Linear Algebra (0031)
Lecture Notes

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1 Week 1

2 Week 2

2.1 Multiplication with Vectors

General rule for $m \times n$ matrix A and n-dimensional column vector $\mathbf{x}$,

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \sum_{j=1}^{n} a_{2j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}x_j \end{bmatrix}$$

Note: A can be multiplied on the right by vector $\mathbf{x}$ only when the number of columns in A is equal to the dimension of $\mathbf{x}$.

2.2 Multiplication as Linear Combination

Multiplication can be expressed as linear combination. For $m \times n$ matrix A and n-dimensional column vector $\mathbf{x}$,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Note: This is very important view of multiplication whice will be useful throughout this course.

2.3 Matrix Multiplication

Generalisation for $m \times n$ matrix A and $n \times p$ matrix B

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=1}^{n} a_{1k}b_{k1} & \sum_{k=1}^{n} a_{1k}b_{k2} & \cdots & \sum_{k=1}^{n} a_{1k}b_{kp} \\ \sum_{k=1}^{n} a_{2k}b_{k1} & \sum_{k=1}^{n} a_{2k}b_{k2} & \cdots & \sum_{k=1}^{n} a_{2k}b_{kp} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{mk}b_{k1} & \sum_{k=1}^{n} a_{mk}b_{k2} & \cdots & \sum_{k=1}^{n} a_{mk}b_{kp} \end{bmatrix} \in \mathbb{R}^{m \times p}$$

Alternative representaion:

$$AB = \begin{bmatrix} \underline{\mathbf{a}}_1 \\ \underline{\mathbf{a}}_2 \\ \vdots \\ \underline{\mathbf{a}}_m \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{a}}_1\mathbf{b}_1 & \underline{\mathbf{a}}_1\mathbf{b}_2 & \cdots & \underline{\mathbf{a}}_1\mathbf{b}_p \\ \underline{\mathbf{a}}_1\mathbf{b}_2 & \underline{\mathbf{a}}_2\mathbf{b}_2 & \cdots & \underline{\mathbf{a}}_2\mathbf{b}_p \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\mathbf{a}}_m\mathbf{b}_1 & \underline{\mathbf{a}}_m\mathbf{b}_2 & \cdots & \underline{\mathbf{a}}_m\mathbf{b}_p \end{bmatrix}$$

$$= A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$$

- $(m \times n)\,(n \times p) = (m \times p)$
- $AA = A^2, AAA = A^3, \cdots$ (Only if A is square matrix)

Note: Consider $m \times n$ matrix A and $l \times p$ matrix B. A can be multiplied on the right by B only if $n = l$.

Properties of Addition and Multiplication Commutativity, Associativity, Distributivity.

Addition	Multiplication
$A+B=B+A$	$AB \neq BA$
$(A+B)+C=A+(B+C)$	$(AB)C=A(BC)$
$c(A+B)=cA+cB$	$C(A+B)=CA+CB$
	$(A+B)C=AC+BC$

2.4 Transposes

The transpose of A, denoted by A^T, is a matrix such that $\left[A^T\right]_{ij} = A_{ji}$.

Properties

- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Inner product of vectors $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} \in \mathbb{R}$$

Outer product of vectors $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{w} \in \mathbb{R}^n$

$$\mathbf{v} \otimes \mathbf{w} = \mathbf{v}\mathbf{w}^T \in \mathbb{R}^{m \times n}$$

$$\left[\mathbf{v}\mathbf{w}^T\right]_{ij} = v_i w_j$$

generalised to Kronecker product.

2.5 Some Special Matrices

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