Essay 1: Johnson–Lindenstrauss and Fast JL Transforms

Johnson–Lindenstrauss Lemma and Fast JL Transforms  
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The Johnson–Lindenstrauss (JL) lemma is a foundational result in high-dimensional geometry. It states that a set of n points in a high-dimensional Euclidean space can be embedded into a lower-dimensional space of dimension m = O(ε⁻² log n), such that all pairwise Euclidean distances are preserved up to (1±ε).  
  
\*\*Mathematical Statement\*\*   
Let 0 < ε < 1/2 and let X be a set of n points in R^D. There exists a linear map f : R^D → R^m with m = O(ε⁻² log n), such that for all u,v in X:  
  
(1-ε) ||u-v||² ≤ ||f(u)-f(v)||² ≤ (1+ε) ||u-v||².  
  
\*\*Random Projection Proof Sketch\*\*   
Choose an m×D random matrix R with entries sampled from N(0,1/m). For any fixed vector x:  
  
E[||Rx||²] = ||x||².   
By concentration inequalities (Chernoff/Johnson-Lindenstrauss tail bounds), with high probability, ||Rx||² is close to ||x||². A union bound across all n² pairs ensures distance preservation.  
  
\*\*Variants\*\*   
- \*Gaussian JL:\* R\_ij ~ N(0,1/m).   
- \*Rademacher JL:\* R\_ij ∈ {±1/√m}.   
- \*Sparse JL (Achlioptas):\* most entries zero, a few ±1/√s.   
  
\*\*Fast JL Transform (FJLT)\*\*   
JL is expensive when D is large. The FJLT accelerates embedding by combining:   
1. Random sign flips (diagonal ±1).   
2. A fast orthogonal transform (Hadamard or DCT) in O(D log D).   
3. Subsampling m coordinates.  
  
Thus embedding cost reduces from O(nDm) to O(nD log m). The theoretical guarantee remains similar: distances preserved up to ε with m = O(ε⁻² log n).  
  
\*\*Applications\*\*   
- Approximate nearest neighbors.   
- Reducing dimensionality in machine learning.   
- Compressed sensing and randomized linear algebra.  
  
The JL lemma shows that high-dimensional data is not inherently fragile: geometry can be preserved under surprisingly aggressive dimensionality reduction.