1.4 Kinematics

As an example of the theory developed so far we show how vector addition and matrix multiplication provide a natural language for describing the position of robot arms. To fix ideas we consider the arm, in Figure 1, with two links and two joints.

In [1]:

```
# 2-link robot arm↔
```

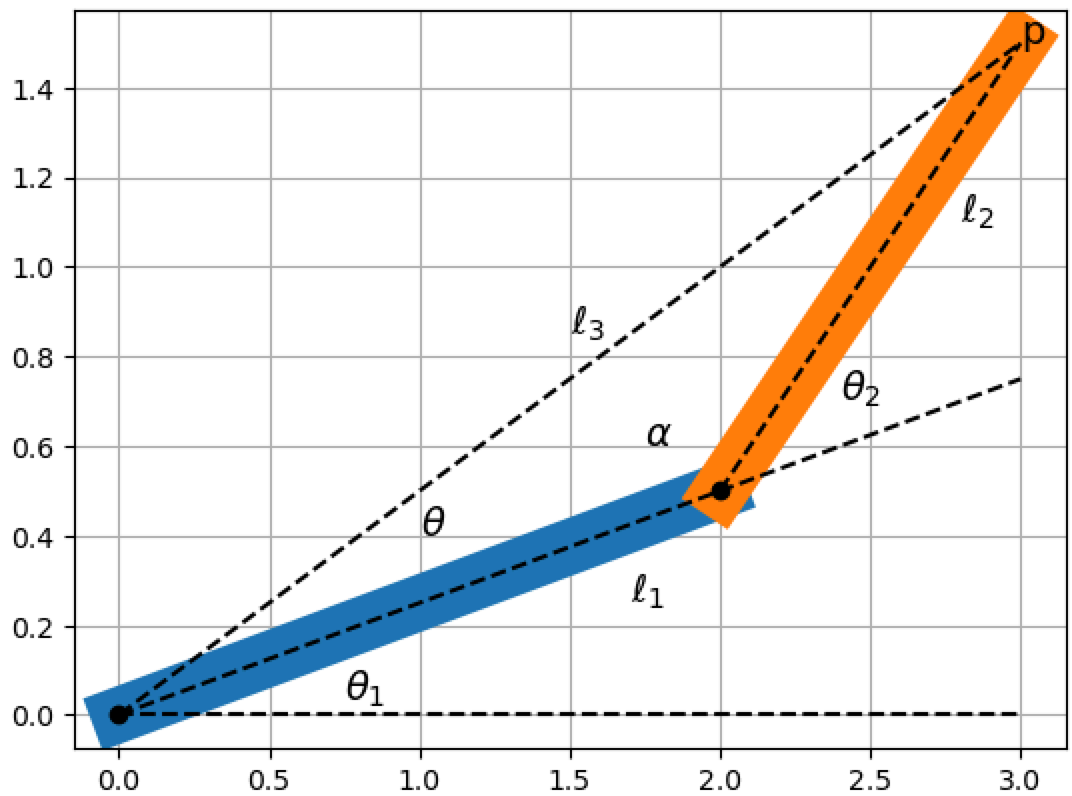

Figure 1 *A robot arm with two links, of lengths ℓ_1 and ℓ_2. The first link makes the angle θ_1 with the positive x-axis. The second link makes the angle θ_2 with the axis of the first link.*

1.4.1 Forward Kinematics

Our initial goal is to express the position, p, of the end of the second link in terms of the link lengths and joint angles.

Working back from p to the first joint, we specify the undeformed positions

$$p_1 = \begin{bmatrix} \ell_1 \\ 0 \end{bmatrix} \quad \text{and} \quad p_2 = \begin{bmatrix} \ell_2 \\ 0 \end{bmatrix}.$$

We first rotate the second link from its **undeformed position** and translate it by the length of the first link and arrive at the intermediate position

$$q = p_1 + K(\theta_2)p_2, \qquad \text{where} \qquad K(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

We then rotate the first link and achieve the final position

$$p = K(\theta_1)q = K(\theta_1)(p_1 + K(\theta_2)p_2) = K(\theta_1)p_1 + K(\theta_1 + \theta_2)p_2$$

We implement this result in the widget below. Please explore!

In [4]:
```
1 # 2-link robot arm widget↔
```

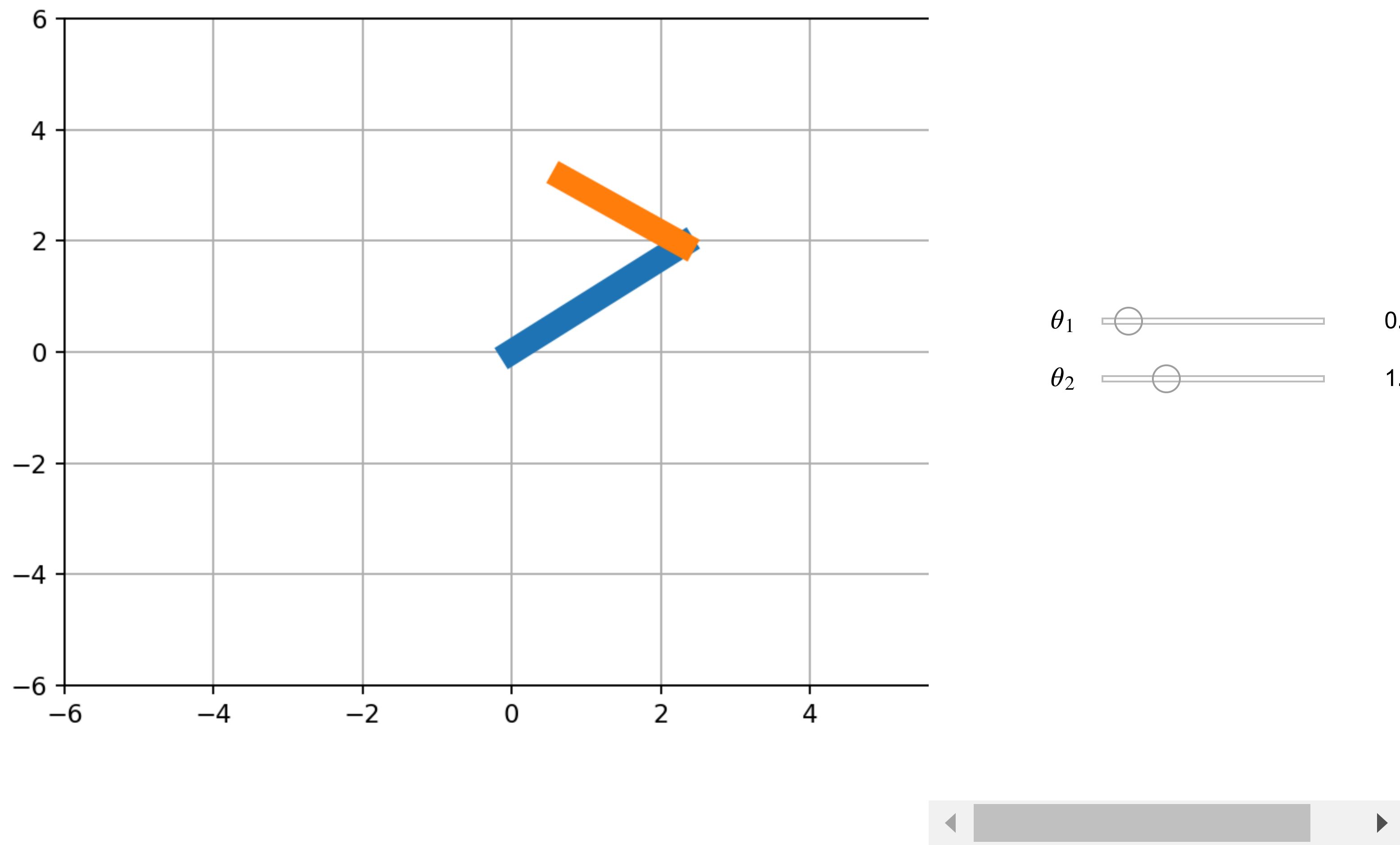

Figure 2 *A two-armed, two-slidered robot widget.*

1.4.2 Inverse Kinematics

Exercise 1 The more common problem in robotics is to be given a target location and to produce the joint angles that get you there. With regard to Figure 1, let q denote the position of joint connecting the two links.

(a) From the physical constraints; $\|q\| = \ell_1$, $\|p - q\| = \ell_2$, and $\|p\| = \ell_3$ deduce that

$$q^T p = \frac{\ell_3^2 + \ell_1^2 - \ell_2^2}{2}$$

(b) Deduce from (a) and the representation, $q^T p = \|q\|\|p\| \cos(\theta)$, achieved in our Vector Orientation (1.1.VectorOrientation.ipynb) notebook, that

$$\cos(\theta) = \frac{\ell_3^2 + \ell_1^2 - \ell_2^2}{2\ell_1\ell_3}$$

(c) Explain why (b) can be solved for θ only when $\ell_2 \geq |\ell_1 - \ell_3|$. What does this inequality say about the physically reachable targets?

(d) Argue that the equation in (b) also follows directly from the Law of Cosines.

(e) With θ in hand now solve for the two joint angles, θ_1 (using $\angle p$) and θ_2 (using the Law of Cosines), as defined in figure 1.

Your solution here.

Please check your answer to this last exercise against the animation produced in the cell below.

In [3]:
```
1 # circle to ellipse↔
```

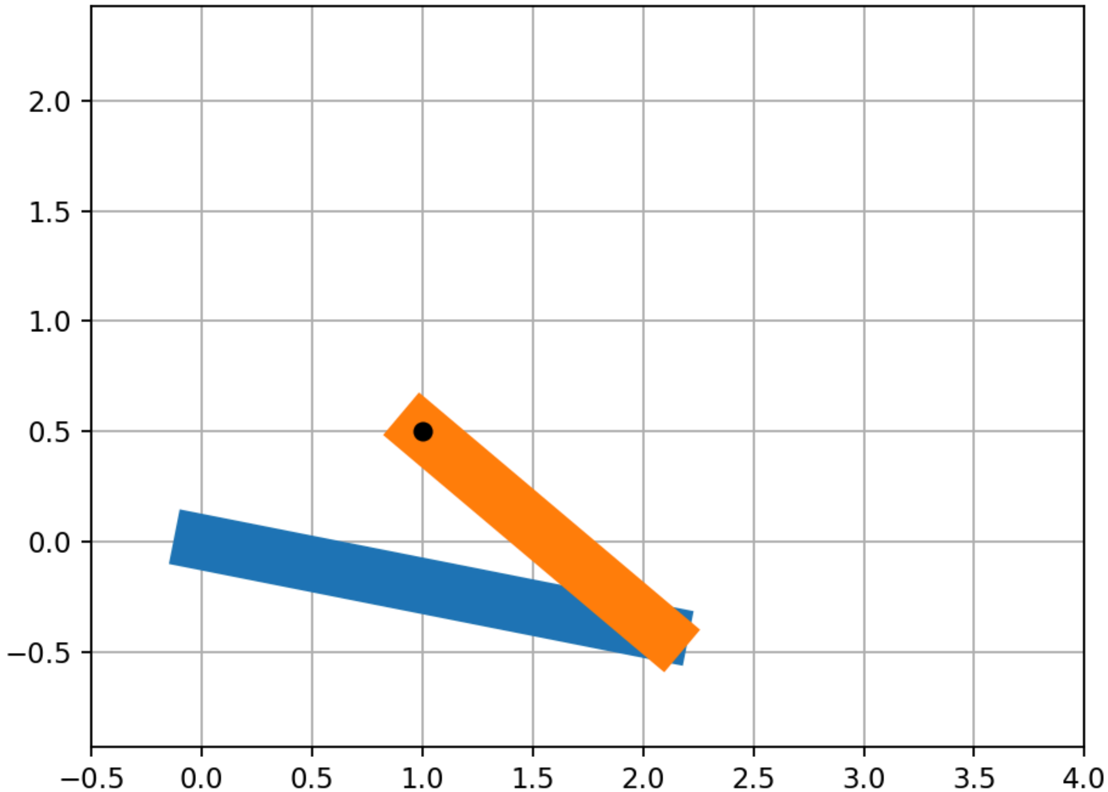

Figure 3 *Animation of the path from rest to target.*

For an interactive 3-arm challenge check out this [scratch game (https://scratch.mit.edu/projects/10607750)](https://scratch.mit.edu/projects/10607750).

Exercise 2 In the animation above we selected, based on Figure 1, the "elbow down" configuration. Please edit it to select the

In [4]:
```
# Your solution here
```

The animation above moves one joint at a time in order to reflect the sequence in which θ_1 and θ_2 are computed in Exercise 1. In reality robots rarely move in such awkward ways. The code below uses the same procedure to move in a straight line from start to stop points.

In [4]:
```
# 2-link robot arm Inverse Kinematics animation between 2 specified points,↔
```

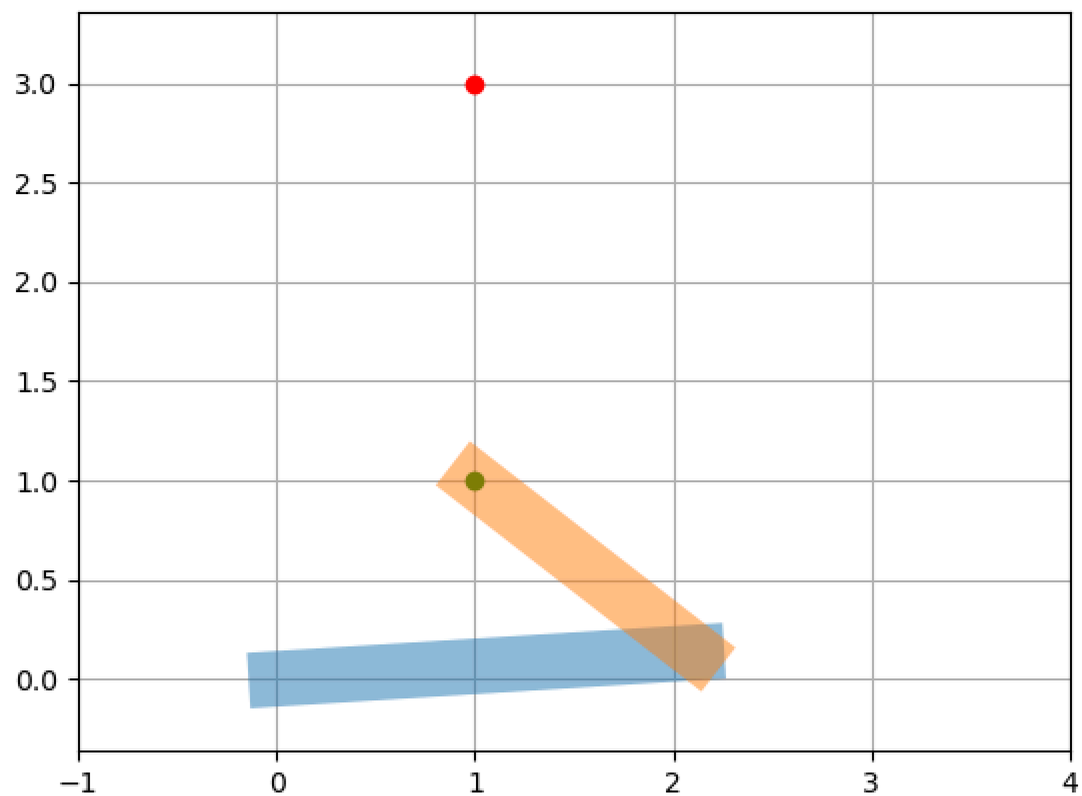

Figure 4 *Animation of the straight line trajectory between specified start and stop points. Please observe the (apparent) similtaneous motion of both joints.*

If instead we ask the end point to travel in a circle we arrive at a marvelous machine.

In [2]:

```
# 2-link robot arm constrained  widget around a circle↔
```

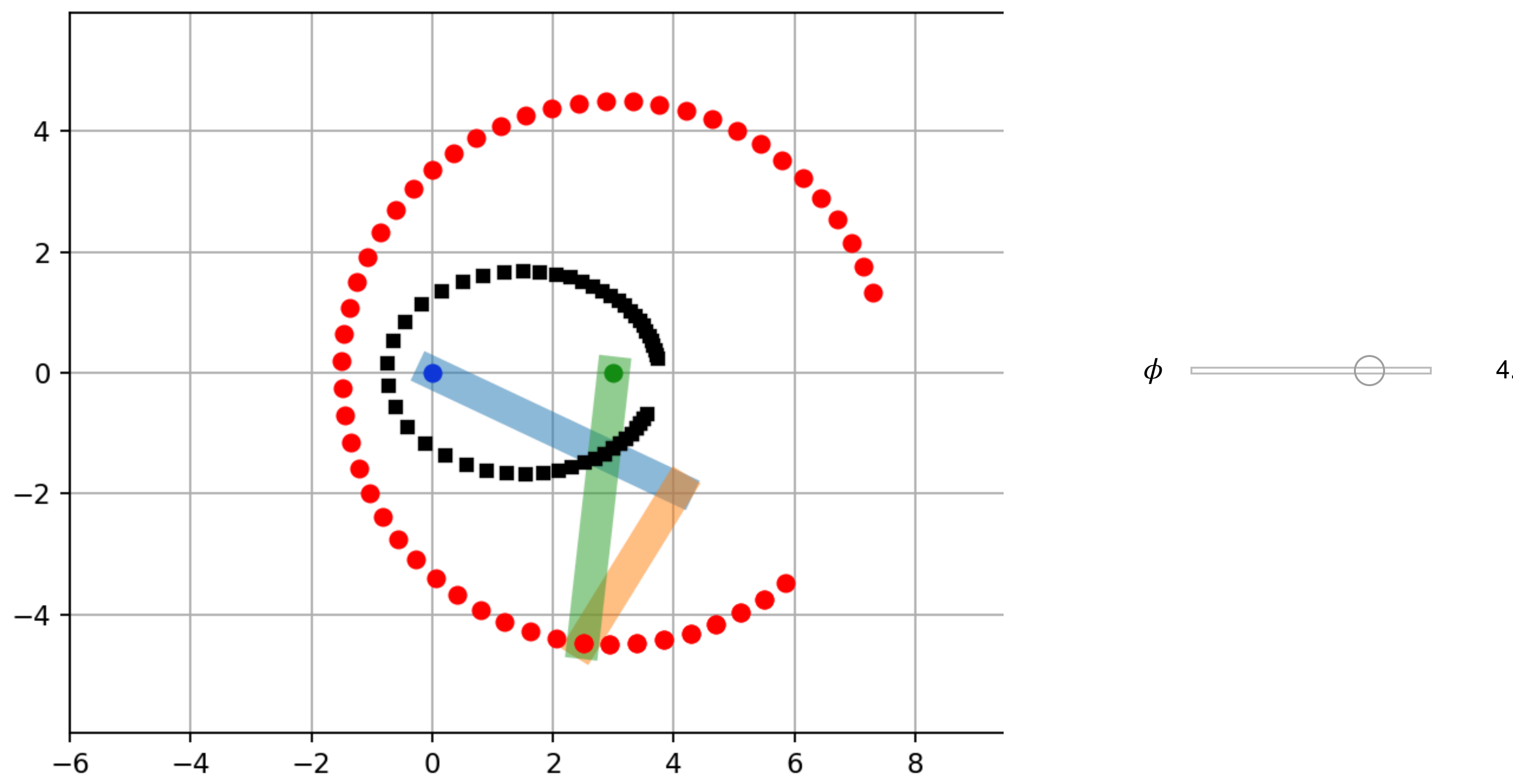

__Figure 5__ We attach (green) link of length ℓ_1 anchored at $(\ell_2, 0)$ and offer a slider that moves this green link about a circle. If we place a sliding joint, and black pen, at the intersection of the blue and green links we see that it traces out an ellipse.

To prove the black point of intersection traces an ellipse, we must prove, with respect to Figure 6 that $a + b$ is constant.

Exercise 3 Show that $a + b = \ell_1$. Hint: argue that the dotted lines are indeed parallel, then argue by symmetry that $\ell_1 - a = b$.

In [35]:
```
# ellipse geometry setup↔
```

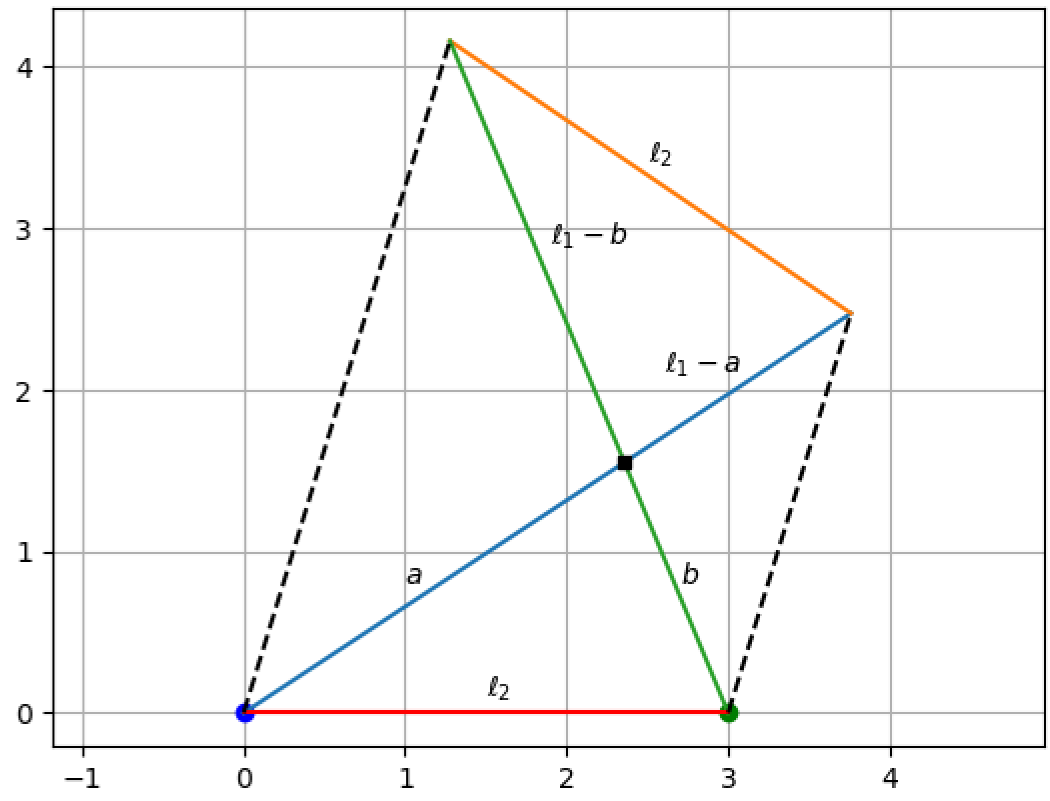

__Figure 6__ Labeled frozen 3-link configuration from the widget above, used in Exercise 3, of our elliptical constructor.

1.4.3 The Peaucellier-Lipkin linkage

We now move from reaching a particular target to achieving a desired motion. The classic problem is to transform linear motion (of say, a piston) to rotational motion (of say, an axle). We begin this exploration with the lovely motion in Figure 7.

In [1]:
```
# illustrate the PL Lemma↔
```

```
HBox(children=(Canvas(toolbar=Toolbar(toolitems=[('Home', 'Reset original view', 'home', 'home'), ('Back', 'Ba…
```

Figure 7 *As p traverses the blue circle (via the θ slider) we observe that q traverses the red line.*

Exercise 4 With reference to Figure 7, if p and q are colinear and $\|p\|\|q\| = k$ then as p traverses a circle through 0 then q traverses a vertical line.

(a) Show that $p(\theta) \equiv (2R\cos^2(\theta), 2R\sin(\theta)\cos(\theta))$ describes a circle of radius R centered at $c \equiv (R, 0)$ as θ moves from $-\pi/2$ to $\pi/2$. Hint: Show that $\|p(\theta) - c\| = R$.

(b) Show that if $q(\theta) = \lambda p(\theta)$ and $\|p(\theta)\|\|q(\theta)\| = k$ then $\lambda = k/\|p\|^2$.

(c) Show that if $q(\theta) = kp(\theta)/\|p\|^2$ then $q(\theta) = (k/(2R),\ k\tan(\theta)/(2R))$ traverses a vertical line as $p(\theta)$ traverses the circle.

Your solution here.

This construction transforms circles to lines (and *vice versa*) with links that change their length with angle. To accomplish this with links of fixed length we introduce the intermediate parallelogram below.

In [8]:

```
1 # illustrate Lemma PL1↔
```

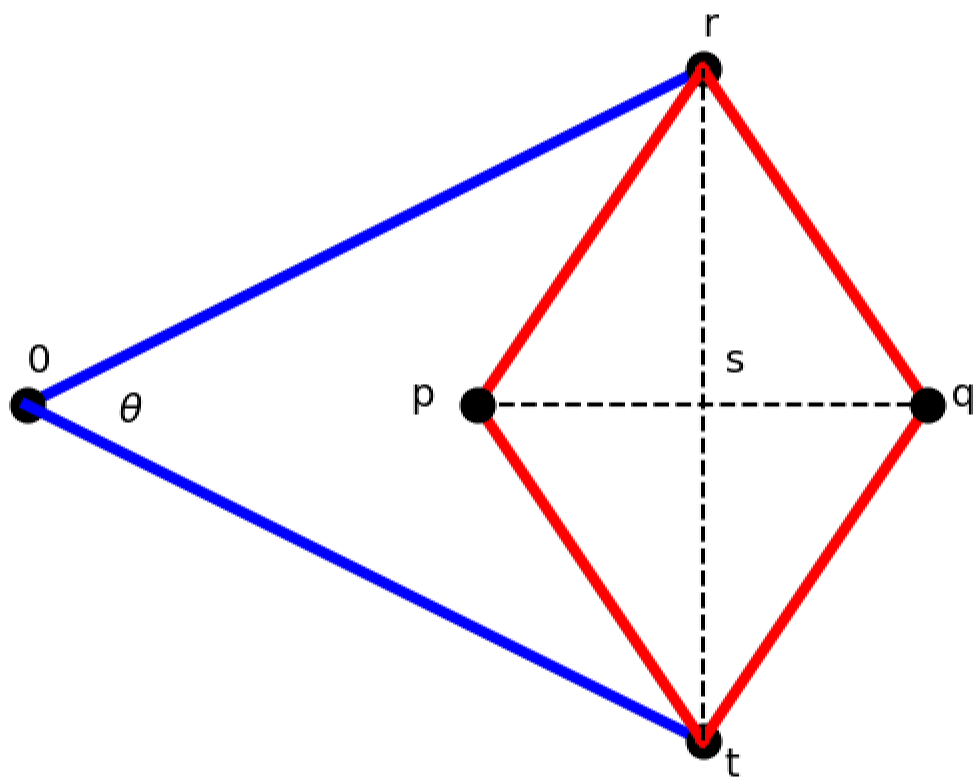

Figure 8 *Six links meeting at five joints. The two blue links have equal length. The four red links have equal length. The black dashed lines will help us analyze this construction.*

Exercise 5 We note by symmetry that p and q are colinear. To reconcile this with our previous construction please show that $\|p\|\|q\| = k$ for some constant $k > 0$, independent of the angle θ.

(a) Use the Pythagorean Theorem to deduce that $\|s\|^2 + \|r-s\|^2 = \|r\|^2$ and $\|p-r\|^2 = \|r-s\|^2 + \|p-s\|^2$.

(b) Deduce from part (a) that $\|r\|^2 - \|p-r\|^2 = (\|s\| - \|p-s\|)(\|s\| + \|p-s\|)$.

(c) Use the figure (and symmetry guaranteed by equality of red lengths) to deduce that $\|s\| - \|p-s\| = \|p\|$ and $\|s\| + \|p-s\| = \|q\|$.

(d) Substitute your findings from part (c) into part (b) and deduce that $\|p\|\|q\| = \|r\|^2 - \|p-r\|^2$ and that this is indeed independent of the angle θ.

Your solution here.

On combining these two results we arrive at the beautiful Peaucellier-Lipkin linkage

In [1]:
```
# illustrate the PL linkage↔
```

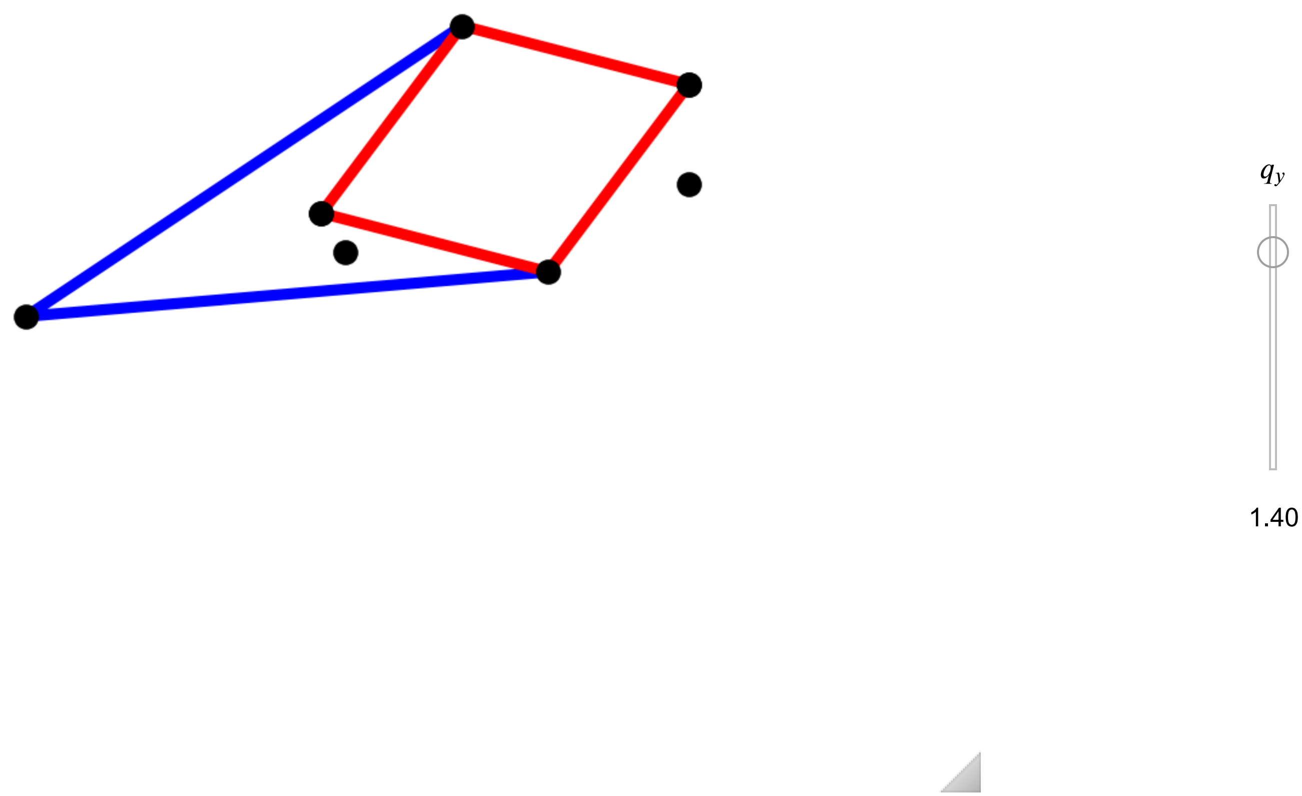

Figure 9 *The Peaucellier-Lipkin Linkage. As you raise and lower the q joint along a vertical line, the p joint traces a segment of a circle.*

Next Section: 1.5 The Projective Line (1.5.ProjectiveLine.ipynb)

In [ ]:

```

```