Linear Algebra (0031)
Problem Set 7 Solutions

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1. Consider the Matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ -2 & 0 & 4 \end{bmatrix}$

(a) Calculate the eigenvalues and eigenvectors of A.

Solution.

$$
\begin{aligned}
|A - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & 1 \\ -1 & 2-\lambda & 1 \\ -2 & 0 & 4-\lambda \end{vmatrix} \\
&= -\lambda^3 + 7\lambda^2 - 16\lambda + 12 \\
&= -(\lambda - 3)(\lambda - 2)^2 \\
&= 0
\end{aligned}
$$

$$
\therefore \lambda_1 = 3,\ \lambda_2 = 2
$$

For every λ, we find its own vectors:

1. $\lambda_1 = 3$

$$
A - \lambda_1 I = \begin{bmatrix} -2 & 0 & 1 \\ -1 & -1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}
$$

2. $\lambda_1 = 2$

$$
A - \lambda_1 I = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -2 & 0 & 2 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
$$

(b) If A has three linearly independent eigenvectors, find the diagonaliation of A

Solution.

$$P = \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$PDP^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ -2 & 0 & 4 \end{bmatrix}$$

(c) What are the eigenvalues and eigenvectors of A^{-1} (if A^{-1} exists).

Solution.

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{6} \\ \frac{1}{6} & 0 & -\frac{1}{6} \\ \frac{1}{3} & 0 & \frac{1}{6} \end{bmatrix}$$

$$|A^T - \lambda I| = -\lambda^3 + \frac{5}{6}\lambda^2 - \frac{1}{6}\lambda$$

$$\lambda_1 = 0, \quad \lambda_2 = \frac{1}{3}, \quad \lambda_3 = \frac{1}{2}$$

1. $\lambda_1 = 0$

$$A - \lambda_1 I = \begin{bmatrix} 2/3 & 0 & -1/6 \\ 1/6 & 0 & -1/6 \\ 1/3 & 0 & 1/6 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

2. $\lambda_2 = \frac{1}{3}$

$$A - \lambda_2 I = \begin{bmatrix} 1/3 & 0 & -1/6 \\ 1/6 & -1/3 & -1/6 \\ 1/3 & 0 & 1/6 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 1/2 \\ -1/4 \\ 1 \end{bmatrix}$$

3. $\lambda_3 = \frac{1}{2}$

$$A - \lambda_3 I = \begin{bmatrix} 1/6 & 0 & -1/6 \\ 1/6 & -1/2 & -1/6 \\ 1/3 & 0 & -1/3 \end{bmatrix} \Rightarrow \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(d) Let $B = A + 3I$, where I is the identity matrix. Find the eigenvalues of B.

Solution.

$$B = \begin{bmatrix} 4 & 0 & 1 \\ -1 & 5 & 1 \\ -2 & 0 & 7 \end{bmatrix}$$

$$\begin{aligned} \begin{vmatrix} 4-\lambda & 0 & 1 \\ -1 & 5-\lambda & 1 \\ -2 & 0 & 7-\lambda \end{vmatrix} &= -\lambda^3 + 16\lambda^2 - 85\lambda + 150 \\ &= -(\lambda-6)(\lambda-5)^2 = 0 \end{aligned}$$

$$\therefore \lambda_1 = 5, \quad \lambda_2 = 6$$

2. Consider a square matrix A. Suppose that A has full column rank. Can A have eigenvalue 0? Justify your answer.

Solution.
If A is a square matrix, the $r(A) + \dim(N(A)) = n$; which is the rank-nullity theorem. The nullity is the dimension of the null space of the matrix, which is all vectors $\mathbf{v}$ of the form:

$$A\mathbf{v} = 0 = 0\mathbf{v}$$

The null space of A is precisely the eigenspace corresponding to eigenvalue 0.

3. The characteristic polynomial equation (CPE) of A is written as $|A - \lambda I| = 0$. Likewise, the characteristic polynomial equation of AT is $|AT - \lambda I| = 0$. Solving CPE gives eigenvalues.

(a) Show that A and A^T have the same eigenvalues (Hint. Take a look at the CPEs of A and A^T, and use the fact that transposing a matrix does not change the determinant)

Solution.
The matrix $(A - \lambda I)^T$ is equals to the matrix $(A^T - \lambda I)$, since the matrix I is symmetric. Thus,

$$|A^T - \lambda I| = |(A - \lambda I)^T| = |A - \lambda I|$$

From above, it is obvious that the eigenvalues are the same for both A and A^T.

4. Consider a matrix $A = \begin{bmatrix} 2 & a \\ 1 & 0 \end{bmatrix}$. Find a condition for A to be diagonalisable. (Hint. A needs to have two linearly independent eigenvectors. Eigenvectors associated to distinct eigenvalues are linearly independent)

Solution.
Matrix A is diagonalisable iff the sum of the dimension of eigenspaces is equal to n.

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 2-\lambda & a \\ 1 & -\lambda \end{vmatrix} \\ &= \lambda^2 - 2\lambda + a \end{aligned}$$

$$\therefore a \neq 1$$

5. Check if the following matrices are positive definite:

(a) $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}$

Solution.

$$|2| = 2$$

$$\begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} = 6$$

$$\begin{vmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{vmatrix} = 30$$

Since all determinants are positive, the matrix (a) is a positive definite.

(b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution.

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} \\ &= \lambda^2 - 5\lambda - 2 \\ &= (\lambda + \frac{\sqrt{33}-5}{2})(\lambda - \frac{\sqrt{33}+5}{2}) \end{aligned}$$

$$\therefore \lambda_1 = \frac{-\sqrt{33}+5}{2}, \quad \lambda_2 = \frac{\sqrt{33}+5}{2}$$

Since all eigenvalues are not positive, the matrix (b) is not a positive definite.

6. Show that $R^T R$ is positive semidefinite for any matrix R.

Solution.
Let $R = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$R^T R = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$$

$$|R^T R - \lambda I| = \begin{vmatrix} 5-\lambda & 11 \\ 11 & 25-\lambda \end{vmatrix}$$

$$\therefore \lambda_1 = 15 + \sqrt{221},\ \lambda_2 = 15 - \sqrt{221}$$

Since all eigenvalues are positive, the matrix $R^T R$ is positive semidefinite.

7. Show that $R^T R$ is positive definite if and only if R has full column rank.

Solution.
Let $R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

$$R^T R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$$

$$|R^T R - \lambda I| = \begin{vmatrix} 2-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix}$$

$$\therefore \lambda_1 = \frac{7+3\sqrt{5}}{2},\ \lambda_2 = \frac{7-3\sqrt{5}}{2}$$

Since all eigenvalues are positive, the matrix $R^T R$ is positive definite.

8. Prove that if $B = M^{-1}AM$, then A and B have the same eigenvalues. (Hint: multiply an eigenvector of B on the right, and then multiply M on the left)

Solution.
Let X_i be an eigenvector of A corresponding to λ_i, and let the eigenvectors be independent of each other.

$$AX_i = \lambda_i X_i$$

Let $M = [X_1\ X_2\ \cdots\ X_n]$ which contains the columns with the eigenvectors of A.
So,

$$\begin{aligned} AM &= [AX_1\ AX_2\ \cdots\ AX_n] \\ &= [\lambda_1 X_1\ \lambda_2 X2\ \cdots\ \lambda_n X_n] \\ &= [X_1\ X_2\ \cdots\ X_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \end{aligned}$$

$$AM = MB$$

B is the diagonal matrix. M is invertible as the eigenvectors are independent and $|M| \neq 0$.
Therefore,

$$M^{-1}AM = B$$