MATH 222 Assignment 4

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1.

2. Consider n distinguishable elements and $n-2$ indistinguishable boxes. Count the number of ways to divide the elements into the boxes such that no box is empty for $n \geq 4$.
   Clearly there are $S(n, n-2)$ such ways.
   There are 2 possible cases for any such distribution as follows:

   (a) There are 3 elements that share a single box. There are $\binom{n}{3}$ ways to pick the 3 elements and since all other indistinguishable boxes must have only 1 element, there are $\binom{n}{3}$ ways total to arrange the elements in this case.

   (b) There are 2 boxes which each contain 2 elements. There are $\binom{n}{4}$ ways to choose the elements and $\frac{\binom{4}{2}}{2} = 3$ ways to distribute the chosen elements into a box for a total of $3\binom{n}{4}$ ways to arrange the elements in this case.

   This makes a total number of ways to distribute the elements of $\binom{n}{3} + 3\binom{n}{4}$
   $\therefore S(n, n-2) = \binom{n}{3} + 3\binom{n}{4}$ for $n \geq 4$.

3. We can use a generating function to get our result.
   Consider the generating function in which the sum of coefficients of x^i with $i \leq 35$ is our result:

$$g(x) = (x^0 + x^1 + x^2 ...)^6 = \frac{1}{(1-x)^6} \tag{1}$$

   If we multiply $g(x)$ by $\frac{1}{1-x} = 1 + x + x^2 + ...$, the coefficient of x^{35} is our result since the new $g(x)$ has coefficients which are cumulative of the old $g(x)$ coefficients. So:

$$\begin{aligned} g(x) &= \frac{1}{(1-x)^6} \cdot \frac{1}{1-x} \\ &= \frac{1}{(1-x)^7} \\ &= \sum_{i=0}^{\infty} \binom{i+7-1}{7-1} x^i \\ &= \sum_{i=0}^{\infty} \binom{i+6}{6} x^i \end{aligned} \tag{2}$$

Clearly then, the coefficient of x^{35}, the number is of integers between 0 and 999999 with digits that sum to no more than 35, is $\binom{41}{6} = 4496388$

4. Consider a hexagonal shaped room with the 6 walls labeled 0,1,2,3,4,5 in order around the room.
   Let c_i be the condition that the walls i and $i+1 \mod 5$ are the same color for $0 \leq i \leq 5$.
   $N = 10^6$
   $N(c_i) = 10^5$
   $N(c_i c_j) = 10^4$
   $N(c_i c_j c_k) = 10^3$
   $N(c_i c_j c_k c_q) = 10^2$
   $N(c_i c_j c_k c_q c_r) = 10$
   $N(c_0 c_1 c_2 c_3 c_4 c_5) = 10$
   $\overline{N} = 10^6 - \binom{6}{1} \cdot 10^5 + \binom{6}{2} \cdot 10^4 - \binom{6}{3} \cdot 10^3 + \binom{6}{4} \cdot 10^2 - \binom{6}{5} \cdot 10 + \binom{6}{6} \cdot 10$
   $\overline{N} = 528450$

5. a For R to be an equivalence relation, it must be reflexive, symmetric, and transitive.
   Reflexive:
   Since for all $x \in S, x^2 > 0$, xRx so R is reflexive.
   Symmetric:
   Since multiplication is commutative, $xy = yx$ so $xRy \implies yRx$ $\forall x, y \in S$ so R is symmetric.
   Transitive:
   For $x, y, z \in S$, if $xy > 0$ then both x and y must have the same sign. Similarly, if $yz > 0$ then both y and z must have the same sign.

Since x, y, z all have the same sign, $yz > 0$ and hence if xRy and yRz, xRz so R is transitive.
Since R is reflexive, symmetric, and transitive, R is an equivalence relation. A partition of R would be the following equivalence classes:
$[1] = \{x \in S | x > 0\}$
$[-1] = \{x \in S | x \leq 0\}$

b A reason why R_2 is not an equivalence relation because it is not reflexive. For example $(-1, -1) \notin R$ since $-1^2 = 1 \nless 0$

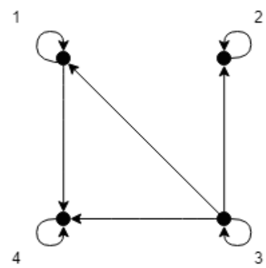
6. a For $R \cap S$ to be a partial order on A it must be reflexive, anti-symmetric, and transitive.
Reflexive:
Since R and S are reflexive, $\forall x \in A, (x, x) \in R$ and $(x, x) \in S$ so by the definition of set intersection, $(x, x) \in R \cap S$ so $R \cap S$ is reflexive.
Anti-Symmetric:
For some $x, y \in A$, if $(x, y) \in R \cap S$ and $(y, x) \in R \cap S$ then $(x, y) \in R$ and $(x, y) \in S$
Similarly $(y, x) \in R$ and $(y, x) \in S$
Since R and S are anti-symmetric, $x = y$ so if $(x, y) \in R \cap S$ and $(y, x) \in R \cap S$ then $x = y$ so $R \cap S$ is anti-symmetric.
Transitive:
Suppose $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$ for some $x, y, z \in A$.
So $(x, y) \in R, (y, z) \in R, (x, y) \in S, (y, z) \in S$ and since R, S are transitive $(x, z) \in R$ and $(x, z) \in S$.
So by the definition of set intersection $(x, z) \in R \cap S$
Since $R \cap S$ is reflexive, anti-symmetric, and transitive, it is a partial order.

b Consider a relation R on A that is both symmetric and anti-symmetric.
Since the relation is symmetric $xRy \implies yRx$ for some $x, y \in A$.
Since the relation is anti-symmetric $xRy \wedge yRx \implies x = y$
Using both definitions we can say $xRy \implies x = x$
Therefore R must be the relation $\{(x,), x \in B, B \subset A\}$ Since B can be any subset of A, there can be many relations on A that are

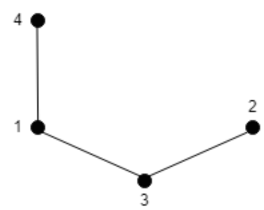
c If R is symmetric and transitive, then R may not be reflexive.

As a counter-example consider $R = \{\}$ which is symmetric and transitive but not reflexive.

7. a



b



c Since a total order most have either xRy or yRx for every $x, y \in A$, and there are 2 of these occurrences in R, between 1 and 2, and 2 and 4.
For each of these pairs there are 2 choices for our total ordering, one for each pair being ordered "after" the other for a total of $2^2 = 4$ total orders that contain the given partial order.

8. The number of equivalence relations on S with exactly 3 equivalence classes is the same as the number of ways to distribute 8 distinguishable elements between 3 indistinguishable boxes.
This number is $S(8, 3) = 966$