QUESTIONS ABOUT MAGNITUDE OF FINITE METRIC SPACES

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To the best of my knowledge the following questions are open. Unless otherwise specified, (A, d) is a finite metric space, tA denotes the metric space (A, td) for $t > 0$, and Z_A denotes the similarity matrix of A.

- Suppose that tA has magnitude for every $t > 0$. Does it follow that A is of negative type?

  Remarks: If Z_{tA} is invertible for every $t > 0$, then we know that A is of negative type, but the hypothesis here is slightly weaker.

- Suppose that A is of negative type. Does it follow that $|tA|$ is nondecreasing in t? What if A embeds in L_1, ℓ_1^n, or ℓ_2^n?

  Remarks: For arbitrary A this is true for sufficiently large t. This is also true if tA is positively weighted for every t; the latter holds for subsets of $\mathbb{R}$ and for ultrametric spaces.

- Suppose that A is of negative type. Does it follow that $|A| \leq \#A$? What if A embeds in L_1, ℓ_1^n, or ℓ_2^n?

  Remarks: A positive answer to the previous question would imply a positive answer to this one.

- Suppose that A embeds in L_1. Does it follow that $\lim_{t\to 0^+} |tA| = 1$?

  Remarks: This is known to be true if A embeds in ℓ_1^n or in ℓ_2^n (and hence if A is finite and embeds in L_2). This is known to be false under only the assumption that A is of negative type: the six-point space A with $|tA| \xrightarrow{t\to 0^+} 6/5$ is of negative type.

  The hypothesis of embeddability in L_1 here is not particularly well-motivated; it's just the most obvious intermediate hypothesis to suggest. I don't know whether the six-point space mentioned above embeds in L_1.

- What (if any) convexity/concavity properties does the magnitude function of A possess, under whatever additional hypotheses?

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