2.1 Vector Orientation, Coordinates and Products

We now make the jump from two to three dimensions. We generalize every notion we saw in the previous notebook, and consider several new ones. More precisely, we explore the inner, outer, and cross products of vectors. This latter product leads naturally to the matrix exponential and the construction of rotation matrices about arbitrary axes. We put these matrices to use in the study of robot arms with six degrees of freedom. We close with a discussion of the projective plane and construct transformations that correct for the persepctive distortion introduced by eye or camera.

We begin with the illustration of a vector $a = (a_x, a_y, a_z)$ in space with respect to a given right-handed orthogonal coordinate system (e_x, e_y, e_z). **Orthogonal** means that the vectors are mutually perpendicular, while **right-handed** means that if, with your right hand, you align your index finger with e_x and your middle finger with e_y then your thumb will align with e_z.

We introduce the **spherical coordinates**

$$\begin{aligned} \theta = &\ \text{polar angle } = \angle(e_z, a), && 0 \le \theta < \pi \\ \phi = &\ \text{azimuthal angle } = \angle(e_x, a^\flat), && 0 \le \phi < 2\pi \\ r = &\ \text{magnitude } = \|a\| = \sqrt{a_x^2 + a_y^2 + a_z^2}, \end{aligned} \tag{1}$$

where $a^\flat = (a_x, a_y, 0)$ is the flattening of a into the (e_x, e_y) plane. In these coordinates, our a vector is expressed as

$$\begin{aligned} a_x &= r\sin(\theta)\cos(\phi) \\ a_y &= r\sin(\theta)\sin(\phi) \\ a_z &= r\cos(\theta) \end{aligned} \tag{2}$$

and illustrated in Figure 2.

In [1]:

```
1 # spherical coordinates↔
```

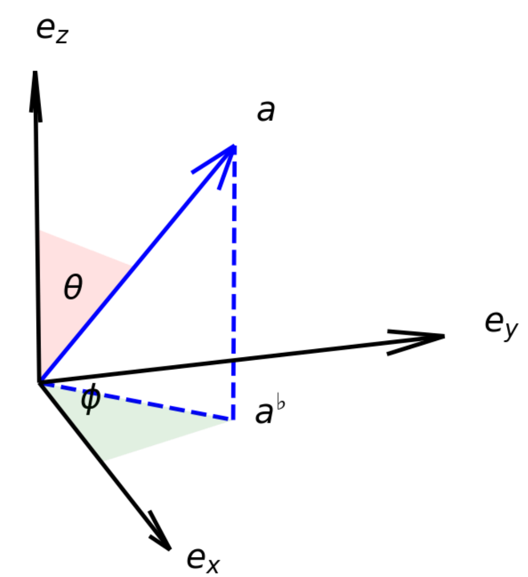

Figure 2 *Representation of $a \in \mathbb{R}^3$ in terms of its **polar angle**, θ, (pink span) **azimuthal angle**, ϕ, (green span) and **magnitude**, r. Please rotate, zoom, and repeat.*

To begin we note the **inner product** of two space vectors remains the sum of their element-wise products. That is

$$a^T b = \begin{bmatrix} a[0] & a[1] & a[2] \end{bmatrix} \begin{bmatrix} b[0] \\ b[1] \\ b[2] \end{bmatrix} = a[0]b[0] + a[1]b[1] + a[2]b[2].$$

In addition, the norm of a remains $\|a\| \equiv (a^T a)^{1/2}$ and the inner product still obeys

$$a^T b = \|a\| \|b\| \cos(\theta) \tag{3}$$

where $\theta \equiv \angle(a, b)$ is the angle from a and b.

The word **inner** here is used by contrast with the **outer product**

$$ab^T = \begin{bmatrix} a[0] \\ a[1] \\ a[2] \end{bmatrix} \begin{bmatrix} b[0] & b[1] & b[2] \end{bmatrix} = \begin{bmatrix} a[0]b[0] & a[0]b[1] & a[0]b[2] \\ a[1]b[0] & a[1]b[1] & a[1]b[2] \\ a[2]b[0] & a[2]b[1] & a[2]b[2] \end{bmatrix} \tag{4}$$

We will use both products throughout the remainder of our work. For our immediate needs we examine the outer product of a vector with itself. With $A = aa^T$ we find that

$$Ax = aa^T x = a(a^T x) = (a^T x)a$$

and hence Ax is simply a mutiple of a. We illustrate the vector a, and the plane perpendicular to it, in Figure 2 below.

Exercise 1 Argue that if $a \in \mathbb{R}^3$ is a unit vector, i.e., $\|a\| = 1$, then $A \equiv aa^T$ obeys $A^2 = A$. In this case we say that A is a **projection** of $\mathbb{R}^3$ onto the line through a.

Your solution here.

Exercise 2 Argue that if $a \in \mathbb{R}^3$ is a unit vector, i.e., $\|a\| = 1$, then $I - aa^T$ is a **projection** of $\mathbb{R}^3$ onto the plane perpendicular to a. Here I denotes the 3-by-3 identity matrix and so $(I - aa^T)x = x - (a^T x)a$. A vector is perpendicular to a when its inner product with a vanishes. Hence it remains only to take the inner product of a and $x - (a^T x)a$.

Your solution here.

We will exhibit two methods for displaying (finite sections) of infinite planes in python. The first, coded in the cell below, uses *meshgrid* to create a grid of planar points and then builds a plane as the height of a linear function of these grid points - where the linear function is built from the requirement that the plane be perpendicular to a given $a \in \mathbb{R}^3$.

In [2]:
```
# plotting lines and planes↔
```

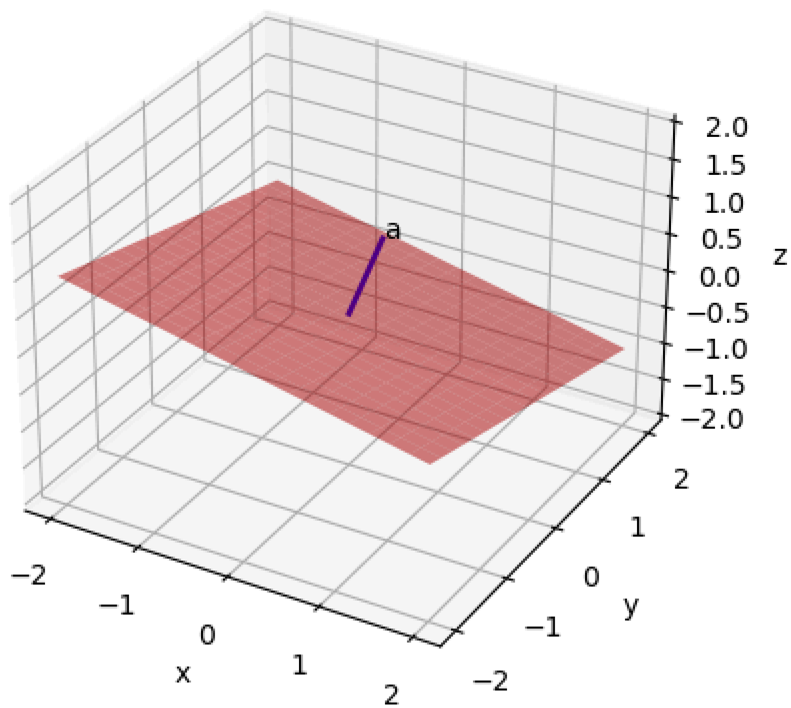

Figure 2 *A unit vector and its perpendicular plane.*

There is a third way of computing the product of two vectors in $\mathbb{R}^3$ that you may have encountered in your Physics course. In particular, the force on a current carrying wire in a magnetic field is proportional to the cross product of current and field. This fact is exploited in the design of the motor in Figure 3.

In [3]:
```
1 # animate a current carrying loop in a magnetic field↔
```

```
HBox(children=(Canvas(toolbar=Toolbar(toolitems=[('Home', 'Reset original view', 'home', 'home'), ('Back', 'Ba…
```

Figure 3 *A simple motor. The blue current flowing in the wire subject to the red magnetic field feels a green force proportional to the cross product the current vector and the magnetic field. Please observe the direction of this force as you rotate the wire by angle θ. In order to keep this wire spinning engineers have constructed a commutator that switches the direction of the current as θ crosses 0. Do you see that this then also switches the direction of the force?*

The **cross product** of u and v is written $u \times v$ and defined as the matrix vector product

$$u \times v \equiv \mathrm{X}(u)v = \begin{pmatrix} 0 & -u_2 & u_1 \\ u_2 & 0 & -u_0 \\ -u_1 & u_0 & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -u_2 v_1 + u_1 v_2 \\ u_2 v_0 - u_0 v_2 \\ -u_1 v_0 + u_0 v_1 \end{pmatrix} \tag{5}$$

The next two exercises develop the key properties of the cross product - as illustrated in Figure 4.

Exercise 3 (a) Show that $u^T(u \times v) = 0$ and $v^T(u \times v) = 0$ and conclude that $u \times v$ is perpendicular to the plane containing u and v.

(b) Confirm that $\mathrm{X}(u)^T = -\mathrm{X}(u)$ and that $\mathrm{X}(u)^2 = uu^T - \|u\|^2 I$.

(c) Use (b) to derive $\|u \times v\|^2 = \|u\|^2\|v\|^2 - (u^T v)^2$.

(d) If $\theta = \angle(u, v)$ is the angle from u to v use (c) and (3) to show that $\|u \times v\| = \|u\|\|v\||\sin\theta|$.

(e) Use (d) and the figure below to conclude that $\|u \times v\|$ is the area (base times height) of the parallelogram with sides u and v.

(f) Use (e) and the figure below to conclude that $|w^T(u \times v)|$ is the volume (area of base times height) of the parallelepiped with sides u, v and w. Hint: Let u and v define the base. Then $u \times v$ is parallel to the height vector obtained by drawing a perpendicular from w to the base.

(g) Based on our work in the previous notebook we note that this parallelepiped is the image of the unit cube by the transformation

$$A = [u, v, w] = \begin{bmatrix} u_0 & v_0 & w_0 \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{bmatrix} \tag{6}$$

As this image has volume $|w^T(u \times v)|$, if the determinant indeed measures volume change (as it did in the plane) then

$$\det(A) = w^T(u \times v) \tag{7}$$

should tell us how to compute the determinant of A. Show that if w is perpendicular to u and v then $\det([u, v, w]) = c\|w\|^2$ for some $c \in \mathbb{R}$. Show that $c > 0$ when w points in the same direction as $u \times v$, while $c < 0$ when w points opposite to $u \times v$.

Your solution here.

Exercise 4 Given

$$u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(a) Compute $u \times v$ by hand, via (5), and check your work in python using np.cross(u,v)

(b) Compute $\det(A)$ by hand, via (7), and check your work in python using np.linalg.det(A) and A=np.column_stack((u,v,w)) to set up A.

In [32]:
```
print(np.cross([1,2,3],[4,5,6]))
```

```
[-3  6 -3]
```

In [4]:

```
1 # cross product illustration↔
```

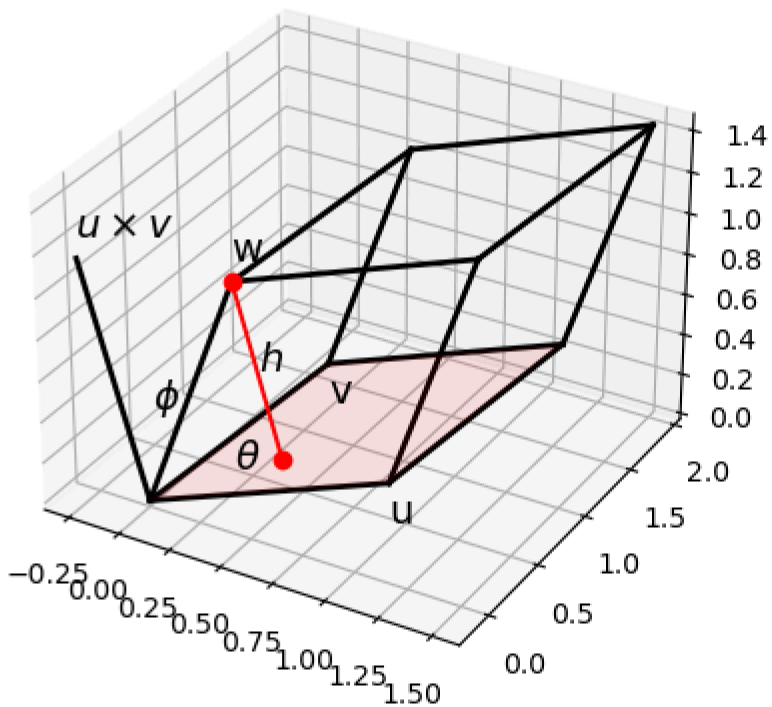

Figure 4 *The geometry of the cross product. Here $\theta = \angle(u, v)$ and we note that $u \times v$ is perpendicular to the plane defined by u and v. The actual direction of $u \times v$ is uniquely specified by the **right hand rule:** If, on your right hand, u aligns with your index finger, v with your middle finger, then $u \times v$ will align with your thumb. To facilitate the computation of volume of the parallelpiped we have shaded the base (light red), illustrated the height, h, and denoted by ϕ the angle between w and $u \times v$.*

From the figure caption above and Exercise 3(g) we deduce a criterion for right-handedness that will be very useful in coming sections.

Proposition 1 *If (u, v, w) is a collection of three vectors in $\mathbb{R}^3$ and w is perpendicular to both u and v then (u, v, w) obeys the right-hand-rule if, and only if, $\det([u, v, w]) > 0$.*

In the planar case, where the matrix $M = [u, v]$ expressed in terms of columns, we noted that $\det(M)$ vanished only u and v are colinear. In moving to our space setting, where $A = [u, v, w]$ we note from (7) that $u \times v \neq 0$ if u and v are not colinear. If $u \times v \neq 0$ and it is perpendicular to the plane containing u and v. As such, if w does not lie in this plane then $w^T(u \times v) \neq 0$. Hence, if u is not a multiple of v and w is not a linear combination of u and v then $\det([u, v, w]) \neq 0$. These conditions on u, v, w are more succinctly expressed in terms of

Definition 1 *We say that three vectors, u, v, w in $\mathbb{R}^3$ are **linearly independent** when the only numbers c_0, c_1, c_2 for which*

$$c_0 u + c_1 v + c_2 w = 0 \tag{8}$$

*is the triple of zeros $c_0 = c_1 = c_2 = 0$. If we lay these vectors into columns of $A = [u, v, w]$, and group the scalars into the column $c = [c_0, c_1, c_2]$ we notice that as $c_0 u + c_1 v + c_2 w = Ac$ we arrive at the equivalent condition; u, v, w in $\mathbb{R}^3$ are **linearly independent** in $\mathbb{R}^3$ when*

$$\text{the only } c \text{ for which } Ac = 0 \text{ is the vector } c = 0. \tag{9}$$

*Finally, given the discussion that prompted this definition, we arrive at the third equivalent condition; u, v, w in $\mathbb{R}^3$ are **linearly independent** in $\mathbb{R}^3$ when*

$$\det([u, v, w]) \neq 0 \tag{10}$$

Exercise 5 Use (10) and (7) to determine whether or not

$$u = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent.

This should help define the characteristc polynomial of $A = [u, v, w]$.

$$\begin{aligned}
\det(A-\lambda I) &= (w-\lambda e_3)^T(u-\lambda e_1)\times(v-\lambda e_2) \\
&= (w-\lambda e_3)^T(u\times v-\lambda(e_1\times v-u\times e_2)+\lambda^2 e_1\times e_2) \\
&= w^T(u\times v)-\lambda(e_3^T(u\times v)+w^T(e_1\times v-u\times e_2))+\lambda^2(e_3^T(e_1\times v-u\times e_2)+w^T e_1\times e_2)-\lambda^3 e_3^T(e_1\times \\
&= \det(A)-\lambda(e_3^T(u\times v)+w^T(e_1\times v-u\times e_2))+\lambda^2\mathrm{tr}(A)-\lambda^3
\end{aligned}$$

The eigenvalues, say $\lambda_1, \lambda_2, \lambda_3$, of A are the roots of this cubic and so the Fundamental Theorem of Algebra requires

$$\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda) = \lambda_1\lambda_2\lambda_3-\lambda(\lambda_2\lambda_3+\lambda_1\lambda_3+\lambda_1\lambda_2)+\lambda^2(\lambda_1+\lambda_2+\lambda_3)-\lambda^3 \tag{12}$$

On reconciling these expressions we find

Proposition 4 *if* $A \in \mathbb{M}_3(\mathbb{R})$ *then* $\mathrm{tr}(A)$ *is the sum of its eigenvalues and* $\det(A)$ *is the product of its eigenvalues.*

We also recall that the determinant of a product of 2-by-2 matrices is the product of their determinants. Let's confirm that here

Proposition 5 *If* $A \in \mathbb{M}_3(\mathbb{R})$ *and* $B \in \mathbb{M}_3(\mathbb{R})$ *then* $\det(AB) = \det(A)\det(B)$*.*

Proof: We express $A = [u, v, w]$ and $B = [x, y, z]$ and so $AB = [Ax, Ay, Az]$ so

$$\begin{aligned}
\det(AB) &= (Az)^T(Ax\times Ay) \\
&= (Az)^T((x_1u+x_2v+x_3w)\times(y_1u+y_2v+y_3w)) \\
&= z^TA^T((x_1y_2-x_2y_1)(u\times v)+(x_1y_3-x_3y_1)(u\times w)+(x_2y_3-x_3y_2)(v\times w)) \\
&= z^T((x_1y_2-x_2y_1)w^T(u\times v)+(x_1y_3-x_3y_1)v^T(u\times w)+(x_2y_3-x_3y_2)u^T(v\times w)) \\
&= w^T(u\times v)z^T(x\times y) = \det(A)\det(B).
\end{aligned} \tag{13}$$