2.3 Kinematics

The Programmable Universal Manipulation Arm (PUMA) of Figure 1 is a common 6-degree-of-freedom industrial robotic arm.

In [3]:
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# 6R PUMA↔
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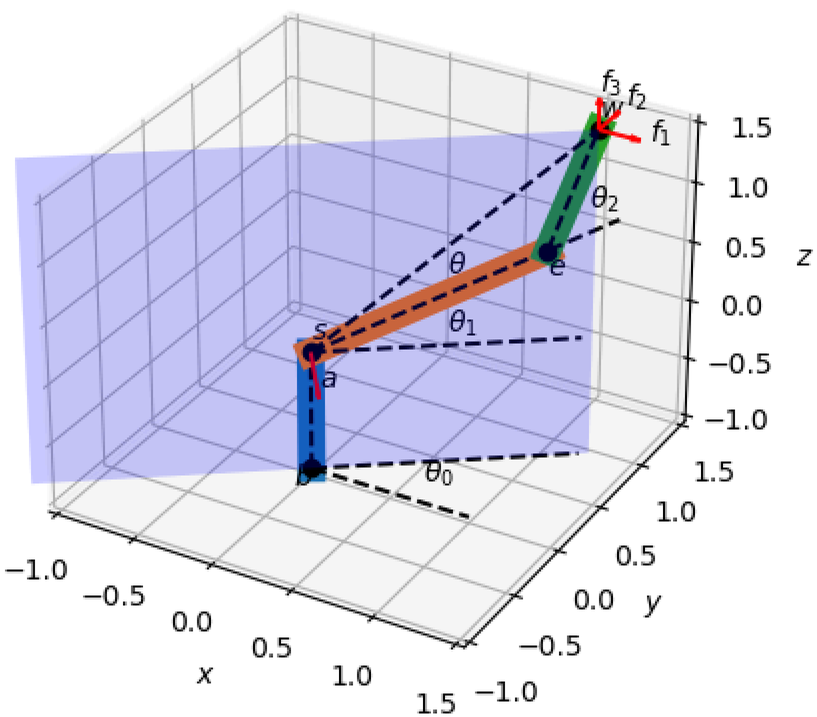

Figure 1 *A 6 degree-of-freedom PUMA robot. It swings by θ_0 at its base, b, by θ_1 at its shoulder, s, by θ_2 at its elbow, e, and by an additional 3 degrees-of-freedom at it's wrist, w. The orange link has length ℓ_1 and the green link has length ℓ_2.*

In terms of the stated angles and known link lengths, the positions of the marked joints in Figure 1 are

$$s = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad e = \begin{bmatrix} \ell_1 \cos(\theta_0)\cos(\theta_1) \\ \ell_1 \sin(\theta_0)\cos(\theta_1) \\ \ell_1 \sin(\theta_1) \end{bmatrix}, \qquad w = \begin{bmatrix} \ell_1 \cos(\theta_0)\cos(\theta_1) + \ell_2 \cos(\theta_0)\cos(\theta_1+\theta_2) \\ \ell_1 \sin(\theta_0)\cos(\theta_1) + \ell_2 \sin(\theta_0)\cos(\theta_1+\theta_2) \\ \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_1+\theta_2) \end{bmatrix} \tag{1}$$

The **inverse problem** of moving from a known configuration, $\{\overline{\theta}_0, \overline{\theta}_1, \overline{\theta}_2, \overline{f}_1, \overline{f}_2, \overline{f}_3\}$, to a target configuration, $\{w, f_1, f_2, f_3\}$ is solved in three steps.

In the **first** step we find the base angle θ_0 by taking the ratio of the first two components in w. In particular

$$\frac{w_1}{w_0} = \tan(\theta_0) \tag{2}$$

We then rotate the base by $\delta_0 \equiv \theta_0 - \overline{\theta}_0$ about the z-axis. This places the wrist in the proper plane and its frame at

$$(f_x, f_y, f_z) \equiv (K(e_z,\delta_0)\overline{f}_1, K(e_z,\delta_0)\overline{f}_2, K(e_z,\delta_0)\overline{f}_3) \tag{3}$$

In the **second** step we recognize that links 1 and 2 lie in a plane and that rotation *in* this plane corresponds to rotation about the axis

$$a = \begin{bmatrix} \sin(\theta_0) \\ -\cos(\theta_0) \\ 0 \end{bmatrix} \tag{4}$$

perpendicular to this plane. We may determine the intermediate angle θ as in our inverse solution of the planar robot arm,

$$\cos(\theta) = \frac{\ell_3^2 + \ell_1^2 - \ell_2^2}{2\ell_1\ell_3} \tag{5}$$

where $\ell_3 = \|w\|$ is the distance from shoulder to wrist. With θ in hand it is a simple matter to deduce the shoulder and elbow angles, θ_1 and θ_2, from Figure 1 via the Law of Cosines. We then rotate the shoulder by $\theta_1 - \overline{\theta}_1$ and the elbow by $\theta_2 - \overline{\theta}_2$. This action rotates the step-1 wrist frame, by $\delta \equiv (\theta_1 + \theta_2) - (\overline{\theta}_1 + \overline{\theta}_2)$ about the axis, a of (4). As such, the wrist frame becomes $(K(a,\delta)K(e_z,\delta_0)\overline{f}_1, K(a,\delta)K(e_z,\delta_0)\overline{f}_2, K(a,\delta)K(e_z,\delta_0)\overline{f}_3)$.

For the **third** and final step we use the composite rotation, $\mathcal{K}$, of (???) to map the step-2 wrist frame to the target wrist frame, (f_1, f_2, f_3).

Exercise 1 Implement these three steps, in code, in moving the PUMA from $\{0, 0, 0, e_x, e_y, e_z\}$ to $\{w, -e_x, e_y, -e_z\}$ where

$$w = \begin{bmatrix} 1/2 \\ 1/2 \\ 2 \end{bmatrix} \tag{6}$$

assuming $\ell_1 = 1.5$ and $\ell_2 = 1$.

(a) Find θ_0 and apply the rotation matrix $K(e_z, \delta_0)$.

(b) Find $\theta, \theta_1, \theta_2$, and apply the rotation matrix $K(a, \delta)$.

(c) Find and apply the composite rotation, $\mathcal{K}$, using the last code cell in the previous section, that brings the wrist frame to $(-e_x, e_y, -e_z)$.

(d) Confirm your calculations against the animation below.

In [2]:

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1 # 3D Inverse Kinematics Animation↔
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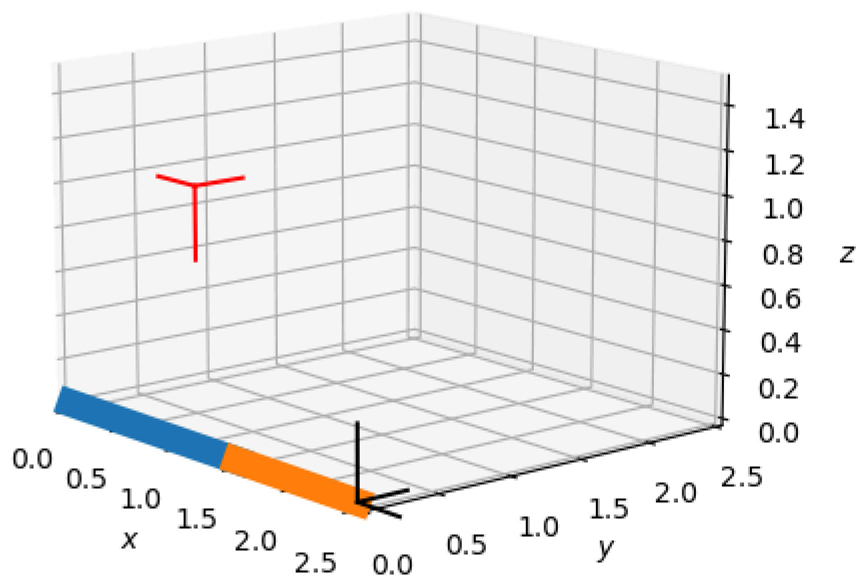

Figure 3 *Animation of the solution of the 3D inverse kinematics problem. The straight-arm base configuration reaches the red target orientation along the green path via the three steps outlined above.*