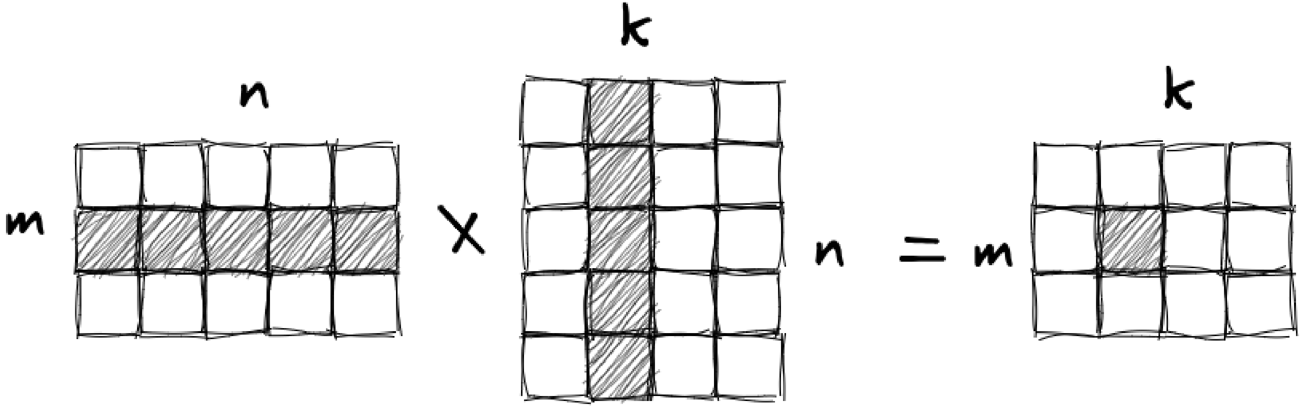
Linear algebra
Vectors and Matrices



Week 4

Middlesex University Dubai. Winter ‘24, CST4050
Instructor: Dr. Ivan Reznikov

Plan

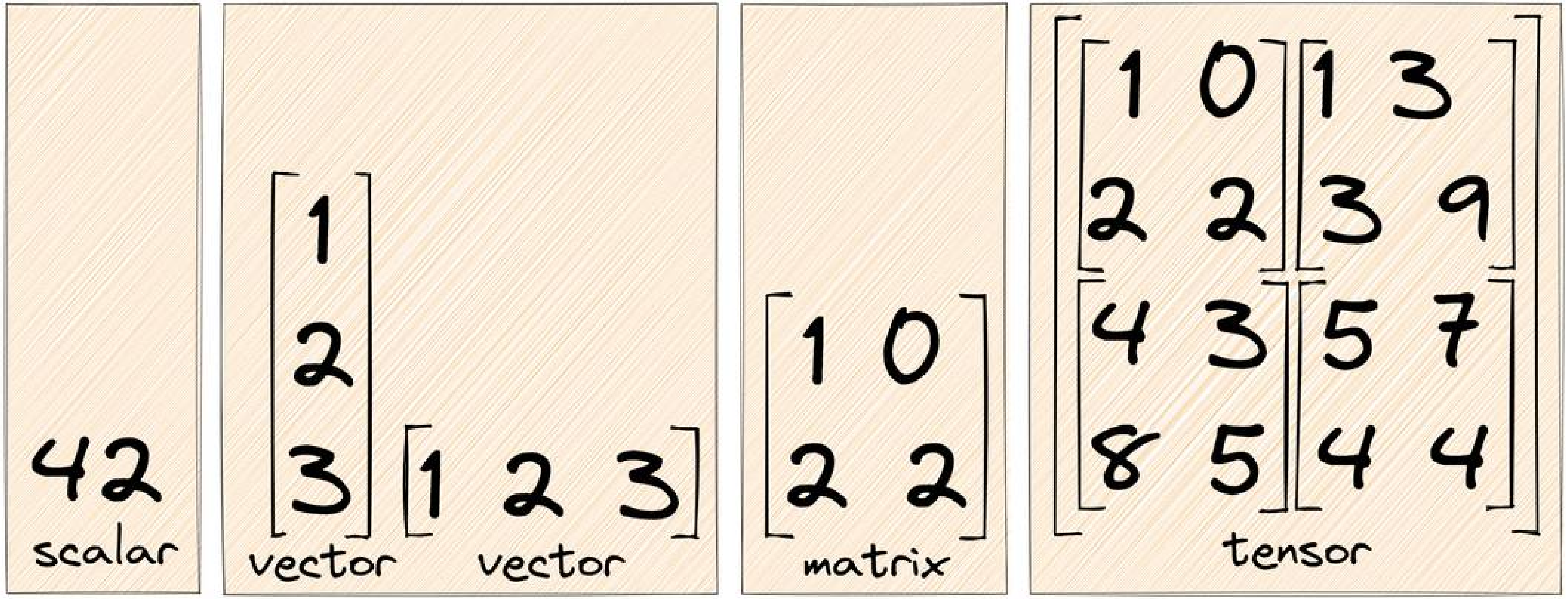
- Vectors
- Matrices
- Determinant

Why need linear algebra?

All data science, especially machine learning, depends on linear algebra and statistics. Knowing math is crucial to understanding how neural networks and other ML algorithms work.

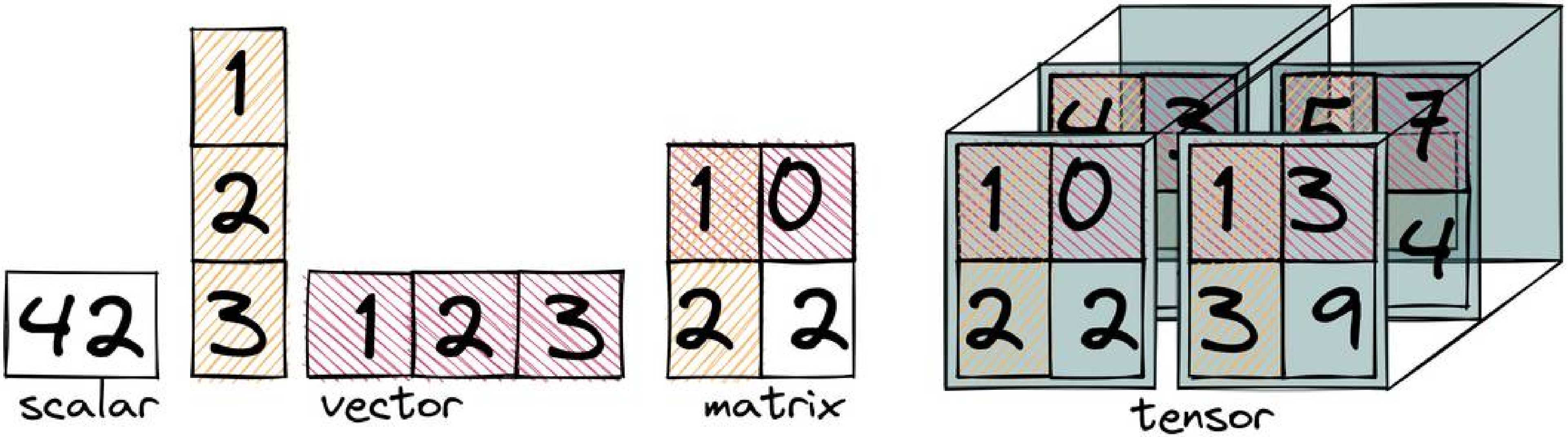
$$
\begin{aligned}
\frac{1}{F(p;\xi)} = {} & 1+\frac{\alpha}{\pi p^2}\int_0^\infty dk\,\frac{k^2F(k;\xi)}{k^2+\mathcal{M}^2(k;\xi)}\left\{a(k,p)\left[-\xi\left(1-\frac{k^2+p^2}{2kp}\ln\left|\frac{k+p}{k-p}\right|\right)\right]\right. \\
& +b(k,p)\left[2(k^2+p^2)\left(1-\frac{k^2+p^2}{2kp}\ln\left|\frac{k+p}{k-p}\right|\right)-\xi\left(k^2+p^2-\frac{(k^2-p^2)^2}{2kp}\ln\left|\frac{k+p}{k-p}\right|\right)\right] \\
& \left.-c(k,p)\left[2\left(1-\frac{k^2+p^2}{2kp}\ln\left|\frac{k+p}{k-p}\right|\right)-\xi\left(1-\frac{k^2-p^2}{2kp}\ln\left|\frac{k+p}{k-p}\right|\right)\right]\right\},\quad \frac{\mathcal{M}(p;\xi)}{F(p;\xi)} \\
= {} & \frac{\alpha}{\pi}\int_0^\infty dk\,\frac{k^2F(k;\xi)}{k^2+\mathcal{M}^2(k;\xi)}\left\{a(k,p)\,\mathcal{M}(k;\xi)\left[(2+\xi)\frac{1}{kp}\ln\left|\frac{k+p}{k-p}\right|\right]\right. \\
& +b(k,p)\,\mathcal{M}(k;\xi)\left[\frac{2(k^2+p^2)}{kp}\ln\left|\frac{k+p}{k-p}\right|+2(\xi-2)\right] \\
& \left.+c(k,p)\left[\frac{(2+\xi)k^2+(2-\xi)p^2}{2kp}\ln\left|\frac{k+p}{k-p}\right|+(\xi-2)\right]\right\},
\end{aligned}
\tag{15}
$$

Scalar, vector, matrix, tensor



Scalar, vector, matrix, tensor

0D 1D 2D ND



Name the value type

72

$\begin{bmatrix} 1 & 1 & 2 & 3 & 5 & 8 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} [1 \;\; 0] & [2 \;\; 3] \\ [2 \;\; 2] & [3 \;\; 9] \\ [4 \;\; 3] & [5 \;\; 7] \end{bmatrix}$

$\begin{bmatrix} [1] & [3] \\ [2] & [9] \\ [4] & [0] \end{bmatrix}$

$\begin{bmatrix} 0 \end{bmatrix}$

$\begin{bmatrix} [0] \end{bmatrix}$

2D array

Most common form of data organization for machine learning is a 2D array.
Rows represent observations (records, items, data points).
Columns represent attributes (features, variables).
Natural to think of each sample as a vector of attributes, and whole array as a matrix

	age	anaemia	creatinine_phosphokinase	diabetes	ejection_fraction	high_blood_pressure	platelets	
0	75.0	0	582	0	20	1	265000.00	
1	55.0	0	7861	0	38	0	263358.03	
2	65.0	0	146	0	20	0	162000.00	vector K
3	50.0	1	111	0	20	0	210000.00	
4	65.0	1	160	1	20	0	327000.00	vector N

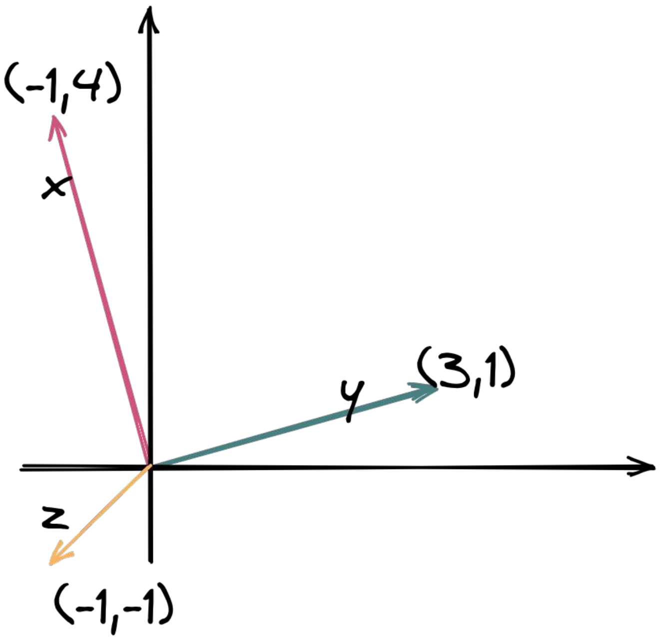
Vectors

Vector is a n-tuple of values (usually real numbers).
A vector can be seen as a point in space or a directed line segment with a magnitude (length) and direction.
Scalar values are defined only by magnitude.

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$X^T = (-1, 4)$$

"transpose"



Vector sum

Let $\boldsymbol{a} = (a_1,\ldots,a_n)^T$ and $\boldsymbol{b} = (b_1,\ldots,b_n)^T$ be two vectors.

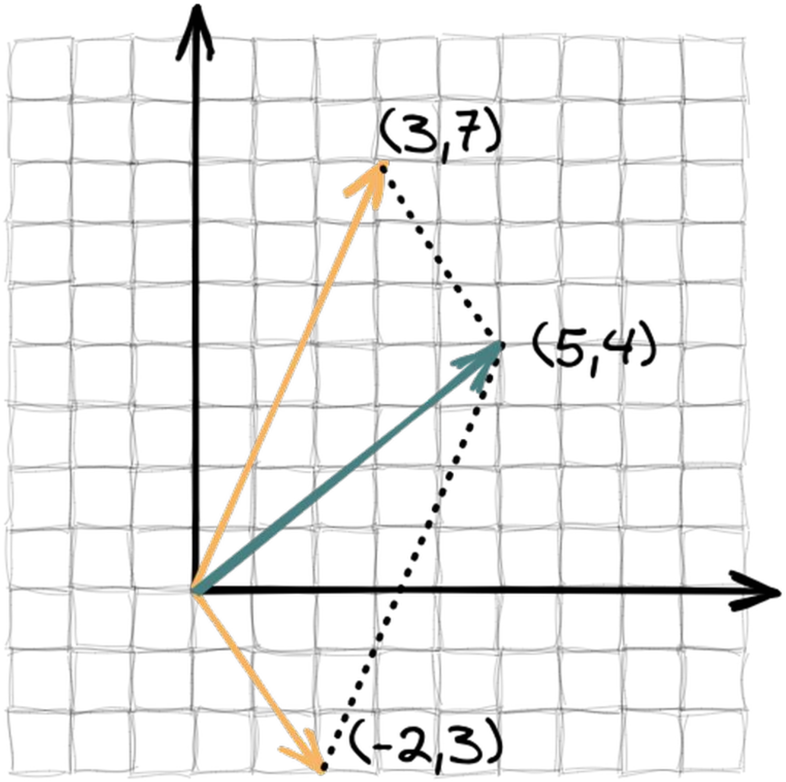
Let vector z = $\boldsymbol{a} + \boldsymbol{b}$

$\boldsymbol{z} = (a_1 + b_1,\ldots,a_n + b_n)^T$

Examples:

$\boldsymbol{a} = (3,7)^T$; $\boldsymbol{b} = (2,-3)^T$; => $\boldsymbol{z} = (5,4)^T$

$\boldsymbol{a} = (3,2,1)^T$; $\boldsymbol{b} = (1,2,0)^T$; => $\boldsymbol{z} = (4,4,1)^T$



Vector multiplication

Let k be scalar, $\boldsymbol{a} = (a_1,\ldots,a_n)^T$

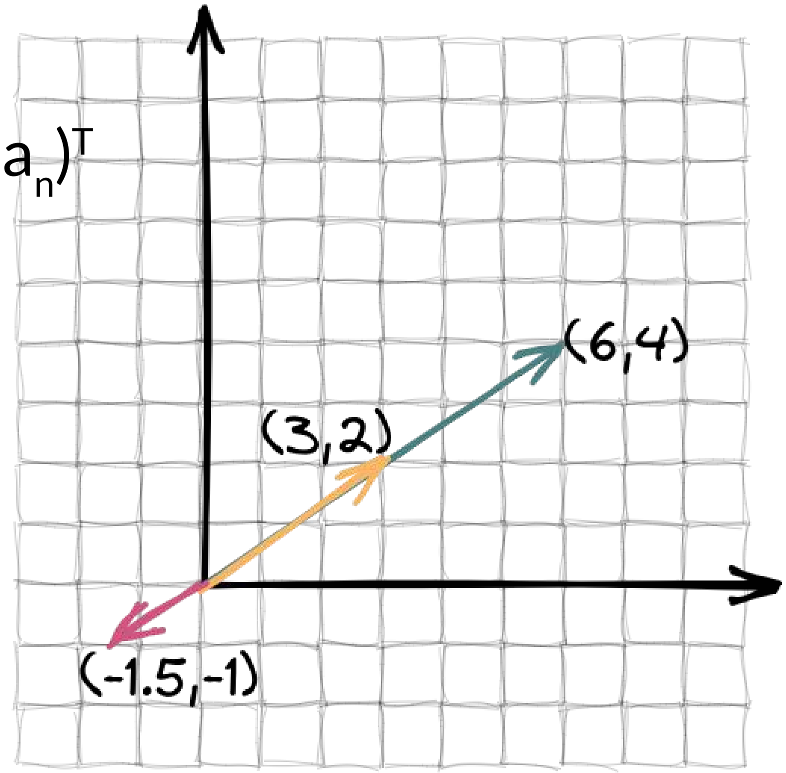
Let vector $z = k \times \boldsymbol{a}$, => $\boldsymbol{z} = (k \times a_1,\ldots,k \times a_n)^T$

<u>Example</u>:

$\boldsymbol{a} = (3,2)^T$, $k = 2$ => $\boldsymbol{z} = (6,4)^T$

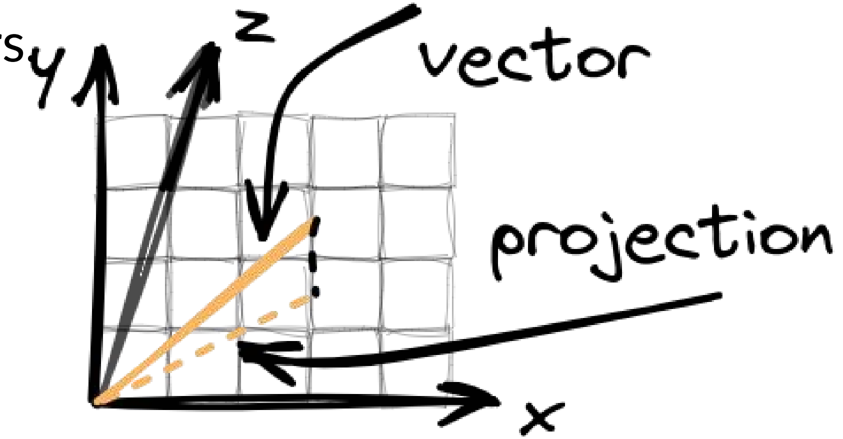
$\boldsymbol{a} = (3,2)^T$, $k = -0.5$ => $\boldsymbol{z} = (-1.5,-1)^T$

$\boldsymbol{a} = (3,7)^T$, $k = 3$ => $\boldsymbol{z} = (9,21)^T$



Vector arithmetic

Let $\boldsymbol{a} = (a_1,\ldots,a_n)^T$ and $\boldsymbol{b} = (b_1,\ldots,b_n)^T$ be two vectors.
Let vector z = $\boldsymbol{a} \times \boldsymbol{b}$



$\boldsymbol{z} = \sum(a_1 \times b_1,\ldots,a_n \times b_n)^T$

<u>Examples</u>:
$\boldsymbol{a} = (3,7)^T$; $\boldsymbol{b} = (2,-3)^T$; => $\boldsymbol{z} = \sum(6,-21)^T = -15$
$\boldsymbol{a} = (3,2,1)^T$; $\boldsymbol{b} = (1,2,0)^T$; => $\boldsymbol{z} = \sum(3,4,0)^T = 7$

Projection: projection of y onto x is a perpendicular line from y onto x (meet at point) and the projection vector is the vector to that point.
$\text{Projection}_{\boldsymbol{a}}(\boldsymbol{b}) = ((\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{a})/(\boldsymbol{a} \times \boldsymbol{a})$

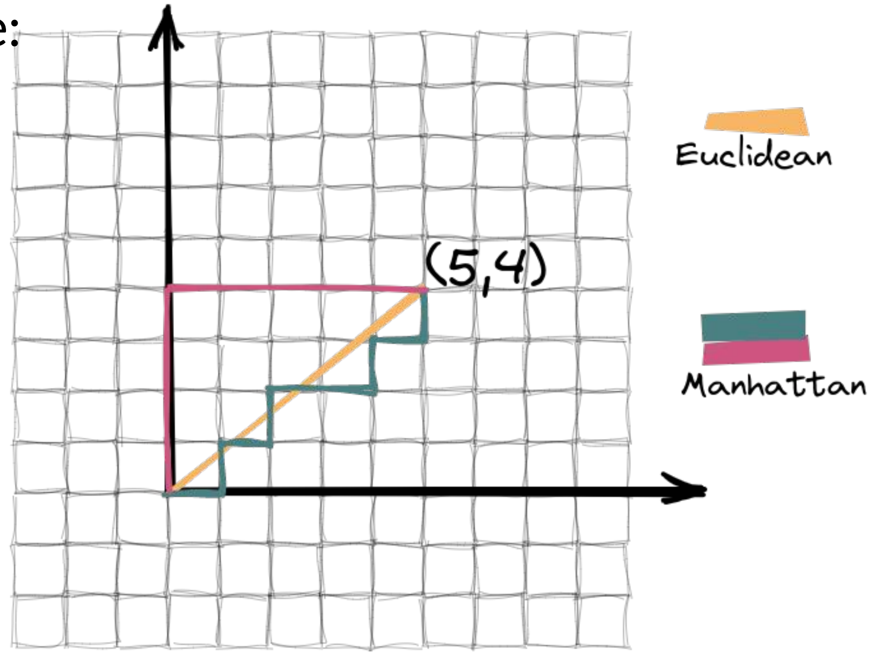
<u>Example</u>:
$\text{Projection}_{(4,3,0)}((25,0,5)) = (25\times4 + 3\times0 + 0\times5)\times(4,3,0)/(4^{2+}3^{2+}0^2) =$
$100\times(4,3,0)/25 = \ = 4\times(4,3,0) = (16,12,0)$

Norm of a vector

Norm of a vector may be understood as distance:

$d(x,y) = ||y - x||$

There are more than one type of distance:
- Eucledian
- Manhattan
- Minkowski
etc



Matrix arithmetic

Definition: an m x n two-dimensional array of values (usually real numbers).
- m rows
- n columns

Matrix referenced by two-element subscript
- 1st element in subscript is row
- 2nd element in subscript is column

=> $a_{\text{row, columns}}$

Vector can be considered as 1D matrix => as vectors
- transpose
- addition
- multiplication by scalar

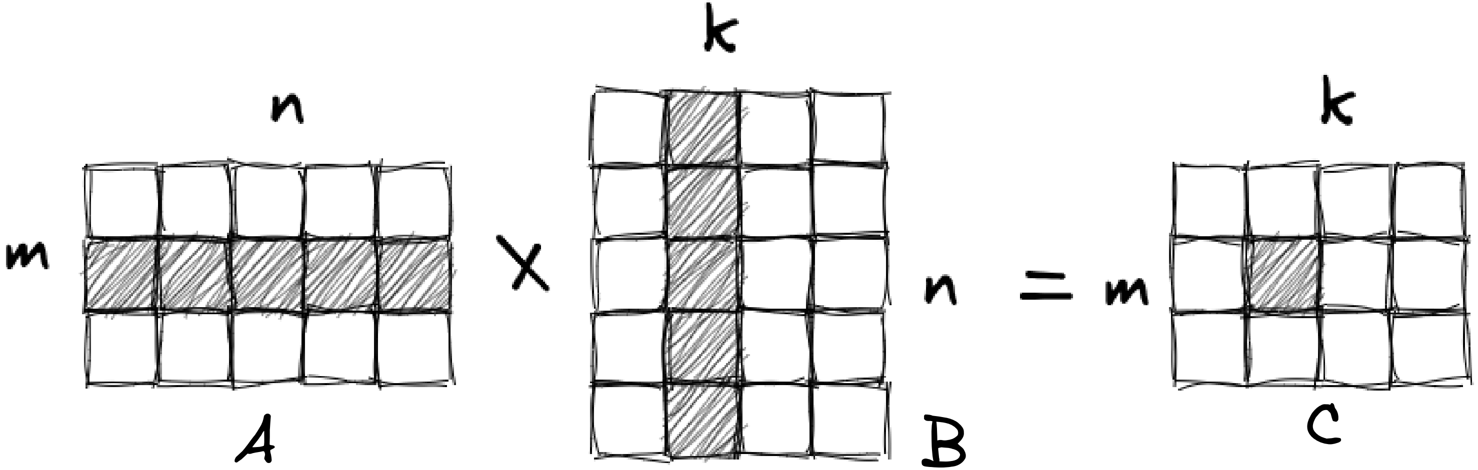
$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,N} \\ \vdots & \ddots & & \vdots \\ a_{N,1} & \cdots & & a_{N,N} \end{pmatrix}$$

Matrix multiplication

Matrix-matrix multiplication is defined as the rows by columns multiplication:

$$c_{i,j} = a_{i,1} \times b_{1,j} + \ldots + a_{i,n} \times b_{n,j} = \sum a_{i,z} \times b_{z,j}$$

A vector-matrix multiplication just a special case of a matrix-matrix multiplication.

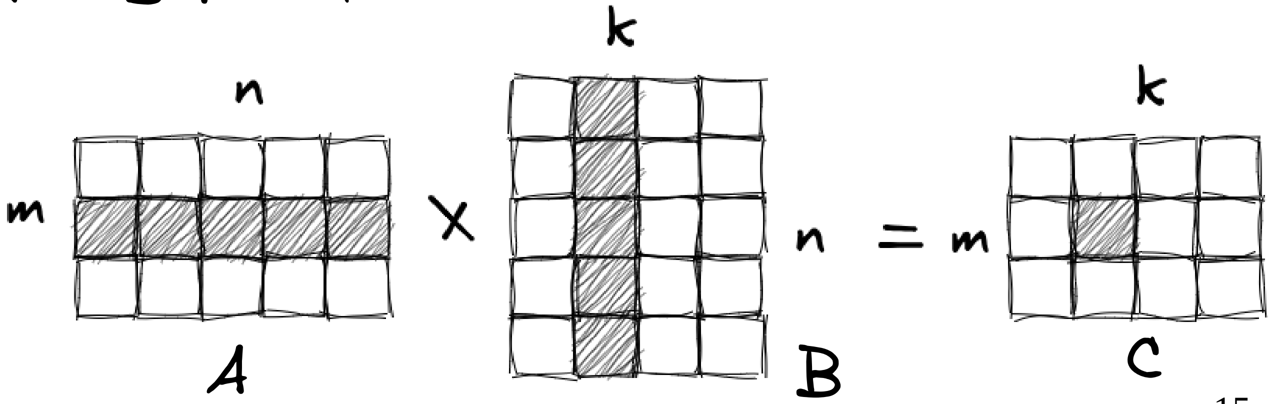


Matrix multiplication

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \quad C = A \times B = \begin{array}{c|ccc} & c_{i,1} & c_{i,2} & c_{i,3} \\ \hline c_1 & 4 & 9 & 12 \\ c_2 & 9 & 13 & 17 \end{array}$$

$$c_{1,3} = 2\times2 + 1\times2 + 2\times3 = 12$$

$$c_{2,1} = 1\times0 + 3\times2 + 3\times1 = 9$$



Matrix multiplication

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} =$$

Step 1

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} =$$

Matrix multiplication

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2*0+ \\ 1*2+ \\ 2*1=4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2*1+ \\ & 1*1+ \\ & 2*3=9 \\ 1*0+ & \\ 3*2+ & \\ 3*1=9 & \end{bmatrix}$$

Step 2, 3

Matrix multiplication

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 9 & \\ 9 & & \end{bmatrix}$$

2*2+
1*2+
2*3 =12

1*1 +
3*1 +
3*3 =13

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 9 & 12 \\ 9 & 13 & 17 \end{bmatrix}$$

Step 4, -1

Matrix multiplication

Matrix multiplication is associative:
$A \times (B \times C) = (A \times B) \times C$

Matrix multiplication is not commutative:
$A \times B \neq B \times A$

Matrix transposition rule:
$(A \times B)^T = B^T \times A^T$

Linear transformation

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \quad x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad C = Ax = \begin{bmatrix} 2 \times a + 1 \times b + 2 \times c \\ 1 \times a + 3 \times b + 3 \times c \end{bmatrix}$$

$$\boxed{\mathbb{R}^3 \to \mathbb{R}^2}$$

T: $\mathbb{R}^n \to \mathbb{R}^m$
T(x) = Ax
Function T is a linear transformation, in fact for each vector **x**,**y** and scalar *c*:

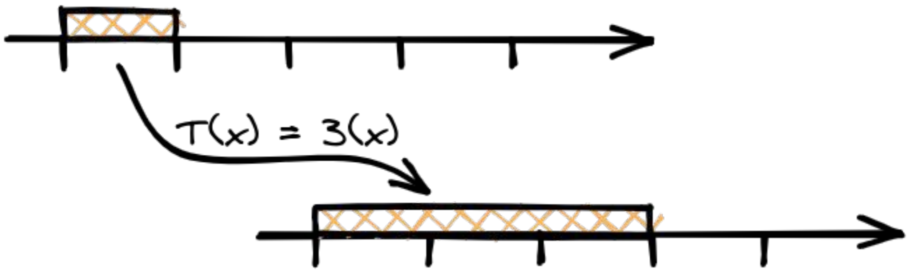
A(**x**+**y**) = A(**x**) + A(**y**)
A(*c***x**) = *c*A(**x**)

Square matrices are important as $\mathbb{R}^n \to \mathbb{R}^n$

1D linear transformation

A one-dimensional linear transformation is a function T(**x**) = *a*(**x**) for some scalar *a*.
To view the one-dimensional case in the same way we view higher dimensional linear transformations, we can view a as a 1×1 matrix.

Example: one-dimensional linear transformation is the function T(**x**) = *3*(**x**) .
A visualization of this function by its graph:

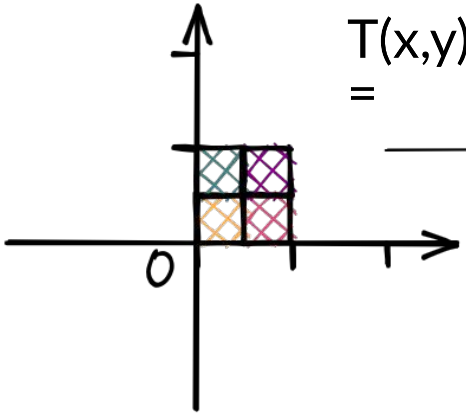


2D linear transformation

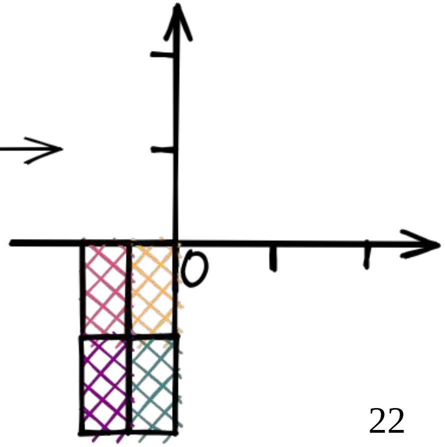
A two-dimensional linear transformation is a function T: ℝ2 → ℝ2 of the form:

$$T(x,y) = (ax+by, cx+dy) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We can write this more succinctly as T(***x***) ¥ A***x***,
where ***x*** = $[x, y]^T$ and A is the 2×2 matrix.

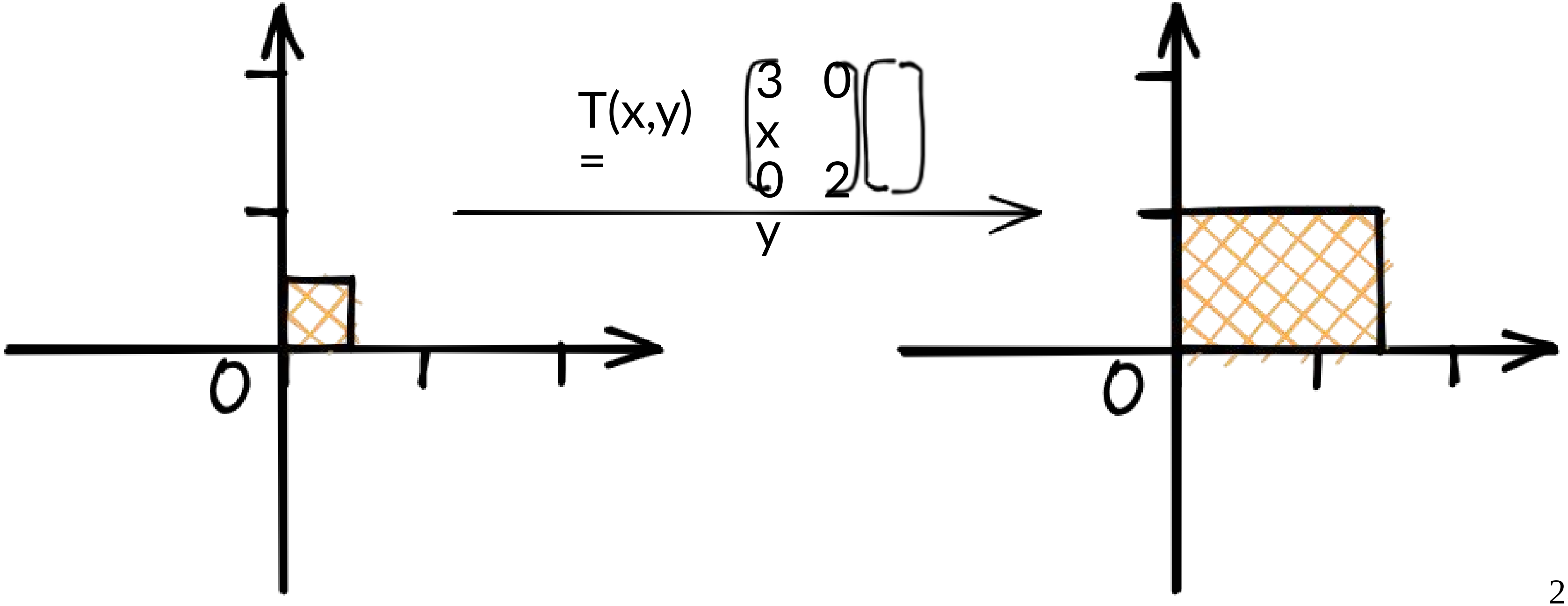


$$T(x,y) = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Determinant

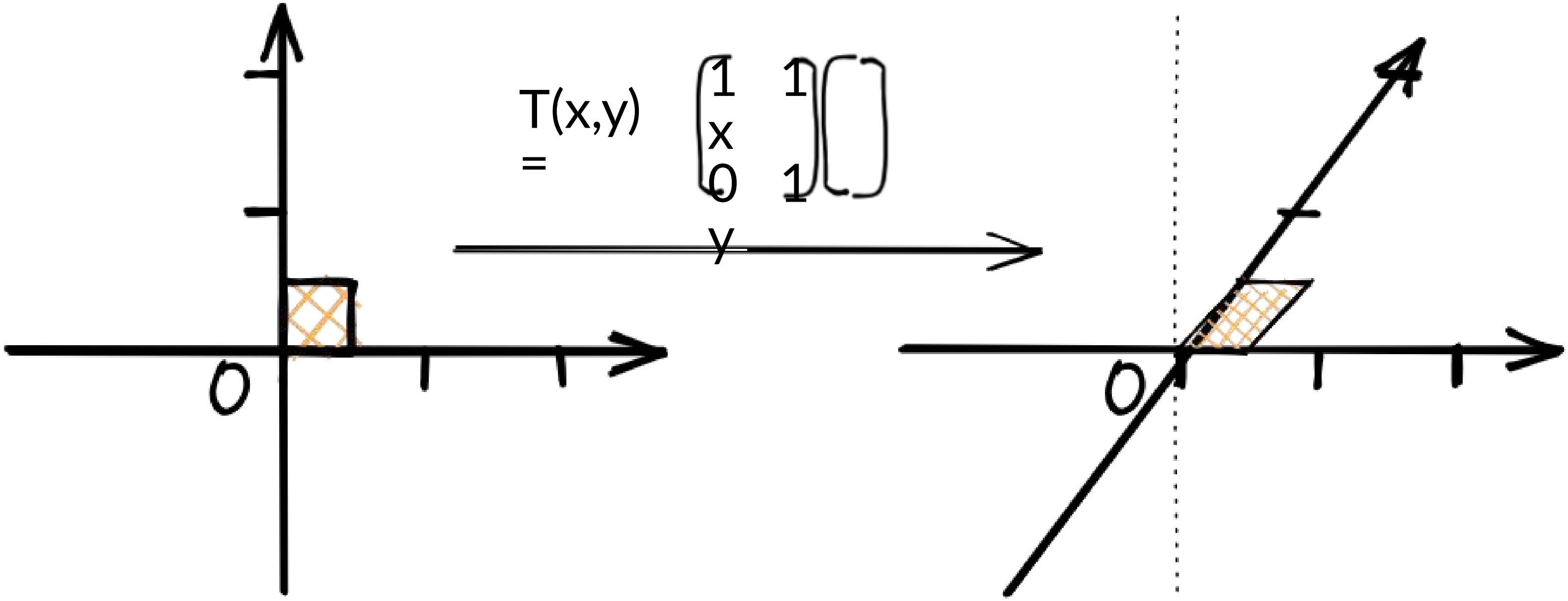
During linear transformations, we perform stretching and squishing some of the dimensions. It would be valuable to determine how our item's area has changed. If we stretch x 3 times and y 2 times, we'll increase the area 6 times.



$$T(x,y) = \begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Determinant

If we don't change the x and y values, no matter how we tilt our item, it's area won't change.



$T(x,y) = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

Determinant

The scaling factor, by which the linear transformation changes items area is called determinant.

$$T(x,y) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad T(x,y) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A) = 6 \qquad \det(A) = 1$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\times\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\times\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\times\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Determinant

In general, in any dimension n, the determinant is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the linear transformation described by the matrix.
The determinant of a matrix **A** is denoted *det*(**A**).

A determinant of a square matrix A can positive, negative or zero.
- Positive determinants denote transformations having a positive area, volume or hyper-volume.
- Negative determinants denote transformations having a negative area, volume or hyper-volume.
- Zero determinants denote transformations having no area, no volume or no hyper-volume

Rank

The maximum number of matrix linearly independent columns (or rows) of a matrix is called the rank of a matrix. The rank of a matrix cannot exceed the number of its rows or columns.

The rank is how many of the rows are "unique", a.k.a not made of other rows. (Same for columns)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\rho(A) = 1$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 1 & 4 & 5 \end{bmatrix} \quad \rho(A) = 2$$

Matrix inversion

The inverse of a number a is such that $a \times a^{-1} = 1$
For example the inverse of 10 is 0.1, as 10×0.1 = 1
The inverse of 5 is 0.2, and the inverse of 0.01 is 100.
The inverse of a matrix **A** is that matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Sometimes there is no inverse at all. In this case we say that the matrix A is not invertible.
A square matrix that is not invertible is called singular.
A square matrix is singular if and only if its determinant is 0.