Practical Project - Formalized Mathematics in Lean

Marc Fares, Stefano Rocca

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1 Mathematical content of the formalization topic: orientability of product manifolds

Orientation of a vector space

Definition 1.1 (Consistently oriented bases for a vector space)**.**
Let V be a real, finite-dimensional vector space.
Consider two bases B and B' for V.
B and B' are said to be **consistently oriented** if the transition matrix $\varphi_{B,B'} : V \to V$ from B to B' has a positive determinant (equivalently, the transition matrix $\varphi_{B',B} : V \to V$ from B' to B has a positive determinant).
Note that this defines an equivalence relation on the set of bases for V.
We call the two equivalence classes, the **orientations** of V; we may then arbitrarily refer to one of them as our **positive** orientation, and to the other as the **negative** one.

Example 1.1 (The standard orientation of a Euclidean space)**.**
The orientation $[e_1, \cdots, e_n]$ containing the standard basis of the Euclidean n-space $\mathbb{R}^n$ is referred to as the **standard orientation** of $\mathbb{R}^n$.

- For $n = 1$, any basis for $\mathbb{R}$ consisting of a positive real number is positively oriented (i.e. $[e_1]$ consists of those bases "pointing to the right").
- For $n = 2$, any (ordered) basis for $\mathbb{R}^2$ consisting of a pair of vectors forming a positive angle of less than π radians (in the counterclockwise direction) is positively oriented.
- For $n = 3$, any (ordered) basis for $\mathbb{R}^3$ satisfying the right-hand rule is positively oriented.

Orientability of a manifold

Definition 1.2 (Orientability of a manifold)**.**
Let M be a connected smooth manifold.
An atlas $\mathcal{A}$ for M is said to be **consistently oriented** if given any pair of overlapping coordinate charts, the transition map between them has a Jacobian with positive determinant.
M is said to be **orientable** if its smooth structure contains (i.e. is generated by) a consistently oriented atlas.
In this case, a choice of such atlas determines a unique orientation on each tangent space to M; the collection of these pointwise orientations is defined to be the **(continuous, or smooth) orientation** of M.

Examples 1.1 (Illustrative examples)**.**

- **Orientable manifold:**
  Any n-sphere $\mathbb{S}^n$ is orientable: the atlas consisting of the stereographic projections is consistently oriented.

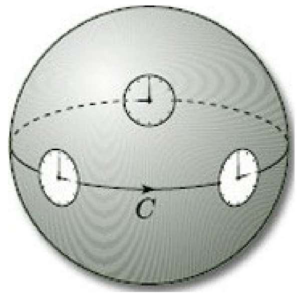

Figure 1: Orientability of $\mathbb{S}^2$

- **Non-orientable manifold:**
  The **Möbius strip** is not orientable: assuming otherwise the existence of a consistently-oriented smoothly-compatible atlas leads to a contradiction after letting a tangent basis do one "trip" around the strip, and comparing it at the beginning and end of the trip.

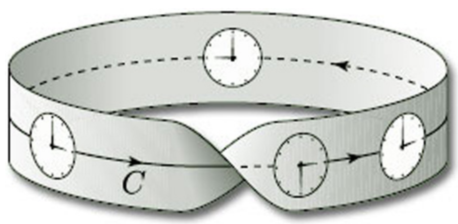

Figure 2: Non-orientability of the Möbius strip

Theorem 1.1 (Orientability under the product)**.**
The product of smooth orientable manifolds is smooth orientable.

Proof idea.
Let M and N be smooth orientable manifolds.
Orientability of $M \times N$ follows directly from orientability of each of M and N, coupled with the following observations:

- An atlas generating the smooth product structure on $M \times N$ is the product atlas (consisting of the charts $(U \times V, \varphi \times \psi)$ where (U, φ) and (V, ψ) are coordinate charts for M and N, respectively), where the transition function from one product chart to the next is the (Cartesian) product of the corresponding transition functions.

- In any matrix notation, the Jacobian of the Cartesian product of two maps is the block diagonal matrix of the Jacobians of the block matrices.
- The determinant of a block diagonal matrix equals the product of the determinants of the block matrices.

□

2 Formalization

We formalized theorem 1.1.

Main results

The main results in our formalization are the following:

- **lemma det_prod:**
  Let E and F be two finite-dimensional real vector spaces.
  Consider endomorphisms $A : E \to E$ and $B : F \to F$.
  Then
  $$\det(A \oplus B) = \det A \cdot \det B$$
  where $A \oplus B : E \oplus F \to E \oplus F$ is defined by $(v, w) \mapsto (Av, Bw)$.

  The determinant of the direct sum of endomorphisms equals the product of their corresponding determinants.

- **lemma orientationPreserving_of_prod:**
  Let E and F be two finite-dimensional real vector spaces.
  Consider two functions $f : E \to E$ and $g : F \to F$.
  Assume f and g are orientation-preserving on subsets $U \subset E$ and $V \subset F$, respectively (i.e. differentiable with positive Jacobian determinant).
  Then $f \times g$ is orientation-preserving on $U \times V \subset E \oplus F$ (where $f \times g : E \oplus F \to E \oplus F$ is defined by $(x, y) \mapsto (f(x), g(y))$.)

  The Cartesian product of orientation preserving functions on finite-dimensional vector spaces is orientation preserving.

- **theorem orientableGroupoid_prod:**
  Let H and H' be models with corners (i.e. embedded submanifolds with corners of Euclidean spaces).
  Let $S \subset H$ and $S' \subset H'$ be two subsets.
  Consider smooth, orientation-preserving homeomorphisms $\tau : S \to \tau(S)$ and $\tau' : S' \to \tau(S')$.
  Then $\tau \times \tau' : S \times S' \to \tau(S) \times \tau'(S')$ is a smooth, orientation-preserving homeomorphism.

  The Cartesian product of smooth, orientation-preserving partial homeomorphisms on model spaces is a smooth, orientation-preserving partial homeomorphism on the model product space.

- **theorem orientableManifold_of_product:**
  Let M and N be smooth orientable manifolds.
  Then $M \times N$ is a smooth orientable manifold.

  The product of two orientable smooth manifolds is an orientable smooth manifold.

Open problem

Having completed all the goals we had set out to reach at the beginning of our work on the project (namely, to prove closedness of orientability under the product in the category of smooth real manifolds), we set out to prove the converse result, i.e. that the product of smooth manifolds is smooth only if the factor manifolds are smooth themselves. This is still work in progress.

References

We have mainly relied on our knowledge of linear algebra and basic differential geometry. For a formal, abstract definition of orientability which suits the one understood by Lean, we have used John Lee's *Introduction to Smooth Manifolds*.