First steps towards a formalization of Forcing

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Zermelo-Fraenkel Set Theory

Set theory arose at the early twentieth century as one possible foundation for mathematics.

More formally, set theory is a first order theory (ZFC) with no constants and one relational symbol $\in$.

To say that ZFC is a possible foundation for mathematics means that any mathematical theorem, say T, can be deduced from ZFC:

$$ZFC \vdash T$$

Cantor's Question and Gödel's and Cohen's answers

Cantor posed the *Continuum Hypothesis* (*CH*):

> *Every uncountable subset of* $\mathbb{R}$ *is equipotent with* $\mathbb{R}$.

Gödel showed that *CH* is *relatively consistent* with *ZFC*.

Later, Cohen showed that also the negation of *CH* is relatively consistent with *ZFC*. For this he invented the technique of *forcing*.

Some preliminaries

To say that T is *relatively consistent* with ZFC can be understood in two ways:

1. Construct a model M' of $ZFC + T$ from a model M for ZFC.
2. Alternatively, deduce a contradiction in ZFC from a contradiction in $ZFC + T$.

Cohen's proofs construct a model $ZFC + \neg CH$ assuming a countable and transitive model (ctm).

- M is a *transitive* model of ZFC if $x \in M$ implies $x \subset M$.
- M is *countable* if it is equipotent with $\mathbb{N}$.

Forcing

- Given a ctm M and a *generic* set G *for* M, one constructs a new ctm $M[G]$.
- One defines a poset $(P, \leq)$, where $P \in M$. The existence of a generic filter G follows from Rasiowa-Sikorski.
- $M[G] = \{\mathrm{val}(G, x) \mid x \in M\}$
- Depending on the definition of P, $M[G]$ will satisfy CH or $\neg CH$.

Our goal: formalize forcing

Questions?

- Why forcing? Has not been formalized yet.
- Related work. Paulson formalized Gödel's proof.
- Which proof assistant? Because of the previous answer: Isabelle/ZF.

Axiomatization of ZF in Isabelle/FOL

- Sets are terms of type `i` and formulas have type `o`.
- Axioms of *ZFC* are postulated as axioms of Isabelle: *extension:* $A = B \longleftrightarrow A \subseteq B \wedge B \subseteq A$
- A theorem of *ZFC* is a *lemma* of Isabelle/ZF:

```
lemma equalityI : "[| A ⊆ B;  B ⊆ A |] ==> A = B"
by (rule extension [THEN iffD2], rule conjI)
```

- In order to speak of formulas and satisfaction (models), Paulson coded formulas (and much more) as sets.
- He also defined relativized (to a class) versions of the axioms

$upair\_ax(M) == \forall x[M].\ \forall y[M].\exists z[M].\ upair(M,x,y,z)$

First steps towards Forcings

Our strategy

1. Be as modular as possible, using *locales*.
2. Define interfaces to deal with models of *ZFC*.
3. Exploit the modularity of forcing to divide the work.

What have we done?

Existence of a generic filter

- Proved a principle of dependent choice.
- Using that principle, proved Rasiowa-Sikorski.
- If M is a ctm and $P \in M$ is a poset, then there exists a generic filter for M.

Definition of $M[G]$

- Defined *names* for $x \in M$ and val.
- Defined $M[G]$, assuming M ctm and G generic for M.
- Proved $M \subseteq M[G]$.

$M[G] \models ZFC$

Axiom by axiom

- One of the hardest part is transfering truth on M to truth on $M[G]$.
- We proved *upair_ax(##M)* implies *upair_ax(##M[G])*
- $M[G] \models$ *separation* (almost).

Future work: The next steps

Soon

- $x \in M$ implies $\check{x} \in M$.
- Define the poset we are interested in ($Add(\omega, \kappa)$).
- Prove that $M[G]$ satisfies union, foundation, infinity.

Not so soon

- Define the forcing relation ($\Vdash$).
- Prove the fundamental theorem of forcing.
- $M[G] \models ZFC$.
- If P is ccc, then cardinals are preserved in $M[G]$.
- Prove that for any generic G for $Add(\omega, \aleph_2)$, then $M[G] \models ZFC + \neg CH$.

Questions?

Thank you!