Orientability of product smooth manifolds

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Orientation of vector spaces

Let V be a real finite-dimensional vector space, and $\mathcal{B}$ and $\mathcal{B}'$ two (ordered) bases of V. There exists a unique linear map

$$f : V \to V$$

sending $\mathcal{B}$ to $\mathcal{B}'$ and its associated matrix $\mathcal{M}_{\mathcal{B}}^{\mathcal{B}'}(f)$ is the usual **change-of-basis matrix**.

Definition

$\mathcal{B}$ and $\mathcal{B}'$ are said to be **coherently oriented** if $det \mathcal{M}_{\mathcal{B}}^{\mathcal{B}'}(f) > 0$.

This induces an equivalence relation

$$\mathcal{B} \sim \mathcal{B}' \iff \mathcal{B} \text{ and } \mathcal{B}' \text{ are coherently oriented}.$$

- $\mathcal{M}_{\mathcal{B}}^{\mathcal{B}} = id_n$ (reflexivity)
- $\mathcal{M}_{\mathcal{B}'}^{\mathcal{B}} = (\mathcal{M}_{\mathcal{B}}^{\mathcal{B}'})^{-1}$ (symmetry)
- $\mathcal{M}_{\mathcal{B}}^{\mathcal{B}''} = \mathcal{M}_{\mathcal{B}'}^{\mathcal{B}''} \mathcal{M}_{\mathcal{B}}^{\mathcal{B}'}$ (transitivity)

And this gives two equivalence classes on the set of bases of V.

Definition

An orientation for V is the choice of one such equivalence class.
$(V, [\mathcal{B}])$ is an **oriented vector space**.

In the same vein, given two oriented vector spaces of the same dimension, $(V, [\mathcal{B}_V])$ and $(W, [\mathcal{B}_W])$.

Definition

A linear map $f : V \to W$ is said to be **orientation preserving** if $detF_{\mathcal{B}_V}^{\mathcal{B}_W} > 0$ and **orientation reversing** if $detF_{\mathcal{B}_V}^{\mathcal{B}_W} < 0$.

Orientation and smooth maps in $\mathbb{R}^n$

Let $\alpha : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^n$. This induces a linear map on tangent spaces at each point $p \in U$,

$$d_p\alpha : T_pU \to T_{\alpha(p)}V$$

called the differential of α at p.
Now, choosing a base for T_pU and $T_{\alpha(p)}V$, the matrix associated to $d_p\alpha$ is $Jac_p(\alpha)$ and it is called the **Jacobian matrix** of α at p.

Definition

We say that α is orientation preserving (resp. reversing) at p if $detJac_p(\alpha) > 0$ (resp < 0). α is orientation preserving (resp. reversing) if it is so at every $p \in U$.

Orientation of smooth manifolds

Let M be a (nice) topological space.

Definition

A **oriented smooth atlas** $\mathcal{A}^o$ on M is a collection of charts $(U_i, \varphi_i : U_i \to V_i \subseteq \mathbb{R}^n)$ such that:

- each φ_i is an homeomorphism.
- $\forall p \in M$, $\exists (U_i, \varphi_i)$ such that $p \in U_i$.
- $\forall (U_i, \varphi_i), (U_j, \varphi_j)$ there are smooth transition maps

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

- $detJac(\varphi_i \circ \varphi_j^{-1}) > 0$ i.e. the transition maps are orientation preserving.

We call $(M, \mathcal{A}^o)$ an oriented smooth manifold.

Main theorem

We are ready to state the theorem we proved in our project:

Theorem

Let M and N be two oriented smooth manifolds of dimension m and n respectively. Then $M \times N$ is a oriented smooth manifold of dimension $n + m$.

Proof strategy

We have to show that $\mathcal{A}_M \times \mathcal{A}_N := \{\varphi_i \times \psi_j : U_i \times V_j \to \mathbb{R}^m \times \mathbb{R}^n\}$ is an oriented smooth atlas.

- $\varphi_i \times \psi_j$ are homeomorphisms because are componentwise homeomorphisms.
- Any point $(p, q) \in M \times N$ is covered by charts, since p is covered in $\mathcal{A}_M$ and q is covered in $\mathcal{A}_N$.
- Transition functions are smooth since $(\varphi \times \psi) \circ (\varphi' \times \psi')^{-1} = (\varphi \circ \varphi'^{-1}) \times (\psi \circ \psi'^{-1})$ which is componentwise smooth.

Hence $\mathcal{A}_M \times \mathcal{A}_N$ is a smooth atlas.

$\mathcal{A}_M \times \mathcal{A}_N$ is also oriented.

- $d_{(p,q)}(\varphi \circ \varphi'^{-1} \times \psi \circ \psi'^{-1}) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ gives that

$$Jac_{(p,q)}((\varphi \circ \varphi'^{-1}) \times (\psi \circ \psi'^{-1})) = \begin{pmatrix} Jac_p(\varphi \circ \varphi'^{-1}) & 0 \\ 0 & Jac_q(\psi \circ \psi'^{-1}) \end{pmatrix}$$

- $det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = detA \cdot detB$ hence

$$detJac_{(p,q)}((\varphi \circ \varphi'^{-1}) \times (\psi \circ \psi'^{-1})) =$$

$$= detJac_p(\varphi \circ \varphi'^{-1}) \cdot detJac_q(\psi \circ \psi'^{-1}) > 0$$