MATH 222 Assignment 4

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1. Since f is onto from a set to the same set, each onto function can be considered a permutation of A.
   Let c_i be the condition that $f(i) = i$ for $i = 2, 4, 6, 8$
   For each satisfied condition, there will be one less element that can be permuted.
   Consider the number of elements that satisfy the conditions i, j, k
   $N = 9!$
   $N(c_i) = 8!$
   $N(c_i c_j) = 8!$
   $N(c_i c_j c_k) = 8!$
   $N(c_2 c_4 c_6 c_8) = 8!$

   So the number of onto functions $\overline{N}$ is:
   $\overline{N} = 9! - \binom{4}{1} \cdot 8! + \binom{4}{2} \cdot 7! - \binom{4}{3} \cdot 6! + \binom{4}{4} \cdot 5!$

2. Consider n distinguishable elements and $n-2$ indistinguishable boxes. Count the number of ways to divide the elements into the boxes such that no box is empty for $n \geq 4$.
   Clearly there are $S(n, n-2)$ such ways.
   There are 2 possible cases for any such distribution as follows:

   (a) There are 3 elements that share a single box. There are $\binom{n}{3}$ ways to pick the 3 elements and since all other indistinguishable boxes must have only 1 element, there are $\binom{n}{3}$ ways total to arrange the elements in this case.

   (b) There are 2 boxes which each contain 2 elements. There are $\binom{n}{4}$ ways to choose the elements and $\frac{\binom{4}{2}}{2} = 3$ ways to distribute the chosen elements into a box for a total of $3\binom{n}{4}$ ways to arrange the elements in this case.

   This makes a total number of ways to distribute the elements of $\binom{n}{3} + 3\binom{n}{4}$
   $\therefore S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$ for $n \geq 4$.

3. We can use a generating function to get our result.
   Consider the generating function in which the sum of coefficients of x^i

with $i \leq 35$ is our result:

$$\begin{aligned} g(x) &= (x^0 + x^1 + ... + x^9)^6 \\ &= (\frac{1 - x^{9+1}}{(1-x)})^6 \\ &= \frac{(1-x^{10})^6}{(1-x)^6} \end{aligned}$$

If we multiply $g(x)$ by $\frac{1}{1-x} = 1 + x + x^2 + ...$, the coefficient of x^{35} is our result since the new $g(x)$ has coefficients which are cumulative of the old $g(x)$ coefficients. So:

$$\begin{aligned} g(x) &= \frac{(1-x^{10})^6}{(1-x)^6} \cdot \frac{1}{1-x} \\ &= \frac{(1-x^{10})^6}{(1-x)^7} \\ &= \sum_{i=0}^{6} \binom{6}{i} (-1)^i (x^{10i}) \cdot \sum_{i=0}^{\infty} \binom{i+7-1}{7-1} x^i \\ &= \sum_{i=0}^{6} \binom{6}{i} (-1)^i (x^{10i}) \cdot \sum_{i=0}^{\infty} \binom{i+6}{6} x^i \end{aligned}$$

Since the only coefficients that matter for us are the ones for x^{35}, we can remove all other elements in the previous product to get:

$$\begin{aligned} &\binom{6}{0} x^0 \cdot \binom{35+6}{6} x^{35} - \binom{6}{1} x^{10} \binom{25+6}{6} x^{25} \\ &+ \binom{6}{2} x^{20} \binom{15+6}{6} x^{15} - \binom{6}{3} x^{30} \binom{5+6}{6} x^5 \end{aligned}$$

Therefore the coefficient of x^{35}, the number is of integers between 0 and 999999 with digits that sum to no more than 35, is $4496388 - 4416686 + 814960 - 9240 = 883422$

4. Consider a hexagonal shaped room with the 6 walls labeled 0,1,2,3,4,5 in order around the room.
   Let c_i be the condition that the walls i and $i+1 \mod 5$ are the same color for $0 \leq i \leq 5$.
   When no conditions are satisfied, there are 6 contiguous colors representing the room, each which can be one of 10 colors. For each

satisfied condition, the number of contiguous colors representing the room decreases by one because if the condition is adjacent to another condition, only one additional wall will be restricted in color (to be the color of the adjacent group of walls), and if the condition is not adjacent to any other condition it restricts 2 walls, but creates 1 new group of colors for $2 - 1 = 1$ less contiguous colors.

Consider the number of elements that satisfy the conditions i, j, k, q, r

$N = 10^6$

$N(c_i) = 10^5$

$N(c_i c_j) = 10^4$

$N(c_i c_j c_k) = 10^3$

$N(c_i c_j c_k c_q) = 10^2$

$N(c_i c_j c_k c_q c_r) = 10$

$N(c_0 c_1 c_2 c_3 c_4 c_5) = 10$

$\overline{N} = 10^6 - \binom{6}{1} \cdot 10^5 + \binom{6}{2} \cdot 10^4 - \binom{6}{3} \cdot 10^3 + \binom{6}{4} \cdot 10^2 - \binom{6}{5} \cdot 10 + \binom{6}{6} \cdot 10$

$\overline{N} = 528450$

So the number of ways to paint the 6 walls is 528450

5. a For R to be an equivalence relation, it must be reflexive, symmetric, and transitive.

   Reflexive:

   Since for all $x \in S, x^2 > 0$, xRx so R is reflexive.

   Symmetric:

   Since multiplication is commutative, $xy = yx$ so $xRy \implies yRx$ $\forall x, y \in S$ so R is symmetric.

   Transitive:

   For $x, y, z \in S$, if $xy > 0$ then both x and y must have the same sign. Similarly, if $yz > 0$ then both y and z must have the same sign.

   Since x, y, z all have the same sign, $yz > 0$ and hence if xRy and yRz, xRz so R is transitive.

   Since R is reflexive, symmetric, and transitive, R is an equivalence relation. A partition of R would be the following equivalence classes:

   $[1] = \{x \in S | x > 0\}$

   $[-1] = \{x \in S | x \leq 0\}$

   b A reason why R_2 is not an equivalence relation because it is not reflexive. For example $(-1, -1) \notin R$ since $-1^2 = 1 \nless 0$

6. a For $R \cap S$ to be a partial order on A it must be reflexive, anti-symmetric, and transitive.

Reflexive:

Since R and S are reflexive, $\forall x \in A, (x,x) \in R$ and $(x,x) \in S$ so by the definition of set intersection, $(x,x) \in R \cap S$ so $R \cap S$ is reflexive.

Anti-Symmetric:

For some $x, y \in A$, if $(x,y) \in R \cap S$ and $(y,x) \in R \cap S$ then $(x,y) \in R$ and $(x,y) \in S$

Similarly $(y,x) \in R$ and $(y,x) \in S$

Since R and S are anti-symmetric, $x = y$ so if $(x,y) \in R \cap S$ and $(y,x) \in R \cap S$ then $x = y$ so $R \cap S$ is anti-symmetric.

Transitive:

Suppose $(x,y) \in R \cap S$ and $(y,z) \in R \cap S$ for some $x, y, z \in A$.

So $(x,y) \in R, (y,z) \in R, (x,y) \in S, (y,z) \in S$ and since R, S are transitive $(x,z) \in R$ and $(x,z) \in S$.

So by the definition of set intersection $(x,z) \in R \cap S$

Since $R \cap S$ is reflexive, anti-symmetric, and transitive, it is a partial order.

b Consider a relation R on A that is both symmetric and anti-symmetric.

Since the relation is symmetric $xRy \implies yRx$ for some $x, y \in A$.

Since the relation is anti-symmetric $xRy \wedge yRx \implies x = y$

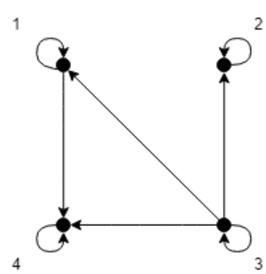
Using both definitions we can say $xRy \implies x = x$

Therefore R must be the relation $\{(x,), x \in B, B \subset A\}$ Since B can be any subset of A, there can be many relations on A that are

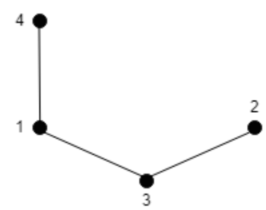
c If R is symmetric and transitive, then R may not be reflexive.

As a counter-example consider $R = \{\}$ which is symmetric and transitive but not reflexive.

7. a



b



c Since a total order most have either xRy or yRx for every $x, y \in A$, and there are 2 of these occurrences in R, between 1 and 2, and 2 and 4.
For each of these pairs there are 2 choices for our total ordering, one for each pair being ordered "after" the other for a total of $2^2 = 4$ total orders that contain the given partial order.

8. The number of equivalence relations on S with exactly 3 equivalence classes is the same as the number of ways to distribute 8 distinguishable elements between 3 indistinguishable boxes.
This number is $S(8, 3) = 966$