MATH 222 Assignment 1

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1. Let n_2 and n_2 be the number of vertices in G of degree 2 and 3 respectively.
   We know that the sum of the degrees of all vertices is twice the number of edges so:

$$\sum_{v \in V} deg(v) = 2|E|$$
$$5 \cdot 11 + 4 \cdot 14 + 3n_3 + 2n_2 = 2 \cdot 78$$
$$3n_3 + 2n_2 = 45 \tag{1}$$

   We can also sum the total number of vertices:

$$11 + 14 + n_3 + n_2 = 45$$
$$n_2 = 20 - n_3 \tag{2}$$

   Using (1) and (2) we get:

$$3n_3 + 2(20 - n_3) = 45$$
$$n_3 = 5$$

   $\therefore$ There are 5 vertices in G with degree 3.

2. Since K_n is a complete graph, all vertices are adjacent to all other vertices so a path of length p is isomorphic to any order p subgraph of K_n, provided $n \geq p$.
   Since there are $\binom{n}{p}$ order p sub-graphs of K_n, there will be $\binom{7}{4} = 35$ sub-graphs of K_7 that are isomorphic to P_4.

3. Assume the graph G is not connected.
   Therefore there is at least 2 distinct connected subgraphs G_1, G_2 of order n_1, n_2.
   It follows that $\forall v \in G_1, deg(v) \leq n_1 - 1$ and $\forall v \in G_2, deg(v) \leq n_2 - 1$.
   If we choose vertices $x \in G_1$ and $y \in G_2$ then:

$$deg(x) + deg(y) \geq 17$$
$$n_1 - 1 + n_2 - 1 \geq 17$$
$$n_1 + n_2 - 2 \geq 17$$
$$18 - 2 \geq 17$$
$$16 \geq 17 \tag{3}$$

But $16 \leq 17$, so (3) is a contradiction.
$\therefore G$ must be connected.

4. a) Trivially for $v = \{1\}$ the only u s.t. $v \cup u = S$ is the complete set S and $S \setminus \{1\}$.
   $\therefore deg(\{1\}) = 2$

   b) More generally for $v =$a subset of S, v is adjacent to all $u = S \setminus w$ $\forall w \subseteq S$.
   Therefore v is adjacent to $2^{|v|}$ vertices when $v \neq S$ and $2^{|v|} - 1$ vertices when $v = S$. This is to stop the self loop on v when $v = S$ so that G remains a graph.
   $\therefore deg(\{1, 2\}) = 2^2 = 4$

   c) Since:

$$\sum_{v \in V} deg(v) = 2|E| \tag{4}$$

   And because there are $\binom{|S|}{i}$ with the same degree $\forall i \in \mathbb{N}$ where $i \leq |S|$:

$$\sum_{v \in V} deg(v) = (\sum_{i=0}^{|S|-1} \binom{|S|}{i} \cdot 2^i) + 2^{|v|} - 1 \tag{5}$$

   Using (4) and (5):

$$1 + 12 + 60 + 160 + 240 + 192 + 63 = 2|E|$$
$$|E| = 364$$

5. a) A bipartite graph G on 5 vertices must have one partition that contains 3 vertices that are not adjacent.
   Since the 3 vertices on that partition are not adjacent, $\overline{G}$ will have those 3 vertices be all adjacent.
   Since these 3 vertices are adjacent in $\overline{G}$, for $\overline{G}$ to be bipartite, 2 of these vertices will be in the same partition, but since all 3 are adjacent, there will be an edge within the partition.
   $\therefore$ There cannot be a graph G on 5 vertices s.t. G and $\overline{G}$ are both bipartite.

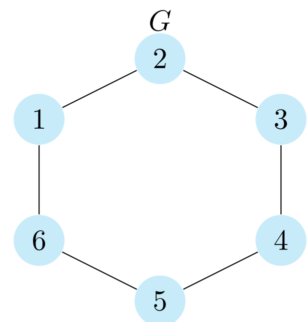
b) Because the complement of G contains all edges in K_n that are not in $E(G)$:

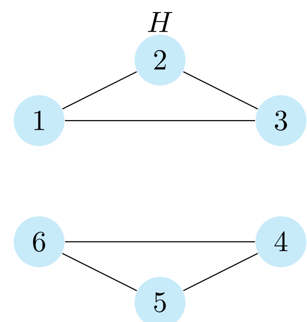
$$|E(G)| = \frac{|E(K_n)|}{2}$$

$$|E(G)| = \frac{n(n-1)}{4} \tag{6}$$

Since $|E(G)|$ must be an integer, from (6) we can see that $n(n-1)$ must be divisible by 4.
It can easily be seen that this is only possible when either $n = 4k$ or $n - 1 = 4k \iff n = 4k + 1$ for some positive integer k.

6. As G is a connected graph, and H is not, G and H cannot be isomorphic.



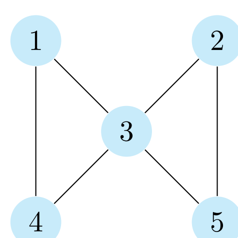


$\therefore$ If G and H have the same number of vertices and the same number of edges and both are 2-regular, G and H are not necessarily isomorphic.

7. Lets assume that G does not have a cycle. Therefore we can start a path from any vertex and never repeat a vertex without repeating an edge.
   Because every vertex has a degree greater than 2, and G does not have any cycles, there will always be an un-traversed edge to a new vertex at every vertex in the path.
   This implies that the G must be infinite.// Since G is a graph and we consider graphs to be finite, G must contain a cycle if every vertex of G has degree of at least 2.

8. a) Since $K_{m,n}$ is bipartite, for it to have a Hamiltonian cycle each vertex in the cycle must take a path to a node in the other partition before returning to the original vertex in the starting partition. It is clear that this requires that $m = n$ for this to occur because if $m \neq n$ the not all vertices could be in the cycle.
   Since $K_{m,n}$ has an Euler circuit, we know that every vertex has even degree.
   For this to be true both m and n must be even as all vertices in $K_{m,n}$ will have either degree m or n.
   $\therefore$ If $K_{m,n}$ has both a Euler circuit and a Hamilton cycle, then $m = n$ and n is even.

   b) As we can draw a graph that contains a Euler circuit but not a Hamiltonian cycle, the proposition is disproven.



   c) The proposition is false as shown in (a), $K_{3,3}$ has a Hamiltonian cycle but no Euler circuit because $m = n$, but n is not even.