NOTES ON "APPROXIMATING THE CONVEX HULL VIA METRIC SPACE MAGNITUDE" BY FUNG, BUNCH, AND DICKINSON

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1. More direct proof of Lemma 1

Lemma 1 follows from the following slightly more general statement.

Suppose that $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ is a positive definite matrix, and that

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{1}$$

where I am abusing notation and letting 1 stand for vectors of different sizes whose entries are all 1. Then

$$1^T A^{-1} 1^T = 1^T x + 1^T y - y^T (M/A) y, \tag{2}$$

where $M/A = C - B^T A^{-1} B$ denotes the Schur complement of A in M.

Proof. By (1) we have $Ax + By = 1$ and $B^T x + Cy = 1$. We therefore have

$$y^T C y = y^T (1 - B^T x) = y^T 1 - (By)^T x = y^T 1 - (1 - Ax)^T x = y^T 1 - 1^T x + x^T A x$$

and

$$y^T B^T A^{-1} B y = (1^T - x^T A) A^{-1} (1 - Ax) = 1^T A^{-1} 1 - 2(1^T x) + x^T A x.$$

Subtracting these identities, we obtain

$$y^T (M/A) y = 1^T x + 1^T y - 1^T A^{-1} 1,$$

which is equivalent to (2). □

2. Corrected version of Proposition 2

In the proof of Proposition 2, the statement following "Next, we know that" is incorrect, since $A/A_{\overline{P}}$ is not diagonal in general. Thus formula (9) in the paper and the final formula in the proof are not correct as stated. They can be corrected using the observation that

$$\lambda_{\min} \|w[P]\|_2^2 \le w[P]^T A/A_{\overline{P}} w[P] \le \lambda_{\max} \|w[P]\|_2^2, \tag{3}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of $A/A_{\overline{P}}$, respectively.

Using (3), one can deduce as in the proof of Proposition 2 that

$$|X| - \lambda_{\min} \sum_{y \in Y} w_X(y)^2 \ge |X \setminus Y| \ge |X| - \lambda_{\max} \sum_{y \in Y} w_X(y)^2. \tag{4}$$

A statement more similar to the one in the paper can be deduced from (4) by observing that

$$N_Y \min\{w_X(y)^2 \mid y \in Y\} \le \sum_{y\in Y} w_X(y)^2 \le N_Y \max\{w_X(y)^2 \mid y \in Y\}$$

and that $\lambda_{\max} \le \operatorname{tr}(A/A_{\overline{P}}) \le |P|$. In particular, it follows that

$$|X| - N_Y \lambda_{\min} \min\{w_X(y)^2 \mid y \in Y\} \ge |X \setminus Y| \ge |X| - N_Y^2 \max\{w_X(y)^2 \mid y \in Y\}. \tag{5}$$

Unfortunately, I don't know of a simple effective lower bound on $\lambda_{\min}(A/A_{\overline{P}})$, but this doesn't appear to be very important for the purposes of the paper.

However, the discussion in section 3.3 becomes simpler if we proceed directly from (4) instead of from (5). In particular, this makes it easier to deal with the issue that x_{β_t} may vary with t.

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