Linear Algebra (0031)
Problem Set 7 Solutions

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1. Consider the Matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ -2 & 0 & 4 \end{bmatrix}$

(a) Calculate the eigenvalues and eigenvectors of A.

Solution.

$$\det(A - \lambda I) = 0$$

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ -1 & 2-\lambda & 1 \\ -2 & 0 & 4-\lambda \end{vmatrix} \\ &= -\lambda^3 + 7\lambda^2 - 16\lambda + 12 \\ &= -(\lambda - 3)(\lambda^2 - 4\lambda + 4) \\ &= -(\lambda - 3)(\lambda - 2)^2 \\ &= 0 \end{aligned}$$

$$\therefore \lambda_1 = 3,\ \lambda_2 = 2$$

(b) If A has three linearly independent eigenvectors, find the diagonalization of A

(c) What are the eigenvalues and eigenvectors of A^{-1} (if A^{-1} exists).

(d) Let $B = A + 3I$, where I is the identity matrix. Find the eigenvalues of B.

2. Consider a square matrix A. Suppose that A has full column rank. Can A have eigenvalue 0? Justify your answer.

3. The characteristic polynomial equation (CPE) of A is written as $|A - \lambda I| = 0$. Likewise, the characteristic polynomial equation of AT is $|AT - \lambda I| = 0$. Solving CPE gives eigenvalues.

(a) Show that A and A^T have the same eigenvalues (Hint. Take a look at the CPEs of A and A^T, and use the fact that transposing a matrix does not change the determinant)

4. Consider a matrix $A = \begin{bmatrix} 2 & a \\ 1 & 0 \end{bmatrix}$. Find a condition for A to be diagonalisable. (Hint. A needs to have two linearly independent eigenvectors. Eigenvectors associated to distinct eigenvalues are linearly independent)

5. Check if the following matrices are positive definite:

(a) $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

6. Show that $R^T R$ is positive semidefinite for any matrix R.

7. Show that $R^T R$ is positive definite if and only if R has full column rank.

8. Prove that if $B = M^{-1}AM$, then A and B have the same eigenvalues. (Hint: multiply an eigenvector of B on the right, and then multiply M on the left)