From the 1D profile to amplitude distribution

Let’s consider matched density distribution ρ(x, px) in the the 2D phase space.

Starting from the x-projection of ρ, ρ_x,we want to determine the amplitude distribution, ρ_r(r= $\sqrt{x^2 + \mathrm{px}^2}$).

In[85]:=

```
$Assumptions = x ∈ Reals;
InverseAbelTransform[A_, r_] := - (1 / Pi) Integrate[
   Derivative[1][A][x] / Sqrt[x^2 - r^2], {x, r, Infinity}, Assumptions → r ≥ 0]
AbelTransform[f_, x_] := 2 Integrate[r f[r] / Sqrt[r^2 - x^2], {r, x, Infinity}]
```

In[89]:=

```
exampleFunction[x_] := 1/Sqrt[2 π] Exp[-x^2/2]
```

$$\text{exampleFunction[x\_] :=}\ \frac{1}{\sqrt{2\,\pi}}\ \text{Exp}\left[-\ \frac{\text{x}\,\hat{}\,2}{2}\right]$$

In[90]:=

```
inverseAbelGaussian = Simplify[InverseAbelTransform[exampleFunction, r]]
```

$$\texttt{Print}\left[\texttt{"Inverse Abel Transform of } \frac{1}{\sqrt{2\,\pi}}\texttt{Exp[-}\frac{\texttt{x}\,\texttt{^}\,\texttt{2}}{2}\texttt{]: ", inverseAbelGaussian}\right]$$

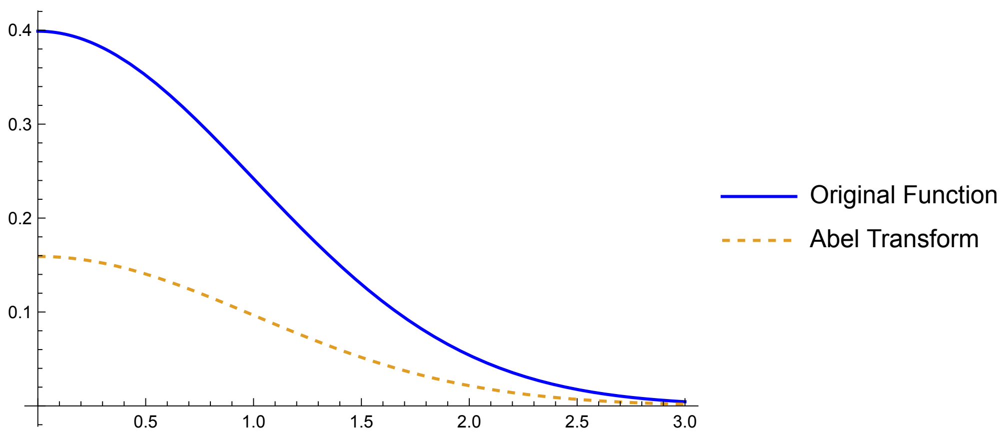
```
Plot[{exampleFunction[r], inverseAbelGaussian}, {r, 0, 3},
 PlotLegends → {"Original Function", "Abel Transform"}, PlotStyle → {Blue, Dashed}]
```

Out[90]=

$$\frac{e^{-\frac{r^2}{2}}}{2\,\pi} \quad \text{if } r > 0$$

$$\texttt{Inverse Abel Transform of } \frac{1}{\sqrt{2\,\pi}}\texttt{Exp[-}\frac{\texttt{x}\,\texttt{^}\,\texttt{2}}{2}\texttt{]: } \quad \frac{e^{-\frac{r^2}{2}}}{2\,\pi} \quad \text{if } r > 0$$

Out[92]=



In[93]:=

$$\texttt{exampleFunction[r\_] :=} \frac{e^{-\frac{r^2}{2}}}{2\,\pi}$$

```
abelGaussian = Simplify[AbelTransform[exampleFunction, x]]
```

Out[94]=

$$\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\,\pi}} \quad \text{if } x > 0$$

In[218]:=

```
x = RandomVariate[NormalDistribution[0, 1], 1 000 000];
px = RandomVariate[NormalDistribution[0, 1], 1 000 000];
Show[
 Histogram[√(x² + px²), 100, "PDF"],
 Plot[2 * Pi * r * inverseAbelGaussian, {r, 0, 5},
  PlotLegends → {"Original Function", "Abel Transform"}, PlotStyle → {Blue, Dashed}]
]
```

Out[220]=