Thesis Notes

Stephen Liu

September 4, 2019

TODO

1. Clean up writing so far.

   Outline:

   (a) Paths with two flat steps in them. Explain that we only have two big cases (well, three, but by symmetry two).

   (b) For each case give the additional primitives of M and asymmetric M.

   (c) mu simplification - identify which cases we now have, but don't need to write out their values explicitly.

   (d) Give proof that we only need to consider sigma.

   (e) Then actually simplify sigma and write down what that final expression gives us.

The problem

Let $d = 2m+1$ where m is a natural number. Denote by B_2^d the d-dimensional Euclidean ball and tB_2^d the d-dimensional ball with radius t for $t > 0$. In general we are interested in the asymptotic behavior of the magnitude function of the odd dimensional Euclidean ball as we shrink it down to a point. Meckes in [Mec19] showed that

$$\frac{d}{dt}\mathrm{Mag}(tB_2^d)\big|_{t=0} = \frac{1}{2}V_1(B_2^d)$$

where $V_1(B_2^d)$ is the first instrinsic volume of the Euclidean ball. We are interested computing the value of

$$\frac{d^2\mathrm{Mag}(tB_2^d)}{dt^2}\big|_{t=0}.$$

We will follow the same approach that Meckes used for the result above, with many of the techniques introduced by Willerton in [Wil17].

Explicit magnitude functions for low dimensions

Barcelo and Carbery in [BC16] explicitly calculated the magnitude functions of Euclidean balls in dimensions 3,5,7:

$$\mathrm{Mag}(tB_2^3) = \frac{t^3}{3!} + t^2 + 2t + 1$$
$$\mathrm{Mag}(tB_2^5) = \frac{t^6 + 18t^5 + 135t^4 + 525t^3 + 1080t^2 + 1080t + 360}{5!(t+3)}$$
$$\mathrm{Mag}(tB_2^7) = \frac{t^7}{7!}$$
$$+ \frac{\frac{1}{180}t^9 + \frac{2}{15}t^8 + \frac{3}{2}t^7 + \frac{31}{3}t^6 + \frac{189}{4}t^5 + 145t^4 + \frac{1165}{4}t^3 + 360t^2 + 240t + 60}{t^3 + 12t^2 + 48t + 60}$$

So taking second derivatives and evaluating at $t = 0$, we have

$$\frac{d\mathrm{Mag}^2(tB_2^3)}{dt^2}\Big|_{t=0} = 2$$
$$\frac{d\mathrm{Mag}^2(tB_2^5)}{dt^2}\Big|_{t=0} = \frac{38}{9} = 4.222\ldots$$
$$\frac{d\mathrm{Mag}^2(tB_2^7)}{dt^2}\Big|_{t=0} = \frac{162}{25} = 6.48$$

We will use these values as sanity checks to compare against throughout our calculations.

Schröder paths

Willerton in [Wil17] gave an expression for the magnitude function of odd dimensional balls in terms of collections of Schröder paths. We introduce them below:

Definition.

- A **Schröder path** is a finite directed path in the integer lattice in which each step $(x, y) \in \mathbb{Z}^2$ is either an **ascent** to $(x+1, y+1)$, a **descent** $(x-1, y-1)$ or a **flat step** $(x+2, y)$ (note the advance by *two* spaces in the horizontal direction).

- Fix $k \geq 0$. A **disjoint k-collection** is a family of Schröder paths from $(-i, i)$ to (i, i) for each $0 \leq i \leq k$ such that no node in $\mathbb{Z}^2$ is contained in two of the paths (the paths are disjoint).
- We denote by X_k the set of all disjoint k-collections and by X_k^j the set of disjoint k-collections with exactly j flat steps.

When thinking about what kinds of disjoint k-collections we can have in X_k^j for some fixed j, it is often useful to think about what a path needs to look like for increasing values of i. For example, consider the set X_k^0, that is, the set of disjoint k-collections with exactly 0 flat steps. At $i = 0$ we have the single dot at $(0,0)$ and at $i = 1$, since we are allowed no flat steps, the only possible path we can have is the path made up of one ascent followed immediately by one descent. Then for $i = 2$, the disjointness condition and the presence of the earlier path at height $i = 1$ ensures that the only possible path we can have is the path made up of two ascents followed by two descents. We continue this argument for successive values of i. We will call the path at height i consisting of i ascents followed by i descents a **V-path at height** i (because they look like upside down V's). So it turns out that X_k^0 consists only of one collection, denoted σ_{roof}^k, which is made up entirely of V-paths for each $0 \leq i \leq k$.

Let σ be a disjoint k-collection in X_k. For each path in σ we associate a weighting to each step τ in the path by the following:

$$\omega_j(\tau) = \begin{cases} 1 & \text{if } \tau \text{ is an ascent,} \\ t & \text{if } \tau \text{ is a flat step,} \\ y+1-j & \text{if } \tau \text{ is a descent from height } y \text{ to height } y-1. \end{cases}$$

For a collection $\sigma \in X_k$ the **(total) weight of** σ, denoted by $\omega_j(\sigma)$ is the product of all the weightings of each step of a path in σ, that is,

$$\omega_j(\sigma) = \prod_{\tau \in \sigma} \omega_j(\tau)$$

Note that if $\sigma \in X_k^\ell$, that is σ has exactly ℓ flat steps, then $\omega_j(\sigma)$ will take the form ct^ℓ where c is the product of all the weights on descents in σ. We will only use the weightings ω_0 and ω_2 for our purposes, that is, a descent starting at height y will be weighted by either $y+1$ and $y-1$ respectively.

Consider a V-path δ at height i, then we have

$$\omega_0(\delta) = \frac{(2i+1)!}{(i+1)!}$$
$$\omega_2(\delta) = \frac{(2i-1)!}{(i-1)!}$$

and since σ^k_{roof} consists only of V-paths, we have

$$\omega_0\left(\sigma^k_{\text{roof}}\right) = \prod_{i=0}^{k} \frac{(2i+1)!}{(i+1)!}$$
$$\omega_2\left(\sigma^k_{\text{roof}}\right) = \prod_{i=1}^{k} \frac{(2i-1)!}{(i-1)!}$$

We will call a path δ starting at height i consisting of $i-1$ ascents, followed by a flat step, followed by $i-1$ descents a **flat step path at height** i. Note that the weightings for this kind of path are given by

$$\omega_0(\delta) = \frac{(2p)!}{(p+1)!}$$
$$\omega_2(\delta) = \frac{(2p-1)!}{(p-1)!}$$

We are interested in these weightings on collections of Schröder paths because Willerton in [Wil17] showed the following:

Theorem 1 ([Wil17], Corollary 27)**.** *Let $d = 2m+1$ be odd. Then*

$$Mag\left(tB_2^d\right) = \frac{\sum\limits_{\sigma\in X_{m+1}} \omega_2(\sigma)}{d! \sum\limits_{\sigma\in X_{m-1}} \omega_0(\sigma)} = \frac{\sum\limits_{\sigma\in X_{m+1}} \prod\limits_{\tau\in\sigma} \omega_2(\tau)}{d! \sum\limits_{\sigma\in X_{m-1}} \prod\limits_{\tau\in\sigma} \omega_0(\tau)}$$

for all $t > 0$.

As mentioned before, $\omega_j(\sigma)$ are of the form ct^ℓ where ℓ is the number of flat steps in σ and c is a constant, so the numerator and the denominator in the expression above are both polynomials in t. We will denote the function in the numerator by $N(t)$ and the function in the denominator (without the extra $d!$) by $D(t)$. Put more succinctly, we have

$$\mathrm{Mag}\left(tB_2^d\right) = \frac{N(t)}{d!D(t)}.$$

Taking the second derivative

From now on, when writing down the value of a function evaluated at zero, for convenience we will omit the "(0)" part, that is, we write N for $N(0)$ and N' for $N'(0)$ and similarly for $D(0)$ and $D'(0)$. For higher derivatives we divide by the order of the derivative, that is, we denote $\frac{1}{2}N''(0)$ by N'' and $\frac{1}{2}D''(0)$ by D''. The point of this is that N'' and D'' are the values of the coefficients of the second order terms in $N(t)$ and $D(t)$ respectively. In Theorem 28 of [Wil17], Willerton showed the following identity

$$N = d!D$$

and Meckes in the proof of Theorem 4 of [Mec19] showed that

$$N'D - ND' = Vd!D^2$$

where V is shorthand for $\frac{1}{2}V_1\left(B_2^d\right)$. Now we evaluate

$$\frac{d^2}{dt^2}\mathrm{Mag}(tB_2^d)\big|_{t=0}$$

Applying the quotient rule and using the two identities above, we have

$$
\begin{aligned}
\frac{d^2}{dt^2}\mathrm{Mag}(tB_2^d)\Big|_{t=0} &= \frac{d}{dt}\left(\frac{d}{dt}\mathrm{Mag}(tB_2^d)\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\frac{d}{dt}\frac{N(t)}{d!D(t)}\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\frac{d!D(t)N'(t)-N(t)d!D'(t)}{d!^2D(t)^2}\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\frac{D(t)N'(t)-N(t)D'(t)}{d!D(t)^2}\right)\Big|_{t=0} \\
&= \frac{d!D(t)^2[D(t)N''(t)-N(t)D''(t)]-[D(t)N'(t)-N(t)D'(t)]d!2D(t)D'(t)}{d!^2D(t)^4}\Big|_{t=0} \\
&= \frac{D(t)[D(t)N''(t)-N(t)D''(t)]-[D(t)N'(t)-N(t)D'(t)]2D'(t)}{d!D(t)^3}\Big|_{t=0} \\
&= \frac{D(t)^2N''(t)-D(t)N(t)D''(t)-2D'(t)D(t)N'(t)+2D'(t)^2N(t)}{d!D(t)^3}\Big|_{t=0} \\
&= \frac{2D^2N''-2DD'N'-2DND''+2ND'^2}{d!D^3} \\
&= \frac{2D^2N''-2DND''-2D'(DN'-D'N)}{d!D^3} \\
&= \frac{2D^2N''-2DND''-2D'(Vd!D^2)}{d!D^3} \\
&= 2\left[\frac{DN''-ND''}{d!D^2}\right]-2V\left[\frac{D'}{D}\right] \\
&= 2\left[\frac{DN''-d!DD''}{d!D^2}\right]-2V\left[\frac{D'}{D}\right] \\
&= 2\left[\frac{N''-d!D''}{d!D}\right]-2V\left[\frac{D'}{D}\right] \\
&= 2\left[\frac{N''-d!D''}{N}\right]-2V\left[\frac{D'}{D}\right]
\end{aligned}
$$

Where the factors of 2 in front of the terms containing a second derivative are because of the factor of $\frac{1}{2}$ that we introduced earlier in our notation for N'' and D''. To continue simplifying this expression, we have to give explicit expressions for the terms involved:

$$N = \sum_{\sigma \in X^0_{m+1}} \prod_{\tau \in \sigma} \omega_2(\tau) = \prod_{\tau \in \sigma^{m+1}_{\text{roof}}} \omega_2(\tau), \quad D = \sum_{\sigma \in X^0_{m-1}} \prod_{\tau \in \sigma} \omega_0(\tau) = \prod_{\tau \in \sigma^{m-1}_{roof}} \omega_0(\tau)$$

$$N' = t^{-1} \sum_{\sigma \in X^1_{m+1}} \prod_{\tau \in \sigma} \omega_2(\tau), \quad D' = t^{-1} \sum_{\sigma \in X^1_{m-1}} \prod_{\tau \in \sigma} \omega_0(\tau)$$

$$N'' = t^{-2} \sum_{\sigma \in X^2_{m+1}} \prod_{\tau \in \sigma} \omega_2(\tau), \quad D'' = t^{-2} \sum_{\sigma \in X^2_{m-1}} \prod_{\tau \in \sigma} \omega_0(\tau)$$

This boils down to a counting problem to do with disjoint collections of Schröder paths with exactly k flat steps for $k = 0, 1, 2$.

Simplifying the $\frac{D'}{D}$ Term

By above we have,

$$D = \prod_{\tau \in \sigma^{m-1}_{roof}} \omega_0(\tau) = \prod_{k=0}^{m-1} \frac{(2k+1)!}{(k+1)!}$$

and

$$D' = t^{-1} \sum_{\sigma \in X^1_{m-1}} \prod_{\tau \in \sigma} \omega_0(\tau)$$

so we want to count how many disjoint $(m-1)$-collections σ have exactly one flat step in them.

Let $\sigma \in X^1_{m-1}$. Let δ be the path in σ containing the single flat step at height, say, p. By the same reasoning as when discussing X^0_k earlier, the paths at height less than p must be V-paths. But then the disjointness condition ensures that δ at height p must have the flat step be centered from $(-1, p)$ to $(1, p)$, so δ is a flat step path as defined above with weighting given as earlier. For the path directly above δ, we can either have a V-path as before or, because of the extra space provided by the flat step just below we can have a path consisting of p ascents, one descent, one ascent and then p descents. Then for the next path above, the disjointness condition ensures that this path can only be either a path of a similar form or a V-path. We will call the path at height i consisting of $i-1$ ascents, one descent, one ascent and then i descents a **M-path at height** i. So in σ, after the flat step path at height p we will have some number of M-paths followed by some number of V-paths. Note that after we have switched to V-paths we cannot have any other paths above because of the disjointness condition. This allows us to characterize all the disjoint $(m-1)$-collections in X^1_{m-1}:

fix $1 \le p \le m-1$ and $0 \le q \le m-1-p$, then the disjoint $(m-1)$-collection $\sigma_{p,q}^{m-1}$ from the bottom up, is composed of $p-1$ V-paths, followed by a flat step path at height p, followed by q M-paths followed by V-paths up to height $m-1$. Then Meckes in [Mec19] observed that

$$X_{m-1}^{1} = \bigcup_{\substack{1\le p\le m-1 \\ 0\le q\le m-1-p}} \sigma_{p,q}^{m-1}$$

Let δ be a M-path at height k, then we have

$$\omega_0(\delta) = \frac{(2k)!(2k)}{(k+1)!}$$
$$\omega_2(\delta) = \frac{(2k-1)!(2k-1)}{(k-1)!}$$

where the extra factor on the numerator comes from the additional descent we have in δ. The ω_2 weighting will become relevant later when we evaluate N.

Since we can recognize X_{m-1}^{1} as a union of disjoint $(m-1)$-collections with a specific form, we can write down an explicit expression for D':

$$D' = \sum_{\substack{1\le p\le m-1 \\ 0\le q\le m-1-p}} \left(\prod_{k=0}^{p-1} \frac{(2k+1)!}{(k+1)!}\right)\left(\frac{(2p)!}{(p+1)!}\right)\left(\prod_{k=p+1}^{p+q} \frac{(2k)!(2k)}{(k+1)!}\right)\left(\prod_{k=p+q+1}^{m-1} \frac{(2k+1)!}{(k+1)!}\right)$$

We can try to simplify the quotient D'/D: For each summand depending on p, q in the quotient D'/D, we have

$$\frac{\prod_{k=0}^{p-1} \frac{1}{(k+1)!} \frac{1}{p!} \prod_{k=p+1}^{p+q} \frac{1}{(k+1)!} \prod_{k=p+q+1}^{m} \frac{1}{(k+1)!} \prod_{k=0}^{p-1} (2k+1)!(2p)! \prod_{k=p+1}^{p+q} (2k)!(2k) \prod_{k=p+q+1}^{m} (2k+1)!}{\prod_{k=0}^{m} \frac{1}{(k+1)!} \prod_{k=0}^{m} (2k+1)!}$$

We can cancel the product of all the $\frac{1}{(k+1)!}$'s since on the top we also have a product of $\frac{1}{(k+1)!}$'s from 0 up to m. This gives us

$$\frac{\prod_{k=0}^{p-1} (2k+1)!(2p)! \prod_{k=p+1}^{p+q} (2k)!(2k) \prod_{k=p+q+1}^{m} (2k+1)!}{\prod_{k=0}^{m} (2k+1)!}$$

We can further cancel all the $(2k+1)!$'s from $k = 0$ to $p-1$ and from $p+q+1$ to m:

$$\frac{(2p)!\prod_{k=p+1}^{p+q}(2k)!(2k)}{\prod_{k=p}^{p+q}(2k+1)!} = \frac{(2p)!\prod_{k=p+1}^{p+q}(2k)!(2k)}{(2p+1)!\prod_{k=p+1}^{p+q}(2k+1)!}$$
$$= \frac{1}{2p+1}\left(\prod_{k=p+1}^{p+q}\frac{2k}{2k+1}\right)$$

So we have
$$\frac{D'}{D} = \sum_{\substack{1\le p\le m-1\\ 0\le q\le m-1-p}} \frac{1}{2p+1}\prod_{k=p+1}^{p+q}\left(1-\frac{1}{2k+1}\right)$$

The $N'' - d!D''$ Term

Now we want to look at the
$$\frac{N'' - d!D''}{N}$$
term in our second derivative. We first need to consider how many disjoint k-collections there are with exactly 2 flat steps. We can already rule out the case where a disjoint k-collection has 2 flat steps on the same path. This is because any path below the one with the flat steps needs to V-path and there's then not enough room for any path above it to start flat stepping at any coordinate other than the centre one. This then also tells us that for any disjoint k-collection with 2 flat steps, the 2 flat steps will be on separate paths and moreover the first flat step will be centred.

The second flat step can either be centred or offset by one to the right or the left. We also have some rules governing the paths that do not contain a flat step. As mentioned before, the only kind of path below the first one with a flat step must be upside down V's. They can't be M's since those will bump into the V's underneath it. The second is that M's can only follow the flat step path (for the same reason) and they can only go under the upside down V's, again for the same reason. Finally if we have the second flat step be off-centred, then directly above it we can have asymmetric M's followed by M's followed by upside down V's. We can't have asymmetric M's any later the M's or upside down V's will bump into them. So in general we have two cases that we need to consider:

(a) Two centred flat steps. Below the first flat step are upside down V's, above it and above the second flat step are some number of M's followed by some number of upside down V's

(b) The first flat step is centred, the second is offset to the right or left by one. Below the first flat step are some number of upside down V's, above the first flat step are some number of M's and **no** upside down V's. Above the second flat step are some number of asymmetric M's followed by some number of M's followed by some number of upside down V's.

If a disjoint k-collection is of type (a), we'll denote it $\sigma^k_{p_1,p_2,q_1,q_2}$ where p_1, p_2 indicate where the first and second flat steps are respectively, and q_1, q_2 indicate how many M's there are above the first and second flat steps respectively. If a disjoint k-collection is of type (b), we'll denote it $L^k_{p_1,p_2,q_1,q_2}$ or $R^k_{p_1,p_2,q_1,q_2}$ where p_1 indicates where the first flat step is, p_2 indicates where the off-centred second flat step is, q_1 indicates how many asymmetric M's above p_2 there are and q_2 indicates how many M's above the asymmetric M's there are. **Be careful to note that the q indexes for σ and L collections mean different things.** Note that for a given p_1, p_2, q_1, q_2, the product of all the weights in the left and right case are the same, that is, we have

$$\prod_{\tau \in L^k_{p_1,p_2,q_1,q_2}} \delta_2(\tau) = \prod_{\tau \in R^k_{p_1,p_2,q_1,q_2}} \delta_2(\tau)$$

since the right and left case are totally symmetric. The same is true if we use δ_0 as well. For brevity, we denote:

$$\delta_2\left(\sigma^k_{p_1,p_2,q_1,q_2}\right) := \prod_{\tau \in \sigma^k_{p_1,p_2,q_1,q_2}} \delta_2(\tau)$$

$$\delta_2\left(L^k_{p_1,p_2,q_1,q_2}\right) := \prod_{\tau \in L^k_{p_1,p_2,q_1,q_2}} \delta_2(\tau)$$

Let's write down what these are. For an upside down V path at height k, the product of the weights (given by δ_2) on that path is given by

$$\frac{(2k-1)!}{(k-1)!}$$

An M at height k is given by

$$\frac{(2k-2)!}{(k-1)!}(2k-2)$$

an asymmetric M by

$$\frac{(2k-2)!}{(k-1)!}(2k-3)$$

and a flat step at p_i by

$$\frac{(2p_i-2)!}{(p_i-1)!}$$

So we have

$$\delta_2\left(\sigma^{m+1}_{p_1,p_2,q_1,q_2}\right) = \left(\prod_{k=0}^{p_1-1}\frac{(2k-1)!}{(k-1)!}\right)\left(\frac{(2p_1-2)!}{(p_1-1)!}\right)\left(\prod_{k=p_1+1}^{p_1+q_1}\frac{(2k-2)!}{(k-1)!}(2k-2)\right)\left(\prod_{k=p_1+q_1+1}^{p_2-1}\frac{(2k-1)!}{(k-1)!}\right)$$
$$\left(\frac{(2p_2-2)!}{(p_2-1)!}\right)\left(\prod_{k=p_2+1}^{p_2+q_2}\frac{(2k-2)!}{(k-1)!}(2k-2)\right)\left(\prod_{k=p_2+q_2+1}^{m+1}\frac{(2k-1)!}{(k-1)!}\right)$$

and we have

$$\delta_2\left(L^{m+1}_{p_1,p_2,q_1,q_2}\right) = \left(\prod_{k=0}^{p_1-1}\frac{(2k-1)!}{(k-1)!}\right)\left(\frac{(2p_1-2)!}{(p_1-1)!}\right)\left(\prod_{k=p_1+1}^{p_2-1}\frac{(2k-2)!}{(k-1)!}(2k-2)\right)\left(\frac{(2p_2-2)!}{(p_2-1)!}\right)$$
$$\left(\prod_{k=p_2+1}^{p_2+q_1}\frac{(2k-2)!}{(k-1)!}(2k-3)\right)\left(\prod_{k=p_2+q_1+1}^{p_2+q_1+q_2}\frac{(2k-2)!}{(k-1)!}(2k-2)\right)\left(\prod_{k=p_2+q_1+q_2+1}^{m+1}\frac{(2k-1)!}{(k-1)!}\right)$$

μ simplification

Summing all these up over p_1, p_2, q_1, q_2 would give us N'', and we would have to do the same with δ_0 to compute D'' and then try to simplify, which seems like a lot. We use the same μ trick as in [Mec19] to view the paths given in D'' as embedded in N'' and exclude them to simplify the subtraction instead. The set $X^2_{m+1} \setminus \mu\left(X^2_{m-1}\right)$ excludes the disjoint $m+1$-collections that can be viewed as embedded disjoint $m-1$-collections. So we are only counting disjoint $(m+1)$-collections which either have

1. The first flat step at $p_1 = 1$.
2. The first flat step is at $p_1 \geq 2$. The second flat step is at $p_2 = m$ and either an M or an asymmetric M above p_2.
3. The first flat step is at $p_1 \geq 2$. The second flat step is at $p_2 = m + 1$.
4. The two flat steps between 2 and $m - 1$ but with all M's above p_2 (as in the σ cases) or only asymmetric M's and M's above p_2 (as in the L cases).

So we have that the value of $N'' - d!D''$ is given by $\sigma + 2L$ where

$$\begin{aligned}\sigma = &\sum_{\substack{p_1=1\\p_1+1\le p_2\le m+1\\0\le q_1\le p_2-p_1-1\\0\le q_2\le m+1-p_2}} \delta_2\left(\sigma^{m+1}_{p_1,p_2,q_1,q_2}\right) + \sum_{\substack{2\le p_1\le m-1\\p_2=m\\0\le q_1\le p_2-p_1-1\\q_2=m+1-p_2}} \delta_2\left(\sigma^{m+1}_{p_1,p_2,q_1,q_2}\right)\\ &+ \sum_{\substack{2\le p_1\le m\\p_2=m+1\\0\le q_1\le p_2-p_1-1\\q_2=m+1-p_2}} \delta_2\left(\sigma^{m+1}_{p_1,p_2,q_1,q_2}\right) + \sum_{\substack{2\le p_1\le m\\p_1+1\le p_2\le m-1\\0\le q_1\le p_2-p_1-1\\q_2=m+1-p_2}} \delta_2\left(\sigma^{m+1}_{p_1,p_2,q_1,q_2}\right)\end{aligned}$$

which simplifying the indices becomes

$$\begin{aligned}\sigma = &\sum_{\substack{p_1=1\\2\le p_2\le m+1\\0\le q_1\le p_2-2\\0\le q_2\le m+1-p_2}} \delta_2\left(\sigma^{m+1}_{p_1,p_2,q_1,q_2}\right) + \sum_{\substack{2\le p_1\le m-1\\p_2=m\\0\le q_1\le m-p_1-1\\q_2=1}} \delta_2\left(\sigma^{m+1}_{p_1,p_2,q_1,q_2}\right)\\ &+ \sum_{\substack{2\le p_1\le m\\p_2=m+1\\0\le q_1\le m-p_1\\q_2=0}} \delta_2\left(\sigma^{m+1}_{p_1,p_2,q_1,q_2}\right) + \sum_{\substack{2\le p_1\le m\\p_1+1\le p_2\le m-1\\0\le q_1\le p_2-p_1-1\\q_2=m+1-p_2}} \delta_2\left(\sigma^{m+1}_{p_1,p_2,q_1,q_2}\right)\end{aligned}$$

and

$$\begin{aligned}L = &\sum_{\substack{p_1=1\\p_1+1\le p_1\le m+1\\0\le q_1\le m+1-p_2\\0\le q_2\le m+1-p_2-q_1}} \delta_2\left(L^{m+1}_{p_1,p_2,q_1,q_2}\right) + \sum_{\substack{2\le p_1\le m-1\\p_2=m\\0\le q_1\le 1\\q_2=1-q_1}} \delta_2\left(L^{m+1}_{p_1,p_2,q_1,q_2}\right)\\ &+ \sum_{\substack{2\le p_1\le m\\p_2=m+1\\q_1=0\\q_2=0}} \delta_2\left(L^{m+1}_{p_1,p_2,q_1,q_2}\right) + \sum_{\substack{2\le p_1\le m\\p_1+1\le p_2\le m-1\\0\le q_1\le m+1-p_2\\q_2=m+1-p_2-q_1}} \delta_2\left(L^{m+1}_{p_1,p_2,q_1,q_2}\right)\end{aligned}$$

which simplifying indices becomes

$$
\begin{aligned}
L = & \sum_{\substack{p_1=1 \\ 2\le p_1\le m+1 \\ 0\le q_1\le m+1-p_2 \\ 0\le q_2\le m+1-p_2-q_1}} \delta_2\left(L^{m+1}_{p_1,p_2,q_1,q_2}\right) + \sum_{\substack{2\le p_1\le m-1 \\ p_2=m \\ 0\le q_1\le 1 \\ q_2=1-q_1}} \delta_2\left(L^{m+1}_{p_1,p_2,q_1,q_2}\right) \\
& + \sum_{\substack{2\le p_1\le m \\ p_2=m+1 \\ q_1=0 \\ q_2=0}} \delta_2\left(L^{m+1}_{p_1,p_2,q_1,q_2}\right) + \sum_{\substack{2\le p_1\le m \\ p_1+1\le p_2\le m-1 \\ 0\le q_1\le m+1-p_2 \\ q_2=m+1-p_2-q_1}} \delta_2\left(L^{m+1}_{p_1,p_2,q_1,q_2}\right)
\end{aligned}
$$

Putting this in Matlab, we compute the compute the second order term evaluated at 0 of the magnitude function for odd-dimensional Euclidean balls (see Figure 1).

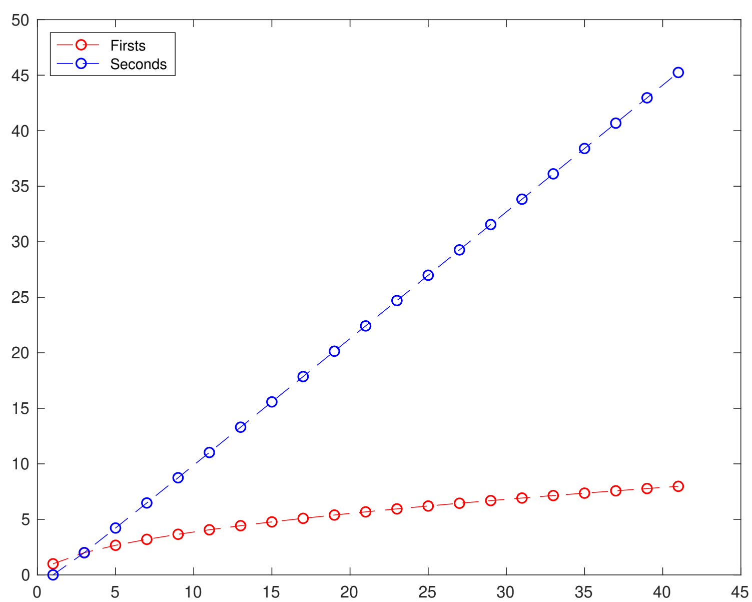

Figure 1: The first and second order terms of the magnitude of odd-dimensional Euclidean balls.

As we can see, the second order term looks to be linear in the dimension (as the first order term turns out to be proportional to the square root of the dimension). In fact, the Convex Magnitude Conjecture (which was shown to be false in [BC16] with the explicit computation of the magnitude function of the 5-dimensional Euclidean ball) suggests that the second order term would be just m. Let's see what we get when we actually simplify our expression above.

$\sigma = L$

We show that in fact the values of σ and L above are equal. By abuse of notation, denote σ to be the set of disjoint $(m+1)$-collections that we are considering when computing the number σ above and similarly for L. We will show that there is a bijection of sets $f : \sigma \to L$ that preserves the product of the weights.

Let δ be a disjoint $(m+1)$-collection in σ. In general, δ will have its first flat step at p_1 and second flat step at p_2. In between the two flat steps there will be q_1 M's followed by $p_2 - p_1 - q_1$ number of upside down V's. We'll call the height at which this first upside down V path appears to be v. Then we define $f(\delta)$ to be a disjoint $(m+1)$-collection in L where we take δ and replace the upside down V path at v with a off-centred flat step path and all the paths from $v+1$ up to p_2 are replaced with asymmetric M's. If δ had no upside down V's in between the first two flat steps, then just replace the second flat step with an off-centred flat step. Clearly $f(\delta)$ is a unique disjoint $(m+1)$-collection in L and this defines a well-defined function of sets from σ to L. We can see that f preserves the product of the weights on δ: each upside down V path starting at a height of say, h had product of weights $\frac{(2h-1)!}{(h-1)!}$ and this path was replaced by an asymmetric M path with product of weights $\frac{(2h-2)!}{(h-1)!}$ but on the asymmetric M path of height $h+1$ we have an extra factor of $(2(h+1)-3) = (2h+2-3) = (2h-1)$ so in total we also have product of weights $\frac{(2h-1)!}{(h-1)!}$. Note that this also applies to the flat step we introduced at height v: it had product of weights $\frac{(2v-2)!}{(v-1)!}$ but the asymmetric M just above it gives an extra factor of $(2v-1)$ so we're fine. Finally the flat step at p_2 in δ had product of weights $\frac{(2p_2-2)!}{(p_2-1)}$ which is the same as the product of the weights in the asymmetric M we introduced at p_2. So we see that f preserves weighting.

Now we define a function $g : L \to \sigma$. Let δ instead be a disjoint $(m+1)$-collection in L. The collection δ has a second off-centred flat step at height p_2 with q_1 number of asymmetric M's above it. Then we define $g(\delta)$ to be the disjoint $(m+1)$-collection where we replace the second flat step at p_2 and all the asymmetric M's above it with upside down V's except for the last one, which we turn into a centred flat step (ie. at height $p_2 + q_1$. The paths above $p_2 + q_1$ will be either upside down V's or M's and so this gives us a disjoint $(m+1)$-collection in σ. Clearly, g, f are inverse to each other, giving us a bijection $\sigma \to L$. Since f preserves products of weights, this also gives us that $\sigma = L$ as values.

σ Term

So by above, we have

$$\frac{N'' - d!D''}{N} = \frac{3\sigma}{N}$$

The denominator N is given by

$$N = \sum_{\sigma \in X^0_{m+1}} \prod_{\tau \in \sigma} \delta_2(\tau) = \prod_{\tau \in X^{m+1}_{\text{roof}}} \delta_2(\tau) = \prod_{k=0}^{m+1} \frac{(2k-1)!}{(k-1)!}$$

Earlier we had split σ into four smaller sums based on which case they fell into. Consider a disjoint $(m+1)$-collection in the first case, that is, we fix $p_1 = 1$, $2 \le p_2 \le m+1$, $0 \le q_1 \le p_2 - 2$, $0 \le q_2 \le m+1-p_2$. Then we have

$$\begin{aligned}
\frac{\sigma'}{N} &= \frac{\left(\prod\limits_{k=2}^{q_1+2}(2k-2)!(2k-2)\right)\left(\prod\limits_{k=q_1+2}^{p_2-1}(2k-1)!\right)(2p_2-2)!\left(\prod\limits_{k=p_2+1}^{p_2+q_2}(2k-2)!(2k-2)\right)\left(\prod\limits_{k=p_2+q_2+1}^{m+1}(2k-1)!\right)}{\left(\prod\limits_{k=0}^{m+1}(2k-1)!\right)} \\
&= \frac{\left(\prod\limits_{k=2}^{q_1+2}(2k-2)!(2k-2)\right)(2p_2-2)!\left(\prod\limits_{k=p_2+1}^{p_2+q_2}(2k-2)!(2k-2)\right)}{\left(\prod\limits_{k=2}^{q_1+2}(2k-1)!\right)\left(\prod\limits_{k=p_2}^{p_2+q_2}(2k-1)!\right)} \\
&= \frac{1}{2p_2-1}\left(\prod_{k=2}^{q_1+2}\frac{(2k-2)}{(2k-1)}\right)\left(\prod_{k=p_2+1}^{p_2+q_2}\frac{(2k-2)}{(2k-1)}\right)
\end{aligned}$$

So for the first case, we have

$$\frac{\sigma_1}{N} = \sum_{\substack{2 \le p_2 \le m+1 \\ 0 \le q_1 \le p_2 - 2 \\ 0 \le q_2 \le m+1-p_2}} \frac{1}{2p_2-1}\left(\prod_{k=2}^{q_1+2}\frac{(2k-2)}{(2k-1)}\right)\left(\prod_{k=p_2+1}^{p_2+q_2}\frac{(2k-2)}{(2k-1)}\right)$$

We similarly find expressions for the other three smaller sums.

Consider a disjoint $(m+1)$-collection in the second case, that is, we fix $2 \le p_1 \le m-1$,$p_2 = m$,$0 \le q_1 \le m-p_1-1$, $q_2 = 1$. Then we have

$$\frac{\sigma'}{N} = \frac{1}{2p_1-1}\left(\prod_{k=p_1+1}^{p_1+q_1}\frac{(2k-2)}{(2k-1)}\right)\left(\frac{2m}{(2m-1)(2m+1)}\right)$$

and for the whole second case, we have

$$\frac{\sigma_2}{N} = \sum_{\substack{2\le p_1\le m-1\\ 0\le q_1\le m-p_1-1}}\frac{1}{2p_1-1}\left(\prod_{k=p_1+1}^{p_1+q_1}\frac{(2k-2)}{(2k-1)}\right)\left(\frac{2m}{(2m-1)(2m+1)}\right)$$

Consider a disjoint $(m+1)$-collection in the third case. Then we have

$$\frac{\sigma'}{N} = \frac{1}{2p_1-1}\left(\prod_{k=p_1+1}^{p_1+q_1}\frac{(2k-2)}{(2k-1)}\right)\left(\frac{1}{2m+1}\right)$$

and for the whole third case, we have

$$\frac{\sigma_3}{N} = \sum_{\substack{2\le p_1\le m\\ 0\le q_1\le m-p_1}}\frac{1}{2p_1-1}\left(\prod_{k=p_1+1}^{p_1+q_1}\frac{(2k-2)}{(2k-1)}\right)\left(\frac{1}{2m+1}\right)$$

Consider a disjoint $(m+1)$-collection in the fourth case. Then we have

$$\frac{\sigma'}{N} = \frac{1}{2p_1-1}\left(\prod_{k=p_1+1}^{p_1+q_1}\frac{(2k-2)}{(2k-1)}\right)\frac{1}{2p_2-1}\left(\prod_{k=p_2+1}^{m+1}\frac{(2k-2)}{(2k-1)}\right)$$

and for the whole fourth case, we have

$$\frac{\sigma_4}{N} = \sum_{\substack{2\le p_1\le m\\ p_1+1\le p_2\le m-1\\ 0\le q_1\le p_2-p_1-1}}\frac{1}{2p_1-1}\left(\prod_{k=p_1+1}^{p_1+q_1}\frac{(2k-2)}{(2k-1)}\right)\frac{1}{2p_2-1}\left(\prod_{k=p_2+1}^{m+1}\frac{(2k-2)}{(2k-1)}\right)$$

Putting It All Together

So all together we have

$$\frac{d^2}{dt^2}\mathrm{Mag}(tB_2^d)\Big|_{t=0} =$$

$$6\sum_{\substack{2\le p_2\le m+1\\0\le q_1\le p_2-2\\0\le q_2\le m+1-p_2}}\frac{1}{2p_2-1}\left(\prod_{k=2}^{q_1+2}\frac{(2k-2)}{(2k-1)}\right)\left(\prod_{k=p_2+1}^{p_2+q_2}\frac{(2k-2)}{(2k-1)}\right)+$$

$$6\sum_{\substack{2\le p_1\le m-1\\0\le q_1\le m-p_1-1}}\frac{1}{2p_1-1}\left(\prod_{k=p_1+1}^{p_1+q_1}\frac{(2k-2)}{(2k-1)}\right)\left(\frac{2m}{(2m-1)(2m+1)}\right)+$$

$$6\sum_{\substack{2\le p_1\le m\\0\le q_1\le m-p_1}}\frac{1}{2p_1-1}\left(\prod_{k=p_1+1}^{p_1+q_1}\frac{(2k-2)}{(2k-1)}\right)\left(\frac{1}{2m+1}\right)+$$

$$6\sum_{\substack{2\le p_1\le m\\p_1+1\le p_2\le m-1\\0\le q_1\le p_2-p_1-1}}\frac{1}{2p_1-1}\left(\prod_{k=p_1+1}^{p_1+q_1}\frac{(2k-2)}{(2k-1)}\right)\frac{1}{2p_2-1}\left(\prod_{k=p_2+1}^{m+1}\frac{(2k-2)}{(2k-1)}\right)-$$

$$V_1\sum_{\substack{1\le p\le m-1\\0\le q\le m-1-p}}\frac{1}{2p+1}\prod_{k=p+1}^{p+q}\left(\frac{2k}{2k+1}\right)$$

Things to look at.

1. Try to combine e+v and lemma 6.38 approach.
2. Try to apply negative type assumption to eigenvalue approach.
3. See what we can say about eigenvalues of matrices of negative type.

Bounding the magnitude of a space of negative type by its cardinality.

Eigenvalues

Note that for a finite positive definite metric space

$$\#A = |I_A| = \sup_{v\neq 0} \frac{\left(\sum v(a)\right)^2}{v^\star I_A v} = \sup_{v\neq 0} \frac{\left(\sum v(a)\right)^2}{\|v\|^2}$$

We want to see if

$$|tA| = \sup_{v\neq 0} \frac{\left(\sum v(a)\right)^2}{v^\star Z_{tA} v} \leq |I_A|$$

or equivalently if

$$v^\star Z_{tA} v \geq \|v\|^2$$

Now since Z_{tA} is positive definite, we can factorize $Z_{tA} = B^\star B$ where B is Hermitian. So we have

$$v^\star Z_{tA} v = v^\star B^\star B v = \langle Bv, Bv\rangle = \|Bv\|^2$$

so equivalently we have

$$\|Bv\|^2 \geq \|v\|^2$$

Now by the Spectral decomposition, we have

$$\|Bv\|^2 = \|U\Lambda U^\star v\|^2 = \|\Lambda U^\star v\|^2 = \sum_{a\in tA} |\lambda_a y_a|^2 \quad \text{where } y = U^\star v$$

So the question is whether

$$\sum_{a\in tA} |\lambda_a y_a|^2 \geq \sum_{a\in tA} |v_a|^2$$

So we want to understand the eigenvalues of the similarity matrix Z_{tA}. Some theorems and tools to look at are

1. Gersgorin Disc Theorem
2. Courant-Fischer Theorem
3. Work by Barcello-Carbery
4. Understand the matrix exponential more

e + v decomposition

Another approach that Professor Meckes suggested was in looking at the supremum expression for PD metric spaces to decompose things in terms of vectors $e + v$:

$$|A| = \sup_{x \neq 0} \frac{\left(\sum x(a)\right)^2}{x^T Z x}$$

Now let $\#A = n$ and let e be the all 1's vector. Then we can decompose $x = e + v$ where $e^T v = 0$. Then we have

$$\sum x(a) = e^T x = e^T (e + v) = e^T e + e^T v = e^T e = n$$

and so we have

$$\begin{aligned} |A| &= \sup_{x \neq 0} \frac{\left(\sum x(a)\right)^2}{x^T Z x} \\ &= \sup_{\substack{v \in \mathbb{R}^A \\ e^T v = 0}} \frac{n^2}{e^T Z e + 2 e^T Z v + v^T Z v} \qquad \text{Symmetricity of } Z \\ &= \frac{n^2}{e^T Z e + \inf\limits_{e^T v = 0} \left(2 e^T Z v + v^T Z v\right)} \\ &= \frac{n^2}{e^T Z e + \inf\limits_{e^T v = 0} \left(-\frac{(e^T Z v)^2}{v^T Z v}\right)} \\ &= \frac{n^2}{e^T Z e - \sup\limits_{e^T v = 0} \left(\frac{(e^T Z v)^2}{v^T Z v}\right)} \end{aligned}$$

So $|A| \leq n$ is equivalent to the condition that

$$n \leq e^T Z e - \sup_{e^T v = 0} \frac{(e^T Z v)^2}{v^T Z v}$$

which is equivalent to

$$\sup_{e^T v=0} \frac{(e^T Z v)^2}{v^T Z v} \leq e^T Z e - n = \sum_{a,b} e^{-d(a,b)} - n = \sum_{a\neq b} e^{-da,b)}$$

which is equivalent to

$$(e^T Z v)^2 \leq (e^T Z e - n)(v^T Z v) \quad \text{whenever } e^T v = 0$$

This looks kind of like a sort of Cauchy-Schwarz formula on the subspace $e^T v = 0$.

Lemma 6.38 Approach

Proposition (Lemma 6.38 of Zhan: Characterizing Negative Type)**.** *Let A be a real symmetric matrix of order n. Then the following are equivalent:*

1. *A is conditionally positive semidefinite:*

$$x^T A x \geq 0 \quad \textit{for all } x \in \Omega := \{x \in \mathbb{R}^n : \sum x_i = 0\}$$

2. *There is a $y \in \mathbb{R}^n$ such that the matrix $(a_{ij} - y_i - y_j)$ is positive semidefinite.*

3. *The Hadamard exponential $e^{\circ A}$ is infinitely divisible, that is, $e^{\circ \alpha A}$ is positive semidefinite for all $\alpha > 0$ (what we mean by negative type).*

In our terms we have the matrix $\tilde{Z} = [-td(a,b)]_{a,b\in A}$. Then the following are equivalent:

1. $\tilde{Z}$ is conditionally positive semidefinite

$$x^T \tilde{Z} x \geq 0 \quad \text{for all } x \text{ such that } \sum x_i = 0$$

2. there is a $y \in \mathbb{R}^n$ such that the matrix $(\tilde{Z}_{ij} - y_i - y_j) = (-td(i,j) - y_i - y_j)$ is positive semidefinite

3. $e^{\circ \tilde{Z}}$ is positive semidefinite for all $t > 0$ (i.e. A is of negative type).

For $x \in \mathbb{R}^n$, define

$$\tilde{x} = x - n^{-1}(e^T x)e = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} - \begin{bmatrix} \frac{\sum x_i}{n} \\ \vdots \\ \frac{\sum x_i}{n} \end{bmatrix}$$

Need to see that $\tilde{x} \in \Omega$ (Not sure here yet):

$$\sum \tilde{x}_i = \sum \left(x_i - \frac{\sum x_i}{n} \right) = \sum \left(\frac{n x_i - \sum x_i}{n} \right)$$

The idea is to look at this vector y and see if it gives us anything. Let e be the all 1's vector. Define

$$y = n^{-1} \tilde{Z} e - \frac{1}{2} n^{-2} (e^T \tilde{Z} e) e$$

Then since $\tilde{Z}$ is conditionally positive semidefinite, we have

$$0 \le \tilde{x}^T \tilde{Z} \tilde{x} = x^T (\tilde{Z} - y e^T - e y^T) x = x^T \tilde{Z} x - x^T y e^T x - x^T e y^T x$$

for all $x \in \mathbb{R}^n$.

If we substitute the expression we have for y into the inequality above, we have

$$\begin{aligned}
0 &\le x^T \tilde{Z} x - x^T y e^T x - x^T e y^T x \\
&= x^T \tilde{Z} x - x^T \left(n^{-1} \tilde{Z} e - \frac{1}{2} n^{-2} (e^T \tilde{Z} e) e \right) e^T x - x^T e \left(n^{-1} \tilde{Z} e - \frac{1}{2} n^{-2} (e^T \tilde{Z} e) e \right)^T x
\end{aligned}$$

Note that $e^T x = x^T e = \sum x_i$. We'll denote this sum $X := \sum x_i$. Also note that $e^T \tilde{Z} e = \sum_{a,b \in A} -td(a,b)$. We'll denote this sum $K := \sum_{a,b \in A} -td(a,b)$. Then the inequality above turns into

$$\begin{aligned}
0 &\le x^T \tilde{Z} x - x^T y e^T x - x^T e y^T x \\
&= x^T \tilde{Z} x - x^T \left(n^{-1} \tilde{Z} e - \frac{1}{2} n^{-2} (e^T \tilde{Z} e) e \right) e^T x - x^T e \left(n^{-1} \tilde{Z} e - \frac{1}{2} n^{-2} (e^T \tilde{Z} e) e \right)^T x \\
&= x^T \tilde{Z} x - x^T \left(n^{-1} \tilde{Z} e - \frac{1}{2} n^{-2} K e \right) X - X \left(n^{-1} \tilde{Z} e - \frac{1}{2} n^{-2} K e \right)^T x
\end{aligned}$$

Note also that the vector

$$n^{-1} \tilde{Z} e = n^{-1} \begin{bmatrix} -td(1,1) & \cdots & -td(1,n) \\ \vdots & \ddots & \vdots \\ -td(n,1) & \cdots & -td(n,n) \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = n^{-1} \begin{bmatrix} \sum_{a \in A} -td(a,1) \\ \vdots \\ \sum_{a \in A} -td(a,n) \end{bmatrix}$$

which means something.

References

[BC16] Juan Antonio Barcelo and Anthony Carbery. On the magnitudes of compact sets in euclidean spaces. *Amer. J. Math*, 2016.

[Lei11] Tom Leinster. The magnitude of metric spaces. *Documenta Mathematica*, 2011.

[Mec19] Mark W. Meckes. On the magnitude and intrinsic volumes of a convex body in euclidean space. *Mathematika*, 2019.

[Wil17] Simon Willerton. The magnitude of odd balls via hankel determinants of reverse bessel polynomials. *arXiv*, 2017.