Linear Algebra (0031)
Problem Set 4 Solutions

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1. (a) Compute the ranks of $A,\ B$.

Solution.

$$A = \begin{bmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & -1 & 6 \\ 1 & 2 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore r(A) = 2$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore r(B) = 1$$

(b) Find the bases of $C(A),\ N(A^T),\ C(A^T),\ N(A)$.

Solution.

$$A = \begin{bmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & -1 & 6 \\ 1 & 2 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore C(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

We can get the basis by just using the pivot columns from the original matrix.

$$A^T = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & -1 & 2 \\ 6 & 2 & 6 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let $x_1 = -\frac{3}{2}x_4 - \frac{1}{2}x_3, x_2 = \frac{1}{2}x_4 - \frac{3}{2}x_3$.

$$\mathbf{x} = \begin{bmatrix} -\frac{3}{2}x_4 - \frac{1}{2}x_3 \\ \frac{1}{2}x_4 - \frac{3}{2}x_3 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} x_4$$

$$\therefore N(A^T) = \left\{ \begin{bmatrix} -\frac{1}{2} \\ -\frac{3}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$A^T = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & -1 & 2 \\ 6 & 2 & 6 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore C(A^T) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & -1 & 6 \\ 1 & 2 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} -4x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix} x_3$$

$$\therefore N(A) = \left\{ \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix} \right\}$$

(c) Find the dimensions of $C(A),\ N(A^T),\ C(A^T),\ N(A)$.

$$\begin{aligned} dim(C(A)) &= 2 \\ dim(N(A^T)) &= 2 \\ dim(C(A^T)) &= 2 \\ dim(N(A)) &= 1 \end{aligned}$$

(d) What is the rank of $\begin{bmatrix} A \\ A \end{bmatrix}$? Compare with the rank of A. Prove or disprove if your result is true in general.

$$\begin{bmatrix} A \\ A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & -1 & 6 \\ 1 & 2 & 8 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & -1 & 6 \\ 1 & 2 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore r\left(\begin{bmatrix} A \\ A \end{bmatrix}\right) = 2 = r(A)$$

Since value of the rank is equals to the number of pivots, It will be true for any matrix A.

(e) What is the rank of $\begin{bmatrix} B \\ A^T B \end{bmatrix}$? Compare with the rank of B. Prove or disprove if your result is true in general.

$$A^T B = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & -1 & 2 \\ 6 & 2 & 6 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 15 \\ 3 & 6 & 9 \\ 22 & 44 & 66 \end{bmatrix}$$

$$\begin{bmatrix} B \\ A^T B \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 5 & 10 & 15 \\ 3 & 6 & 9 \\ 22 & 44 & 66 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore r\left(\begin{bmatrix} B \\ A^T B \end{bmatrix}\right) = 1 = r(B)$$

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(f) What is the rank of $\begin{bmatrix} A & B \end{bmatrix}$?

$$\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 6 & 1 & 2 & 3 \\ 1 & -1 & 2 & 1 & 2 & 3 \\ 2 & -1 & 6 & 1 & 2 & 3 \\ 1 & 2 & 8 & 1 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$r\left(\begin{bmatrix} A & B \end{bmatrix}\right) = 3$$

(g) Let $C = \begin{bmatrix} 1 & 5 & 3 \\ 1 & 1 & 3 \\ 1 & 4 & 3 \\ 1 & 7 & 3 \end{bmatrix}$. What is the rank of $\begin{bmatrix} A & C \end{bmatrix}$? Compare the results in parts (f) and (g), and explain why.

$$\begin{bmatrix} A & C \end{bmatrix} = \begin{bmatrix} 1 & 1 & 6 & 1 & 5 & 3 \\ 1 & -1 & 2 & 1 & 1 & 3 \\ 2 & -1 & 6 & 1 & 4 & 3 \\ 1 & 2 & 8 & 1 & 7 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 4 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore r\left(\begin{bmatrix} A & C \end{bmatrix}\right) = 3$$

(h) Compute the rank of $A+B$. Is it greater than $r(A)+r(B)$?

$$A+B=\begin{bmatrix}2&3&9\\2&1&5\\3&1&9\\2&4&11\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

$$\therefore r(A+B)=3$$

$$\therefore r(A+B)=r(A)+r(B)$$

(i) Compute the rank of $A+C$. Is it greater than $r(A)+r(C)$?

$$A+C=\begin{bmatrix}2&6&9\\2&0&5\\3&3&9\\2&9&11\end{bmatrix}\Rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

$$\therefore r(A+C)=3$$

$$\therefore r(A+C)<r(A)+r(C)(\because r(C)=2)$$

2. (a) Find the bases of S_1 and S_2.

$$S_1=\left\{\begin{bmatrix}a\\2\\1\\1\end{bmatrix},\begin{bmatrix}2\\b\\2\\3\end{bmatrix}\right\}$$

$$S_2=\left\{\begin{bmatrix}1\\1\\c\\1\end{bmatrix},\begin{bmatrix}1\\-1\\1\\d\end{bmatrix}\right\}$$

(b) Determine the values of a,b,c,d so that S_1 and S_2 are orthogonal complement of each other.

$$\begin{bmatrix}a&2&1&1\\2&b&2&3\end{bmatrix}\begin{bmatrix}1&1\\1&-1\\c&1\\1&d\end{bmatrix}=\begin{bmatrix}a+c+3&a+d-1\\b+2c+5&-b+3d+4\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\therefore a=\frac{6}{5},\quad b=\frac{17}{5},\quad c=-\frac{21}{5},\quad d=-\frac{1}{5}$$