2.2 Rotations

To prepare for rotations in space we consider the planar rotations of the previous notebook from a more analytical viewpoint. In particular, with reference to Figure 1, we observe that

$$\begin{aligned} x_0(\theta) &= r\cos(\theta), \\ x_1(\theta) &= r\sin(\theta) \end{aligned} \tag{1}$$

traces out a circle of radius r in the (x_0, x_1) plane as θ travels from 0 to 2π. On differentiating this pair with respect to θ we arrive at the tangent vector with components

$$\begin{aligned} x_0'(\theta) &= -r\sin(\theta), \\ x_1'(\theta) &= r\cos(\theta) \end{aligned} \tag{2}$$

This vector is most naturally illustrated when translated to the point at which it is tangent.

In [1]:
```
1 # planar rotation with tangent↔
```

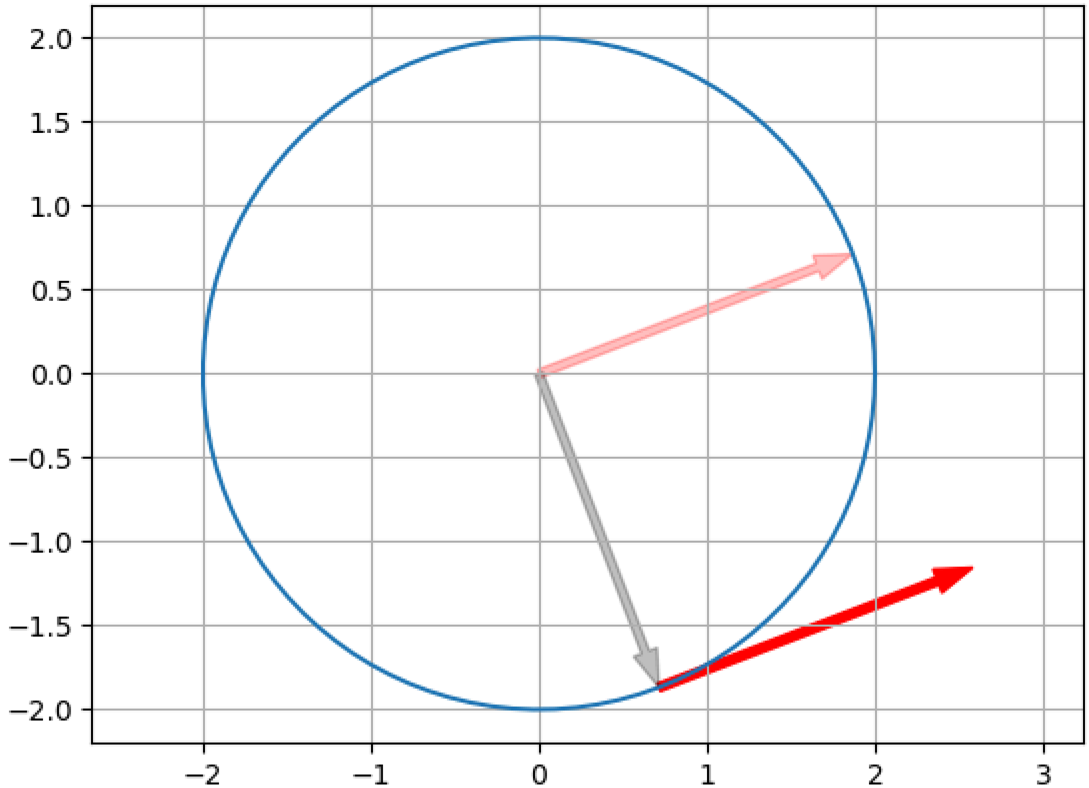

Figure 1 *At a random point, x in gray, on the blue circle we graph the tangent vector, x', in light red, and it's parallel translate in dark red. Rerun to see new tangent.*

Exercise 1 Regarding (1) and (2), please confirm that $\|x(\theta)\| = r$, and $\|x'(\theta)\| = r$ and $x^T(\theta)x'(\theta) = 0$ for each $0 \le \theta < 2\pi$.

This exercise and Figure 1 indicate that x' is simply rotation of x by $\pi/2$, i.e.,

$$x'(\theta) = Sx(\theta), \qquad \text{where} \qquad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{3}$$

We solve this differential equation by analogy with the scalar case. In particular, as the solution to $y'(t) = 5y(t)$ is $y(t) = \exp(5t)y(0)$, we expect the solution of (3) to be

$$x(\theta) = \exp(\theta S)x(0) \tag{4}$$

where

$$\exp(\theta S) = I + \theta S + (\theta S)^2/2! + (\theta S)^3/3! + (\theta S)^4/4! + (\theta S)^5/5! + \cdots \tag{5}$$

where I is the 2-by-2 identity matrix. This looks complicated, until you realize that $S^2 = -I$.

Exercise 2 Please show that

$$S^{2m-1} = (-1)^{m+1}S \quad \text{and} \quad S^{2m} = (-1)^m I \quad m = 1, 2, \ldots$$

and deduce that

$$\begin{aligned}\exp(\theta S) &= I + \theta S - \theta^2 I/2! - \theta^3 S/3! + \theta^4 I/4! + \theta^5 S/5! + \cdots \\ &= (1 - \theta^2/2! + \theta^4/4! - \cdots)I + (\theta - \theta^3/3! + \theta^5/5! - \cdots)S \\ &= \cos(\theta)I + \sin(\theta)S \\ &= K(\theta) \quad \text{from our previous notebook.}\end{aligned}$$

To **recap**, our return to planar rotations has revealed that

$$\boxed{K(\theta) = \exp(\theta S)} \tag{6}$$

where S is the matrix that transforms points to tangents along the circle of rotation. This will be our clue to exploring rotations in space. Given a unit vector $a \in \mathbb{R}^3$ and vector $x \in \mathbb{R}^3$ we illustrate in Figure 2 the counterclockwise rotation of x about a by angle θ, together with its translated tangent vector

In [1]:
```
# an illustration of rotation about an axis↔
```

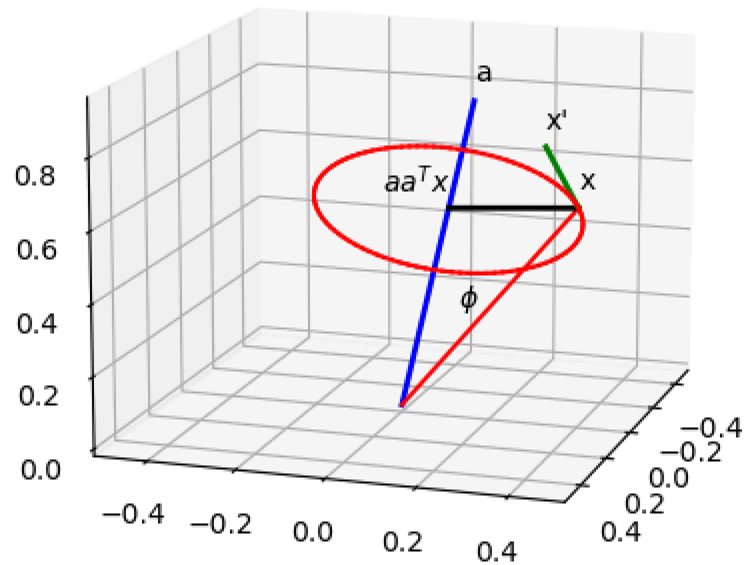

Figure 2 *An axis defined by a blue unit vector, a. We choose a red vector x that makes the angle ϕ with a. We rotate x about a and trace out the red circle centered at $aa^T x$ (the projection of x onto a). At x we also plot the black normal (radial) vector to the circle and the green tangent vector, x'.*

We learn from this figure that the tangent vector, x', is perpendicular to the plane spanned by a and x, obeys the right-hand-rule, and that its length is $\|x\| \sin(\phi)$. As such, the tangent vector at x is the cross product of a and x. That is,

$$x'(\theta) = a \times x = X(a)x \tag{7}$$

and **so $X(a)$ is the matrix that transforms points to tangents during a rotation.** Based on (6) we therefore expect the associated rotation matrix to be

$$K(a,\theta) \equiv \exp(\theta X(a)) = I + \theta X(a) + (\theta X(a))^2/2 + (\theta X(a))^3/3! + (\theta X(a))^4/4! + \cdots$$

Now, thanks to Exercise [ex:CP0] we know that powers of $X(a)$ divide neatly between even and odd that in turn reveal $\sin(\theta)$ and $\cos(\theta)$.

Exercise 3 Please show that

$$X(a)^{2m-1} = (-1)^{m+1}X(a) \quad \text{and} \quad X(a)^{2m} = (-1)^m(I - aa^T) \quad m = 1, 2, \ldots$$

and so

$$K(a,\theta) = \exp(\theta X(a)) = I + \sin(\theta)X(a) + (\cos(\theta) - 1)(I - aa^T) = aa^T + \sin(\theta)X(a) + \cos(\theta)(I - aa^T)$$

This deserves a box. The counterclockwise rotation by θ about the unit-length axis $a \in \mathbb{R}^3$ is

$$\boxed{K(a,\theta) = aa^T + \sin(\theta)X(a) + \cos(\theta)(I - aa^T)} \tag{8}$$

Let us produce these matrices when a is one of the coordinate axes

$$e_x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{9}$$

Exercise 4 Please confirm that

$$K(e_x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad K(e_y,\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}, \tag{10}$$
$$K(e_z,\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and in each case show that its determinant, via Equation ???, is 1 and its trace is $1 + 2\cos\theta$.

Your solution here.

These coordinate rotations, ([10](#)), indeed have the look of the planar rotations of our previous notebook. Let's now show that $K(a,\theta)$ indeed leaves a unmolested, that its transpose reverses the rotation, that composite rotations correspond to sums of angles, and that it preserves length (norm) and angle (with respect to a).

Exercise 5 Use Exercise ([ex:CP0]) to conclude that

(i) $K(a,\theta)a = a$

(ii) $K(a,\theta)^T = K(a,-\theta)$

(iii) $K(a,\theta)K(a,\phi) = K(a,\theta+\phi)$

(iv) $\|K(a,\theta)x\| = \|x\|$, for each $x \in \mathbb{R}^3$ and $\theta \in (0,2\pi)$.

(v) $\angle(a,x) = \angle(a,K(a,\theta)x)$, for each $x \in \mathbb{R}^3$ and $\theta \in (0,2\pi)$.

(vi) $K'(a,\theta) = X(a)K(a,\theta)$ where prime, as above, denotes $d/d\theta$.

(vii) $\mathrm{tr}(K(a,\theta)) = 1 + 2\cos(\theta)$

Your solution here.

These exercises may be worked by direct confirmation. Though the determinant of $K(a,\theta)$ may also be evaluated from scratch it is very tedious and far from illuminating. We will instead deduce it from the **Euler Product Formula**

Proposition 1 *Given $a \in S^2$ and $\theta \in \mathbb{R}$ there exist angles α, β, and γ such that*

$$K(a,\theta) = K(e_x,\gamma)K(e_z,\beta)K(e_x,\alpha) \tag{11}$$

Proof: Given $A = K(a,\theta)$ choose α such that $B \equiv AK(e_x,\alpha)$ has $b_{1,3} = 0$, via $-a_{1,2}\sin(\alpha) + a_{1,3}\cos(\alpha) = 0$. Next, note that for arbitrary β the matrix $C \equiv BK(e_z,\beta)$ enjoys $c_{1,3} = 0$. Now choose β so that $c_{1,2} = 0$, via $-b_{1,1}\sin(\beta) + b_{1,2}\cos(\beta) = 0$. Now as the rows of C are unit vectors we require $c_{1,1}^2 = 1$. If $c_{1,1} = -1$ then we can make it one by replacing β with $\beta + \pi$. Now, with $c_{1,1} = 1$ it follows that $C = K(e_x,\gamma)$ for some γ. But $C = BK(e_z,\beta) = AK(e_x,\alpha)K(e_z,\beta)$ and so $A = K(e_x,\gamma)K(e_z,-\beta)K(e_x,-\alpha)$. **End of Proof.**

This together with the fact proved in our previous notebook that determinants of products are products of determinants reveals

Corollary 2 $\det(K(a,\theta)) = 1.$

Exercise 6 Show that the eigenvalues of $K(a,\theta)$ are 1 and $\exp(\pm i\theta)$.

Your solution here.

Our next application of rotations will be to map one right-handed **orthonormal** frame, (f_x, f_y, f_z), to another, (f_1, f_2, f_3). By orthonormal we mean the three vectors are each unit vectors, and each is perpendicular to the other two. We illustrate these two frames in Figure 3.

```
In [1]:  1 # transformation of frames↔
```

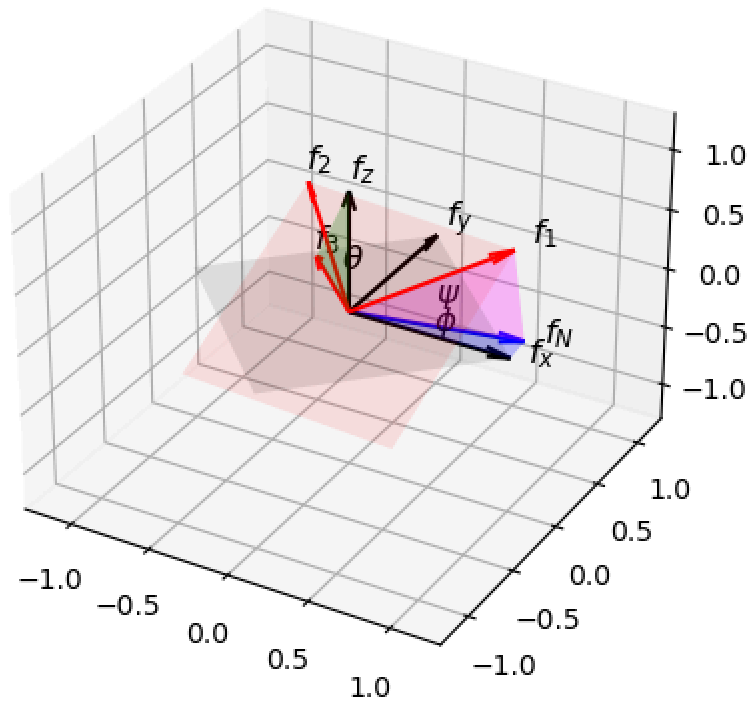

Figure 3 *Transformation of the given frame* (f_x, f_y, f_z) *into the target frame* (f_1, f_2, f_3)*. Please rotate and zoom to explore the three angles between the two frames. The black* (f_x, f_y) *plane intersects the red* (f_1, f_2) *plane along the blue* $f_N = f_z \times f_3 / \| f_z \times f_3 \|$ *direction. We denote by* ϕ *the angle from* f_x *to* f_N *looking down* f_z*. We paint this span blue and note, as* f_x *and* f_N *lie in the black plane, that*

$$f_N = K(f_z, \phi) f_x \tag{12}$$

We next denote by θ the angle from f_z to f_3 looking down f_N. We paint this span green and note that

$$f_3 = K(f_N, \theta) f_z \tag{13}$$

We finally denote by ψ the angle from f_N to f_1 looking down f_3. We paint this span purple and note, as f_N and f_1 lie in the red plane, that

$$f_1 = K(f_3, \psi) f_N \tag{14}$$

We collect these individual rotations into the composite

$$\mathcal{K} \equiv K(f_3, \psi) K(f_N, \theta) K(f_z, \phi) \tag{15}$$

and establish that $\mathcal{K}$ indeed transforms (f_x, f_y, f_z) to (f_1, f_2, f_3).

Exercise 7 (a) Please confirm that $\mathcal{K} f_x = f_1$ and $\mathcal{K} f_z = f_3$.

(b) Please use $(AB)^T = B^T A^T$ and Exercise 5 to deduce that $\mathcal{K}^T \mathcal{K} = I$.

(c) Use (b) to show that $\mathcal{K} f_y$ is a unit vector perpendicular to **both** f_1 and f_3 and so can only be f_2 or $-f_2$. Use Proposition [prop:frameOri] to rule out the latter option.

Your solution here.

With (15) in hand we can now generalize Prop.1 to a form needed in our study of quantum computation.

Proposition 3 *Given $a \in S^2$ and $\theta \in \mathbb{R}$ and two orthogonal axes, $u, w \in S^2$, there exist angles α', β', and γ' such that*

$$K(a, \theta) = K(u, \gamma') K(w, \beta') K(u, \alpha') \tag{16}$$

Proof: Complete the pair (u, w) to an orthonormal frame (u, v, w) and choose $\mathcal{K}$ in (15) to be the rotation of (u, v, w) to the standard frame (e_x, e_y, e_z) and compose and expand

$$\mathcal{K} K(a, \theta) \mathcal{K}^{-1} = K(e_x, \gamma) K(e_z, \beta) K(e_x, \alpha) \tag{17}$$

using Prop.1. Now unwrap and judicially interpose $\mathcal{K}\mathcal{K}^{-1}$ terms

$$K(a, \theta) = (\mathcal{K}^{-1} K(e_x, \gamma) \mathcal{K})(\mathcal{K}^{-1} K(e_z, \beta) \mathcal{K})(\mathcal{K}^{-1} K(e_x, \alpha) \mathcal{K}). \tag{18}$$

To see that the first triple is rotation about u it suffices to show that it fixes u. As $\mathcal{K}u = e_x$ we find

$$\mathcal{K}^{-1}K(e_x,\gamma)\mathcal{K}u = \mathcal{K}^{-1}K(e_x,\gamma)e_x = \mathcal{K}^{-1}e_x = u, \tag{19}$$

and it follows that $\mathcal{K}^{-1}K(e_x,\gamma)\mathcal{K} = K(u,\gamma')$ for some γ' and, by the same token, $\mathcal{K}^{-1}K(e_x,\alpha)\mathcal{K} = K(u,\alpha')$ for some α'. Finally, as $\mathcal{K}w = e_z$ it then follows that $\mathcal{K}^{-1}K(e_z,\beta)\mathcal{K} = K(w,\beta')$ for some β'. **End of Proof.**

For our work in robotics we will need to automate the generation of the frame transformer in (15). The only nonstandard part of the calculation lies in the computing the angles between two vectors while looking down a third vector. To quantify this we use Proposition [prop:frameOri] to determine whether or not the three vectors of concern obey the right-hand-rule. In particular, we multiply the putative angle (obtained via (???)) by the sign of the determinant of the three directions.

The code below implements this logic on random reference and target frames and prints the errors committed in reaching the target

In [70]:

```
# frame-2-frame transformation. To keep the code clean and reusable we limit it to straight numerics↔
```

```
error in f1 =  5.489222091146175e-15
error in f2 =  5.4922430125786724e-15
error in f3 =  3.7238012298709097e-16
```

Exercise 9 Explain in words how this code constructs frames and why they are each orthonormal.