Linear Algebra (0031)
Problem Set 2 Solutions

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1. (a) Find the 3×3 elimination matrix E_{21} that subtracts 2 times row 1 from row 2.

Solution.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Find the 3×3 elimination matrix E_{31} that subtracts -3 times row 1 from row 3.

Solution.

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

(c) Calculate $E_1 = E_{31}E_{21}$. What does E_1 do in terms of elimination? Based on your interpretation of what E_1 does, calculate the inverse of E_1.

Solution.

$$\begin{aligned} E_1 &= E_{31}E_{21} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \end{aligned}$$

My interpretation of E_1 is to subtract 2 times row 1 from row 2 and subtract -3 times row 1 from row 3. So, inverse of E_1 should add 2 times row 1 from row 2 and add -3 times row 1 from row 3.

$$\therefore E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

(d) Find the 3×3 elimination matrix E_{32} that subtracts 5 times row 2 from row 3.

Solution.

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

(e) Calculate the inverse of E_{32}.

Solution. As the E_{32} subtracts 5 times row 2 from row 3, inverse of E_{32} should add 5 times row 2 from row 3.

$$\therefore E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

(f) Calculate $E = E_{32}E_{31}E_{21}$ and interpret what E does in terms of elimination.

Solution.

$$\begin{aligned} E &= E_{32}E_{31}E_{21} \\ &= E_{32}E_1 (\because \text{(c) and Associative Property}) \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & -5 & 1 \end{bmatrix} \end{aligned}$$

My interpretation of E is to subtract 2 times row 1 from row 2, add 7 times row 1 from row 3 and subtract 5 times row 2 from row 3.

(g) In order to calculate the inverse of E, use the following two methods:

- Use the properties of inverse matrix as $E^{-1} = (E_{32}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$

  Solution.

$$
\begin{aligned}
E^{-1} &= (E_{32}E_{31}E_{21})^{-1} \\
&= E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5 & 1 \end{bmatrix}
\end{aligned}
$$

- Use the interpretation of what E does, and derive the inverse that reverts what E does.

  Solution. As the interpretation of matrix E,

$$
\therefore E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5 & 1 \end{bmatrix}
$$

Do the results in i and ii coincide? Yes.

2.

Solution.

$$
\begin{aligned}
LU &= A \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ -2 & 1 & 2 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 2 \\ -2 & 1 & 2 & 3 \end{array}\right] \\
&= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ -2 & 1 & 2 & 3 \end{array}\right] \\
&= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 4 & 5 \end{array}\right] \\
&= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 5 \end{array}\right]
\end{aligned}
$$

$$
\therefore \mathbf{x} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}
$$

3.

Solution. Let $A = \begin{bmatrix} 1 & \alpha \\ \alpha & \beta \end{bmatrix}$ $(\because \text{(a)})$.

$$\begin{bmatrix} 1 & \alpha \\ \alpha & \beta \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \alpha \\ 0 & \beta - \alpha^2 \end{bmatrix}$$

$$\therefore \beta - \alpha^2 = \frac{5}{2} (\because \text{(a)})$$

$$\therefore 1 + 2\alpha + \beta = 15 (\because \text{(b)})$$

$$\begin{cases} \beta - \alpha^2 = \frac{5}{2} \\ 2\alpha + \beta = 14 \end{cases}$$

$$\therefore \alpha = -1 - \frac{5}{\sqrt{2}}, \beta = 16 + 5\sqrt{2} \text{ or } \alpha = \frac{5}{\sqrt{2}} - 1, \beta = 16 - 5\sqrt{2}$$

$$\therefore A = \begin{bmatrix} 1 & -1 - \frac{5}{\sqrt{2}} \\ -1 - \frac{5}{\sqrt{2}} & 16 + 5\sqrt{2} \end{bmatrix} \text{ or } \begin{bmatrix} 1 & \frac{5}{\sqrt{2}} - 1 \\ \frac{5}{\sqrt{2}} - 1 & 16 - 5\sqrt{2} \end{bmatrix}$$

4. (a) Apply Gaussian elimination to solve $A\mathbf{x} = \mathbf{b}$.

Solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 3 & 4 & 4 & 32 \\ 6 & 8 & 9 & 67 \end{array}\right]$$
$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 1 & 5 \\ 6 & 8 & 9 & 67 \end{array}\right]$$
$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 1 & 5 \\ 0 & 2 & 3 & 13 \end{array}\right]$$
$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\therefore \mathbf{x} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

(b) Let $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = PA$ and $\mathbf{c} = P\mathbf{b}$. Find a solution to $B\mathbf{x} = \mathbf{c}$ ”without” solving the equation directly.

First, P shifts row 1 to row 2, row 2 to row 3 and row 3 to row 1.

$$\therefore B = \begin{bmatrix} 6 & 8 & 9 \\ 1 & 1 & 1 \\ 3 & 4 & 4 \end{bmatrix}$$

and,

$$\therefore \mathbf{c} = \begin{bmatrix} 67 \\ 9 \\ 32 \end{bmatrix}$$

Therefore, $B\mathbf{x} = \mathbf{c}$ is same with row exchanged $A\mathbf{x} = \mathbf{b}$. As row exchange does not change the value of $\mathbf{x}$,

$$\therefore \mathbf{x} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

5. (a) $\begin{bmatrix} 2 & 2 \\ 14 & 17 \end{bmatrix}$

Solution. Subtract 7 times row 1 from row 2.

$$\therefore U = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$$

$$\therefore L = \begin{bmatrix} 1 & 0 \\ 7 & 1 \end{bmatrix}$$

And the pivots are 2 and 3.

$$\begin{aligned} &\left[\begin{array}{cc|cc} 2 & 2 & 1 & 0 \\ 14 & 17 & 0 & 1 \end{array}\right] \\ \Rightarrow &\left[\begin{array}{cc|cc} 1 & 1 & \frac{1}{2} & 0 \\ 14 & 17 & 0 & 1 \end{array}\right] \\ \Rightarrow &\left[\begin{array}{cc|cc} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 3 & -7 & 1 \end{array}\right] \\ \Rightarrow &\left[\begin{array}{cc|cc} 1 & 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{7}{3} & \frac{1}{3} \end{array}\right] \\ \Rightarrow &\left[\begin{array}{cc|cc} 1 & 0 & \frac{17}{6} & -\frac{1}{3} \\ 0 & 1 & -\frac{7}{3} & \frac{1}{3} \end{array}\right] \end{aligned}$$

Therefore, inverse matrix is $\begin{bmatrix} \frac{17}{6} & -\frac{1}{3} \\ -\frac{7}{3} & \frac{1}{3} \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 & 1 \\ 4 & 1 & 1 \\ 2 & 5 & 3 \end{bmatrix}$

Solution. Let $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Subtract 2 times row 1 from row 2.

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 2 & 5 & 3 \end{bmatrix}$$

$$l_{21} = 2$$

Subtract row 1 from row 3.

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 6 & 2 \end{bmatrix}$$

$$l_{31} = 1$$

Subtract 2 times row 2 from row 3.

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$l_{32} = 2$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

And the pivots are $2, 3$ and 4.

$$
\left[\begin{array}{ccc|ccc} 2 & -1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 1 & 0 & 1 & 0 \\ 2 & 5 & 3 & 0 & 0 & 1 \end{array}\right]
$$

$$
\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 4 & 1 & 1 & 0 & 1 & 0 \\ 2 & 5 & 3 & 0 & 0 & 1 \end{array}\right]
$$

$$
\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 3 & -1 & -2 & 1 & 0 \\ 0 & 6 & 2 & -1 & 0 & 1 \end{array}\right]
$$

$$
\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 6 & 2 & -1 & 0 & 1 \end{array}\right]
$$

$$
\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 4 & 3 & -2 & 1 \end{array}\right]
$$

$$
\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right]
$$

$$
\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{12} & \frac{1}{3} & -\frac{1}{12} \\ 0 & 1 & 0 & -\frac{5}{12} & \frac{1}{6} & \frac{1}{12} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right]
$$

Therefore, inverse matrix is $\begin{bmatrix} -\frac{1}{12} & \frac{1}{3} & -\frac{1}{12} \\ -\frac{5}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$

6. (a) Is it possible for A to have only one pivot? If so, find the values of a,b,c,d so that A has only one pivot

Solution.

$$\begin{bmatrix}1&1&1\\a&1&1\\2&b&2\\d&1&c\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&1\\0&1-a&1-a\\0&b-2&0\\0&1-d&c-d\end{bmatrix}$$

To A have only one pivot, $a=1, b=2, c=1, d=1$.

(b) Given $d=2$, let k be the minimum possible number of pivots A can have. Find the condition for A to have exactly k pivots.

Solution.

$$\begin{bmatrix}1&1&1\\a&1&1\\2&b&2\\2&1&c\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&1\\0&1-a&1-a\\0&b-2&0\\0&-1&c-2\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&1\\0&-1&c-2\\0&1-a&1-a\\0&b-2&0\end{bmatrix}$$

If $a=1, b=2$, the minimum $k=2$.

(c) Given $d=2$, find the condition (if exists) for A to have exactly $k+1$ pivots.

Solution.

$$\begin{bmatrix}1&1&1\\a&1&1\\2&b&2\\2&1&c\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&1\\0&-1&c-2\\0&1-a&1-a\\0&b-2&0\end{bmatrix}\Rightarrow\begin{bmatrix}1&1&1\\0&-1&c-2\\0&0&1-a+(1-a)(c-2)\\0&b-2&0\end{bmatrix}$$

If $a\neq 1, b=2$, the value of k will be 3.

(d) Given $d=2$, find the condition (if exists) for A to have exactly $k+2$ pivots.

Solution. It does not exists.

(e) Given $d=2$, find the condition (if exists) for A to have exactly $k+3$ pivots.

Solution. It does not exists.

7. (a)

Solution.

$$\left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 2&1&0&0&0&1&0&0\\ 1&0&1&0&0&0&1&0\\ 2&0&0&1&0&0&0&1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&1&0&0&-2&1&0&0\\ 0&0&1&0&-1&0&1&0\\ 0&0&0&1&-2&0&0&1 \end{array}\right]$$

The inverse of the matrix is $\begin{bmatrix} 1&0&0&0\\ -2&1&0&0\\ -1&0&1&0\\ -2&0&0&1 \end{bmatrix}$

(b)

Solution.

$$\left[\begin{array}{ccc|ccc} \sin\theta & -\cos\theta & 0 & 1&0&0\\ \cos\theta & \sin\theta & 0 & 0&1&0\\ 0&0&1&0&0&1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} \sin\theta & -\cos\theta & 0 & 1&0&0\\ 0 & \frac{1}{\sin\theta} & 0 & -\frac{\cos\theta}{\sin\theta}&1&0\\ 0&0&1&0&0&1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} \sin\theta & 0 & 0 & 1-\cos^2\theta & \sin\theta\cos\theta & 0\\ 0 & \frac{1}{\sin\theta} & 0 & -\frac{\cos\theta}{\sin\theta}&1&0\\ 0&0&1&0&0&1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1&0&0 & \sin\theta & \cos\theta & 0\\ 0&1&0 & -\cos\theta & \sin\theta & 0\\ 0&0&1&0&0&1 \end{array}\right]$$

The inverse of the matrix is $\begin{bmatrix} \sin\theta & \cos\theta & 0\\ -\cos\theta & \sin\theta & 0\\ 0&0&1 \end{bmatrix}$

(c)

Solution.

$$\left[\begin{array}{ccc|ccc} 0&1&0&1&0&0\\ 0&0&1&0&1&0\\ 1&0&0&0&0&1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1&0&0&0&0&1\\ 0&1&0&1&0&0\\ 0&0&1&0&1&0 \end{array}\right]$$

The inverse of the matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

(d)

Solution.

$$\left[\begin{array}{ccc|ccc} a & 0 & 0 & 1 & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & 0 & 0 & 1 \end{array}\right]$$
$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{a} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{b} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{c} \end{array}\right]$$

The inverse of the matrix is $\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

(e)

Solution.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$
$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$
$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 & 1 \end{array}\right]$$
$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & -2 & 1 \end{array}\right]$$

The inverse of the matrix is $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & -2 & 1 \end{bmatrix}$

8.

Solution. If P^{-1} equals to P^T, $PP^T = I$ will hold ($\because PP^{-1} = I$).

$$\begin{aligned}(PP^T)_{ij} &= \sum_{k=1}^{n} P_{ik}P^T_{kj} \\ &= \sum_{k=1}^{n} P_{ik}P_{jk} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}\end{aligned}$$

and the formula above is exactly same with the definition of identity matrix.

$$\therefore PP^T = I$$

$$\therefore P^{-1} = P^T$$