this case we write

$$\lim_{n \rightarrow \infty} x_n = A \tag{3.1}$$

and call  $A$  the **limit** of  $(x_n)_{n=0}^{\infty}$ .

##### Lemma 3.4: Uniqueness of the Limit

A convergent sequence  $(x_n)_{n=0}^{\infty}$  has exactly one limit.

*Proof.* Let  $A, B \in \mathbb{R}$  be limits of  $(x_n)_{n=0}^{\infty}$ . Fix  $\varepsilon > 0$ . Then there exists  $N_A, N_B \in \mathbb{N}$  such that  $|x_n - A| < \varepsilon$  for all  $n \geq N_A$  and  $|x_n - B| < \varepsilon$  for all  $n \geq N_B$ . We define  $N := \max\{N_A, N_B\}$ . Then it holds that

$$|A - B| \leq |A - x_N| + |x_N - B| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, it follows that  $A = B$ . □

#### 3.2 Convergent Subsequences and Accumulation Points

##### Definition 3.5: Subsequences

Let  $(x_n)_{n=0}^{\infty}$  be a sequence. A **subsequence** is of the form  $(x_{n_k})_{k=0}^{\infty}$ , where  $(n_k)_{k=0}^{\infty}$  is a strictly increasing sequence of non-negative integers, i.e.,  $n_{k+1} > n_k \forall k \in \mathbb{N}$ .

**Remark 3.6.** Since  $n_{k+1} > n_k$  for all  $k \in \mathbb{N}$  it follows by induction that  $n_k \geq k$  for all  $k \in \mathbb{N}$ .

*Proof.* For  $k = 0$  we have that  $n_0 \geq 0$ , because  $(n_k)_{k=0}^{\infty}$  is a sequence of non-negative integers. So the condition is fulfilled. For the inductive step we want to show that the condition holds for  $k + 1$  under the assumption that the condition is true for  $k$ . Because  $(n_k)_{k=0}^{\infty}$  is also a strictly increasing sequence, we have that  $n_{k+1} > n_k \geq k$ . Additionally since  $n_k \in \mathbb{N}$ , we have that  $n_{k+1} \geq n_k + 1$ . So it follows that  $n_{k+1} \geq n_k + 1 \geq k + 1$ , which proves the condition for  $k + 1$ . □

**Lemma 3.7: Subsequences of Convergent Sequences are Convergent**

Let  $(x_n)_{n=0}^{\infty}$  be a sequence converging to  $A \in \mathbb{R}$ . Then every subsequence  $(x_{n_k})_{k=0}^{\infty}$  also converges to  $A$ .

*Proof.* Let  $(x_n)_{n=0}^{\infty}$  be a sequence converging to  $A \in \mathbb{R}$ . Fix  $\varepsilon > 0$ . Since  $(x_n)_{n=0}^{\infty}$  converges to  $A$ , there exists  $N \in \mathbb{N}$  such that  $|x_n - A| < \varepsilon \forall n \geq N$ . As by Remark 3.6 we know that  $n_k \geq k$  for all  $k \in \mathbb{N}$ . Therefore for all  $k \geq N$  it holds that  $|x_{n_k} - A| < \varepsilon$ .  $\square$

**Definition 3.8: Accumulation Points of Sequences**

Let  $(x_n)_{n=0}^{\infty}$  be a sequence in  $\mathbb{R}$ . A point  $A \in \mathbb{R}$  is an **accumulation point** of  $(x_n)_{n=0}^{\infty}$  if

$$\forall \varepsilon > 0 \forall N \in \mathbb{N} \exists n \geq N : |x_n - A| < \varepsilon.$$

**Proposition 3.9: Subsequences and Accumulation Points**

Let  $(x_n)_{n=0}^{\infty}$  be a sequence in  $\mathbb{R}$ . A point  $A$  is an accumulation point of  $(x_n)_{n=0}^{\infty}$  if and only if there exists a convergent subsequence of  $(x_n)_{n=0}^{\infty}$  with limit  $A$ .

*Proof.* First assume that  $A \in \mathbb{R}$  is an accumulation point of  $(x_n)_{n=0}^{\infty}$ . We construct  $(n_k)_{k \geq 0}$  recursively:

- first, apply the definition of accumulation point with  $N = 1$  and  $\varepsilon = 1 = 2^0$  to find  $n_0 \geq 1$  with  $|x_{n_0} - A| \leq 2^0$ ,
- second, apply the definition the definition of accumulation point with  $N = n_0 + 1$  and  $\varepsilon = 2^{-1}$  to find  $n_1 \geq n_0 + 1$  with  $|x_{n_1} - A| \leq 2^{-1}$ ,
- more in general given  $n_{k-1}$ , we apply the definition of accumulation point with  $N = n_{k-1} + 1$  and  $\varepsilon = 2^{-k}$  to find  $n_k \geq n_{k-1} + 1$  with  $|x_{n_k} - A| \leq 2^{-k}$ .

Now given  $\varepsilon > 0$  choose  $N$  such that  $2^{-N} < \varepsilon$ . Then for all  $k \geq N$  we have that

$$|x_{n_k} - A| \leq 2^{-k} \leq 2^{-N} < \varepsilon,$$

so  $\lim_{k \rightarrow \infty} x_{n_k} = A$ .

Conversely, assume that there exists a subsequence  $(x_{n_k})_{k=0}^{\infty}$  converging to  $A$ . Fix  $\varepsilon > 0$  and  $N \in \mathbb{N}$ . Since  $\lim_{k \rightarrow \infty} x_{n_k} = A$ , there exists  $N_0$  such that  $|x_{n_k} - A| < \varepsilon$  for all  $k \geq N_0$ . Hence if we choose  $k = \max\{N_0, N\}$ , because  $n_k \geq n$  (recall Remark 3.6) we have that  $n_k \geq N$  and  $|x_{n_k} - A| < \varepsilon$ . Thus  $A$  is an accumulation point.  $\square$

**Corollary 3.10: Infinitely Many Terms Near an Accumulation Point**

If  $A \in \mathbb{R}$  is an accumulation point of  $(x_n)_{n=0}^{\infty}$ , then for every  $\varepsilon > 0$  there are infinitely many  $n$  with  $x_n \in (A - \varepsilon, A + \varepsilon)$ .

*Proof.* By Proposition 3.9, there exists a subsequence  $(x_{n_k})_{k=0}^{\infty}$  with  $\lim_{k \rightarrow \infty} x_{n_k} = A$ . Hence for every  $\varepsilon > 0$  there exists  $K$  such that  $x_{n_k} \in (A - \varepsilon, A + \varepsilon)$  for all  $k \geq K$ , providing infinitely many elements of the sequence inside the interval  $(A - \varepsilon, A + \varepsilon)$ .  $\square$

### Corollary 3.11: Accumulation Points of Convergent Sequences

*convergent sequence has exactly one accumulation point, namely its limit.*

## 3.3 Addition, Multiplication and Inequalities

### Proposition 3.12: Limits and Operations

Let  $(x_n)_{n=0}^{\infty}$  and  $(y_n)_{n=0}^{\infty}$  be sequences converging to  $A, B \in \mathbb{R}$  respectively. Then:

1. The sequence  $(x_n + y_n)_{n=0}^{\infty}$  converges to  $A + B$ .
2. The sequence  $(x_n y_n)_{n=0}^{\infty}$  converges to  $AB$ .
3. Given  $\alpha \in \mathbb{R}$ , the sequence  $(\alpha x_n)_{n=0}^{\infty}$  converges to  $\alpha A$ .
4. Suppose  $x_n \neq 0$  for all  $n \in \mathbb{N}$  and  $A \neq 0$ . Then the sequence  $(x_n^{-1})_{n=0}^{\infty}$  converges to  $A^{-1}$ .

### Proposition 3.13: Limits and Inequalities

Let  $(x_n)_{n=0}^{\infty}$  and  $(y_n)_{n=0}^{\infty}$  be sequences converging to  $A, B \in \mathbb{R}$  respectively.

1. If  $A < B$ , then there exists  $N \in \mathbb{N}$  such that  $x_n < y_n$  for all  $n \geq N$ .
2. If there exists  $N \in \mathbb{N}$  such that  $x_n \leq y_n$  for all  $n \geq N$ , then  $A \leq B$ .

**Remark 3.14.** In Proposition 3.13 even if we assume that  $x_n < y_n$  for all  $n \in \mathbb{N}$ , we cannot conclude that  $A < B$ . For example take

$$x_n = \frac{1}{n}, \quad y_n = \frac{1}{n}.$$

Then we have that  $x_n < y_n$  for all  $n \in \mathbb{N}$  but  $A = B = 0$ .

### Lemma 3.15: Sandwich Lemma

Let  $(x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}, (z_n)_{n=0}^{\infty}$  be sequences such that for some  $N \in \mathbb{N}$ , we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N.$$

Suppose that both  $(x_n)_{n=0}^{\infty}$  and  $(z_n)_{n=0}^{\infty}$  converge to the same limit. Then  $(y_n)_{n=0}^{\infty}$  also converges, and we have that

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n.$$

*Proof.* Let  $(x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}, (z_n)_{n=0}^{\infty}$  be sequences such that for some  $N_0 \in \mathbb{N}$ , we have that

$$x_n \leq y_n \leq z_n \quad \forall n \geq N_0.$$

Additionally suppose that  $(x_n)_{n=0}^{\infty}$  and  $(z_n)_{n=0}^{\infty}$  converge to  $A \in \mathbb{R}$ . Fix  $\varepsilon > 0$ . Since  $(x_n)_{n=0}^{\infty}, (z_n)_{n=0}^{\infty}$  converge to  $A$  there exists  $N_x, N_z \in \mathbb{N}$  such that

$$\begin{aligned} A - \varepsilon &< x_n < A + \varepsilon \quad \forall n \geq N_x \\ A - \varepsilon &< z_n < A + \varepsilon \quad \forall n \geq N_z. \end{aligned}$$



So we choose  $N := \max\{N_0, N_x, N_z\}$ . Then we have that

$$A - \varepsilon < x_n \leq y_n \leq z_n < A + \varepsilon \quad \forall n \geq N,$$

which shows that  $\lim_{n \rightarrow \infty} y_n = A$ . □

#### Definition 3.16: Bounded Sequences

A sequence  $(x_n)_{n=0}^{\infty}$  is called **bounded** if there exists a real number  $M \geq 0$  such that

$$|x_n| \leq M \quad \forall n \in \mathbb{N}.$$

#### Lemma 3.17: Convergent Sequences are Bounded

*Every convergent sequence is bounded.*

*Proof.* Let  $(x_n)_{n=0}^{\infty}$  be a sequence converging to  $A \in \mathbb{R}$ . Let  $\varepsilon = 1$ . Then, by convergence of  $(x_n)_{n=0}^{\infty}$ , there exists  $N$  such that  $|x_n - A| \leq 1$  for all  $n \geq N$ . So we have that

$$|x_n| = |x_n - A + A| \leq |x_n - A| + |A| \leq 1 + |A| \quad \forall n \geq N.$$

We choose

$$M = \max(|x_0|, |x_1|, \dots, |x_{N-1}|, 1 + |A|).$$

Then  $|x_n| \leq M$  for all  $n \in \mathbb{N}$  as desired. □

#### Definition 3.18: Monotone Sequences

A sequence  $(x_n)_{n=0}^{\infty}$  is called:

- **(monotonically) increasing** if  $m > n \Rightarrow x_m \geq x_n$ ,
- **strictly (monotonically) increasing** if  $m > n \Rightarrow x_m > x_n$ ,
- **(monotonically) decreasing** if  $m > n \Rightarrow x_m \leq x_n$ ,
- **strictly (monotonically) decreasing** if  $m > n \Rightarrow x_m < x_n$ .

If a sequence is decreasing or increasing we call it monotone. If a sequence is strictly increasing or strictly decreasing then we call it strictly monotone.

**Remark 3.19.** *An equivalent formulation of monotone sequences can be given using only successive terms:*

- $(x_n)_{n=0}^{\infty}$  is increasing if  $x_{n+1} \geq x_n$  for all  $n$ ,
- $(x_n)_{n=0}^{\infty}$  is strictly increasing if  $x_{n+1} > x_n$  for all  $n$ ,
- $(x_n)_{n=0}^{\infty}$  is decreasing if  $x_{n+1} \leq x_n$  for all  $n$ ,
- $(x_n)_{n=0}^{\infty}$  is strictly decreasing if  $x_{n+1} < x_n$  for all  $n$ .

### Theorem 3.20: Convergence of Monotone Sequences

A monotone sequence  $(x_n)_{n=0}^{\infty}$  converges if and only if it is bounded. More precisely, let  $X = \{x_n \mid n \in \mathbb{N}\}$  denote the set of points in the sequence.

- If  $(x_n)_{n=0}^{\infty}$  is increasing, then  $\lim_{n \rightarrow \infty} x_n = \sup(X)$ ,
- if  $(x_n)_{n=0}^{\infty}$  decreasing, then  $\lim_{n \rightarrow \infty} x_n = \inf(X)$ .

*Proof.* If  $(x_n)_{n=0}^{\infty}$  converges Lemma 3.17 says that it is bounded.

Conversely, let  $(x_n)_{n=0}^{\infty}$  be a bounded monotone sequence. Wlog assume that  $(x_n)_{n=0}^{\infty}$  is increasing (otherwise consider  $(-x_n)_{n=0}^{\infty}$ ). Since  $(x_n)_{n=0}^{\infty}$  is bounded from above, the set  $X = \{x_n \mid n \in \mathbb{N}\}$  has a supremum, that we'll call  $A = \sup(X)$ .

By definition of  $A$ :

- (i)  $x_n \leq A \quad \forall n \in \mathbb{N}$ ,
- (ii)  $\forall \varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $x_N > A - \varepsilon$ .

Then, for all  $n \geq N$  using (ii) and monotonicity, we have that  $x_n \geq x_N > A - \varepsilon$ . Then using (i), we conclude that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N.$$

□

## 3.4 Superior and Inferior Limits

Let  $(x_n)_{n=0}^{\infty}$  be a bounded sequence. To study its behavior for large  $n$  it is useful to look at its tails

$$X_{\geq n} = \{x_k \mid k \geq n\} \subseteq \mathbb{R}.$$

The concept of limits can be restated using the tails of a sequence, i.e., the sequence  $(x_n)_{n=0}^{\infty}$  converges to  $A \in \mathbb{R}$  if and only if, for every  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that  $X_N \subseteq (A - \varepsilon, A + \varepsilon)$ .

However, since not every sequence has a limit we now introduce a related notion (the **superior** and **inferior limits**), which always exist for bounded sequences.

For each  $n \in \mathbb{N}$ , define

$$s_n = \sup(X_{\geq n}) = \sup_{k \geq n} x_k, \quad i_n = \inf(X_{\geq n}) = \inf_{k \geq n} x_k.$$

Since  $X_{\geq m} \subset X_{\geq n}$ , whenever  $m > n$ , we have that

$$i_n \leq i_m \leq s_m \leq s_n \quad \forall m > n.$$

Thus,  $(s_n)_{n=0}^{\infty}$  is a monotonically decreasing sequence, while  $(i_n)_{n=0}^{\infty}$  is a monotonically increasing sequence. Moreover, since  $(x_n)_{n=0}^{\infty}$  is bounded both  $(s_n)_{n=0}^{\infty}$  and  $(i_n)_{n=0}^{\infty}$  are bounded as well. Hence by Theorem 3.20, both sequences converge. Their limits will be called the *superior* and the *inferior limit* of  $(x_n)_{n=0}^{\infty}$  respectively.

Note that, since  $x_n \in X_{\geq n}$ , we have that

$$i_n \leq x_n \leq s_n \quad \forall n \in \mathbb{N}. \tag{3.2}$$

**Definition 3.21: Superior and Inferior Limits**

Let  $(x_n)_{n=0}^{\infty}$  be a bounded sequence in  $\mathbb{R}$ . The numbers

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k \right), \quad \liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} x_k \right)$$

are called the **superior** and **inferior limit** of  $(x_n)_{n=0}^{\infty}$  respectively. From Equation 3.2 and Proposition 3.13, we have

$$\liminf_{n \rightarrow \infty} x_n \leq \limsup_{n \rightarrow \infty} x_n.$$

**Lemma 3.22: Convergence and Superior/Inferior Limits**

A bounded sequence  $(x_n)_{n=0}^{\infty}$  in  $\mathbb{R}$  converges if and only if

$$\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n.$$

*Proof.* For every  $n \in \mathbb{N}$ , define

$$i_n = \inf_{k \geq n} x_k, \quad s_n = \sup_{k \geq n} x_k,$$

and set

$$I = \lim_{n \rightarrow \infty} i_n = \liminf_{n \rightarrow \infty} x_n, \quad S = \lim_{n \rightarrow \infty} s_n = \limsup_{n \rightarrow \infty} x_n.$$

First suppose that  $I = S$ . Since  $i_n \leq x_n \leq s_n$  (see Equation 3.2), the Sandwich Lemma 3.15 implies that the sequence  $(x_n)_{n=0}^{\infty}$  converges, and its limit equals  $I = S$ .

Conversely, assume that  $(x_n)_{n=0}^{\infty}$  converges to  $A \in \mathbb{R}$ . Given  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$A - \varepsilon < x_n < A + \varepsilon \quad \forall n \geq N.$$

Then for all  $n \geq N$ , the same inequalities holds for  $i_n$  and  $s_n$ , i.e.,

$$A - \varepsilon \leq i_n \leq s_n \leq A + \varepsilon.$$

Taking limits and using Proposition 3.13, we obtain

$$A - \varepsilon \leq I \leq S \leq A + \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, it follows that  $A = I = S$ , which proves the result.  $\square$

**Theorem 3.23: Superior and Inferior Limits are Accumulation Points**

Let  $(x_n)_{n=0}^{\infty}$  be a bounded sequence and let  $A = \limsup_{n \rightarrow \infty} x_n$ . Then  $A$  is an accumulation point of  $(x_n)_{n=0}^{\infty}$ , and for every  $\varepsilon > 0$  the following hold:

1. only finitely many elements satisfy  $x_n \geq A + \varepsilon$ ;
2. infinitely many elements satisfy  $A - \varepsilon < x_n < A + \varepsilon$ .

An analogous statement holds for the inferior limit.

*Proof.* Since the sequence  $(s_n)_{n=0}^{\infty}$  is monotonically decreasing and converges to  $A$ , given  $\varepsilon > 0$ , there

exists  $N_0 \in \mathbb{N}$  such that

$$A \leq s_n < A + \varepsilon \quad \forall n \geq N_0. \quad (3.3)$$

We first prove that  $A$  is an accumulation point.

Fix  $N \in \mathbb{N}$  and set  $N_1 = \max\{N, N_0\}$ . Since  $s_{N_1} = \sup_{k \geq N_1} x_k$ , there exists  $n_1 \geq N_1 \geq N_0$  such that

$$s_{N_1} - \varepsilon < x_{n_1} \leq s_{N_1}.$$

Thus, combining this bound with Equation 3.3 we obtain

$$A - \varepsilon < s_{N_1} - \varepsilon < x_{n_1} \leq s_{N_1} < A + \varepsilon.$$

This construct shows that for any  $\varepsilon > 0$  and any  $N \in \mathbb{N}$ , there exists  $n_1 \geq N$  such that  $A - \varepsilon < x_{n_1} < A + \varepsilon$ . Thus  $A$  is an accumulation point for  $(x_n)_{n=0}^\infty$ .

We now prove 1. and 2.. From Equation 3.3 we have  $x_n < A + \varepsilon$  for all  $n \geq N_0$ , so only finitely many terms satisfy  $x_n \geq A + \varepsilon$ . This shows 1..

Also since  $A$  is an accumulation point, it follows from Corollary 3.10 that infinitely many terms of the sequence lie within any interval  $(A - \varepsilon, A + \varepsilon)$ .  $\square$

#### Corollary 3.24: Bounded Sequences have Convergent Subsequences

*Every bounded sequence has at least one accumulation point and therefore possesses a convergent subsequence.*

*Proof.* By Theorem 3.23, the number

$$A = \limsup_{n \rightarrow \infty} x_n$$

is always an accumulation point of  $(x_n)_{n=0}^\infty$ . Moreover, by Proposition 3.9, every accumulation point is the limit of a convergent subsequence. Hence every bounded sequence admits at least one convergent subsequence.  $\square$

### 3.5 Cauchy Sequences

#### Definition 3.25: Cauchy Sequences

A sequence  $(x_n)_{n=0}^\infty$  is called a **Cauchy sequence** if for every  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$  such that

$$|x_n - x_m| < \varepsilon \quad \forall n, m \geq N.$$

#### Lemma 3.26: Cauchy Sequences are Bounded

*Every Cauchy sequence is bounded.*

*Proof.* By definition, there exists  $N \in \mathbb{N}$  such that

$$|x_n - x_N| \leq 1 \quad \forall n \geq N.$$

Hence, for  $n \geq N$ , we have  $|x_n| \leq 1 + |x_N|$ . Now, define

$$M = \max\{|x_0|, |x_1|, \dots, |x_{N-1}|, 1 + |x_N|\}.$$

Then,  $|x_n| \leq M$  for all  $n \in \mathbb{N}$ , so  $(x_n)_{n=0}^\infty$  is bounded.  $\square$

### Theorem 3.27: Convergence and Cauchy Sequences

*A sequence  $(x_n)_{n=0}^\infty$  of real numbers converges if and only if it is a Cauchy sequence.*

*Proof.* Suppose first that  $(x_n)_{n=0}^\infty$  converges to some  $A \in \mathbb{R}$ , and let us prove that  $(x_n)_{n=0}^\infty$  is a Cauchy sequence.

Given  $\varepsilon > 0$ , choose  $N \in \mathbb{N}$  such that

$$|x_n - A| < \frac{\varepsilon}{2} \quad \forall n \geq N.$$

Then for all  $n, m \geq N$ , we have that

$$|x_n - x_m| \leq |x_n - A| + |x_m - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence  $(x_n)_{n=0}^\infty$  is a Cauchy sequence.

Viceversa, let  $(x_n)_{n=0}^\infty$  be a Cauchy sequence. Since it is bounded (by Lemma 3.26), Corollary 3.24 implies that there exists a subsequence  $(x_{n_k})_{k=0}^\infty$  converging to some  $A \in \mathbb{R}$ . Given  $\varepsilon > 0$ , choose  $N_0 \in \mathbb{N}$  such that

$$|x_n - x_m| < \frac{\varepsilon}{2} \quad \forall n, m \geq N_0,$$

and choose  $N_1 \in \mathbb{N}$  such that

$$|x_{n_k} - A| < \frac{\varepsilon}{2} \quad \forall k \geq N_1.$$

Let  $N = \max\{N_0, N_1\}$ . Since  $n_N \geq N$  (see Remark 3.6), for all  $n \geq N$  we have

$$|x_n - A| \leq |x_n - x_{n_N}| + |x_{n_N} - A| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus,  $(x_n)_{n=0}^\infty$  converges to  $A$ .  $\square$

## 3.6 Improper Limits

We now extend the notion of limit to allow the **improper limit values**  $+\infty$  (often abbreviated as  $\infty$ ) and  $-\infty$ .

### Definition 3.28: Improper Limits

Let  $(x_n)_{n=0}^\infty$  be a sequence in  $\mathbb{R}$ . We say  $(x_n)_{n=0}^\infty$  **diverges to**  $+\infty$ , and we write

$$\lim_{n \rightarrow \infty} x_n = +\infty,$$

if for every  $M > 0$  there exists  $N \in \mathbb{N}$  such that  $x_n > M$  for all  $n \geq N$ .

Similarly,  $(x_n)_{n=0}^\infty$  **diverges to**  $-\infty$  if for every  $M > 0$  there exists  $N \in \mathbb{N}$  such that  $x_n < -M$  for all  $n \geq N$ . In both cases, we say that  $(x_n)_{n=0}^\infty$  has an **improper limit**.

An unbounded sequence doesn't need to diverge to  $+\infty$  or  $-\infty$ . For instance, the sequence  $x_n = (-1)^n n$ , is unbounded but neither diverges to  $+\infty$  nor to  $-\infty$ .

The notion of improper limit allows us to extend the definitions of superior and inferior limits to

*unbounded* sequences. If  $(x_n)_{n=0}^{\infty}$  is not bounded from above, then

$$\sup_{k \geq n} x_k = +\infty \quad \forall n \in \mathbb{N},$$

and we write

$$\limsup_{n \rightarrow \infty} x_n = +\infty.$$

If  $(x_n)_{n=0}^{\infty}$  is bounded from above but not from below, then we define

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (\sup_{k \geq n} x_k),$$

where the right-hand side is a real limit if the decreasing sequence  $\sup_{k \geq n} x_k$  is bounded, and the improper limit  $-\infty$  otherwise. The definition of the inferior limit extends analogously.

### 3.7 Sequences of Complex Numbers

Informally, a **sequence of complex numbers** is just like a sequence of real numbers, except that each term is a complex number instead of a real one. Thus, we study ordered lists  $(z_0, z_1, \dots)$ , where  $z_n : \mathbb{N} \rightarrow \mathbb{C}$ . As in the real case, we are mainly interested in their convergence, divergence and limit behavior.

To analyze sequences in  $\mathbb{C}$ , it is often sufficient to consider separately the corresponding sequences of real and imaginary parts in  $\mathbb{R}$ .

#### Definition 3.29: Sequences of Complex Numbers

A sequence of complex numbers  $(z_n)_{n=0}^{\infty}$ , where

$$z_n = x_n + iy_n,$$

is said to **converge** to a limit  $A + iB \in \mathbb{C}$  if the two sequences of real numbers  $(x_n)_{n=0}^{\infty}$  and  $(y_n)_{n=0}^{\infty}$  converge to  $A$  and  $B$ , respectively. In this case, we write

$$\lim_{n \rightarrow \infty} z_n = A + iB.$$

We say that  $(z_n)_{n=0}^{\infty}$  **diverges to  $\infty$**  if the sequence of moduli  $(|z_n|)_{n=0}^{\infty}$  diverges to  $+\infty$ , i.e.,

$$\lim_{n \rightarrow \infty} |z_n| = \lim_{n \rightarrow \infty} \sqrt{x_n^2 + y_n^2} = +\infty.$$

**Remark 3.30.** As for sequences of real numbers, one can consider subsequences of sequences  $\mathbb{C}$ . Given a strictly increasing sequence of non-negative integers  $(n_k)_{k=0}^{\infty}$ , the corresponding subsequence is

$$(z_{n_k})_{k=0}^{\infty} = (x_{n_k} + iy_{n_k})_{k=0}^{\infty}.$$

## 4 Functions of one Real Variable

In this chapter we study real-valued functions defined on subsets of  $\mathbb{R}$ , typically intervals. The central concept is *continuity*.

## 4.1 Real valued functions

### 4.1.1 Boundedness and Monotonicity

For a non-empty set  $D \subseteq \mathbb{R}$ , the set of **real-valued** functions on  $D$  is

$$\mathcal{F}(D) = \{f \mid f : D \rightarrow \mathbb{R}\}.$$

For  $f_1, f_2 \in \mathcal{F}(D)$ ,  $\alpha \in \mathbb{R}$ , and  $x \in D$  we define

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \quad (\alpha f_1)(x) = \alpha f_1(x), \quad (f_1 f_2)(x) = f_1(x) f_2(x).$$

Given  $\alpha \in \mathbb{R}$ , we write  $f \equiv \alpha$  for the constant function  $x \mapsto \alpha$  on  $D$ .

**Remark 4.1.** *With the operations above,  $\mathcal{F}(D)$  is a commutative ring (the additive identity is  $f \equiv 0$  and the multiplicative identity is  $f \equiv 1$ ).*

A point  $x \in D$  is a **zero** of  $f \in \mathcal{F}(D)$  if  $f(x) = 0$ . The **zero set** of  $f$  is  $\{x \in D \mid f(x) = 0\}$ . We order  $\mathcal{F}(D)$  pointwise: for  $f_1, f_2 \in \mathcal{F}(D)$ ,

$$\begin{aligned} f_1 \leq f_2 &\Leftrightarrow f_1(x) \leq f_2(x) \quad \forall x \in D, \\ f_1 < f_2 &\Leftrightarrow f_1(x) < f_2(x) \quad \forall x \in D. \end{aligned}$$

We say that  $f \in \mathcal{F}(D)$  is **non-negative** if  $f \geq 0$ , and **positive** if  $f > 0$ .

#### Definition 4.2: Bounded Functions

Let  $D \neq \emptyset$  and  $f : D \rightarrow \mathbb{R}$ . We say that  $f$  is **bounded from above** if there exists  $M > 0$  such that

$$f(x) \leq M \quad \forall x \in D.$$

We say that  $f$  is **bounded from below** if there exists  $M > 0$  such that

$$f(x) \geq -M \quad \forall x \in D.$$

We say that  $f$  is **bounded** if it is both bounded from above and from below. Equivalently,  $f$  is bounded if there exists  $M > 0$  such that

$$|f(x)| \leq M \quad \forall x \in D.$$

#### Definition 4.3: Monotone Functions

Let  $D \subseteq \mathbb{R}$  and  $f : D \rightarrow \mathbb{R}$ . The function  $f$  is:

1. **increasing** if  $x < y \Rightarrow f(x) \leq f(y) \quad \forall x, y \in D$ ;
2. **strictly increasing** if  $x < y \Rightarrow f(x) < f(y) \quad \forall x, y \in D$ ;
3. **decreasing** if  $x < y \Rightarrow f(x) \geq f(y) \quad \forall x, y \in D$ ;
4. **strictly decreasing** if  $x < y \Rightarrow f(x) > f(y) \quad \forall x, y \in D$ .

We call  $f$  **monotone** if it is increasing or decreasing, and **strictly monotone** if it is strictly increasing or strictly decreasing.

### 4.1.2 Continuity

#### Definition 4.4: Continuous Functions

Let  $D \subseteq \mathbb{R}$  and  $f : D \rightarrow \mathbb{R}$ . We say that  $f$  is **continuous at**  $x_0 \in D$  if for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

We say that  $f$  is **continuous on**  $D$  if it is continuous at every point of  $D$ .

**Remark 4.5.** It suffices to verify the implication above for small  $\varepsilon$ . Precisely, assume there exists  $\varepsilon_0 > 0$  such that for every  $\varepsilon \in (0, \varepsilon_0]$  there is a  $\delta > 0$  such that

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon.$$

Then  $f$  is continuous at  $x_0$ .

Indeed, for  $\varepsilon_0 > \varepsilon$  we can choose the number  $\delta > 0$  corresponding to  $\varepsilon$  to get

$$\forall x \in D, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \varepsilon < \varepsilon_0.$$

In other words, if  $\delta$  works for  $\varepsilon$ , then it works for all  $\varepsilon_0 > \varepsilon$ .

#### Definition 4.6: Restriction

Let  $D \subseteq \mathbb{R}$  and  $f : D \rightarrow \mathbb{R}$ . For any  $D' \subseteq D$  the **restriction** of  $f$  to  $D'$  is the function  $f|_{D'} : D' \rightarrow \mathbb{R}$  defined by

$$f|_{D'}(x) = f(x) \quad \forall x \in D'.$$

We regard  $f|_{D'}$  and  $f$  as different functions unless  $D' = D$ .

#### Proposition 4.7: Combination of Continuous Functions

Let  $D \subseteq \mathbb{R}$ , and let  $f_1, f_2 : D \rightarrow \mathbb{R}$  be continuous at  $x_0 \in D$ . Then  $f_1 + f_2$ ,  $f_1 f_2$ , and  $\alpha f_1$  (for any  $\alpha \in \mathbb{R}$ ) are continuous at  $x_0$ .

*Proof.* We first prove the result for the sum. Let  $\varepsilon > 0$ . Since  $f_1$  and  $f_2$  are continuous at  $x_0$ , there exists  $\delta_1, \delta_2 > 0$  such that for all  $x \in D$ ,

$$|x - x_0| < \delta_1 \Rightarrow |f_1(x) - f_1(x_0)| < \frac{\varepsilon}{2}, \quad |x - x_0| < \delta_2 \Rightarrow |f_2(x) - f_2(x_0)| < \frac{\varepsilon}{2}.$$

So, choosing  $\delta = \min \delta_1, \delta_2$ , for  $|x - x_0| < \delta$  we get

$$|(f_1 + f_2)(x) - (f_1 + f_2)(x_0)| \leq |f_1(x) - f_1(x_0)| + |f_2(x) - f_2(x_0)| < \varepsilon,$$

which shows that  $f_1 + f_2$  is continuous at  $x_0$ .

For the product, note that

$$\begin{aligned} |f_1(x)f_2(x) - f_1(x_0)f_2(x_0)| &= |f_1(x)f_2(x) - f_1(x_0)f_2(x) + f_1(x_0)f_2(x) - f_1(x_0)f_2(x_0)| \\ &\leq |f_1(x)f_2(x) - f_1(x_0)f_2(x)| + |f_1(x_0)f_2(x) - f_1(x_0)f_2(x_0)| \\ &= |f_2(x)||f_1(x) - f_1(x_0)| + |f_1(x_0)||f_2(x) - f_2(x_0)|. \end{aligned}$$



Now, first choose  $\delta_0 > 0$  such that  $|x - x_0| < \delta_0$  implies  $|f_2(x) - f_2(x_0)| < 1$ , so that

$$|x - x_0| < \delta_0 \quad \Rightarrow \quad |f_2(x)| < 1 + |f_2(x_0)|.$$

Then choose  $\delta_1, \delta_2 > 0$  such that

$$\begin{aligned} |x - x_0| < \delta_1 &\Rightarrow |f_1(x) - f_1(x_0)| < \frac{\varepsilon}{2(1 + |f_2(x_0)|)}, \\ |x - x_0| < \delta_2 &\Rightarrow |f_2(x) - f_2(x_0)| < \frac{\varepsilon}{2(1 + |f_1(x_0)|)}. \end{aligned}$$

So choosing  $\delta = \min \delta_0, \delta_1, \delta_2$ , for  $|x - x_0| < \delta$  we get

$$\begin{aligned} |f_1(x)f_2(x) - f_1(x_0)f_2(x_0)| &< |f_2(x)| \frac{\varepsilon}{2(1 + |f_2(x_0)|)} + |f_1(x_0)| \frac{\varepsilon}{2(1 + |f_1(x_0)|)} \\ &< (1 + |f_2(x_0)|) \frac{\varepsilon}{2(1 + |f_2(x_0)|)} + |f_1(x_0)| \frac{\varepsilon}{2(1 + |f_1(x_0)|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

thus  $f_1f_2$  is continuous at  $x_0$ .

Finally, the statement about  $\alpha f_1$  follows by choosing  $f_2 \equiv \alpha$  (a constant function) and using the product case proved above: since  $f_1$  and  $f_2$  are continuous at  $x_0$ , their product  $f_1f_2 = \alpha f_1$  is continuous at  $x_0$ .  $\square$

#### Definition 4.8: Sum and Product Notation

Let  $n \in \mathbb{N}$  and  $a_0, a_1, \dots, a_n \in \mathbb{R}$ . We use the notation

$$\sum_{j=0}^n a_j = a_0 + a_1 + \dots + a_n, \quad \prod_{j=0}^n a_j = a_0 \cdot a_1 \cdot \dots \cdot a_n.$$

Here  $a_j$  is a **summand** in the sum and a **factor** in the product;  $j$  is the **index** (or **running variable**). If  $J$  is a finite set and numbers  $(a_j)_{j \in J}$  are given, we write

$$\sum_{j \in J} a_j, \quad \prod_{j \in J} a_j.$$

By convention, for the empty index set  $\emptyset$ ,

$$\sum_{j \in \emptyset} a_j = 0, \quad \prod_{j \in \emptyset} a_j = 1.$$

#### Proposition 4.9: Composition of Continuous Functions

Let  $D_1, D_2 \subseteq \mathbb{R}, x_0 \in D_1$  and  $f : D_1 \rightarrow D_2$  be continuous at  $x_0$ . If  $g : D_2 \rightarrow \mathbb{R}$  is continuous at  $f(x_0)$ , then  $g \circ f : D_1 \rightarrow \mathbb{R}$  is continuous at  $x_0$ . In particular, the composition of continuous functions is continuous.

*Proof.* Let  $\varepsilon > 0$ . By continuity of  $g$  at  $f(x_0)$ , there exists  $\eta > 0$  such that

$$\forall y \in D_2, \quad |y - f(x_0)| < \eta \quad \Rightarrow \quad |g(y) - g(f(x_0))| < \varepsilon.$$

By continuity of  $f$  at  $x_0$ , there exists  $\delta > 0$  such that

$$\forall x \in D_1, \quad |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \eta.$$

Combining the implications gives, for any  $x \in D_1$ ,

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - f(x_0)| < \eta \quad \Rightarrow \quad |g(f(x)) - g(f(x_0))| < \varepsilon. \quad \square$$

**Remark 4.10.** Applying Proposition 4.9 with  $g(y) = |y|$ , we see that if  $f : D \rightarrow \mathbb{R}$  is continuous, then  $x \mapsto |f(x)|$  is continuous.

### 4.1.3 Sequential Continuity

#### Definition 4.11: Notation for Limits of Sequences

Let  $(x_n)_{n=0}^\infty \subseteq \mathbb{R}$  and  $\bar{x} \in \mathbb{R}$ . We write

$$x_n \rightarrow \bar{x} \quad \text{or} \quad x_n \xrightarrow{n \rightarrow \infty} \bar{x}$$

to mean

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

#### Theorem 4.12: Continuity = Sequential Continuity

Let  $D \subseteq \mathbb{R}$ ,  $f : D \rightarrow \mathbb{R}$ , and  $\bar{x} \in D$ . Then  $f$  is continuous at  $\bar{x}$  if and only if for every sequence  $(x_n)_{n=0}^\infty \subseteq D$  with  $x_n \rightarrow \bar{x}$  we have  $f(x_n) \rightarrow f(\bar{x})$ .

*Proof.* ' $\Rightarrow$ ': First Assume that  $f$  is continuous at  $\bar{x}$ . Then, given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\forall x \in D, \quad |x - \bar{x}| < \delta \quad \Rightarrow \quad |f(x) - f(\bar{x})| < \varepsilon.$$

Also, since  $x_n \rightarrow \bar{x}$ , there exists  $N \in \mathbb{N}$  such that

$$n \geq N \quad \Rightarrow \quad |x_n - \bar{x}| < \delta.$$

Thus,

$$n \geq N \quad \Rightarrow \quad |f(x_n) - f(\bar{x})| < \varepsilon,$$

which implies that the sequence  $(f(x_n))_{n=0}^\infty$  converges to  $f(\bar{x})$ .

' $\Leftarrow$ ': To prove the converse, assume that  $f$  is not continuous at  $x_0$ . This means that there exists  $\varepsilon > 0$  such that, for every  $\delta > 0$ , there is  $x \in D$  with

$$|x - \bar{x}| < \delta \quad \text{and} \quad |f(x) - f(\bar{x})| \geq \varepsilon.$$

Now, for every  $n \in \mathbb{N}$ , we apply this property with  $\delta = 2^{-n}$  to find a point  $x_n \in D$  such that

$$|x_n - \bar{x}| < 2^{-n} \quad \text{and} \quad |f(x_n) - f(\bar{x})| \geq \varepsilon$$

Then the sequence constructed in this way satisfies  $x_n \rightarrow \bar{x}$  but  $f(x_n) \not\rightarrow f(\bar{x})$ .  $\square$

**Remark 4.13.** The proof above shows that if  $f : D \rightarrow \mathbb{R}$  is not continuous at  $\bar{x}$ , then there exists  $\varepsilon > 0$  and a sequence  $(x_n)_{n=0}^{\infty} \subseteq D$  with  $x_n \rightarrow \bar{x}$  such that  $|f(x_n) - f(\bar{x})| \geq \varepsilon$  for all  $n \in \mathbb{N}$ . This is useful to show that a function  $f$  is not continuous at  $\bar{x}$ .

## 4.2 Continuous Functions

### 4.2.1 Intermediate Value Theorem

In this section we prove a fundamental theorem that formalizes the idea that the graph of a continuous function on an interval is a continuous curve, and thus cannot make any jumps.

#### Theorem 4.14: Intermediate Value Theorem

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function with  $f(a) \leq f(b)$ . Then, for every real number  $c$  with  $f(a) \leq c \leq f(b)$ , there exists  $\bar{x} \in [a, b]$  such that  $f(\bar{x}) = c$ .

*Proof.* Fix  $c \in [f(a), f(b)]$ . Then define

$$X = \{x \in [a, b] \mid f(x) \leq c\}.$$

Since  $a \in X$  and  $X \subseteq [a, b]$ , the set is non-empty and bounded from above. By Theorem 2.35, its supremum

$$\bar{x} = \sup(X) \in [a, b]$$

exists. We now use the continuity of  $f$  at  $x_0$  to show that  $f(\bar{x}) = c$ .

Since  $\bar{x}$  is the supremum of  $X$ , for each  $n \geq 0$ , we can find a point  $x_n \in [\bar{x} - 2^{-n}, \bar{x}]$ . Then  $|x_n - \bar{x}| \leq 2^{-n}$ , hence  $x_n \rightarrow \bar{x}$ . Also, by the definition of  $X$ , we have  $f(x_n) \leq c$ . Thus, by Theorem 4.12 (continuity of  $f$  along sequences),

$$\lim_{n \rightarrow \infty} f(x_n) = f(\bar{x}).$$

And Proposition 3.13 yields  $\lim_{n \rightarrow \infty} f(x_n) \leq c$ . Therefore,  $f(\bar{x}) \leq c$ .

Suppose, by contradiction,  $f(\bar{x}) < c$  and set  $\varepsilon := c - f(\bar{x}) > 0$ . By continuity at  $\bar{x}$ , there exists  $\delta > 0$  such that for all  $x \in [a, b]$

$$|x - \bar{x}| < \delta \quad \Rightarrow \quad |f(x) - f(\bar{x})| < \varepsilon,$$

hence  $f(x) < f(\bar{x}) + \varepsilon = c$ . Therefore, by the definition of  $X$ ,

$$(\bar{x} - \delta, \bar{x} + \delta) \cap [a, b] \subseteq X.$$

Moreover, since  $f(\bar{x}) < c \leq f(b)$ , we cannot have  $\bar{x} = b$ ; hence  $\bar{x} < b$ . Because  $\bar{x} < b$ , the interval  $(\bar{x}, \bar{x} + \delta) \cap [a, b] \subseteq X$  is non-empty. Pick

$$y \in (\bar{x}, \bar{x} + \delta) \cap [a, b] \subseteq X.$$

Then  $y \in X$  and  $y > \bar{x}$ , which contradicts the defining property of the supremum:  $\bar{x}$  is an upper bound of  $X$ , and  $X$  cannot contain elements larger than  $\bar{x}$ . This contradiction shows that  $f(\bar{x}) \geq c$ . Together with  $f(\bar{x}) \leq c$  proved above, we conclude that  $f(\bar{x}) = c$ , as desired.  $\square$

### Theorem 4.15: Inverse Function Theorem

*Let  $I$  be an interval and  $f : I \rightarrow \mathbb{R}$  a continuous strictly monotone function. Then  $f(I)$  is an interval, and the mapping  $f : I \rightarrow f(I)$  has a continuous strictly monotone inverse function  $f^{-1} : f(I) \rightarrow I$ .*

*Proof.* We may assume that  $I$  is non-empty and not a single point. Also, w.l.o.g, suppose  $f$  is strictly increasing (otherwise replace  $f$  with  $-f$ ).

Let  $J = f(I)$ . Since  $f$  is strictly monotone it is injective. Also, since by definition  $J = f(I)$ , it is surjective, hence bijective. Therefore there exists a unique inverse  $g = f^{-1} : J \rightarrow I$ .

Because  $f$  is strictly increasing, we have

$$x_1 < x_2 \quad \Leftrightarrow \quad f(x_1) < f(x_2) \quad \forall x_1, x_2 \in I. \quad (4.1)$$

(Note: here we have equivalence in the statements because  $f$  is both injective and strictly increasing) Defining  $y_1 = f(x_1)$  and  $y_2 = f(x_2)$ , this is equivalent to

$$y_1 < y_2 \quad \Leftrightarrow \quad g(y_1) < g(y_2) \quad \forall y_1, y_2 \in J$$

Thus,  $g$  is strictly increasing.

To show that  $J$  is an interval,  $y_1, y_2 \in J$ , and assume w.l.o.g that  $y_1 < y_2$ . Since,  $J = f(I)$ , Equation 4.1 implies that  $y_1 = f(x_1), y_2 = f(x_2)$  for some  $x_1, x_2 \in I$  with  $x_1 < x_2$ . Now by the Intermediate Value Theorem 4.14 applied to  $f : [x_1, x_2] \rightarrow \mathbb{R}$ , we have that all values  $c \in [y_1, y_2]$  are in the image of  $f : [x_1, x_2] \rightarrow \mathbb{R}$ , i.e.,

$$[y_1, y_2] \subseteq f([x_1, x_2]) \subseteq J.$$

Since,  $y_1, y_2$  were two arbitrary points in  $J$ , this proves that  $J$  is an interval.

It remains to show that  $g = f^{-1}$  is continuous. Fix  $\bar{y} \in J$  and suppose, by contradiction, that  $g$  is not continuous at  $\bar{y}$ . Then by Remark 4.13, there exists  $\varepsilon > 0$  and a sequence  $(y_n)_{n=0}^{\infty} \subseteq J$  such that

$$y_n \longrightarrow \bar{y} \quad \text{but} \quad |g(y_n) - g(\bar{y})| \geq \varepsilon \quad \forall n \in \mathbb{N}. \quad (4.2)$$

Set  $x_n = g(y_n) \in I$  and  $\bar{x} = g(\bar{y}) \in I$ . Then for every  $n \in \mathbb{N}$ , either  $x_n \leq \bar{x} - \varepsilon$  or  $x_n \geq \bar{x} + \varepsilon$ . In particular, at least one of these cases must occur infinitely often. W.l.o.g, assume  $x_n \leq \bar{x} - \varepsilon$  for infinitely many  $n$ , and extract a subsequence  $(x_{n_k})_{k=0}^{\infty}$  with  $x_{n_k} \leq \bar{x} - \varepsilon$  for all  $k$ . Since,  $I$  is an interval,  $\bar{x} - \varepsilon \in I$ , and by strict monotonicity of  $f$  we obtain

$$y_{n_k} = f(x_{n_k}) \leq f(\bar{x} - \varepsilon) < f(\bar{x}) = \bar{y}.$$

Then Proposition 3.13 gives (recall  $y_n \longrightarrow \bar{y}$ , see 4.2)

$$\bar{y} = \lim_{k \rightarrow \infty} y_{n_k} \leq f(\bar{x} - \varepsilon) < f(\bar{x}) = \bar{y},$$

a contradiction. Hence,  $g$  is continuous. □

## 4.3 Continuous Functions on Compact Intervals

In this section we show that continuous functions on **bounded closed** intervals, called **compact intervals**, enjoy special properties.

## 4.3.1 Boundedness and Extrema

## Lemma 4.16: Compactness

Let  $[a, b]$  be a compact interval, and let  $(x_n)_{n=0}^{\infty}$  be a sequence contained in  $[a, b]$ . Then there exists a subsequence  $(x_{n_k})_{k=0}^{\infty}$  such that

$$\lim_{k \rightarrow \infty} x_{n_k} = \bar{x} \quad \text{for some } \bar{x} \in [a, b].$$

*Proof.* Since  $(x_n)_{n=0}^{\infty}$  is bounded (as it lies in  $[a, b]$ ), Corollary 3.24 ensures the existence of a convergent subsequence  $(x_{n_k})_{k=0}^{\infty}$ . Let  $\bar{x}$  denote its limit. Because  $a \leq x_{n_k} \leq b$  for all  $k$ , Proposition 3.13 yields  $a \leq \bar{x} \leq b$ .  $\square$

## Theorem 4.17: Boundedness

Let  $[a, b]$  be compact interval, and let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Then  $f$  is bounded.

*Proof.* Assume by contradiction that  $f$  is unbounded. Then, for every  $n \in \mathbb{N}$ , there exists  $x_n \in [a, b]$  such that  $|f(x_n)| \geq n$ . By Lemma 4.16, there is a subsequence  $(x_{n_k})_{k=0}^{\infty}$  converging to some  $\bar{x} \in [a, b]$ .

Since  $f$  is continuous, so is  $|f|$  (recall Remark 4.10), therefore  $|f(x_{n_k})| \rightarrow |f(\bar{x})| \in \mathbb{R}$ . This contradicts  $|f(x_{n_k})| \geq n_k \rightarrow \infty$ , so  $f$  must be bounded.  $\square$

**Exercise 4.18.** Find examples of:

1. a continuous but unbounded function on a bounded open interval.

$$f : (0, 1) \rightarrow \mathbb{R}, \quad x \mapsto \frac{1}{x}.$$

2. a continuous but unbounded function on an unbounded closed interval.

$$f : [0, \infty) \rightarrow \mathbb{R}, \quad x \mapsto x.$$

3. an unbounded function on a compact interval but discontinuous at only one point.

$$f : [0, 1] \rightarrow \mathbb{R}, \quad x \mapsto \begin{cases} \frac{1}{x}, & \text{for } x \neq 0 \\ a \in \mathbb{R}, & \text{for } x = 0. \end{cases}$$

## Definition 4.19: Extreme Values

Let  $D \subseteq \mathbb{R}$  and  $f : D \rightarrow \mathbb{R}$ .

- We say that  $f$  takes its **maximum value** at  $x_0 \in D$  if  $f(x) \leq f(x_0)$  for all  $x \in D$ . Then  $f(x_0)$  is the **maximum** of  $f$ .
- We say that  $f$  takes its **minimum value** at  $x_0 \in D$  if  $f(x) \geq f(x_0)$  for all  $x \in D$ . Then  $f(x_0)$  is the **minimum** of  $f$ .

Maxima and minima are called **extreme values** or **extrema**.

### Theorem 4.20: Extreme Value Theorem

Let  $[a, b]$  be a compact interval, and let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Then  $f$  attains both its minimum and its maximum.

*Proof.* Theorem 4.17 guarantees that  $f$  is bounded, or equivalently, that  $f([a, b]) \subseteq \mathbb{R}$  is a bounded subset of  $\mathbb{R}$ . Thus, Theorem 2.35 implies that

$$S := \sup f([a, b])$$

exists. By definition of the supremum, for each  $n \in \mathbb{N}$  there exists  $y_n \in f([a, b])$  such that  $S - 2^{-n} \leq y_n \leq S$ . Hence,  $y_n \rightarrow S$ . Also, since  $y_n \in f([a, b])$ , there exists  $x_n \in [a, b]$  such that  $f(x_n) = y_n$ .

Now, by Lemma 4.16, we can find a subsequence  $(x_{n_k})_{k=0}^{\infty}$  such that  $x_{n_k} \rightarrow \bar{x} \in [a, b]$ . By continuity of  $f$ , we have that

$$f(\bar{x}) = \lim_{k \rightarrow \infty} f(x_{n_k}) = \lim_{k \rightarrow \infty} y_{n_k} = S,$$

so  $f$  attains its maximum at  $\bar{x}$ .

Applying the same reasoning to  $-f$  shows that  $f$  also attains its minimum.  $\square$

### 4.3.2 Uniform Continuity

#### Definition 4.21: Uniform Continuity

Let  $D \subseteq \mathbb{R}$ . A function  $f : D \rightarrow \mathbb{R}$  is **uniformly continuous** if, for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon \quad \forall x, y \in D.$$

**Remark 4.22.** The difference between the usual definition of continuity and the one of uniform continuity lies in how the choice of  $\delta$  depends on the points considered.

For a function that is continuous at each  $x_0 \in D$ , the  $\delta$  in the definition may depend on both  $\varepsilon$  and  $x_0$ : for every  $\varepsilon > 0$  and each  $x_0$ , we can find a  $\delta = \delta(\varepsilon, x_0)$  that works near  $x_0$ .

Uniform continuity is stronger: there exists a single  $\delta = \delta(\varepsilon)$  that works **simultaneously** for all  $x, y \in D$ . In other words, the control on the variation of  $f$  does not deteriorate as we move along the domain. This property is automatically satisfied on compact intervals for continuous functions, as we will prove below.

### Theorem 4.23: Uniform Continuity on Compact Intervals

Let  $[a, b]$  be a compact interval, and  $f : [a, b] \rightarrow \mathbb{R}$  continuous on  $[a, b]$ . Then  $f$  is uniformly continuous.

*Proof.* Assume, by contradiction, that  $f$  is not uniformly continuous on  $[a, b]$ . Then there exists  $\varepsilon > 0$  such that for every  $\delta > 0$  one can find  $x, y \in [a, b]$  with

$$|x - y| < \delta \quad \text{and} \quad |f(x) - f(y)| \geq \varepsilon.$$

Taking  $\delta = 2^{-n}$  for each  $n \in \mathbb{N}$ , we obtain sequences  $(x_n)_{n=0}^{\infty}$  and  $(y_n)_{n=0}^{\infty}$  in  $[a, b]$  with

$$|x_n - y_n| < 2^{-n} \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \varepsilon. \quad (4.3)$$

By Lemma 4.16, the sequence  $(x_n)_{n=0}^\infty$  has a subsequence  $(x_{n_k})_{k=0}^\infty$  converging to some  $\bar{x} \in [a, b]$ . Then

$$|y_{n_k} - \bar{x}| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - \bar{x}| < 2^{-n_k} + |x_{n_k} - \bar{x}| \xrightarrow{k \rightarrow \infty} 0,$$

so  $y_{n_k} \rightarrow \bar{x}$  as well. Thus, by continuity of  $f$  and Theorem 4.12, we have that

$$\lim_{k \rightarrow \infty} f(x_{n_k}) = \lim_{k \rightarrow \infty} f(y_{n_k}) = f(\bar{x}),$$

therefore,

$$|f(x_{n_k}) - f(y_{n_k})| \leq |f(x_{n_k}) - f(\bar{x})| + |f(\bar{x}) - f(y_{n_k})| \xrightarrow{k \rightarrow \infty} 0,$$

which contradicts Equation 4.3. Hence,  $f$  is uniformly continuous on  $[a, b]$ .  $\square$

#### Definition 4.24: Lipschitz Continuity

Let  $D \subseteq \mathbb{R}$ , and  $f : D \rightarrow \mathbb{R}$ . We say that  $f$  is **Lipschitz continuous** if there exists  $L \geq 0$  such that

$$|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in D.$$

#### Lemma 4.25: Lipschitz Continuity $\Rightarrow$ Uniform Continuity

Let  $D \subseteq \mathbb{R}$ , and  $f : D \rightarrow \mathbb{R}$  be a Lipschitz continuous function. Then  $f$  is uniformly continuous.

*Proof.* Let  $D \subseteq \mathbb{R}$  and assume that  $f : D \rightarrow \mathbb{R}$  is a Lipschitz continuous function. Then there exists  $L \geq 0$  such that

$$|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in D.$$

Now, fix  $\varepsilon > 0$ . We assume that  $L \neq 0$  (otherwise the result follows immediately) and choose  $\delta = \frac{\varepsilon}{L}$ . Because of the Lipschitz continuity of  $f$ , we have that for all  $x, y \in D$  it holds that

$$\begin{aligned} |x - y| < \delta = \frac{\varepsilon}{L} &\Leftrightarrow L|x - y| < \varepsilon \\ \Rightarrow |f(x) - f(y)| &\leq L|x - y| < \varepsilon, \end{aligned}$$

which shows that  $f$  is also uniformly continuous.  $\square$

## 4.4 Example: Exponential and Logarithmic Functions

### 4.4.1 Definition of the Exponential Function

#### Lemma 4.26: Bernoulli's Inequality

For all  $a \in \mathbb{R}$  with  $a \geq -1$  and all  $n \in \mathbb{N}$  with  $n \geq 1$ , it holds that

$$(1 + a)^n \geq 1 + na.$$

*Proof.* We proceed by induction. For  $n = 1$  we have  $(1 + a)^1 = 1 + a = 1 + 1 \cdot a$ .

Now assume that the inequality holds for some  $n \geq 1$ . Since  $1 + a \geq 0$  by assumption, we find

$$(1 + a)^{n+1} = (1 + a)^n(1 + a) \geq (1 + na)(1 + a) = 1 + na + a + na^2 \geq 1 + (n + 1)a,$$

which establishes the induction step and completes the proof.  $\square$

**Proposition 4.27: Existence of the Exponential**

Let  $x \in \mathbb{R}$ . The sequence  $(a_n)_{n=1}^{\infty}$  defined by

$$a_n = \left(1 + \frac{x}{n}\right)^n$$

is convergent, and its limit is a positive real number.

**Lemma 4.28: Monotonicity**

Given  $x \in \mathbb{R}$ , let  $n_0 \in \mathbb{N}$  satisfy  $n_0 \geq 1$  and  $n_0 > -x$ . Then the sequence  $(a_n)_{n=n_0}^{\infty}$  defined in Proposition 4.27 is increasing.

**Definition 4.29: Exponential Function**

The **exponential function**  $\exp : \mathbb{R} \rightarrow \mathbb{R}_{>0}$  is defined by

$$\exp(x) = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n \quad \forall x \in \mathbb{R}.$$

**Corollary 4.30: Growth of the Exponential**

Given  $n \in \mathbb{N}$  with  $n \geq 1$ , the exponential function satisfies

$$\exp(x) \geq \left(1 + \frac{x}{n}\right)^n \quad \forall x > -n.$$

*Proof.* By Lemma 4.28 and Definition 4.29, for  $x > -n$  we have

$$a_n \leq a_{n+1} \leq \dots \leq \exp(x).$$

□

**4.4.2 Properties of the Exponential Function****Theorem 4.31: Properties of the Exponential Function**

The exponential function  $\exp : \mathbb{R} \rightarrow \mathbb{R}_{>0}$  is bijective, strictly increasing, and continuous. Moreover,

$$\begin{aligned} \exp(0) &= 1, \\ \exp(-x) &= \exp(x)^{-1}, \\ \exp(x+y) &= \exp(x)\exp(y), \end{aligned}$$

for all  $x, y \in \mathbb{R}$ .

**4.4.3 The Natural Logarithm****Definition 4.32: Logarithm**

The unique inverse function

$$\log : \mathbb{R}_{>0} \rightarrow \mathbb{R}$$

of the bijective map  $\exp : \mathbb{R} \rightarrow \mathbb{R}_{>0}$  is called the **logarithm**.



## Corollary 4.33: Properties of the Logarithm

The logarithm  $\log : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  is strictly increasing, continuous, and bijective. Moreover,

$$\begin{aligned}\log(1) &= 0, \\ \log(a^{-1}) &= -\log(a), \\ \log(ab) &= \log(a) + \log(b),\end{aligned}$$

for all  $a, b \in \mathbb{R}_{>0}$ .

The logarithm defined here is also called the **natural logarithm** to distinguish it from logarithms with another **base**  $a > 1$  (for instance  $a = 10$  or  $a = 2$ ). For any  $a > 1$ , we define

$$\log_a(x) = \frac{\log(x)}{\log(a)} \quad \forall x > 0.$$

Unless stated otherwise,  $\log(x)$  always denotes the natural logarithm, i.e., the logarithm to base  $e$ .

We can now define powers with arbitrary real exponents. For  $a > 0$  and  $x \in \mathbb{R}$  we set

$$a^x = \exp(x \log(a)).$$

## 4.5 Limits of Functions

We consider functions  $f : D \rightarrow \mathbb{R}$  defined on a subset  $D \subseteq \mathbb{R}$ , and we wish to define the limit of  $f(x)$  as  $x \in D$  approaches a point  $x_0 \in \mathbb{R}$ . Typical examples include  $D = \mathbb{R}$ ,  $D = [0, 1]$  or  $D = (0, 1)$ , with  $x_0 = 0$  in each case.

### 4.5.1 Limit in the Vicinity of a Point

Let  $D \subseteq \mathbb{R}$  be non-empty, and let  $x_0 \in \mathbb{R}$  be such that

$$D \cap (x_0 - \delta, x_0 + \delta) \neq \emptyset \quad (4.4)$$

for all  $\delta > 0$ . Whenever this holds, we say that  $x_0$  is an **accumulation point** of  $D$ . Note that if  $x_0 \in D$ , then Equation 4.4 is automatically satisfied.

Condition 4.4 ensures that there exists a sequence of points in  $D$  converging to  $x_0$ .

#### Definition 4.34: Limit of a Function

Let  $f : D \rightarrow \mathbb{R}$ , and  $x_0$  be an accumulation point of  $D$ . A number  $L \in \mathbb{R}$  is called the **limit of  $f(x)$  as  $x \rightarrow x_0$**  if, for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$|x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon \quad \forall x \in D.$$

In general, the limit of  $f(x)$  as  $x \rightarrow x_0$  may not exist. However, if it exists, it is uniquely determined. Hence we speak of *the* limit and write

$$\lim_{x \rightarrow x_0} f(x) = L$$

to indicate the limit exists and is equal to  $L$ . Informally, this means that the function values  $f(x)$  are arbitrarily close to  $L$  whenever  $x \in D$  is sufficiently close to  $x_0$ .

The limit of a function satisfies properties analogous to those of Proposition 3.13. More precisely, if  $f, g$  are functions on  $D$  such that

$$\lim_{x \rightarrow x_0} f(x) = L_1 \quad \text{and} \quad \lim_{x \rightarrow x_0} g(x) = L_2,$$

then

$$\lim_{x \rightarrow x_0} (f + g)(x) = L_1 + L_2, \quad \lim_{x \rightarrow x_0} (f \cdot g)(x) = L_1 \cdot L_2.$$

Moreover,  $f \leq g$  implies  $L_1 \leq L_2$ , and the sandwich lemma holds: if  $f \leq h \leq g$  and  $L_1 = L_2$  then  $\lim_{x \rightarrow x_0} h(x) = L_1 = L_2$ .

**Remark 4.35.** Let  $f : D \rightarrow \mathbb{R}$  be a function. If  $x_0 \in D$ , then  $f$  is continuous at  $x_0$  if and only if  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ .

Suppose that  $x_0 \in D$  is an accumulation point of  $D \setminus \{x_0\}$ . Let  $f : D \rightarrow \mathbb{R}$ , and consider the restriction  $f|_{D \setminus \{x_0\}}$ . It may happen that  $f$  is discontinuous at  $x_0$ , but the limit

$$L = \lim_{x \rightarrow x_0} f|_{D \setminus \{x_0\}}(x) \tag{4.5}$$

nevertheless exists. In this case, the point  $x_0$  is called a **removable discontinuity** of  $f$ , and one also writes

$$L = \lim_{\substack{x \rightarrow x_0 \\ x \neq x_0}} f(x). \tag{4.6}$$

If we now define

$$\tilde{f}(x) = \begin{cases} f(x), & x \in D \setminus \{x_0\}, \\ L, & x = x_0, \end{cases} \tag{4.7}$$

then  $\tilde{f}$  is continuous at  $x_0$ . In other words, we can remove the discontinuity of  $f$  by redefining its value at  $x_0$  to be  $L$ .

If instead  $x_0 \notin D$  but the limit in Equation 4.6 exists, we call the function  $\tilde{f}$  defined in Equation 4.7 the **continuous extension** of  $f$  to  $D \cup \{x_0\}$ .

Arguing as in the proof of Theorem 4.12, we obtain the following result.

#### Lemma 4.36: Limit and Sequences

Let  $f : D \rightarrow \mathbb{R}$ . Then  $L = \lim_{x \rightarrow \bar{x}} f(x)$  if and only if, for every sequence  $(x_n)_{n=0}^\infty \subseteq D$  converging to  $\bar{x}$ , one has  $\lim_{n \rightarrow \infty} f(x_n) = L$ .

We now state a result describing the behaviour of limits under composition with a continuous function.

#### Proposition 4.37: Limit and Composition

Let  $E \subseteq \mathbb{R}$ , and let  $f : D \rightarrow E$  be such that the limit  $L = \lim_{x \rightarrow \bar{x}} f(x)$  exists and belongs to  $E$ . If  $g : E \rightarrow \mathbb{R}$  is continuous at  $L$ , then

$$\lim_{x \rightarrow \bar{x}} g(f(x)) = g(L).$$

*Proof.* Let  $(x_n)_{n=0}^\infty \subseteq D$  be a sequence converging to  $\bar{x}$ . By Lemma 4.36, we have  $\lim_{n \rightarrow \infty} f(x_n) = L$ . Since  $g$  is continuous at  $L$ , Theorem 4.12 gives  $\lim_{n \rightarrow \infty} g(f(x_n)) = g(L)$ . Because  $(x_n)_{n=0}^\infty$  was arbitrary, using Lemma 4.36 again, we conclude that  $\lim_{x \rightarrow \bar{x}} g(f(x)) = g(L)$ .  $\square$

We now introduce conventions for improper limits of functions, in analogy with improper limits for sequences.

#### Definition 4.38: Improper Limits

Let  $f : D \rightarrow \mathbb{R}$ , and let  $x_0$  be an accumulation point of  $D$ . We say that  $f$  **diverges to  $+\infty$  as  $x \rightarrow x_0$** , and write

$$\lim_{x \rightarrow x_0} f(x) = +\infty,$$

if for every  $M > 0$ , there exists  $\delta > 0$  such that

$$\forall x \in D : |x - x_0| < \delta \Rightarrow f(x) \geq M.$$

Analogously,  $f$  **diverges to  $-\infty$  as  $x \rightarrow x_0$**  and we write  $\lim_{x \rightarrow x_0} f(x) = -\infty$ , if for every  $M > 0$ , there exists  $\delta > 0$  such that

$$\forall x \in D : |x - x_0| < \delta \Rightarrow f(x) \leq -M.$$

#### 4.5.2 One-Sided Limits

It is often useful to consider limits taken from one side only and to allow  $x_0$  to be  $\pm\infty$  as well. To this end, let  $x_0 \in \mathbb{R}$  be such that

$$D \cap (x_0, x_0 + \delta) \neq \emptyset \tag{4.8}$$

for every  $\delta > 0$ . In this case, we say that  $x_0$  is a **right-hand accumulation point** of  $D$ . Analogously, if

$$D \cap (x_0 - \delta, x_0) \neq \emptyset \tag{4.9}$$

for every  $\delta > 0$ , we say that  $x_0$  is a **left-hand accumulation point** of  $D$ .

#### Definition 4.39: One-Sided Limits

Let  $f : D \rightarrow \mathbb{R}$ , and let  $x_0 \in \mathbb{R}$  be a right-hand accumulation point of  $D$ . A number  $L \in \mathbb{R}$  is called the **right-hand limit** of  $f$  at  $x_0$  if, for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$x \in D \cap (x_0, x_0 + \delta) \Rightarrow |f(x) - L| < \varepsilon.$$

In this case we write  $L = \lim_{x \rightarrow x_0^+} f(x)$ . We also allow improper one-sided limits. We say that

$$\lim_{x \rightarrow x_0^+} f(x) = +\infty$$

if for every  $M > 0$ , there exists  $\delta > 0$  such that

$$x \in D \cap (x_0, x_0 + \delta) \Rightarrow f(x) \geq M.$$

Similarly,  $\lim_{x \rightarrow x_0^+} f(x) = -\infty$  means that, for every  $M > 0$ , there exists  $\delta > 0$  such that

$$x \in D \cap (x_0, x_0 + \delta) \Rightarrow f(x) \leq -M.$$

The **left-hand limit** is defined analogously, considering a left-hand accumulation point of  $D$  and writing  $\lim_{x \rightarrow x_0^-} f(x)$ .

Next, we define the notion of limit at infinity.

#### Definition 4.40: Limits at Infinity

Let  $f : D \rightarrow \mathbb{R}$ , and assume that  $D \cap (R, \infty) \neq \emptyset$  for every  $R > 0$ . A number  $L \in \mathbb{R}$  is called the **limit of  $f$  as  $x \rightarrow +\infty$**  if, for every  $\varepsilon > 0$ , there exists  $R > 0$  such that

$$x \in D \cap (R, \infty) \Rightarrow |f(x) - L| < \varepsilon.$$

We say that  $f$  **diverges to  $+\infty$  as  $x \rightarrow +\infty$**  if, for every  $M > 0$ , there exists  $R > 0$  such that

$$x \in D \cap (R, \infty) \Rightarrow f(x) \geq M.$$

The corresponding definition for  $x \rightarrow -\infty$  and diverges to  $-\infty$  are analogous.

Limits at  $+\infty$  can be converted into right-hand limits at 0 via inversion. Given  $f : D \rightarrow \mathbb{R}$  as above, define

$$E = \{x > 0 \mid x^{-1} \in D\}, \quad g : E \rightarrow \mathbb{R}, \quad g(x) = f(x^{-1}).$$

Then

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow 0^+} g(x),$$

so one limit exists if and only if the other does.

#### Definition 4.41: One-Sided Continuity and Jumps

Let  $f : D \rightarrow \mathbb{R}$  and  $x_0 \in D$ . If  $\lim_{x \rightarrow x_0^+} f(x)$  exists and equals  $f(x_0)$ , then  $f$  is **continuous from the right** at  $x_0$ . **Continuity from the left** is defined similarly. We call  $x_0$  a **jump point** if both one-sided limits exist but are different, i.e.,

$$L_- := \lim_{x \rightarrow x_0^-} f(x) \in \mathbb{R}, \quad L_+ := \lim_{x \rightarrow x_0^+} f(x) \in \mathbb{R}, \quad L_- \neq L_+.$$

### 4.5.3 Landau Notation

We introduce two standard notations that compare the asymptotic behaviour of a function to that of another function. (often called *relative asymptotics*).

#### Definition 4.42: Big-O at a Point

Let  $f, g : D \rightarrow \mathbb{R}$ , and let  $x_0$  be an accumulation point of  $D$ . We write

$$f(x) = O(g(x)) \quad \text{as } x \rightarrow x_0$$

if there exists  $M > 0$  and  $\delta > 0$  such that

$$x \in D \cap (x_0 - \delta, x_0 + \delta) \Rightarrow |f(x)| \leq M|g(x)|.$$

We then say that  $f$  is a **Big-O** of  $g$  as  $x \rightarrow x_0$ .

If  $g(x) \neq 0$  for all  $x$  sufficiently close to  $x_0$  (with  $x \in D$ ), then

$$f(x) = O(g(x)) \quad \text{as } x \rightarrow x_0 \quad \Leftrightarrow \quad \frac{f(x)}{g(x)} \text{ is bounded near } x_0.$$

#### Definition 4.43: Big-O at Infinity

Let  $f, g : D \rightarrow \mathbb{R}$ , and assume  $D \cap (R, \infty) \neq \emptyset$  for every  $R > 0$ . We write

$$f(x) = O(g(x)) \quad \text{as } x \rightarrow +\infty$$

if there exists  $M > 0$  and  $R > 0$  such that

$$x \in D \cap (R, \infty) \quad \Rightarrow \quad |f(x)| \leq M|g(x)|.$$

The definition for  $x \rightarrow -\infty$  is analogous.

The big-O notation hides the precise bound by an *implicit constant*  $M$ , which is often irrelevant for the argument one is interested in.

#### Example

- if  $f$  and  $g$  are bounded and continuous near  $x_0$  with  $g(x_0) \neq 0$ , then  $f(x) = O(g(x))$  as  $x \rightarrow x_0$ .
- As  $x \rightarrow 0$ , one has  $x^2 = O(x)$ , but  $x \neq O(x^2)$  (since  $x/x^2$  is unbounded near 0).
- As  $x \rightarrow +\infty$ ,  $\frac{3x^3}{x^3+3} = O(1)$ , but  $\frac{3x^3}{x^3+3} \neq O(x^\alpha)$  for  $\alpha < 0$ .

As discussed above, the big-O means that  $f$  is bounded by a multiple of  $g$ . One may also consider a stronger condition, namely that  $f$  is asymptotically negligible with respect to  $g$ . This leads to the following definition.

#### Definition 4.44: Little-O at a Point

Let  $f, g : D \rightarrow \mathbb{R}$ , and let  $x_0$  be an accumulation point of  $D$ . We write

$$f(x) = o(g(x)) \quad \text{as } x \rightarrow x_0$$

if, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$x \in D \cap (x_0 - \delta, x_0 + \delta) \quad \Rightarrow \quad |f(x)| \leq \varepsilon|g(x)|.$$

We then say that  $f$  is a **little-o** of  $g$  as  $x \rightarrow x_0$ .

If  $g(x) \neq 0$  for all  $x$  near  $x_0$  (with  $x \in D$ ), then

$$f(x) = o(g(x)) \quad \text{as } x \rightarrow x_0 \quad \Leftrightarrow \quad \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0.$$

Moreover,  $f(x) = o(g(x)) \Rightarrow f(x) = O(g(x))$ .

**Definition 4.45: Little-o at Infinity**

Let  $f, g : D \rightarrow \mathbb{R}$ , and assume that  $D \cap (R, \infty) \neq \emptyset$  for every  $R > 0$ . We write

$$f(x) = o(g(x)) \quad \text{as } x \rightarrow +\infty$$

if, for every  $\varepsilon > 0$  there exists  $R > 0$  such that

$$x \in D \cap (R, \infty) \quad \Rightarrow \quad |f(x)| \leq \varepsilon |g(x)|.$$

The definition for  $x \rightarrow -\infty$  is analogous.

**Example**

- $x = o(x^2)$  as  $x \rightarrow +\infty$ , and  $x^2 = o(x)$  as  $x \rightarrow 0$

- For any  $\alpha < 1$ ,

$$\frac{3x^3}{2x^2 + x^{10}} = o(|x|^\alpha) \quad \text{as } x \rightarrow 0,$$

but not for  $\alpha \geq 1$ . Indeed,

$$\left| \frac{3x^3}{|x|^\alpha 2x^2 + x^{10}} \right| = |x|^{1-\alpha} \frac{3}{2 + x^8} \rightarrow 0 \quad \text{as } x \rightarrow 0,$$

whenever  $\alpha < 1$ .

In computations, one often uses Landau symbols as placeholders. Writing

$$f(x) = o(g(x)) \quad \text{as } x \rightarrow x_0$$

means there is a function  $h : D \rightarrow \mathbb{R}$  with  $h(x) = o(g(x))$  as  $x \rightarrow x_0$ . Similarly for big-O.

**Example**

Polynomial division gives, as  $x \rightarrow +\infty$ ,

$$\frac{x^3 - 7x^2 + 6x + 2}{x^2} = x - 7 + O\left(\frac{1}{x}\right) = x - 7 + o(1) = x + O(1) = x + o(x).$$

**4.6 Sequences of Functions****4.6.1 Pointwise Convergence****Definition 4.46: Sequences of Functions**

A **sequence** of real-valued on a subset  $D \subseteq \mathbb{R}$  is a family of functions  $f_n : D \rightarrow \mathbb{R}$  indexed by  $\mathbb{N}$ . The function  $f_n$  is called the **n-th element** of the sequence. One often writes  $(f_n)_{n \in \mathbb{N}}$ ,  $(f_n)_{n=0}^\infty$ , or  $(f_n)_{n \geq 0}$  for a sequence of functions.

**Definition 4.47: Pointwise Convergence**

Let  $D \subseteq \mathbb{R}$ , and let  $(f_n)_{n=0}^\infty$  be a sequence of functions  $f_n : D \rightarrow \mathbb{R}$ . Let  $f : D \rightarrow \mathbb{R}$  be another function. We say that  $(f_n)_{n=0}^\infty$  **converges pointwise** to  $f$ , if for every  $x \in D$ , the sequence of real numbers  $(f_n(x))_{n=0}^\infty$  converges to  $f(x)$ , i.e., for every  $x \in D$  and for every  $\varepsilon > 0$ , there exists  $n \in \mathbb{N}$  such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N.$$

In this case,  $f$  is called the **pointwise limit** of the sequence  $(f_n)_{n=0}^\infty$ .

**Remark 4.48.** Note that, in the definition of pointwise convergence the index  $N$  may depend on both  $x$  and  $\varepsilon$ . In other words, for each point  $x \in D$  we examine the convergence of  $(f_n)_{n=0}^\infty$  to  $f$  separately.

**4.6.2 Uniform Convergence****Definition 4.49: Uniform Convergence**

Let  $D \subseteq \mathbb{R}$ , and let  $(f_n)_{n=0}^\infty$  be a sequence of functions  $f_n : D \rightarrow \mathbb{R}$ . Let  $f : D \rightarrow \mathbb{R}$  be another function. We say that  $(f_n)_{n=0}^\infty$  **converges uniformly** to  $f$  on  $D$  if, for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N, \forall x \in D.$$

**Remark 4.50.** Note that, in the definition of uniform convergence the index  $N$  may only depend on  $\varepsilon$ , and therefore the condition has to hold for all  $x \in D$  for the sequence of functions  $(f_n)_{n=0}^\infty$  to converge uniformly to  $f$  on  $D$ .

**Remark 4.51.** Let  $D \subseteq \mathbb{R}$ , and  $(f_n)_{n=0}^\infty$  be a sequence of functions  $f_n : D \rightarrow \mathbb{R}$  converging uniformly to  $f : D \rightarrow \mathbb{R}$  on  $D$ . Then the sequence of functions  $(f_n)_{n=0}^\infty$  also converges pointwise to  $f$ .

**Theorem 4.52: Continuity under Uniform Convergence**

Let  $D \subseteq \mathbb{R}$ , and let  $(f_n)_{n=0}^\infty$  be a sequence of continuous functions converging uniformly to  $f : D \rightarrow \mathbb{R}$ . Then  $f$  is continuous.

*Proof.* To prove that  $f$  is continuous, we fix  $\bar{x} \in D$  and show that  $f$  is continuous at  $\bar{x}$ . Given  $\varepsilon > 0$ , the uniform convergence of  $f_N$  to  $f$  provides  $N \in \mathbb{N}$  such that

$$|f_N(y) - f(y)| < \frac{\varepsilon}{3} \quad \forall y \in D.$$

Also, since  $f_N$  is continuous at  $\bar{x}$ , there exists  $\delta > 0$  such that

$$|x - \bar{x}| < \delta \quad \Rightarrow \quad |f_N(x) - f_N(\bar{x})| < \frac{\varepsilon}{3}.$$

Then, for  $|x - \bar{x}| < \delta$ , we have

$$|f(x) - f(\bar{x})| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(\bar{x})| + |f_N(\bar{x}) - f(\bar{x})| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Which shows that  $f$  is continuous at  $\bar{x}$ . Since,  $\bar{x}$  is arbitrary,  $f$  is continuous on  $D$ .  $\square$