MATH 222 Assignment 2

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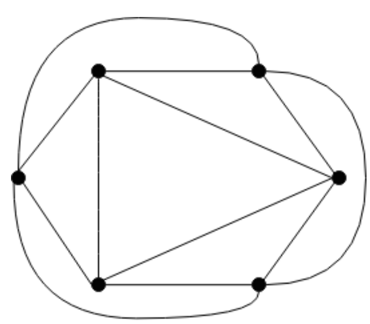
1. Consider a tree T with some vertex v of degree 4.
   Since a tree cannot have any cycles, every path from v taking a unique edge incident to v must eventually terminate in at least one leaf.
   Therefore T must have at least 4 leaves.
   Now lets consider if T has another vertex u of degree 3.
   Since T is a tree, there must be a single v-u path in T.
   And since there must be a single v-u path and there can be no cycles, exactly one of the edges incident to v must yield a v-u path and exactly one of the edges incident to u must yield a u-v path.
   So there are exactly 3 paths from v taking a unique edge incident to v that terminate in at least one leaf, and similarly there are exactly 2 paths from u taking a unique edge incident to u that terminate in at least one leaf.
   $\therefore T$ contains at least 5 leaves.

2. For a tree $T = (V, E)$, there must have a single u-v path in T $\forall u, v \in V$.
   For a trail to be Eulerian it would need to include every vertex in T, including all leaves.
   Since a leaf only has one edge incident to it, including a leaf in the trail (not including endpoints) would result in taking the edge twice, violating a property of an Eulerian trail.
   Since leaves at the endpoints only need the adjacent edge to be traversed once, they can be in the Eulerian trail.
   Similarly if T only has 2 leaves, a Eulerian trail would simply be the $u - v$ path between them.
   $\therefore T$ has a Eulerian trail iff T has exactly 2 leaves.

3.



Example of a 4-regular plane graph with 6 vertices.

Since $|V| - |E| + R = 2 \iff 6 - 12 + 8 = 2$ holds, we can verify that this example holds.

4. We can use the theorem that $|E| \leq 3|V| - 6$ for any planar graph G with $|V| \geq 3$.
   Assume that both G and $\overline{G}$ are planar.
   Therefore for G:

$$\begin{aligned} |E| &\leq 3|V| - 6 \\ |E| &\leq 3|11| - 6 \\ |E| &\leq 27 \end{aligned}$$

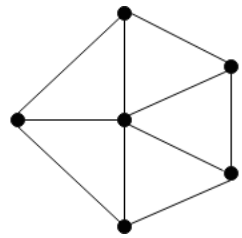
   And therefore for $\overline{G}$, since the maximum number of edges in $\overline{G}$ is the difference between the number of edges in K_{11} and the maximum number of edges in G :

$$\begin{aligned} |E| &\leq 3|V| - 6 \\ \frac{11(11-1)}{2} - 27 &\leq 3|11| - 6 \\ 28 &\leq 27 \end{aligned}$$

   Which is a contradiction.
   $\therefore$ either G or $\overline{G}$ must be non-planar.

5. First we prove that if $\deg(v) < k$ and G is k-colorable then $G - v$ is k-colorable.
   Since G is k-colorable using Brooks' theorem, we know that the maximum degree of a vertex in G must be $k - 1$.
   Since the maximum degree of $G - v$ is less or equal the maximum degree of G because some edges in G may have been removed, therefore the maximum degree of $G - v$ must be $k - 1$.
   $\therefore$ if G is k-colorable, $G - v$ must be k-colorable.
   Now we prove that if $G - v$ is k-colorable then G is k-colorable.
   Using Brooks's theorem the maximum degree of $G - v$ is $k - 1$.
   And since $v < k$, G will have a maximum degree of k since each of the vertices in $G - v$ will have at most one more edge in G.
   Since G has a maximum degree of k, using Brooks' theorem again we can easily see that G will be k-colorable.
   $\therefore$ G is k-colorable iff $G - v$ is k-colorable.

6. a) As a counter-example, consider C_5 in which $\chi(C_5) = 3$ yet contains no subgraph isomorphic to K_3.
   b) Consider the Wheel graph W_6 with $\chi(G) = 4$.



While W_6 has a subraph homeomorphic to K_4, it has no subgraph isomorphic to K_4.
$\therefore$ the statement is false.

c) Since a $\chi(G) = 2$, G must be bipartite.
Since for G to be planar, using kuratowskis' theorem, G must have no subgraph that is homeomorphic to K_5 or $K_{3,3}$.
Now consider $G = K_{6,6}$
We know a bipartite graph $K_{m,n}$ has a Hamiltonian cycle iff $m = n$ so G has a Hamiltonian cycle.
We know that a graph with at least 2 vertices has an Eulerian circuit iff it is connected and every vertex is of even degree, so clearly G has a Eulerian circuit.
Clearly $K_{3,3}$ is an isomorphic subgraph of $K_{6,6}$ so $K_{3,3}$ is a homeomorphic subgraph to $K_{6,6}$.
$\therefore G = K_{6,6}$ is not planar.
$\therefore$ the statement is false