For these exercises, we will consider the linear transport equation:

$$u_t + au_x = 0, \quad x \in [-1, 1],\ t > 0 \tag{1}$$
$$u(x, 0) = f(x)$$

with periodic boundary conditions. The discontinuous Galerkin scheme is given by:
Find $u_h \in V_h^p$ *such that*

$$((u_h)_t, v_h)_{I_j} - (au_h, (v_h)_x)_{I_j} + (a\hat{u}_h v_h^-)|_{x_{j+1/2}} - (a\hat{u}_h v_h^+)|_{x_{j-1/2}} = 0 \tag{2}$$

for all $v_h \in V_h^p$. Here, V_h^p denotes an approximation basis consisting of piecewise polynomials of degree less than or equal to p. For these exercises the local Lax-Friedrichs flux will be used. This is defined as

$$\hat{f}_{j+1/2} = \frac{1}{2}(f(u_{j+1/2}^+) + f(u_{j+1/2}^-) - \alpha(u_{j+1/2}^+ - u_{j+1/2}^-)), \quad \alpha = \max_{u_{j+1/2}^+, u_{j+1/2}^-} |f'(u)|. \tag{3}$$

1. Prove the following
Proposition: *Consider the linear transport equation given in Equation (1) and assume that the initial condition is smooth,* $f \in C^{p+2}$*. Then the approximation using DG scheme (2) with numerical flux (3) gives a* $(p+1)^{th}$ *order accurate solution. That is,*

$$||u - u_h||_0 \leq C(\Delta x)^{p+1},$$

where $C = \tilde{C}||f(x)||_{p+2}$.

Considering the semi-discrete variational formulation in (2), we can sum over all elements $I_j,\ j = 1, \ldots, N$ to result in

$$\sum_{j=1}^{N}((u_h)_t, v_h)_{I_j} - \sum_{j=1}^{N}(au_h, (v_h)_x)_{I_j} + \sum_{j=1}^{N}\left[(a\hat{u}_h v_h^-)|_{x_{j+1/2}} - (a\hat{u}_h v_h^+)|_{x_{j-1/2}}\right] = 0.$$

If we sum up the flux terms, we can simplify the two terms as follows:

$$\begin{aligned}
\sum_{j=1}^{N}\left[(a\hat{u}_h v_h^-)|_{x_{j+1/2}} - (a\hat{u}_h v_h^+)|_{x_{j-1/2}}\right] &= a\hat{u}_{h,3/2}v_{h,3/2}^- - a\hat{u}_{h,1/2}v_{h,1/2}^+ + a\hat{u}_{h,5/2}v_{h,5/2}^- - a\hat{u}_{h,3/2}v_{h,3/2}^+ + \\
&\quad \ldots + a\hat{u}_{h,N+1/2}v_{h,N+1/2}^- - a\hat{u}_{h,N-1/2}v_{h,N-1/2}^+, \\
&= -a\hat{u}_{h,1/2}^+ v_{h,1/2}^+ + \sum_{j=1}^{N} a\hat{u}_{h,j+1/2}\left(v_{h,j+1/2}^- - v_{h,j+1/2}^+\right) + \\
&\quad \ldots + a\hat{u}_{h,N+1/2}v_{h,N+1/2}^-, \\
&= -a\hat{u}_{h,1/2}^+ v_{h,1/2}^+ - \sum_{j=1}^{N} a\hat{u}_{h,j+1/2}[\![v_h]\!]_{j+1/2} + a\hat{u}_{h,N+1/2}v_{h,N+1/2}^-,
\end{aligned}$$

Given the periodic boundary conditions, we have $\hat{u}_{h,1/2} = \hat{u}_{h,N+1/2}$ and thus

$$\sum_{j=1}^{N} \left[(a\hat{u}_h v_h^-)|_{x_{j+1/2}} - (a\hat{u}_h v_h^+)|_{x_{j-1/2}} \right] = -\sum_{j=1}^{N} a\hat{u}_{h,j+1/2} [\![v_h]\!]_{j+1/2}.$$

The semi-discrete variational DG scheme for the entire domain is hence to find $u_h \in V_h^p$ such that

$$((u_h)_t, v_h)_\Omega - (au_h, (v_h)_x)_\Omega - \sum_{j=1}^{N} a\hat{u}_{h,j+1/2} [\![v_h]\!]_{j+1/2} = 0$$

for all $v_h \in V_h^p$ with initial condition $(u_h, v_h)_\Omega = (f(x), v_h)_\Omega$. The variational formulation of the exact solution is

$$((u)_t, v_h)_\Omega - (au, (v_h)_x)_\Omega - \sum_{j=1}^{N} au(x_{j+1/2}, t) [\![v_h]\!]_{j+1/2} = 0.$$

Taking the difference of the exact and approximation variational formulations, we have

$$((u - u_h)_t, v_h)_\Omega - (a(u - u_h), (v_h)_x)_\Omega - \sum_{j=1}^{N} a(u(x_{j+1/2}, t) - \hat{u}_{h,j+1/2}) [\![v_h]\!]_{j+1/2} = 0.$$

As the difference between the exact solution and the approximation is the error, we define $e = u - u_h$ and thus

$$(e_t, v_h)_\Omega - (ae, (v_h)_x)_\Omega - \sum_{j=1}^{N} a(u(x_{j+1/2}, t) - \hat{u}_{h,j+1/2}) [\![v_h]\!]_{j+1/2} = 0.$$

We can then define a projection Pu with the following conditions

$$\begin{cases} \int_{I_j} (Pu - u) v_h dx = 0 \quad \forall v_h \in \mathbb{P}^{p-1}(I_j) \\ Pu(x_{j+1/2}^-) = u(x_{j+1/2}). \end{cases}$$

We can also break down the error as

$$e = u - u_h = \underbrace{(u - Pu)}_{\varepsilon_h} + \underbrace{(Pu - u_h)}_{e_h}.$$

Finally we will consider our test function v_h be equal to the error between the projection and the approximation, i.e. $v_h = Pu - u_h = e_h$. We'll consider now the variational formulation on e as follows, where we use $\hat{e}_h = u - \hat{u}$:

$$(e_t, e_h)_\Omega - (ae, (e_h)_x)_\Omega - \sum_{j=1}^{N} a(u(x_{j+1/2}, t) - \hat{u}_{h,j+1/2})[\![e_h]\!]_{j+1/2} = 0,$$

$$((\varepsilon_h + e_h)_t, e_h)_\Omega - (a(\varepsilon_h + e_h), (e_h)_x)_\Omega - \sum_{j=1}^{N} a\hat{e}_h(x_{j+1/2})[\![e_h]\!]_{j+1/2} = 0,$$

$$\underbrace{((e_h)_t, e_h)_\Omega}_{i} - \underbrace{(ae_h, (e_h)_x)_\Omega}_{ii} - \underbrace{\sum_{j=1}^{N} a\hat{e}_h(x_{j+1/2})[\![e_h]\!]_{j+1/2}}_{iii} = -\underbrace{((\varepsilon_h)_t, e_h)_\Omega}_{iv} + \underbrace{(a\varepsilon_h, (e_h)_x)_\Omega}_{v}.$$

At this point, let's simplify the five terms:

i. $((e_h)_t, e_h)_\Omega = \frac{1}{2}\frac{d}{dt}(e_h, e_h)_\Omega = \frac{1}{2}\frac{d}{dt}||e_h||_0^2$

ii.

$$\begin{aligned}
(ae_h, (e_h)_x)_\Omega &= \sum_{j=1}^{N} \int_{I_j} ae_h(e_h)_x dx, \\
&= \sum_{j=1}^{N} \int_{I_j} a\frac{\partial}{\partial x}\left(\frac{e_h^2}{2}\right) dx, \\
&= \sum_{j=1}^{N} a\frac{e_h^2}{2}\Big|_{x_{j-1/2}^+}^{x_{j+1/2}^-}, \\
&= \frac{a}{2}\left[e_h^2|_{3/2^-} - e_h^2|_{1/2^+} + e_h^2|_{5/2^-} - e_h^2|_{3/2^+} + \ldots + e_h^2|_{N+1/2^-} - e_h^2|_{N-1/2^+}\right], \\
&= \frac{a}{2}\left[-e_h^2|_{1/2^+} + e_h^2|_{N+1/2}^- - \sum_{j=1}^{N}[\![e_h^2]\!]_{j+1/2}\right], \\
&= -\frac{a}{2}\sum_{j=1}^{N}[\![e_h^2]\!]_{j+1/2}, \\
&= -\frac{a}{2}\sum_{j=1}^{N}\left((e_h^2)_{j+1/2}^+ - (e_h^2)_{j+1/2}^-\right), \\
&= -\frac{a}{2}\sum_{j=1}^{N}(e_h^+ - e_h^-)_{j+1/2}(e_h^+ + e_h^-)_{j+1/2}, \\
&= -\frac{a}{2}\sum_{j=1}^{N}[\![e_h]\!]_{j+1/2}(e_h^+ + e_h^-)_{j+1/2}
\end{aligned}$$

iii. Combined with (ii):

$$-(ae_h,(e_h)_x)-\sum_{j=1}^{N}\hat{e}_h(x_{j+1/2},t)[\![e_h]\!]_{j+1/2}=\frac{a}{2}\sum_{j=1}^{N}[\![e_h]\!]_{j+1/2}(e_h^{+}+e_h^{-})_{j+1/2}+\dots$$
$$-\sum_{j=1}^{N}a(u(x_{j+1/2},t)-\hat{u}(x_{j+1/2},t))[\![e_h]\!]_{j+1/2}$$

At this point, we need to evaluate the flux for either positive a or negative a. Either way, we have $f(u)=au$ and thus $f'(u)=a$ and $\alpha=|a|$. The flux evaluations are hence

$$a>0:\quad a\hat{u}_{j+1/2}=\frac{1}{2}\left(au^{+}_{j+1/2}+au^{-}_{j+1/2}-a(u^{+}_{j+1/2}-u^{-}_{j+1/2})\right)=au^{-}_{j+1/2},$$

$$a<0:\quad a\hat{u}_{j+1/2}=\frac{1}{2}\left(au^{+}_{j+1/2}+au^{-}_{j+1/2}+a(u^{+}_{j+1/2}-u^{-}_{j+1/2})\right)=au^{+}_{j+1/2}.$$

For $-(ii)+(iii)$, we hence have

$$\begin{aligned}
a>0:\quad -(ae_h,(e_h)_x)-\sum_{j=1}^{N}\hat{e}_h(x_{j+1/2},t)[\![e_h]\!]_{j+1/2}&=\frac{a}{2}\sum_{j=1}^{N}[\![e_h]\!]_{j+1/2}(e_h^{+}+e_h^{-})_{j+1/2}+\dots\\
&\quad-\sum_{j=1}^{N}a(u(x_{j+1/2},t)-\hat{u}(x_{j+1/2},t))[\![e_h]\!]_{j+1/2},\\
&=\frac{a}{2}\sum_{j=1}^{N}[\![e_h]\!]_{j+1/2}(e_h^{+}+e_h^{-})_{j+1/2}+\dots\\
&\quad-\sum_{j=1}^{N}a(u(x_{j+1/2},t)-u(x^{-}_{j+1/2},t))[\![e_h]\!]_{j+1/2},\\
&=\frac{a}{2}\sum_{j=1}^{N}[\![e_h]\!]_{j+1/2}(e_h^{+}+e_h^{-})_{j+1/2}+\dots\\
&\quad-\sum_{j=1}^{N}ae_{h}{}^{-}_{j+1/2}[\![e_h]\!]_{j+1/2},\\
&=a\sum_{j=1}^{N}[\![e_h]\!]_{j+1/2}\left((e_h^{+}+e_h^{-})_{j+1/2}-e_h^{-}\right),\\
&=\frac{a}{2}\sum_{j=1}^{N}[\![e_h]\!]^2_{j+1/2},
\end{aligned}$$

and

$$a > 0: \quad -(ae_h, (e_h)_x) - \sum_{j=1}^{N} \hat{e}_h(x_{j+1/2}, t)[\![e_h]\!]_{j+1/2} = \frac{a}{2}\sum_{j=1}^{N}[\![e_h]\!]_{j+1/2}(e_h^+ + e_h^-)_{j+1/2} + \dots$$
$$- \sum_{j=1}^{N} a{e_h}^+_{j+1/2}[\![e_h]\!]_{j+1/2},$$
$$= a\sum_{j=1}^{N}[\![e_h]\!]_{j+1/2}\left((e_h^+ + e_h^-)_{j+1/2} - e_h^- +\right),$$
$$= -\frac{a}{2}\sum_{j=1}^{N}[\![e_h]\!]^2_{j+1/2},$$

and overall for any a,

$$-(ae_h, (e_h)_x) - \sum_{j=1}^{N} \hat{e}_h(x_{j+1/2}, t)[\![e_h]\!]_{j+1/2} = \frac{|a|}{2}\sum_{j=1}^{N}[\![e_h]\!]^2_{j+1/2}$$

iv. $((\varepsilon_h)_t, e_h) \leq \frac{1}{2}\left(\frac{d}{dt}||\varepsilon_h||_0^2 + ||e_h||_0\right) \leq A(\Delta x)^{2(p+1)} + ||e_h||_0^2$

v. $(a\varepsilon_h, (e_h)_x) = a\sum_{j=1}^{N}(u - Pu, (Pu - u_h)_x)_{I_j} = -a\sum_{j=1}^{N}\int_{I_j}(Pu-u)v_h dx = 0 \forall v_h \in \mathbb{P}^{p-1}$

Bringing all the terms together, we hence result in

$$\frac{1}{2}\frac{d}{dt}||e_h||_0^2 + \frac{|a|}{2}\sum_{j=1}^{N}[\![e_h]\!]^2_{j+1/2} \leq A(\Delta x)^{2(p+1)} + ||e_h||_0^2,$$
$$\frac{1}{2}\frac{d}{dt}||e_h||_0^2 \leq A(\Delta x)^{2(p+1)} + ||e_h||_0^2 - \frac{|a|}{2}\sum_{j=1}^{N}[\![e_h]\!]^2_{j+1/2},$$
$$\frac{1}{2}\frac{d}{dt}||e_h||_0^2 \leq A(\Delta x)^{2(p+1)} + ||e_h||_0^2,$$
$$||e_h||_0^2 \leq \int_0^T 2A(\Delta x)^{2(p+1)} dt + \int_0^T 2||e_h||_0^2 dt,$$
$$||e_h||_0^2 \leq 2AT(\Delta x)^{2(p+1)} + \int_0^T 2||e_h||_0^2,$$
$$||e_h||_0^2 \leq 2AT(\Delta x)^{2(p+1)} \exp\left(\int_0^T 2||e_h(t)||_0^2 dt\right),$$
$$||e_h||_0^2 \leq \bar{A}(\Delta x)^{2(p+1)},$$
$$||e_h||_0 \leq \tilde{A}(\Delta x)^{p+1},$$

where I attempted Grönwall's inequality towards the end, but I may have employed it incorrectly. I am not sure if the integral argument of the exponential is correct and how it would lead to $||u_0||^2_{p+2}$. Ultimately, this yields a bound on the norm of e_h. Recall, $u - u_h = e = \varepsilon_h + e_h = (u - Pu) + (Pu - u_h)$. We have found the norm on e_h and $||\varepsilon_h||_0 \leq A(\Delta x)^{p+1}$, from the step in (iv) which I'm not sure how to show to be honest. Ultimately

$$\begin{aligned} ||u-u_h||_0^2 &\leq ||u-Pu||_0^2 + ||Pu-u_h||_0^2, \\ &\leq A(\Delta x)^{p+1} + \tilde{A}(\Delta x)^{p+1}, \\ &\leq C(\Delta x)^{p+1}, \end{aligned}$$

where $C = \tilde{C}||f(x)||_{p+2}$ per the notes. I am not so sure about this last part, but it seems that we should not stop at the norm of e_h but go back to $e = u - u_h$.

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2. In this exercise, we consider the case when $a > 0$ in Equation (1).

(i) Use Mathematica to determine the dispersion and dissipation error for a $p = 2, 3$ DG approximation using the local Lax-Friedrichs flux.

Re-writing the variational formulation of the DG scheme in (2), we have : *Find* $u_h \in V_h^p$ *such that*

$$\int_{I_j} (u_h)_t v_h dx = \int_{I_j} a u_h (v_h)_x dx - (a\hat{u}_h v_h^-)|_{x_{j+1/2}} + (a\hat{u}_h v_h^+)|_{x_{j-1/2}}$$

for all $v_h \in V_h^p$.
Given $a > 0$, it was found above that $\hat{f}_{j+1/2} = a\hat{u}_{j+1/2} = au^-_{j+1/2}$, resulting in the upwind flux. This gives us

$$\int_{I_j} (u_h)_t v_h dx = \int_{I_j} a u_h (v_h)_x dx - (au_h^- v_h^-)|_{x_{j+1/2}} + (au_h^- v_h^+)|_{x_{j-1/2}}$$

We'll consider the Legendre polynomial space and hence

$$u_h(x,t)\Big|_{I_j} = \sum_{k=0}^{p} u_j^{(k)}(t) P^{(k)}(\xi_j), \quad v_h(x) = P^{(m)}(\xi_j).$$

Once we go through the derviations similar to the ones shown in Homework 1, we have

$$\frac{d}{dt}\mathbf{u}_j = \frac{1}{\Delta x}(\tilde{A}\mathbf{u}_j + \tilde{B}\mathbf{u}_{j-1}), \tag{4}$$

where the matrices are $\tilde{A} = 2aM^{-1}A$, $\tilde{B} = 2aM^{-1}B$ (where the factor of 2 comes from the transformation from $x \mapsto \xi$ in the integration) with

$$\begin{aligned} M(m,k) &= \int_{-1}^{1} P^{(k)}(\xi_j) P^{(m)}(\xi_j) d\xi_j, \\ A(m,k) &= \int_{-1}^{1} P^{(k)}(\xi_j) \frac{d}{dx} P^{(m)}(\xi_j) d\xi_j - 1, \\ B(m,k) &= (-1)^m, \qquad \text{for } m,k = 0,1,\ldots,p. \end{aligned}$$

To examine the dispersion and dissipation errors, let's examine the solution in frequency space and hence consider one Fourier mode, $\mathbf{u}_j(t) = \hat{\mathbf{u}}_m(t) e^{imx_j}$. Plugging this Fourier mode in (4), the equation becomes

$$\begin{aligned} \frac{d}{dt}\hat{\mathbf{u}}_m(t) e^{imx_j} &= \frac{1}{\Delta x}\left(\tilde{A}\hat{\mathbf{u}}_m(t)e^{imx_j} + \tilde{B}\hat{\mathbf{u}}_m(t)e^{im(x_j-\Delta x)}\right), \\ \frac{d}{dt}\hat{\mathbf{u}}_m(t) &= \frac{1}{\Delta x}\left(\tilde{A} + \tilde{B}e^{-im\Delta x}\right)\hat{\mathbf{u}}_m(t), \\ &= \frac{m}{m\Delta x}\left(\tilde{A} + \tilde{B}e^{-im\Delta x}\right)\hat{\mathbf{u}}_m(t), \\ &= G(\mu)\hat{\mathbf{u}}_m(t), \end{aligned}$$

where $\mu = m\Delta x,\ G(\mu) = \frac{m}{\mu}(\tilde{A} + \tilde{B}e^{-i\mu})$ is a $(p+1)\times(p+1)$ matrix. Additionally, the exact dispersion relation is $\frac{d}{dt}\hat{\mathbf{u}}_m(t) = im\hat{\mathbf{u}}_m(t)$. Following the approach from Exercise 2, we can find the dispersion and dissipation errors as calculated in the Mathematica notebook "HW2-Problem2.nb".

By finding the eigenvalues of the $G(\mu)$ matrix for $p = 2, 3$, we can find the dispersion and dissipation errors by calculating $im - \lambda_1$, where λ_1 is the physically relevant eigenvalue expanded as $im + \mathcal{O}(\mu^{2p+1})$ and the other eigenvalues expand as $C/\mu + \mathcal{O}(1)$. Here, the advection speed is set to $a = 1$. Referring to results in the notebook, the errors are:

$$\begin{aligned} p = 2: &\quad \frac{m\mu^5}{7200} - \frac{im\mu^6}{42,000} + \mathcal{O}(\mu^7), \\ p = 3: &\quad \frac{m\mu^7}{1,411,200} - \frac{im\mu^8}{11,113,200} + \mathcal{O}(\mu^9) \end{aligned}$$

(ii) What does the real part of your error analysis represent? The imaginary part?

The real part of the error analysis above, or $\mathrm{Re}(im - \lambda_1)$, represents the dissipation error (error in amplitude). For $p = 2, p = 3$, the dissipation error is hence

$$\begin{aligned} \text{Dissipation Error}_{p=2} &= \frac{m\mu^5}{7200} = \mathcal{O}(\mu^5) = \mathcal{O}((\Delta x)^5) \\ \text{Dissipation Error}_{p=3} &= \frac{m\mu^7}{1,411,200} = \mathcal{O}(\mu^7) = \mathcal{O}((\Delta x)^7), \end{aligned}$$

which are both of order $\mathcal{O}((\Delta x)^{2p+1})$.
The imaginary part of the error analysis above, or $\text{Im}(im-\lambda_1)$, represents the dispersion error (error in phase). For $p=2, p=3$, the dispersion error is hence

$$\text{Dispersion Error}_{p=2} = -\frac{m\mu^6}{42,000} = \mathcal{O}(\mu^6) = \mathcal{O}((\Delta x)^6)$$
$$\text{Dispersion Error}_{p=3} = -\frac{m\mu^8}{11,113,200} = \mathcal{O}(\mu^8) = \mathcal{O}((\Delta x)^8),$$

which are both of order $\mathcal{O}((\Delta x)^{2p+2})$.

(iii) From your findings above, what does this mean in terms of wave propagation?

As the dissipation error is of order $\mathcal{O}((\Delta x)^{2p+1})$ compared to the dispersion error of order $\mathcal{O}((\Delta x)^{2p+2})$, the dissipation error dominates, resulting in errors in amplitude leading over errors in phase.

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3. Prove the following proposition:
Proposition: *Consider the DG scheme (2) for the linear advection equation (1) using the local Lax-Friedrichs flux. Further, consider an exact solution on element I_j of the form $\hat{\mathbf{u}}_m(T) = \hat{\mathbf{u}}_m(0)e^{im(x_j-T)}$ (given a wavespeed $a=1$) and write the numerical approximation as $\hat{\mathbf{u}}_m(T) = \hat{\mathbf{u}}_m(0)e^{imx_j-\omega_h T}$, where ω_h is the numerical dispersion. Define $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$. Then we can write*

$$||\mathbf{e}(T)|| \leq \underbrace{C_1 T(\Delta x)^{2p+1}}_{\textit{physically relevant mode}} + \underbrace{C_2(\Delta x)^{p+2}}_{\textit{initial projection}} + \underbrace{C_3 e^{-CT/\Delta x}(\Delta x)^{p+2}}_{\textit{parasitic modes}}, T > 0. \tag{5}$$

Try to use these facts to isolate the terms into three terms: one containing the physically relevant eigenvalue, one containing the initial condition, and one containing the non-physically relevant eigenvalues.

In addition to the exact solution and numerical approximation of $\hat{\mathbf{u}}_m(T)$, we'll use the decomposition of $\hat{\mathbf{u}}_m(0)$, as given in Exercise 2 to be $\hat{\mathbf{u}}_m(0) = \sum_{k=1}^{p+1} c_k \mathbf{v}_k$. We can begin by simplifying the error first before we bound the norm:

$$\begin{aligned}
\mathbf{e} &= \mathbf{u} - \mathbf{u}_h, \\
&= \hat{\mathbf{u}}_m(0)e^{im(x_j-T)} - \hat{\mathbf{u}}_m(0)e^{imx_j-\omega_h T}, \\
&= \sum_{k=1}^{p+1} c_k \mathbf{v}_k e^{im(x_j-T)} - \sum_{k=1}^{p+1} c_k \mathbf{v}_k e^{imx_j-\lambda_k T}, \\
&\leq c_1\mathbf{v}_1 e^{imx_j}(e^{-imT} - e^{-\lambda_1 T}) + \sum_{k=1}^{p+1} c_k \mathbf{v}_k e^{im(x_j-T)} - \sum_{k=2}^{p+1} c_k \mathbf{v}_k e^{imx_j-\lambda_k T}, \\
&= c_1\mathbf{v}_1 e^{imx_j}\left(1 - imT + \frac{(-imT)^2}{2} + \ldots - 1 + \lambda_1 T - \frac{(-\lambda_1 T)^2}{2} + \ldots\right) + \\
&\quad + \sum_{k=1}^{p+1} c_k \mathbf{v}_k e^{im(x_j-T)} - \sum_{k=2}^{p+1} c_k \mathbf{v}_k e^{imx_j-\lambda_k T}, \\
&\approx c_1\mathbf{v}_1 e^{imx_j}(-im + \lambda_1)T + \sum_{k=1}^{p+1} c_k \mathbf{v}_k e^{im(x_j-T)} - \sum_{k=2}^{p+1} c_k \mathbf{v}_k e^{imx_j-\lambda_k T}.
\end{aligned}$$

We can now take the norm as

$$\begin{aligned}
||\mathbf{e}||_0 &\leq \left|\left| c_1\mathbf{v}_1 e^{imx_j}(-im + \lambda_1)T + \sum_{k=1}^{p+1} c_k \mathbf{v}_k e^{im(x_j-T)} - \sum_{k=2}^{p+1} c_k \mathbf{v}_k e^{imx_j-\lambda_k T}\right|\right|_0, \\
&\leq \left|\left| c_1\mathbf{v}_1 e^{imx_j}(-im + \lambda_1)T\right|\right|_0 + \left|\left| \sum_{k=1}^{p+1} c_k \mathbf{v}_k e^{im(x_j-T)}\right|\right|_0 + \left|\left| \sum_{k=2}^{p+1} c_k \mathbf{v}_k e^{imx_j-\lambda_k T}\right|\right|_0, \\
&\leq ||c_1\mathbf{v}_1 e^{imx_j}||_0 |(-im + \lambda_1)T| + \left|\left| \sum_{k=1}^{p+1} c_k \mathbf{v}_k\right|\right|_0 |e^{im(x_j-T)}| + \left|\left| \sum_{k=1}^{p+1} c_k \mathbf{v}_k\right|\right|_0 |e^{imx_j}||e^{-CT/(\Delta x)}|, \\
&= ||c_1\mathbf{v}_1 e^{imx_j}||_0 |(-im + \lambda_1)T| + ||\hat{\mathbf{u}}_m(0)||_0 |e^{im(x_j-T)}| + ||\hat{\mathbf{u}}_m(0)||_0 |e^{imx_j}||e^{-CT/(\Delta x)}|, \\
&= C_1 T(\Delta x)^{2p+1} + C_2(\Delta x)^{p+2} + C_3 e^{-CT/\Delta x}(\Delta x)^{p+2},
\end{aligned}$$

and thus

$$||\mathbf{e}||_0 \leq C_1 T(\Delta x)^{2p+1} + C_2(\Delta x)^{p+2} + C_3 e^{-CT/\Delta x}(\Delta x)^{p+2},$$

where the order on the first term comes from the dominating dissipation error of the physically-relevant eigenvalue found in (ii) and the order on the second and third terms come from the order on the initial projection, $||\hat{\mathbf{u}}_m(0)||_0$. I honestly expected it to be of $p+1$ order, but it seems that in Fourier space we get $p+2$ order (??). I will ask about this in office hours.