Linear Algebra (0031)
Problem Set 0 Solutions

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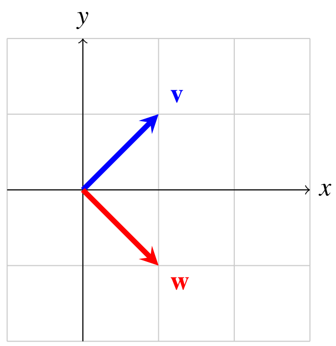
1. (a) Visualise the two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$

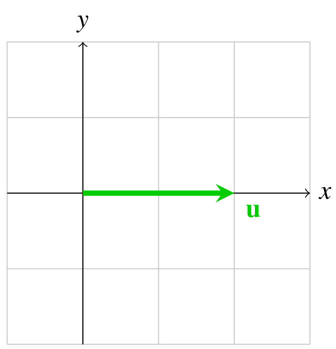
Solution.



(b) Calculate $\mathbf{u} = \mathbf{v} + \mathbf{w}$, and draw $\mathbf{u}$ in $\mathbb{R}^2$

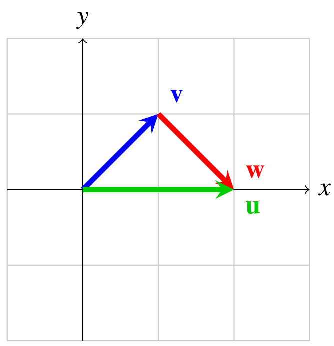
Solution.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$



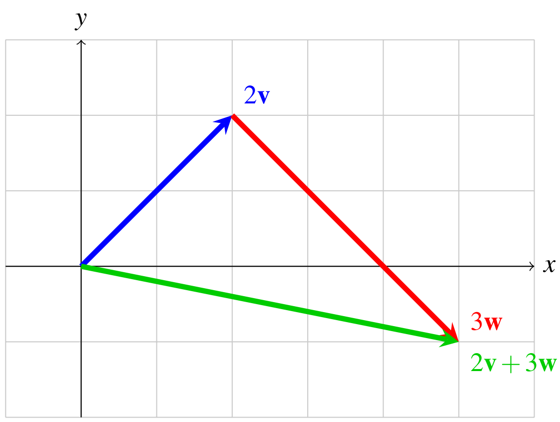
(c) Visualise the operation $\mathbf{v}+\mathbf{w}$.

Solution.



(d) Visualise the two vectors $2\mathbf{v}$ and $3\mathbf{w}$, and also the addition $2\mathbf{v}+3\mathbf{w}$.

Solution.



(e) Consider $\mathbf{u} = c\mathbf{v}+d\mathbf{w}$. Find an example of (c,d) pair such that $u_1 > 0, u_2 > 0$.

Solution.

$$\begin{aligned}\mathbf{u} &= c\begin{bmatrix}1\\1\end{bmatrix} + d\begin{bmatrix}1\\-1\end{bmatrix}\\ &= \begin{bmatrix}c+d\\c-d\end{bmatrix}\end{aligned}$$

example of cases such that $c+d>0$, $c-d>0$

$$\therefore (c,d) = (1,0),(2,1),(2,0),(3,2),(3,1),(3,0)\cdots$$

(f) Repeat part (e) for each of the following cases

Solution.

- $u_1 > 0,\ u_2 < 0$

$$c+d>0,\ c-d<0$$
$$\therefore (c,d)=(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\cdots$$

- $u_1 < 0,\ u_2 > 0$

$$c+d<0,\ c-d>0$$
$$\therefore (c,d)=(-1,-2),(-1,-3),(-1,-4),(-2,-3),(-2,-4),(-2,-5)\cdots$$

- $u_1 < 0,\ u_2 < 0$

$$c+d<0,\ c-d<0$$
$$\therefore (c,d)=(-1,0),(-2,-1),(-2,0),(-3,-2),(-3,-1),(-3,0)\cdots$$

(g) Do you think you can get any point in $\mathbb{R}^2$ with the linear combination $c\mathbf{v}+d\mathbf{w}$? Justify your answer.

Solution.

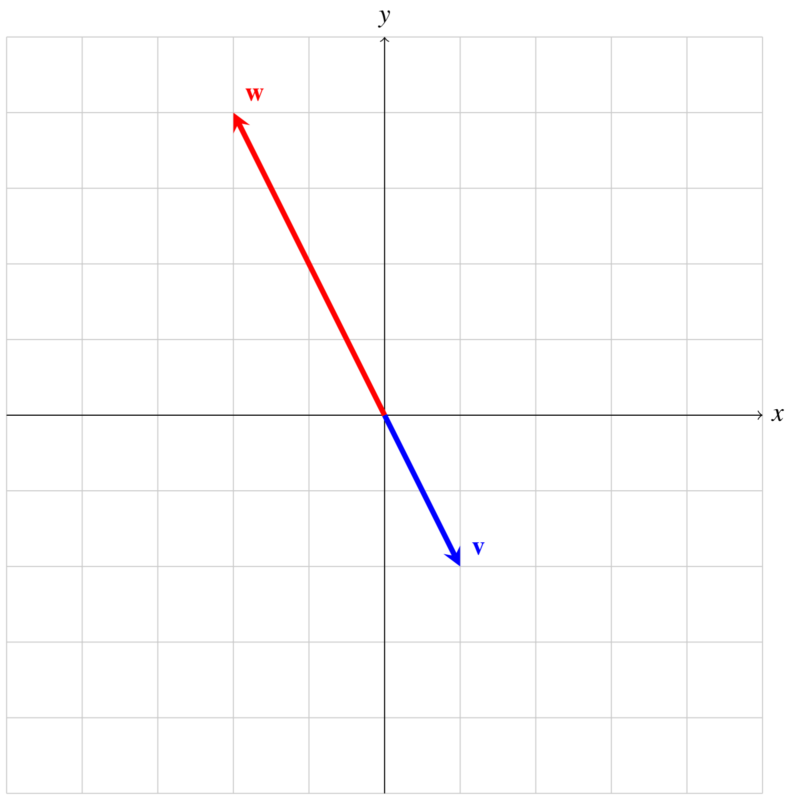
$$\alpha=\frac{1}{2},\ \beta=\frac{1}{2} \quad\longrightarrow\quad \alpha\mathbf{v}+\beta\mathbf{w}=\begin{bmatrix}1\\0\end{bmatrix}$$
$$\gamma=\frac{1}{2},\ \delta=-\frac{1}{2} \quad\longrightarrow\quad \gamma\mathbf{v}+\delta\mathbf{w}=\begin{bmatrix}0\\1\end{bmatrix}$$

Since $c\begin{bmatrix}1\\0\end{bmatrix}+d\begin{bmatrix}0\\1\end{bmatrix}$ can represent any point in $\mathbb{R}^2$, $c\mathbf{v}+d\mathbf{w}$ can also represent any point in $\mathbb{R}^2$. □

(h) Answer the same question in part (g) with new vectors $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

Solution. There's no real root for $c\mathbf{v} + d\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

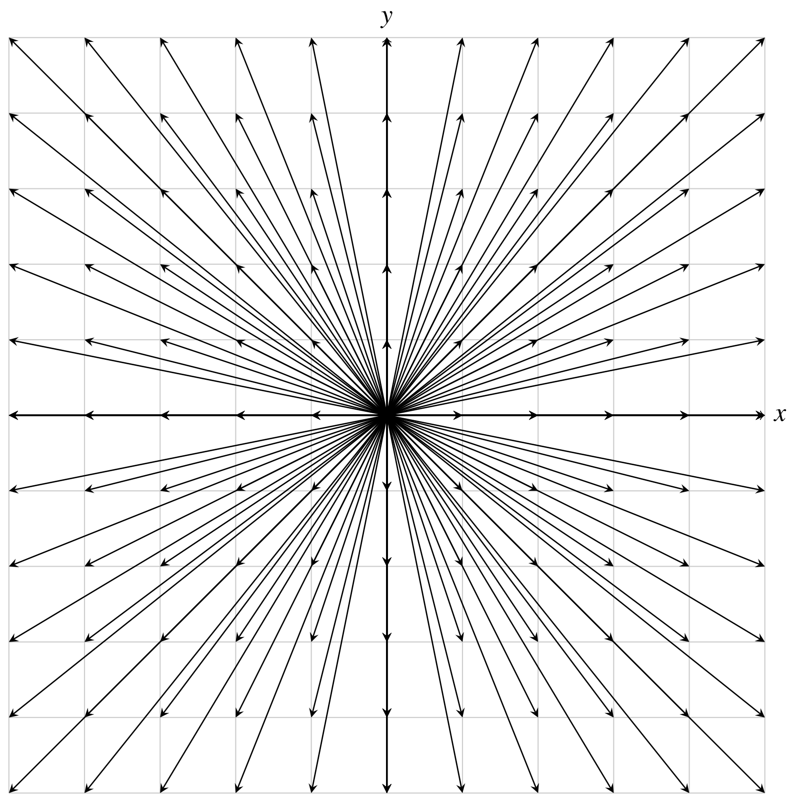
Also,



As you can see, the linear combination of $\mathbf{v}$ and $\mathbf{w}$ can represent only points on the $y = -2x$. So, $c\mathbf{v} + d\mathbf{w}$ cannot represent any point in $\mathbb{R}^2$. □

2. (a) Visualise $c\begin{bmatrix}1\\0\end{bmatrix}+d\begin{bmatrix}0\\1\end{bmatrix}$ for all scalars c,d.

Solution.



Drawn all $c\begin{bmatrix}1\\0\end{bmatrix}+d\begin{bmatrix}0\\1\end{bmatrix}$ $(c,d\in\mathbb{Z},-5\le c,d\le 5)$.

$$\therefore c\begin{bmatrix}1\\0\end{bmatrix}+d\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}c\\d\end{bmatrix}\in\mathbb{R}^2 \quad (\because c,d\in\mathbb{R})$$

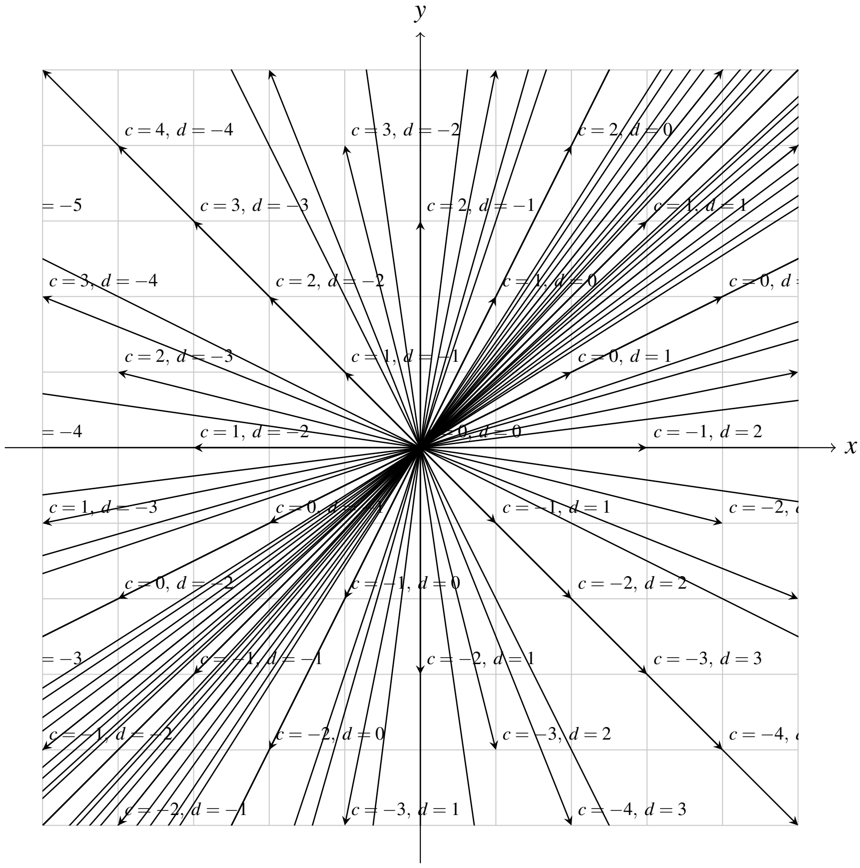
(b) Consider $\mathbf{v}=\begin{bmatrix}1\\2\end{bmatrix}$ and $\mathbf{w}=\begin{bmatrix}2\\1\end{bmatrix}$. Express $\mathbf{v}$ and $\mathbf{w}$ as a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

Solution.

$$\mathbf{v}=\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix}$$
$$\mathbf{w}=2\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}$$

(c) Visualise $c\mathbf{v} + d\mathbf{w}$ for all c, d.

Solution.



Drawn all $c\mathbf{v} + d\mathbf{w} (c, d \in \mathbb{Z}, -5 \le c, d \le 5)$.

3. (a) Calculate $A + B$ for $A = \begin{bmatrix} 0 & -3 & 1 \\ 5 & 7 & -4 \\ 3 & -1 & -3 \\ 7 & -1 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 4 & -3 \\ 2 & 1 & 8 \\ -3 & 7 & 10 \\ 2 & -4 & -2 \end{bmatrix}$.

Solution.

$$A + B = \begin{bmatrix} 0 & -3 & 1 \\ 5 & 7 & -4 \\ 3 & -1 & -3 \\ 7 & -1 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 4 & -3 \\ 2 & 1 & 8 \\ -3 & 7 & 10 \\ 2 & -4 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -2 \\ 7 & 8 & 4 \\ 0 & 6 & 7 \\ 9 & -5 & -7 \end{bmatrix}$$

(b) Let $A = \begin{bmatrix} a & -5 \\ -3 & c \\ e & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & b \\ d & -7 \\ 6 & f \end{bmatrix}$. Find the values of a, b, c, d, e, f so that

$$3A + 2B = \begin{bmatrix} 5 & 10 \\ 15 & 1 \\ 11 & 30 \end{bmatrix}$$

Solution.

$$\begin{aligned} 3A + 2B &= \begin{bmatrix} 3a & -15 \\ -9 & 3c \\ 3e & 12 \end{bmatrix} + \begin{bmatrix} 10 & 2b \\ 2d & -14 \\ 12 & 2f \end{bmatrix} \\ &= \begin{bmatrix} 3a+10 & 2b-15 \\ 2d-9 & 3c-14 \\ 3e+12 & 2f+12 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 10 \\ 15 & 1 \\ 11 & 30 \end{bmatrix} \end{aligned}$$

from above,

$$\begin{cases} 3a + 10 = 5 \\ 2b - 15 = 10 \\ 3c - 14 = 1 \\ 2d - 9 = 15 \\ 3e + 12 = 11 \\ 2f + 12 = 30 \end{cases}$$

$$\therefore a = -\frac{5}{3}, \quad b = \frac{25}{2}, \quad c = 5, \quad d = 12, \quad e = -\frac{1}{3}, \quad f = 9$$

(c) Find the two matrices A and B that satisfy $A+B=\begin{bmatrix}3 & 7\\ 1 & -6\end{bmatrix}$ and $A-B=\begin{bmatrix}9 & -4\\ -5 & -9\end{bmatrix}$.

Solution.

$$(A+B)+(A-B)=2A=\begin{bmatrix}12 & 3\\ -4 & -15\end{bmatrix}$$
$$\therefore A=\begin{bmatrix}6 & \frac{3}{2}\\ -2 & -\frac{15}{2}\end{bmatrix}$$

$$(A+B)-(A-B)=2B=\begin{bmatrix}-6 & 11\\ 6 & 3\end{bmatrix}$$
$$\therefore B=\begin{bmatrix}-3 & \frac{11}{2}\\ 3 & -\frac{3}{2}\end{bmatrix}$$

4. (a) Calculate the inner product $\mathbf{a}\cdot\mathbf{b}$.

Solution.

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b}&=\begin{bmatrix}2\\1\\2\\1\end{bmatrix}\cdot\begin{bmatrix}2\\1\\2\\6\end{bmatrix}\\&=2\times2+1\times1+2\times2+1\times6\\&=4+1+4+6\\&=15\end{aligned}$$

(b) Calculate $||\mathbf{a}||$ and $||\mathbf{b}||$.

Solution.

$$\begin{aligned}||\mathbf{a}||&=\sqrt{2^2+1^2+2^2+1^2}\\&=\sqrt{4+1+4+1}\\&=\sqrt{10}\\||\mathbf{b}||&=\sqrt{2^2+1^2+2^2+6^2}\\&=\sqrt{4+1+4+36}\\&=\sqrt{45}\\&=3\sqrt{5}\end{aligned}$$

(c) Calculate the angle (ranging from 0 to π) between $\mathbf{a}$ and $\mathbf{b}$.

Solution.

$$15 = 15\sqrt{2}\cos\theta \quad (\because \mathbf{a}\cdot\mathbf{b} = ||\mathbf{a}||\cdot||\mathbf{b}||\cos\theta)$$
$$\cos\theta = \frac{1}{\sqrt{2}}$$
$$\therefore \theta = \frac{\pi}{4} \quad (\because 0 \le \theta \le \pi)$$

5. (a) Expand $||\mathbf{v}+\mathbf{w}||^2$.

Solution.

$$||\mathbf{v}+\mathbf{w}||^2 = ||\mathbf{v}||^2 + 2\mathbf{v}\cdot\mathbf{w} + ||\mathbf{w}||^2$$

(b) Expand $(||\mathbf{v}||+||\mathbf{w}||)^2$.

Solution.

$$(||\mathbf{v}||+||\mathbf{w}||)^2 = ||\mathbf{v}||^2 + 2||\mathbf{v}||\cdot||\mathbf{w}|| + ||\mathbf{w}||^2$$

(c) Use the results in parts (a) and (b) to show the triangle inequality.

Solution. By the Cauchy-Schwarz Inequality,

$$\mathbf{v}\cdot\mathbf{w} \le ||\mathbf{v}||\cdot||\mathbf{w}||$$

So,

$$||\mathbf{v}+\mathbf{w}||^2 = ||\mathbf{v}||^2 + 2\mathbf{v}\cdot\mathbf{w} + ||\mathbf{w}||^2 \le ||\mathbf{v}||^2 + 2||\mathbf{v}||\cdot||\mathbf{w}|| + ||\mathbf{w}||^2 = (||\mathbf{v}||+||\mathbf{w}||)^2$$

i.e.,

$$||\mathbf{v}+\mathbf{w}||^2 \le (||\mathbf{v}||+||\mathbf{w}||)^2 \implies ||\mathbf{v}+\mathbf{w}|| \le ||\mathbf{v}||+||\mathbf{w}|| \qquad \square$$

6. (a) Prove that $-1 \le \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} \le 1$.

Solution 1. By the Cauchy-Schwarz Inequality,

$$|\mathbf{v}\cdot\mathbf{w}| \le ||\mathbf{v}||\cdot||\mathbf{w}||$$
$$\frac{|\mathbf{v}\cdot\mathbf{w}|}{||\mathbf{v}||\cdot||\mathbf{w}||} \le 1$$
$$\therefore -1 \le \frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} \le 1 \qquad \square$$

Solution 2.

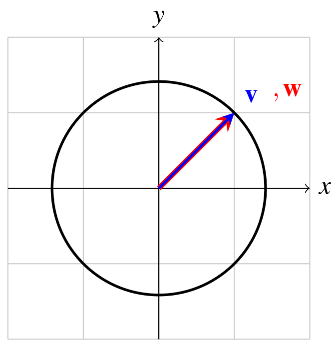
$$\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} = \frac{||\mathbf{v}||\cdot||\mathbf{w}||\cos\theta}{||\mathbf{v}||\cdot||\mathbf{w}||}$$
$$= \cos\theta$$
$$-1 \leq \cos\theta \leq 1$$ □

(b), (c) Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with $a^2 + b^2 = 2$. Find the vector $\mathbf{v}$ for each of the following cases, and visualise the vectors $\mathbf{w}$, and $\mathbf{v}$.

Solution.

i. $\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} = 1$

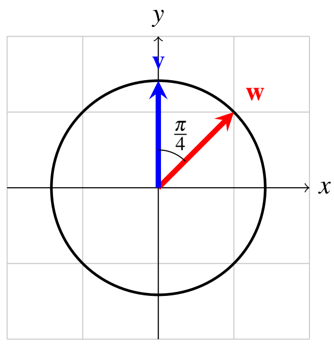
$$\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} = \cos\theta = 1$$
$$\therefore \theta = 0$$



$$\therefore \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

ii. $\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} = \frac{1}{\sqrt{2}}$

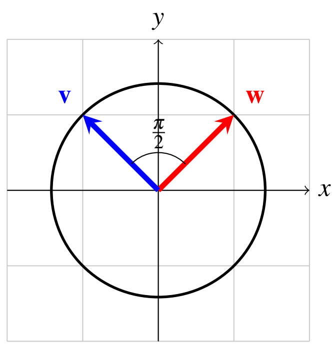
$$\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} = \cos\theta = \frac{1}{\sqrt{2}}$$
$$\therefore \theta = \frac{\pi}{4}$$



$$\therefore \mathbf{v} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$

iii. $\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} = 0$

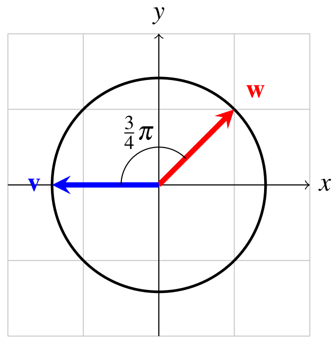
$$\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} = \cos\theta = 0$$
$$\therefore \theta = \frac{\pi}{2}$$



$$\therefore \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

iv. $\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} = -\frac{1}{\sqrt{2}}$

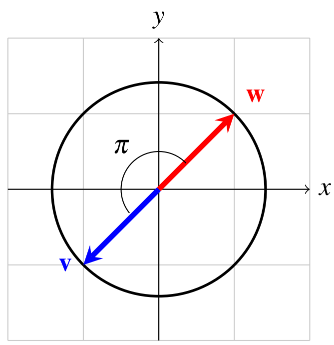
$$\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} = \cos\theta = -\frac{1}{\sqrt{2}}$$
$$\therefore \theta = \frac{3}{4}\pi$$



$$\therefore \mathbf{v} = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}$$

v. $\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} = -1$

$$\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||} = \cos\theta = -1$$
$$\therefore \theta = \pi$$



$$\therefore \mathbf{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

(d) Use the results in parts (b) and (c) to argue that the value $\frac{\mathbf{v}\cdot\mathbf{w}}{||\mathbf{v}||\cdot||\mathbf{w}||}$ is an appropriate metric for representing "similarity" of two vectors in some sense.

Solution. By the $\cos\theta$ value of the angle between two vectors can represent the similarity of two vectors. e.g. *cosine similarity* of two identical vectors is equal to 1. And *cosine similarity* between $\mathbf{v}$ and $-\mathbf{v}$(the angle is totally different) is equal to -1. As the angle of the two vectors increases from 0 to π, the *cosine similarity* value goes to -1 from 1 CONTINUOUSLY as shown in (c). Therefore, I think it can be useful when representing the difference between two data.