CSC 311: Introduction to Machine Learning

CSC311 Summer 2025

University of Toronto

Bernoulli Distribution

- Bernoulli distribution: $\mathbf{X}$ is a random variable with two outcomes. We say that $\mathbf{X}$ follows $Ber(\mu)$ if:

$$P(\mathbf{X} = x) = \mu^x (1 - \mu)^{1-x}, x \in \{0, 1\}$$

- Example: A coin follows Bernoulli distribution.

Mixture of Bernoulli

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- Assume a datapoint x is generated as follows:
  - Choose a cluster z from $\{1, ..., K\}$ such that $p(z = k) = \pi_k$
  - Given z, sample x from $Ber(\mu_k)$.
- We say x follows mixtures of Bernoulli distributions. It pmf can be expressed as:

$$P(x) = \sum_{i=1}^{k} \pi_i \mu_i^x (1 - \mu_i)^{1-x}$$

Maximum Likelihood

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- Training objective: Maximize log likelihood

$$\max_{\pi_k,\mu_k} \sum_{n=1}^{N} \log \sum_{i=1}^{K} \pi_k \mu_k^{x_n} (1-\mu_k)^{1-x_n}$$

- Log inside sum: EM algorithm

E step - compute the posterior

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$$P(z_n = k|x_n) = \frac{P(z_n = k, x_n)}{P(x_n)}$$
$$= \frac{\pi_k \mu_k^{x_n}(1-\mu_k)^{1-x_n}}{\sum_{i=1}^{k} \pi_i \mu_i^{x_n}(1-\mu_i)^{1-x_n}}$$

- z_{nk} can be interpreted as how much we think a cluster k is responsible for generating a datapoint x_n.

M step - optimize the joint log likelihood

- The joint likelihood can be expressed as:

$$\log p(\mathbf{X}, \mathbf{Z}; \mu, \pi)$$
$$= \sum_{n=1}^{N} \sum_{k=1}^{K} z_{nk} (\log \pi_k + x_n \log \mu_k + (1 - x_n) \log(1 - \mu_k))$$

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- Assume the responsibility is known, setting the derivative respect to μ_k to zero, we get:

$$\mu_k = \frac{\sum_{n=1}^{N} z_{nk} x_n}{\sum_{n=1}^{N} z_{nk}}$$

- Interpretation: The mean of component k is equal to the weighted mean of the data, with weighted coefficients proportional to the responsibility.

True or False

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True or False

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True. See slide 46

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False

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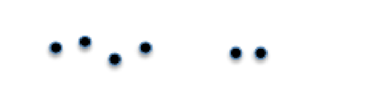
Consider the set of training data below, and two clustering algorithms: K-Means, and a Gaussian Mixture Model (GMM) trained using EM. These two clustering algorithms will produce the same cluster centers (means) for this data set.



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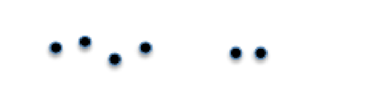
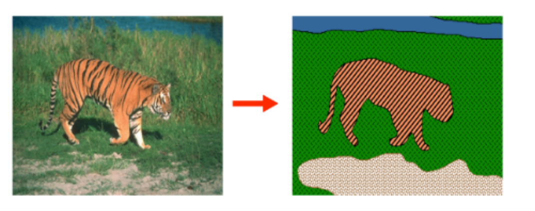


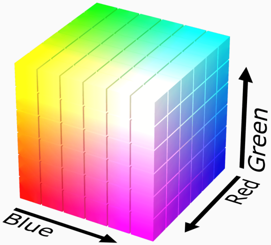
False. In k-means, the means of the clusters are determined by an average of the points assigned to that cluster, but in GMM the means of each cluster are (differently) weighted averages of all points.

Application: EM for image segmentation



Partition an image into regions each of which has a reasonably homogenous visual appearance

RGB image



- Each pixel in an RGB image is a point in 3-dimensional space comprising the intensities of the red, blue and green channels.
- We can think of image segmentation tasks as clustering problems on pixels.
- We can apply EM and k-means for image segmentation.
- More on the colab notebook!