Optimal Task Assignments*

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Abstract

A principal wields task assignment power over her team of agents working on a project with multiple tasks. Effort spent working on a task generates positive spillovers and substitutes for effort at similar tasks but incurs a convex cost. Assigning agents to more tasks mitigates free-riding and shirking, but can cause task overload and prevent agents from completing more tasks. I focus on the class of key-peripheral projects composed of peripheral tasks (linked to one other task) and key tasks (linked to at least two peripheral tasks). The modular assignment, which assigns each agent to a module consisting of a key task and its peripheral tasks, resolves the free-riding problem and implements a unique equilibrium— each agent exerts full effort at his key task. The modular assignment is optimal if every key task is sufficiently specialized. A key task exhibits high specialization if linked to more peripheral tasks than key tasks. Characterizing the optimal task assignment beyond key-peripheral networks requires additional assumptions on the shape of the cost function and project structure.

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1 Introduction

A principal and her team of agents work on a project with multiple tasks. Effort spent completing a task generates positive spillovers and is a strategic substitute for effort spent finishing similar tasks. For example, a piece of code written for a particular application in software development can be referenced by team members creating complementary features.[1] Cleaning a critical dataset or performing document review facilitates progress on subsequent analyses. Creating a standard manual for sample collection aids different laboratory sub-teams in planning various experiments.

The main novelty of this paper is that the principal possesses task assignment power in place of conventional contracting means. In most organizations, a separate compensation and benefits department, not the manager herself, designates salaries and bonus rates for employees. I analyze how a fundamental managerial responsibility in assigning tasks can influence project outcomes.

The incentives for agents are standard. Each agent is evaluated by the number of tasks he completes. An agent finishes a task if he exerts enough effort that together with the sum of efforts at neighboring tasks meets an exogenous completion threshold. Effort provision is subject to a convex cost and for simplicity, I assume perfect effort substitutability between related tasks. A binary evaluation of task completion is practical in many settings. For instance, progress made towards fixing a piece of software that crashes is futile if the principal needs it to be fully debugged for deployment. Cleaning half of a dataset is impractical; the principal needs the whole dataset scrubbed to uphold the integrity of subsequent analyses.

The principal assigns agents to tasks with the objective of completing as many tasks as possible while keeping total team effort costs low. The latter captures the fact that managerial evaluations often assess how well the manager builds, supports, and motivates her team (Hyväri (2006); Chen and Lee (2007)). Managing team effort costs is critical to a favorable performance review for the principal because they measure fatigue, a key contributor to employee burnout and low morale (Maslach and Leiter (2016)).[2] To isolate and focus on the incentive problems in the model, I assume there are as many agents as tasks so there are no understaffing concerns.

The fundamental trade-off involves balancing free-riding and task overload. The principal wants agents to exert high effort at tasks central to the project and take

[1] Code is typically shared on a team-accessible GitHub page.

[2] A 2022 Forbes article reported 67% of survey respondents experiencing some level of burnout resulting from feeling overworked.

advantage of outsized positive effort spillovers, but they will inevitably free-ride off other agents' efforts at related tasks and shirk. This leads to an overall increase in effort costs because agents assigned to related tasks must increase their efforts to compensate. To combat free-riding, the principal should assign more tasks to each agent so they internalize their spillovers. Nevertheless, doing so can overload agents. Assigning too many tasks to each agent can surge effort costs and cause them to leave tasks unfinished. Task overload pushes in the opposite direction—the principal should instead assign each agent to fewer tasks and encourage a balanced effort distribution among agents.

I model the project task structure as a network where links between tasks symbolize effort spillovers and substitutability. Take, for instance, a programming team working on a phone app with the following coding tasks: task A is a call feature, task B is a music playback capability and task C is a sound recording component. Tasks A and B require access to a device's speakers while tasks A and C require control over a device's microphone. Tasks B and C do not have an evident overlap; therefore, the project can be represented as a network with links between tasks A and B, and tasks A and C. Characterizing the optimal task assignment will depend on how the tension between free-riding and task overload manifests in particular network structures.

I focus on a class of networks with a dichotomous classification of tasks: tasks linked to one other task (peripheral tasks) and tasks linked to at least two peripheral tasks (key tasks). I call this class of networks key-peripheral. Key-peripheral networks are simple project structures and include a variety of commonly studied graphs such as stars, connecting stars, hierarchical structures like trees with two leaves for every internal vertex, etc. They model software development projects particularly well because they embody a basic design mechanism— inheritance. Inheritance champions code reuse in constructing parent classes and methods so that subclasses and auxiliary methods can reference them when building specific functionalities.[3] Creating parent classes and building general libraries of functions are key tasks while coding accessory methods and features correspond to peripheral tasks. Excluding classes and methods that are neither key nor peripheral avoids adding unnecessary design complexity and depth of inheritance that make it difficult to predict how the software behaves (see Chidamber and Kemerer (1994) and Subramanyam and Krishnan (2003)). Key-peripheral task structures are prominent in settings beyond software development projects. Cleaning and preparing a

[3]See IBM's primer on inheritance. Figure 4 of the guide displays the code architecture for an elementary bank account system, which is key-peripheral with one key task and two peripheral tasks. While the graph is directed to illustrate inheritance relationships, effort spillovers and substitutability between related coding tasks are bidirectional.

dataset is a key task, and analyses that use different subsets of the data are peripheral tasks.[4] Developing a standard operating procedure for a laboratory instrument is a key task, and different experiments that use said instrument are peripheral tasks.

To tackle optimal assignments (where agents behave strategically), I turn to a familiar concept in organizational economics: modularization.[5] The modular task assignment achieves the delicate balance between mitigating free-riding and task overload by assigning each agent to a module consisting of a key task and all of its peripheral tasks.[6] It resolves the free-rider problem by inducing each agent to exert full effort at his key task to meet the task completion threshold. Agents are assigned to similar tasks, enabling them to effectively internalize their own spillovers and avoid overexertion as a consequence of task overload. Modularization performs remarkably well with regards to implementability and optimality.

The first main result is that the modular assignment implements a unique equilibrium where full effort is played at key tasks and zero effort everywhere else. The equilibrium takes a specific graph theoretic structure— the tasks with full effort form a minimum dominating set. A *dominating set* is a set of tasks that together with their neighboring tasks constitute the whole network; a minimum dominating set is a dominating set of minimum size. Minimum dominating effort profiles, which prescribe full effort at every task in a minimum dominating set and zero elsewhere, exploit effort substitutability and complete the entire project with positive effort at the minimum number of tasks. This language allows us to establish a baseline for what a principal can implement when facing an arbitrary project structure. I provide an algorithm that, given any network, constructs a generalized modular task assignment that implements a minimum dominating effort profile as the unique equilibrium.

The second main result is that the modular assignment is optimal for key-peripheral projects with sufficiently specialized key tasks. The *specialization* of a key task is the ratio of its number of peripheral tasks to the number of key tasks it is linked to. Specialization neatly captures the trade-off between mitigating free-riding and task overload

[4]Free-riding takes the following form: team members clean sections of the dataset relevant to their respective analyses. The analyst in charge of cleaning and preparing the whole dataset free-rides and just compiles the various cleaned subsets of the data together.

[5]The idea of modularization was first popularized by Simon (1965). Since then, it has been a keystone concept in software development and production design. See Parnas (1972) and Garud, Langlois and Kumaraswamy (2003) for more about the history and prevalence of modularization in software development and production design.

[6]A module is analogous to a *package* in software design which typically consists of a parent class and associated subclasses. See Figure 8 in this IBM guide on class diagrams for a diagram.

in key-peripheral projects. Alternate assignments that threaten the optimality of the modular assignment must assign agents to fewer tasks to exploit convexity and reduce costs. However, they can perform poorly because entire sets of peripheral tasks must be left incomplete in equilibrium.

Reducing effort costs involves assigning a different agent to each task within a module and spreading efforts among them: high (but not full) key task effort and low (but nonzero) peripheral task efforts. The total efforts that contribute to completing the key task exceeds the completion threshold, and the key task agent will free-ride off peripheral efforts and shirk. Free-riding can be avoided only if he is assigned an extra neighboring key task, which I call a supporting key task.[7] The sum of efforts played at the supporting key task and its linked tasks must be *exactly* the completion threshold in equilibrium; otherwise, the agent has a profitable deviation by reducing his effort at the original key task. This fragile equilibrium condition assures no peripheral task of the supporting key task can be completed in equilibrium. If it were so, then the efforts played at the peripheral and supporting key task sum to at least the completion threshold. The total efforts that contribute to completing the supporting key task, which includes the effort played at the original key task, is greater than the completion threshold, contradicting equilibrium. All in all, if every key task is sufficiently specialized, then decreasing each agents' task load and reducing costs via convexity is suboptimal because entire sets of peripheral tasks must be sacrificed. The modular assignment is optimal.

High key task specialization is analogous to the concept of *loose coupling* in systems design, which entails a low degree of interdependence between modules within a system. Projects with specialized key tasks/loosely coupled modules are common and possess several appealing properties: easy testability, simple scalability, reduced maintenance costs, and increased immunity to catastrophic cascading failures (Stevens, Myers and Constantine (1999)).

For projects with low key task specialization, characterizing the optimal task assignment depends on the specific cost function. The modular assignment still resolves the free-rider problem, but the principal could achieve a higher payoff by decreasing each agents' task load and reduce costs via convexity in the following manner: designate a key task with low specialization to support reducing costs at multiple neighboring key tasks while sacrificing its own peripheral tasks. Whether or not this scheme is profitable

[7] That is, if there exists a neighboring key task to support this scheme. Assigning him to neighboring peripheral tasks encourages him to exert full effort at the key task, which just replicates the modular assignment payoff.

hinges on the shape of the cost function. I show that if a project contains a key task with specialization below one, then there exists a cost function that makes the modular assignment optimal and another cost function that makes it suboptimal.

Moving beyond key-peripheral projects, I study the performance of the modular task assignment on a broader class of networks that consist of key, peripheral, and intermediary tasks. Intermediary tasks are helper tasks that aid progress at multiple key tasks, but are not linked to any peripheral tasks. An identical null result regarding optimality applies: for any project composed of key, peripheral, and intermediary tasks, there exists a cost function that makes the modular assignment optimal and another cost function that makes it suboptimal. Now, incorporating tasks that are not key, peripheral or intermediary makes the analysis decidedly more difficult. Determining what effort profiles an assignment can implement as equilibria becomes a fickle exercise that depends on the exact task structure.[8] All together, this establishes a tight boundary about the main result. Key-peripheral networks with sufficiently specialized key tasks are a large class of interpretable project structures for which a simple characterization of the optimal task assignment exists *without* additional assumptions regarding the network structure and cost function.

To be transparent, I make two substantive assumptions in the main analysis: perfect substitution of effort between linked tasks and no staffing shortages. Relaxing the first assumption, I show that the modular assignment is optimal (given sufficiently specialized key tasks) so long as the degree of effort substitutability between tasks is sufficiently high. Robustness of the main result is a critical property because related tasks, in practice, may exhibit moderate to high levels of effort substitutability, but not perfect substitutability. With regards to understaffing, the main optimality result still holds so long as the number of agents weakly exceeds the number of key tasks because the modular assignment is still feasible. The analysis remains inconclusive when the team size drops below that number. Whether it is profitable to assign each agent more tasks and overwork them or assign each agent fewer tasks and leave tasks incomplete depends on the specific network structure and cost function.

[8] See Example 4 in Section 4.2.

1.1 Related Literature

This paper contributes to the literature on network games, principal-agent problems, and team theory in organizations.[9] In the networks literature, agents are typically treated as nodes in the network. That notion is decoupled in my model: tasks make up the network, and a principal assigns agents to tasks. The set-up of my model is most similar to Bramoullé and Kranton (2007), who study the provision of a public good that cannot be excluded along links in a network. Equilibria feature free-riding and take the shape of maximal independent sets of the network: agents who exert full effort do not neighbor each other. Introducing task assignments expands the set of equilibria to include minimum dominating effort profiles. Crucially, minimum dominating sets may not always be independent sets. Many recent papers focus on public good provision in networks and discuss the free-rider problem (see Allouch (2015); Kinateder and Merlino (2017, 2023); Elliott and Golub (2019)). In more general analyses, Bramoullé, Kranton and D'Amours (2014) determine the structure of all Nash equilibria in network games that exhibit linear best responses. To tackle the prevalent issue of equilibrium multiplicity, Galeotti et al. (2010) incorporate incomplete information into network games. In a setting with complete information, task assignments can resolve the issue of equilibrium multiplicity. In particular, I show that the modular task assignment implements a unique equilibrium.

Optimal contracts with moral hazard have been the common theme in research on principal-agent problems, starting with the classic papers of Mirrlees (1976) and Holmström (1979). Many variations and extensions have been considered, such as adding a retention rule (Banks and Sundaram (1998)), incorporating dynamics (Holmström (1999)), and endogenizing the information acquisition process (Georgiadis and Szentes (2020)). Holmström (2017) provides an overview of the literature. I do not model uncertainty in my environment; the primary forces that threaten the principal's payoff are free-riding and task overload. Instead of writing contracts or monitoring agents, the principal can only assign tasks. Task assignments resemble delegation, where the principal delegates a set of decisions to the agent from which he makes a choice (e.g. Alonso and Matouschek (2008)). In my model, the decisions an agent faces are linked to each other by way of the underlying project architecture. In Matouschek, Powell and Reich

[9] Assignment problems are ubiquitous in operations research, but the treatment does not allow agents to behave strategically nor incorporate incentives and equilibrium analysis. Ahuja, Magnanti and Orlin (1993) provides a comprehensive overview of classical assignment problems. Approaches to solving assignment problems include rewriting them as network flow problems or graph coloring problems.

(2024), a principal designs communication networks for agents trying to match their actions to exogenous local states and coordinate with other agents residing within the same module of production. The authors focus on how a firm's organizational structure, specifically a modular production network, affects the structure of the optimal communication network. In contrast, the principal in my model chooses a particular task assignment and influences the production design rather than taking it as fixed.

Research on team theory has similarly focused on designing optimal incentives in the presence of moral hazard and conflicts of interest (e.g. Groves (1973); Holmström (1982); Che and Yoo (2001); Georgiadis (2015)). Recent work focuses on how a manager with limited commitment chooses project objectives for a team of agents (Georgiadis, Lippman and Tang (2014)), determines the optimal team size when the project is risky (Fu, Subramanian and Venkateswaran (2016)), finds the optimal team composition (Glover and Kim (2021)) and characterizes the optimal incentive structure for agents with production spillovers (Dasaratha, Golub and Shah (2024)). I contribute to the literature on team theory by studying how a manager should optimally assign tasks to a team of agents where task collaboration is pinned down by the project structure— the optimal assignment is the modular assignment given sufficiently high key task specialization.

2 Model

$N = \{1, ..., n\}$ is a set of agents and $K = \{1, ..., k\}$ is a set of tasks. Tasks are arranged in a network G and links between tasks indicate the presence of effort spillovers and substitutability. Tasks are assigned to agents who decide how much effort to exert from the nonnegative effort space $\mathbb{R}^+$. The principal chooses an assignment $f : N \to 2^K$, where $f(a)$ is the set of tasks assigned to agent a. The set of viable assignment functions is $F = \{f \mid \forall a, d \in N, f(a) \cap f(d) = \emptyset\}$, ensuring no more than one agent is assigned to a task. An assignment f induces an effort game played by agents $\langle M, ((\mathbb{R}^+)^{|f(a)|})_{a \in M}, (\pi_a)_{a \in M} \rangle$:

- $M \subseteq N$ is the set of assigned agents i.e. agents a with $f(a) \neq \emptyset$.
- $(\mathbb{R}^+)^{|f(a)|}$ is the effort space for agent a. He chooses effort levels at each task in $f(a)$; denote e_i as the effort chosen at task i.
- $\pi_a = \sum_{i \in f(a)} \mathbb{1}\{e_i + \sum_{\ell \in G_i} e_\ell \geq b\} - c(\sum_{i \in f(a)} e_i)$ is the payoff for agent a where:
    - b is an exogenous task completion threshold.

- G_i is the set of tasks that neighbor task i in graph G.
- $c : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1, strictly increasing and strictly convex function with $c(0) = 0$ and $c'(0) = 0$.

Agents who remain unassigned receive a payoff of zero. Tasks can also remain unassigned and have zero exerted effort. The total payoff to agent a is the sum of per-task payoffs less the cost of effort, which is a function of his aggregate effort. Completing a task requires at least b units of effort. Tasks are perfect substitutes for each other and the completion threshold can be reached if enough effort is expended on linked tasks. I consider imperfect effort substitutability in Section 5.

Given a task assignment, agents play a Nash equilibrium in the induced effort game. Every agent chooses effort levels to maximize his payoffs in response to the effort provisions of all other agents. The principal chooses a viable assignment $f \in F$ such that an induced Nash equilibrium $\{\hat{e}_i\}_{i=1}^k$ maximizes the total payoff across the entire network:

$$\max_{f \in F} \sum_{i=1}^{k} \mathbb{1}\{\hat{e}_i + \sum_{\ell \in G_i} \hat{e}_\ell \geq b\} - \sum_{a \in M} c\left(\sum_{i \in f(a)} \hat{e}_i\right)$$

2.1 Main Assumptions

Assumption 1. *$c(b) < 1$.*

The payoff to completing a task is greater than the cost of full effort. For an agent assigned to exactly one task, his best response is to work just enough to reach the completion threshold.

Assumption 2. *$n \geq k$.*

There are at least as many agents as tasks and the principal does not have to worry about an understaffed team. Whether the principal can implement a certain effort profile depends only on the agents' strategic behavior. This assumption will be relaxed in Section 6.

2.2 Environment: Key-Peripheral Networks

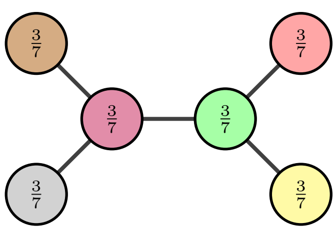
The main analysis focuses on the class of key-peripheral networks.

Definition 1. A **peripheral task** is linked to only one other task. A **key task** has at least two neighboring tasks that are peripheral and the set of all peripheral tasks of a key task is its **peripheral set**. A network is **key-peripheral** if it contains only key and peripheral tasks.

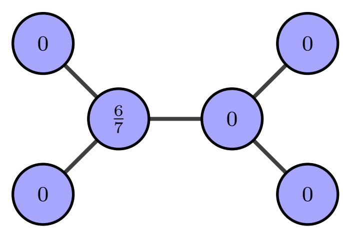
An agent spending effort on a key task expedites progress at many other auxiliary tasks. Peripheral tasks are specialized offshoot tasks— they synergize well with only one other task in the project. Key-peripheral networks, which contain only key and peripheral tasks, model simple task structures and keeps the analysis tractable. It is a flexible and interpretable class of networks that includes star graphs, connecting star graphs, and tree graphs where every branch node (including the root) has at least two leaf nodes. Key-peripheral networks can contain cycles; the definition does not restrict how key tasks are linked to each other.

2.3 Example

Let the net payoff to an agent a assigned to tasks in $f(a)$ is $\sum_{i \in f(a)} \mathbb{1}\{e_i + \sum_{\ell \in G_i} e_\ell \geq \frac{6}{7}\} - (\sum_{i \in f(a)} e_i)^2$. The completion threshold is $\frac{6}{7}$ and the cost of effort is quadratic. Consider the connecting star network, which is key-peripheral.



(a) Problem: Free-Riding



(b) Problem: Task Overload

Figure 1: Free-Riding versus Task Overload on the Connecting Star Graph

Each color represents a different assigned agent. I say an assignment **implements** an effort profile if it is a Nash equilibrium in the induced effort game. The left assignment is the trivial assignment which assigns each agent to one task and the depicted effort profile completes every task while spreading the effort costs among many agents. However, the trivial assignment fails to implement it because of free-riding; key task agents will shirk and reduce their efforts. To alleviate free-riding, the principal can assign every task to the blue agent, as she does in the right assignment. By "selling the project to one agent," the agent now internalizes all of his positive effort spillovers. Unfortunately, he is overloaded and convexity takes hold. The blue agent cannot complete every task because it is simply too costly to do so.[10] The right assignment implements an equilibrium where the agent plays full effort at only one of the key tasks.

2.4 Modular Assignments

Figure 1 illustrates the two extreme assignments. Intuitively, the optimal task assignment should strike a balance between mitigating free-riding and task overload. The modular assignment, as defined next, achieves exactly that.

Definition 2. A **module** is a set of tasks containing a key task and its peripheral set. On key-peripheral networks, a **modular assignment** assigns each agent to one module within the network (until all modules are assigned).

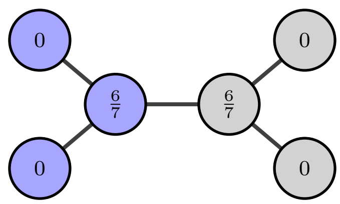


Figure 2: A Modular Assignment on the Connecting Star

Figure 2 shows a modular assignment on the same connecting star project. The principal assigns two agents, blue and gray, to modules consisting of a key task and its associated peripheral tasks. Each agent will in turn choose to exert all of his effort on completing the key task regardless of how the other agent behaves. The modular design obliges each agent to internalize their spillovers, guaranteeing they do not free-ride. While agents are assigned multiple tasks, they are not assigned dissimilar tasks, saving them from being overloaded and enduring high convex costs.

3 Unique Implementability

I first focus on the unique implementability of the modular assignment.

Theorem 1. *The modular assignment implements a unique effort profile that specifies full effort on key tasks and zero effort everywhere else.*

[10] If the cost function was concave, then clearly the principal should always assign all of the tasks to one agent. In this scenario, the first-best coincides with the agent's problem, making the analysis trivial.

Proof. The modular assignment assigns each agent to a module. Each assigned agent plays effort b at his assigned key task i and zero effort at every task in the peripheral set S_i in equilibrium. Assume by contradiction that he chooses some other effort level $e_i \in [0, b)$. Then playing effort $e_h = b - e_i$ at one of his peripheral tasks h ensures the completion of two tasks (i and h) at a cost of $c(e_h + e_i) = c(b)$. But $|S_i| \geq 2$, so there is at least one other assigned task that is left incomplete. Setting $e_i = b$ completes more tasks with a weakly lesser cost. □

The modular assignment rectifies the equilibrium multiplicity problem that is frequently encountered in network games.[11] The unique equilibrium takes a particular graph theoretic structure— a minimum dominating effort profile.

Definition 3. For a graph G and a subset D of the vertex set $V(G)$, let $G[D]$ be the set of vertices of G which are in D or adjacent to a vertex in D. If $G[D] = V(G)$, then D is a **dominating set** of G. A dominating set of the smallest size is a **minimum dominating set**; the size of a minimum dominating set is called the **domination number** $\gamma(G)$.[12] A **minimum dominating effort profile** prescribes effort b at tasks in a minimum dominating set and zero effort everywhere else.

Minimum dominating effort profiles are attractive for two reasons. First, they exploit effort substitutability and complete every task in the project with positive effort exerted at the least number of tasks. Second, minimum dominating effort profiles are universally implementable, not just on key-peripheral networks.

Theorem 2. *Every minimum dominating effort profile of a network is implementable.*

Proof. Fix a network and denote D_{min} as its minimum dominating set. Consider the following constructive algorithm:

[11] The unique equilibrium can be implemented by assigning each agent to a key task and two peripheral tasks instead of the entire peripheral set. Nevertheless, assigning agents to the entire peripheral set matters for the extension on imperfect substitutability. As I will show in Section 5, assigning agents to more peripheral tasks within their module allows lower levels of effort substitutability to support optimality.

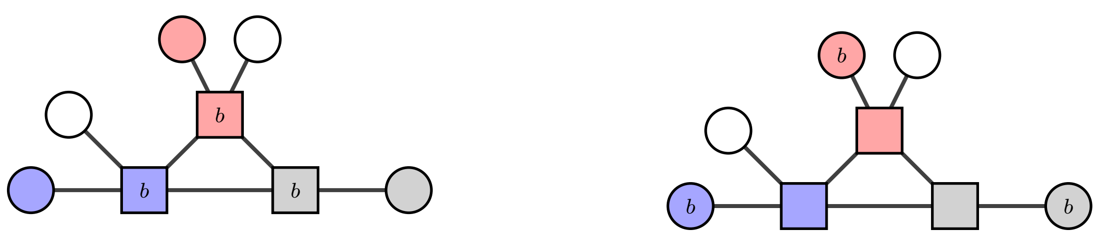
[12] Dominating sets are well-studied by graph theorists. For further reference, see West (2001). In general, finding a minimum dominating set (and therefore the domination number) is an NP-hard problem where a brute force algorithm would have a time complexity of $O(2^n)$. On key-peripheral networks, the unique minimum dominating set is the set of key tasks.

Algorithm 1

1: Assign each task in D_{min} to an unassigned agent
2: **for** each assigned agent **do**
3:     **if** his assigned task i neighbors another task in D_{min} **then**
4:         Assign him to an extra task $j \notin D_{min}$ such that $G_j \cap D_{min} = \{i\}$
5:     **end if**
6: **end for**

An agent assigned to a single task will always play effort b to reap a net payoff of $1 - c(b) > 0$. For an agent assigned to two tasks, playing effort b at his assigned task in D_{min} is an equilibrium action. If he decreases his effort at that task, he loses the task completion payoff of the extra task not in D_{min}. Any deviation that keeps the sum of his efforts across the two tasks equal to b yields the same payoff as before. Such an extra task $j \notin D_{min}$ with $G_j \cap D_{min} = \{i\}$ on line 4 of Algorithm 1 always exists. Suppose not, then every neighboring task of i is also linked to some task in $\ell \in D_{min}$ with $\ell \neq i$. Then $D_{min} \setminus \{i\}$ dominates the graph, which contradicts D_{min} being a minimum dominating set. Thus the assignment created from Algorithm 1 implements a minimum dominating effort profile as a Nash equilibrium. □

Agents assigned to neighboring tasks in a minimum dominating set will free-ride. To counteract this, Algorithm 1 assigns each of those agents to an extra task, inducing them to internalize their spillovers and exert full effort at the task in the minimum dominating effort profile. While every minimum dominating effort profile of any network is implementable, they may not be *uniquely* implementable. The assignment constructed via Algorithm 1 can induce other Nash equilibria that are undesirable. Take the following network where the square tasks constitute a minimum dominating set.



(a) Good Equilibrium   (b) Bad Equilibrium

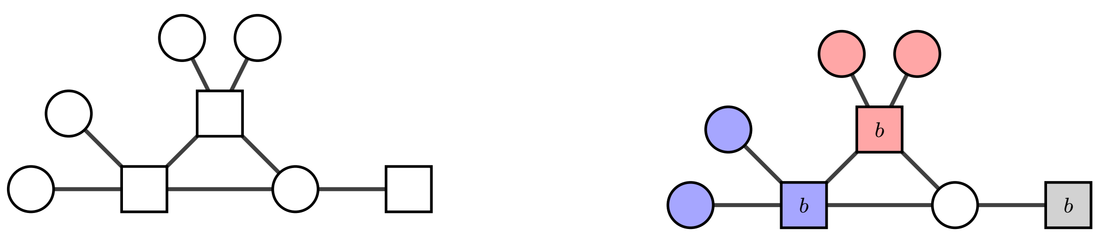
Figure 3: A Minimum Dominating Effort Profile (Not Uniquely Implementable)

Unfortunately, this assignment also implements an undesirable equilibrium where every agent exerts effort level b at their assigned circle tasks instead of the square tasks in the minimum dominating set. This leaves the white circle tasks unfinished. With some adjustments, we can always find a minimum dominating effort profile of a network that is implementable as the unique Nash equilibrium.

Theorem 3. *There exists a minimum dominating effort profile implementable as the unique Nash equilibrium for any network.*

Proof. See Appendix. □

I discuss Theorem 3 in the context of the minimum dominating effort profile of Figure 3, but relegate its proof to the Appendix. There are two issues that threaten uniqueness: (1) every agent is not assigned enough tasks to ensure they always exert full effort at their task in the minimum dominating set and (2) the gray agent's square task only neighbors one other task not in the minimum dominating set. To simultaneously solve both problems, I show that a minimum dominating set with the following special property exists for any network: tasks in the minimum dominating set that neighbor each other are linked to at least two extra tasks with only one neighboring task in the minimum dominating set. Such a minimum dominating set can be constructed via recursion.



(a) Special Minimum Dominating Set

(b) Generalized Modular Assignment

Figure 4: Unique Implementability

The special minimum dominating set is represented by square tasks in Panel (a) of Figure 4. The **generalized modular assignment** implements the corresponding minimum dominating effort profile as the unique equilibrium by assigning agents to two extra peripheral-like tasks if their task in the minimum dominating set neighbors each other.

4 Optimality

Finding an optimal task assignment can be difficult because there are $(n+1)^k$ number of viable assignments, where n is the number of agents and k is the number of tasks.

As we saw in Theorem 1, the modular assignment implements an effort profile where every agent exerts full effort at his assigned key task and zero elsewhere. Every task in the project is completed. Fortunately, we can exploit the key-peripheral structure to dramatically reduce the space of assignments that are comparable to the modular assignment.

Proposition 1. *If an assignment achieves a strictly higher payoff than the modular assignment, then it must implement effort $e_i \in (0, b)$ at some key task i and leave the entire peripheral set of a neighboring key task $j \in G_i$ unfinished.*

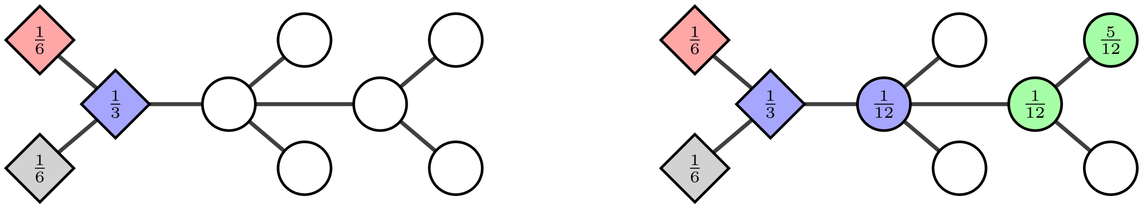
Proof. To achieve a strictly higher payoff than modularization, an assignment must have a lower total cost. It must implement an effort profile with effort $e_i \in (0, b)$ at some key task i because implementing effort $e_i = 0$ is strictly worse and $e_i = b$ just replicates the payoff to modularization. The assignment must complete every peripheral task of a key task i with $e_i \in (0, b)$. If not, then the modular assignment achieves a strictly higher payoff since $c(b) < 1$ by Assumption 1.

Let S_i denote key task i's peripheral set. To complete every task in S_i, each peripheral task must be assigned to agents who each exert effort $b - e_i$. The effort threshold at key task i is at least $e_i + |S_i|(b - e_i) > b$, causing the key task agent to shirk. This can be supported as an equilibrium only if the key task agent is also assigned to some neighboring key task $j \in G_i$ that specifically requires effort level e_i. The completion threshold at key task j must bind b i.e. $e_j + \sum_{\ell \in G_j} e_\ell = b$. If not, the agent will again have a profitable deviation by decreasing his effort at task i. Every peripheral task $\ell \in S_j$ has to be left unfinished in equilibrium since $e_\ell + e_j < e_\ell + e_j + e_i \leq e_j + \sum_{k \in G_j} e_k = b$. $\square$

Proposition 1 pares down the space of potentially better assignments to ones implementing effort profiles that are *locally* different from the minimum dominating effort profile.[13] A direct implication is that the modular assignment is optimal on projects with disconnected key tasks.

Example 1. Take the key-peripheral network shown below with completion threshold $b = \frac{1}{2}$ and a quadratic cost function. Consider an alternate assignment that implements a key task effort of $\frac{1}{3}$ on the diamond module:

[13]This trade-off does not always exist for arbitrary networks. See Example 3 in found later in this section.



(a) Reducing Costs on the Diamond Module

(b) An Equilibrium

All tasks within the diamond module are completed at a cost of $2 \cdot (\frac{1}{6})^2 + (\frac{1}{3})^2$, which is lower than the cost of the modular assignment of $(\frac{1}{2})^2$. Observe that the sum of efforts at the diamond key task exceeds the completion threshold; the blue agent will always shirk. The only way to sustain an effort level of $\frac{1}{3}$ in equilibrium is to assign him to a neighboring key task that *requires* the diamond task effort to be exactly $\frac{1}{2}$. Assigning the blue agent to the neighboring circle key task supports the diamond key task effort of $\frac{1}{3}$ in equilibrium but Proposition 1 provides strict equilibrium restrictions. The sum of efforts at the blue circle key task must be equal to $\frac{1}{2}$, so its peripheral tasks must remain unfinished in equilibrium.

To capture the trade-off between free-riding and task overload in key-peripheral projects, I introduce the specialization of a key task.

Definition 4. The **specialization of key task** i is $\sigma_i = \frac{|S_i|}{|G_i \backslash S_i|}$ where S_i is its peripheral set.

The specialization of a key task is the ratio of its number of peripheral tasks to its number of linked key tasks. If a key task is not linked to any other key tasks, then it has infinite specialization. Modularization is advantageous if every key task is highly specialized because decreasing each agent's task load and reducing costs via convexity necessarily leaves large sets of peripheral tasks incomplete by Proposition 1. In contrast, if a key task exhibits low specialization, then designating said key task to support reducing task loads and spreading effort costs across multiple neighboring modules (and sacrificing its own set of peripheral tasks) could be favorable. This leads to the main result concerning optimality.

Theorem 4. *Let $\mathcal{K}(G)$ be the set of all key tasks in key-peripheral network G. If $\min_{i \in \mathcal{K}(G)} \sigma_i \geq c(b)$, then the modular assignment is optimal. The optimal number of assigned agents is the domination number $\gamma(G)$.*

Proof. Fix an alternate assignment that implements intermediate effort $e_i \in (0, b)$ at some key task i and completes all of its peripheral tasks. Let L be the set of all key tasks with implemented efforts in $(0, b)$ and completed peripheral tasks. Let U be the set of supporting key tasks. U is nonempty and the peripheral sets of key tasks in U are left unfinished in equilibrium by Proposition 1. It is without loss to compare payoffs for modules with key tasks residing in L. The modular assignment does weakly better than the alternate assignment at modules where the key tasks have implemented effort of b or 0. The modular assignment achieves a strictly higher payoff if:

$$\underbrace{\sum_{i\in L}(|S_i|+1) - |L|\cdot c(b)}_{\text{modular assignment payoff}} \geq \underbrace{\sum_{i\in L}(|S_i|+1) + |U|}_{\text{upper bound on alt. assignment payoff}} \iff c(b) \leq \frac{\sum_{i\in U}|S_i|}{|L|}$$

Since every key task in L neighbors a key task in U, we can write:

$$L \subseteq \bigcup_{i\in U}(G_i \setminus S_i) \implies |L| \leq \sum_{i\in U}(|G_i \setminus S_i|)$$

The following inequalities hold:

$$\frac{\sum_{i\in U}|S_i|}{|L|} \geq \frac{\sum_{i\in U}|S_i|}{\sum_{i\in U}(|G_i\setminus S_i|)} \geq \min_{i\in U}\frac{|S_i|}{|G_i\setminus S_i|} \geq \min_{i\in\mathcal{K}(G)}\frac{|S_i|}{|G_i\setminus S_i|} = \min_{i\in\mathcal{K}(G)}\sigma_i$$

If $\min_{i\in\mathcal{K}(G)}\sigma_i \geq c(b)$, then the modular assignment achieves a weakly higher payoff than any assignment that satisfies the necessary condition outlined in Proposition 1. In other words, no assignment achieves a strictly higher payoff than the modular assignment. The modular assignment assigns as many agents as the number of key tasks, which is the graph domination number $\gamma(G)$. □

The modular assignment assigns $\gamma(G)$ agents and if the total number of agents n exceeds $\gamma(G)$, then the principal should keep $n - \gamma(G)$ agents unassigned.[14] The next result finds that as the peripheral sets of each key task gets larger, the modular assignment payoff converges to efficiency. I fully characterize the efficient effort profile in the Appendix.

Proposition 2. *The payoff to the modular assignment converges to efficiency as $|S_i| \to \infty$ for each key task i in a key-peripheral network.*[15]

[14] The principal can still uniquely implement the minimum dominating effort profile by assigning an agent to a key task and two linked peripheral tasks, and so the upper bound on the number of assigned agents is $k - 2\cdot\gamma(G)$ where k is the total number of tasks. Assigning more agents threatens the uniqueness of implementability.

[15] $|S_i| \to \infty$ implies $\min_{i\in\mathcal{K}(G)}\sigma_i \to \infty$ but the converse is not true because $\sigma_i = \infty$ by definition

Proof. The efficient assignment is the trivial assignment because assigning each agent to one task achieves the lowest cost by exploiting convexity. The efficient effort profile solves the following cost minimization:

$$\min_{\{e_i\}_{i=1}^k} \sum_{i=1}^{k} c(e_i) \quad \text{s.t. } e_i + \sum_{\ell \in G_i} e_\ell \geq b \text{ and } e_i \geq 0 \text{ for all } i$$

Characterization of Efficiency: The efficient effort at key task i solves $c'(e_i^*) = |S_i| \cdot c'(b - e_i^*)$. See Appendix.

Now let e_i^* be the efficient effort at key task i. Then the following inequality must hold, where S_i is the peripheral set of key task i:

$$\underbrace{\sum_{i \in \mathcal{K}(G)} |S_i| + 1 - \sum_{i \in \mathcal{K}(G)} c(e_i^*) - \sum_{i \in \mathcal{K}(G)} |S_i| c(b - e_i^*)}_{\text{efficiency}} \geq \underbrace{\sum_{i \in \mathcal{K}(G)} |S_i| + 1 - \gamma(G) c(b)}_{\text{modular payoff}}$$

$$\implies \gamma(G) c(b) \geq \sum_{i \in \mathcal{K}(G)} c(e_i^*) + \sum_{i \in \mathcal{K}(G)} |S_i| c(b - e_i^*)$$

From the characterization of efficiency, $|S_i| \to \infty \implies e_i^* \to b$ and $\sum_{i \in \mathcal{K}(G)} c(e_i^*) \to \gamma(G) c(b)$. Then $\lim_{|S_i| \to \infty} \sum_{i \in \mathcal{K}(G)} |S_i| c(b - e_i^*) = 0$ by the above inequality and the difference between the modular payoff and efficiency vanishes as $|S_i| \to \infty$. □

In the following corollaries, I detail some scenarios where the modular task assignment is always optimal.

Corollary 1. *If the principal needs to complete every task in a key-peripheral project, then the modular assignment is optimal.*

It is realistic to assume settings where the principal must finish the entire project, and managing team effort costs is a secondary objective. Any assignment that achieves a strictly higher payoff must leave tasks incomplete in equilibrium by Proposition 1, so the modular assignment is optimal.

Corollary 2. *If every key task is linked to weakly more peripheral tasks than key tasks, then the modular assignment is optimal.*

If every key task is linked to more peripheral tasks than key tasks, then $\min_{i \in \mathcal{K}(G)} \sigma_i \geq 1$. The sufficient condition of Theorem 4 holds since $c(b) < 1$ by Assumption 1.

whenever $|G_i \setminus S_i| = 0$.

Corollary 3. *The modular assignment is optimal for linear costs and fixed capacity costs of the form:*

$$c(e) = \begin{cases} 0 & 0 \leq e \leq b \\ +\infty & e > b \end{cases}$$

With fixed capacity costs, there are no benefits to dividing tasks among different agents. With linear costs, it is actually harmful to divide tasks within a module among different agents. If an agent assigned to key task i exerts any effort less than b, then the total cost to completing the entire module is at least $c(e_i) + |S_i|c(b - e_i) > c(b)$, which is more expensive than the modular assignment. In either case, the trade-off between free-riding and task overload disappears and the optimal assignment must complete every task. The modular assignment does exactly that. The same reasoning applies to arbitrary networks: the generalized modular assignment, which implements a minimum dominating effort profile as a Nash equilibrium by Theorem 3, is optimal for linear and fixed capacity costs.

4.1 When Sufficiency Fails

Part of the appeal of Theorem 4 is that it is agnostic of the functional form of the cost of effort and relies only on computing $c(b)$ and σ_i. The next result shows that finding the optimal assignment on key-peripheral networks with low specialization relies on explicit computations that depend non-trivially on the shape of the cost function.

Theorem 5. *If* $\min_{i \in \mathcal{K}(G)} \sigma_i < 1$*, then there exists a cost function that makes the modular assignment optimal and another cost function that makes it suboptimal.*

Proof. See Appendix. □

By Theorem 4, if a cost function satisfies $c(b) < \min_{i \in \mathcal{K}(G)} \sigma_i$, then the modular assignment is optimal. However, it is not a necessary condition. Constructing a cost function that makes the modular assignment suboptimal requires more care, but the intuition is simple. The principal should exploit the key task with low specialization to reduce task loads and spread effort costs at many neighboring modules. A cost function that is close to zero for most the interval $[0, b]$ suffices. The following example illustrates.

Example 2. The network in Figure 6 has minimum key task specialization of $\frac{2}{3}$. An alternate assignment designates the key task with the lowest specialization to support intermediate efforts at neighboring key tasks. Each neighboring key task is assigned to the blue agent. Each peripheral task is assigned to a different agent. The peripheral tasks of the blue circle key task are left incomplete.

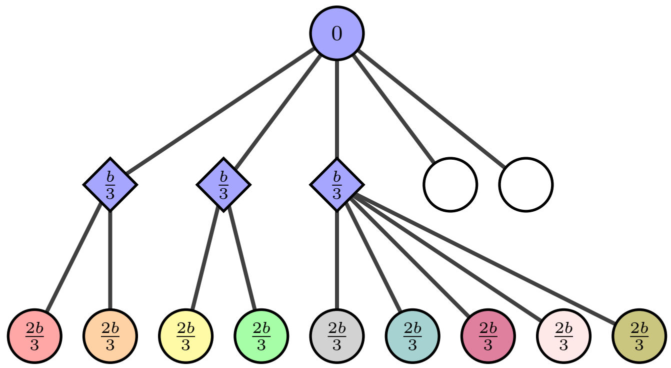


Figure 6: An Alternate Assignment

The total payoff of the implemented equilibrium is $13 - (c(b) + 9 \cdot c(\frac{2b}{3}))$. The payoff from modularization is $15 - 4 \cdot c(b)$, and the alternate assignment does better than the modular assignment if the cost function satisfies $c(b) > \frac{2}{3} + 3 \cdot c(\frac{2b}{3})$. With a completion threshold of $b = 0.99$, the quadratic cost function $c(e) = e^2$ fails the inequality while $c(e) = e^6$ does not. Interestingly, a very convex cost function $c(e) = e^{41}$ fails the inequality yet again because $c(b) = 0.99^{41} \approx 0.6622 < \frac{2}{3}$; the sufficient condition in Theorem 4 is satisfied and the modular assignment is optimal.

4.2 Beyond Key-Peripheral Networks

A natural step towards a general characterization of optimal task assignments is to include intermediary tasks that act as helper tasks for multiple key tasks.

Definition 5. An **intermediary** task is linked to at least two key tasks but not any peripheral tasks.

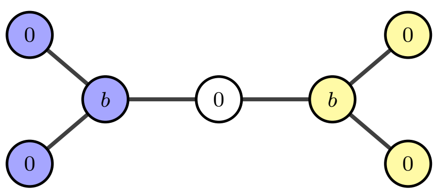
For networks that contain intermediary tasks in addition to key and peripheral tasks, a local reduction in the space of assignments still applies: if an assignment achieves a strictly higher payoff than the modular assignment, then it must implement intermediate effort $e_i \in (0, b)$ at some key task i. For this larger class of project structures, a null result regarding optimality holds.

Theorem 6. *For a network composed of key, peripheral and intermediary tasks, there exists a cost function that makes the modular assignment optimal and another cost function that makes it suboptimal.*

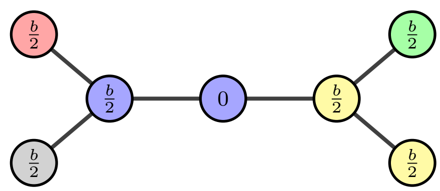
Proof. See Appendix. □

I discuss the intuition behind the proof of Theorem 6, but relegate its finer details to the Appendix. To make the modular assignment optimal, the desired cost function should ensure spreading costs via convexity is futile. A cost function that is sufficiently close to being linear, such as $c(e) = (\frac{e}{b\beta})^{\alpha}$ with β sufficiently large to ensure $c(b) < 1$ and α sufficiently close to (but greater than) 1 does the trick. Concerning suboptimality, I construct a cost function and an assignment that beats modularization similar to the proof of Theorem 5. Example 3 illustrates. The main takeaway is that outside the class of sufficiently specialized key-peripheral networks, characterizing the optimal assignment requires additional assumptions regarding the exact cost function and network structure.

Example 3. The following figure depicts two different assignments on a network with an intermediary task connecting two star graphs.



(a) Modular Assignment



(b) Alternate Assignment

Figure 7: A Network with an Intermediary Task

Each color represents a different assigned agent and white tasks remained unassigned. Notice that the presence of the intermediary task can allow all tasks to be completed in the equilibrium of the alternate assignment. The alternate assignment spreads effort across multiple tasks and achieves a higher payoff if $\frac{c(b)}{4} > c(\frac{b}{2})$. Fixing $b = 0.99$, the cubic cost function $c(e) = e^3$ satisfies this inequality but another cost function $c(e) = e^{4/3}$ fails to do so.

Now, the analysis becomes difficult and fickle upon incorporating tasks that are not key, peripheral or intermediary. Determining whether an effort profile is implementable or not is a delicate exercise that hinges on the exact network structure.

Example 4. Take the network from Figure 7 and add an extra task k linked to the intermediary task and one of the key tasks as shown below.

To beat the modular assignment on this network, an alternate assignment needs to implement intermediate efforts at either key task ℓ or i and finish all of its peripheral

(a) Modular Assignment (b) Tasks to Reduce Costs

Figure 8: A Network with an Extra Task

tasks. I focus on the first case. Implement intermediate effort $e_\ell \in (0, b)$ and assign the peripheral tasks to two different agents who play efforts $b - e_\ell$. As always, the agent assigned to task ℓ needs to be assigned to a supporting neighboring task j with $e_j + \sum_{m \in G_j} e_m = b$ to prevent him from shirking at task ℓ. Since $G_k \cup \{k\} \subset G_j \cup \{j\}$, we can immediately conclude that $e_j + \sum_{m \in G_j} e_m = b$ implies $e_k + \sum_{m \in G_k} e_m < b$. In other words, task k must be left incomplete and cannot be assigned to any agent in equilibrium, lest that agent will have a profitable deviation to try and complete task k. Moreover, effort e_i must be strictly less than b. This leaves two possible assignments, either task i is assigned to the blue agent or another agent:

(a) Blue Agent Task i (b) Yellow Agent Task i

Figure 9: Two Possible Assignments

In the left assignment, $e_j + \sum_{m \in G_j} e_m = b$ so the blue agent is exerting total effort of b for a cost of $c(b)$. In the right assignment, the yellow agent is assigned to a key task with intermediate effort effort $e_i < b$ and must be assigned to a neighboring peripheral task. The yellow agent will also exert total effort of b for a cost of $c(b)$. In either assignment, there is one agent who exerts total effort of b for a cost of $c(b)$ and the alternate assignment achieves an upper bound payoff of $7 - c(b)$. The modular assignment achieves a a higher payoff of $8 - 2c(b)$ since $8 - 2c(b) > 7 - c(b) \iff 1 > c(b)$.

Similar reasoning holds for the other case where the principal implements intermediate effort at key task i, which I detail in the Appendix. Thus, after a long-winded case analysis, we conclude the modular assignment is optimal on this network. It is important to stress that determining which effort profiles are implementable in this network relied specifically on how task k is linked in the network. The analysis changes drastically when we add another task linked to the intermediary task and the other key task.

(a) Modular Assignment

(b) Alternate Assignment

Figure 10: Network with Two Extra Tasks

The alternate assignment beats the modular assignment in this project if $c(b) > 5 \cdot c(\frac{b}{2})$. The addition of an unrelated task at another region of the project structure dramatically transformed the space of implementable effort profiles. In particular, an alternate assignment that reduces task loads and exploits convexity does not leave tasks incomplete in this network. To conclude, determining whether an effort profile is implementable, let alone characterizing the optimal task assignment, relies on how each task is positioned within an arbitrary project structure.

5 Imperfect Substitutability

I introduce an extension of the model that incorporates imperfect substitutability between tasks. I modify the per-task payoff function to be $\mathbb{1}\{e_i + \delta \sum_{\ell \in G_i} e_\ell \geq b\}$ where δ measures the substitutability of effort at linked tasks. For simplicity, I assume the degree of substitutability is constant across all neighboring tasks. High δ values reflect high levels of effort substitutability where $\delta = 1$ represents perfect substitutability, while low δ values reflect low levels of substitutability. If $\delta = 0$, then links between tasks are superfluous and each task can be treated as a singleton.

Under imperfect substitutability, I require effort $\frac{b}{\delta}$ instead of b at tasks in a minimum dominating set. The performance of the modular assignment depends on the value of δ. If δ is very low, then exerting $\frac{b}{\delta}$ at the key task can be expensive for both the principal and the agent. An adequately high δ alleviates this concern.

Theorem 7. *If $\min_{i \in \mathcal{K}(G)} \sigma_i \geq c(\frac{b}{\delta})$ and δ is sufficiently high (which can depend on the cost function and network structure), the modular assignment implements the minimum dominating set and is optimal for the principal.*

Proof. See Appendix. □

Example 5. I provide intuition for how δ plays a role in determining the implementability and optimality of modularization. Consider the connecting star graph where one key task is linked to three peripheral tasks and the other is connected to two peripheral tasks.

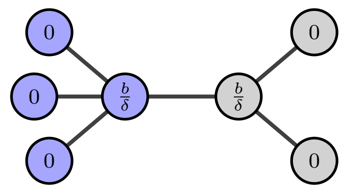


Figure 11: Modular Assignment on the Connecting Star

Implementability: The payoff received by the blue agent is $4 - c(\frac{b}{\delta})$. δ needs to be sufficiently high to prevent the blue agent from having a profitable deviation. We can inspect all possible deviations the blue agent has while holding his number of completed tasks constant. If the blue agent seeks to complete only one task within his assigned module, then the cheapest way to do so is by shirking and free-riding at the key task; δ must satisfy the incentive-compatible (IC) constraint $1 - c(0) \leq 4 - c(\frac{b}{\delta})$. The least expensive way to complete two tasks within his module is by exerting b at a peripheral task and free-riding at the key task; the corresponding constraint is $2 - c(b) \leq 4 - c(\frac{b}{\delta})$. Continuing this pattern, the modular assignment implements the minimum dominating effort profile for the blue agent if δ satisfies the following IC constraints: $4 - c(\frac{b}{\delta}) \geq m - c((m-1)b)$ for $m \in \{1, 2, 3, 4\}$. A δ value that satisfies every IC constraint for the blue agent exists since c is continuous and every inequality holds for $\delta = 1$. Under imperfect substitutability, assigning agents to an entire module has its benefits— a greater gross payoff for agents permits lower values of δ. Ostensibly, a higher δ parameter is needed for more convex cost functions. A computation yields that δ needs to be at least 0.469 to satisfy all IC constraints if $b = \frac{3}{4}$ and a quadratic cost. For a cubic cost and the same completion threshold, δ needs to be greater than 0.558.

For the minimum dominating effort profile to be implementable across the entire network, δ must satisfy the IC constraints for all agents. The sufficient level of δ is higher for the gray agent because he is assigned to a smaller module. The smaller the module, the more expensive it is to exert effort $\frac{b}{\delta}$. Thereby, it is sufficient for δ to satisfy the IC constraints for the agent assigned to the smallest module. Repeating the same calculations for the gray agent with $b = \frac{3}{4}$, δ must be at least 0.600 for a quadratic cost, while for a cubic cost, the lower bound for δ is 0.667.

Optimality: I relegate the finer case analysis to the proof in the Appendix. The crucial insight is that if δ is sufficiently large, then the analysis essentially reduces to that of Theorem 4. Modularization is optimal as long as each key task is sufficiently specialized.

6 Understaffing

What kind of task assignments should the principal employ when her team is understaffed? I relax Assumption 2 and allow there to be fewer agents than tasks in the project i.e. $n < k$. How understaffing affects the principal's optimal assignment depends on how big the project is. If $n \geq \gamma(G)$ where $\gamma(G)$ is the domination number of key-peripheral network G, then the modular assignment is still feasible and Theorem 4 holds. However, if $n < \gamma(G)$, then the comparison across assignments depends on the network structure and costs. Free-riding is less of a problem as task overloading now.

Example 6. If $n = 2$, then using an assignment that assigns the blue and gray agents to some modules completes eight tasks out of ten in the project shown in Figure 12.

(a) Modular Assignment (b) Overloading Modular Assignment

Figure 12: Assignments with $n = 2$

Assignments that spread effort among tasks within modules reap low payoffs when there is understaffing because they require assigning different agents to peripheral tasks. Instead, the principal should utilize an **overloading modular assignment** and assign agents to multiple modules. This accomplishes more tasks but incurs higher costs. If $c(2b) - c(b) \leq 2$, then the gray agent will exert effort level b at both of his key tasks. Moreover, this implies the overloading modular assignment also obtains a higher payoff than the modular assignment.

In the example above, there is only one overloading modular assignment. In larger projects, how modules are divided among agents is central to determining which assignment is best. For instance, in a project with 6 modules and 3 agents, the payoffs to assigning 2 agents to 3 modules each (where each agent exerts effort level $3b$) or 3 agents to 2 modules each (where each agent exerts effort level $2b$) depends on the values of $c(3b)$ and $c(2b)$. As can be seen, the structure of optimal assignments where the team is acutely understaffed i.e. $n < \gamma(G)$ depends on the specific cost function and project.

7 Concluding Remarks

Free-riding is a prevalent issue in network games where links symbolize effort spillovers and substitutability; agents will shirk at central tasks in the project and they should be assigned more tasks. Assigning too many tasks to each agent runs into the problem of task overload. This paper offers an intuitive solution to both problems via the modular assignment on key-peripheral projects. Assigning an agent to a module of related tasks— a key task and its peripheral tasks— ensures that he will prioritize completing the most important task in his work stream. The modular assignment is optimal on key-peripheral networks where key tasks are sufficiently specialized. Key-peripheral projects with highly specialized key tasks are common and embrace two important principles in systems design: inheritance and loose coupling. Characterizing the optimal task assignment beyond key-peripheral projects requires supplementary details regarding the shape of the cost function and the task structure.

There are multiple directions for future research on the task assignment problem. First, in my model, I take the project structure as a primitive. Adding an initial project planning period that precedes the task assignment phase can capture that, in practice, the principal may have some control over project design. Second, agents may have different skill sets more suitable for some tasks than others. Incorporating heterogeneous productivity among agents across tasks can add another dimension to the trade-off— in addition to mitigating free-riding and convexity, the principal should try to match agents to tasks that fit their skill sets. Third, projects may contain tasks that have to be completed in sequence. The underlying task structure thus takes the form of a directed network, and the optimal task assignment will likely depend on the direction of effort spillovers and substitutability. Lastly, some tasks may take longer than others to complete. Embedding the task assignment problem in a dynamic setting can introduce time-dependent free-riding behavior where agents wait until linked tasks are completed before starting their task. Equipping the principal with the ability to set deadlines or project milestones in tandem with task assignment power would be interesting to explore.

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A Omitted Proofs

A.1 Proof of Theorem 3

Proof. Consider Algorithm 2, which takes as input a minimum dominating set and outputs a minimum dominating set.

Algorithm 2

Require: D_{min} is a minimum dominating set
Implement D_{min} via Algorithm 1
for each agent assigned to a pair of tasks $i \in D_{min}$ and $j \notin D_{min}$ **do**
    if $\tilde{D} = (D_{min} \cup \{j\}) \setminus \{i\}$ dominates the network **then**
        return $\tilde{D}$
    end if
end for
return D_{min}

Start with an arbitrary minimum dominating set and recursively call Algorithm 2 until a fixed point (a minimum dominating set) is reached. Note that each call to Algorithm 2 returns a minimum dominating set because $\tilde{D}$ within the loop satisfies $|\tilde{D}| = |D_{min}|$. Since task $j \notin D_{min}$ satisfies $G_j \cap D_{min} = \{i\}$ and $\tilde{D} = (D_{min} \cup \{j\}) \setminus \{i\}$, then $G_j \cap \tilde{D} = \emptyset$. Therefore, the set of agents assigned to a pair of tasks shrinks with each recursive step, ensuring a fixed point is reached in a finite number of steps.

Let $\tilde{D}_{min}$ be the minimum dominating set that results from the recursion of Algorithm 2. Now construct the **generalized modular assignment** via Algorithm 3:

Algorithm 3

Assign each task in $\tilde{D}_{min}$ to an unassigned agent
for each assigned agent **do**
    if his assigned task i neighbors another task in D_{min} **then**
        Assign him to tasks $j, k \notin \tilde{D}_{min}$ such that $G_j \cap \tilde{D}_{min} = G_k \cap \tilde{D}_{min} = \{i\}$
    end if
end for

There exists at least one extra task j such that $G_j \cap \tilde{D}_{min} = \{i\}$ by the same argument in Theorem 2. Suppose that there does not exist another task $k \neq j$ with the same property. Then $(\tilde{D}_{min} \cup \{j\}) \setminus \{i\}$ is a minimum dominating set, which contradicts that $\tilde{D}_{min}$ was the output fixed point of Algorithm 2.

Like before, an agent assigned to a single task will always play effort b to reap a net payoff of $1 - c(b) > 0$. Agents assigned to three tasks will always play effort b at his assigned task $i \in \tilde{D}_{min}$ and zero effort at his two other tasks not in $\tilde{D}_{min}$. If he chooses some other effort level $e_i \in [0, b)$ in equilibrium, then playing effort $e_h = b - e_i$ at one of his other tasks h ensures the completion of two tasks (i and h) at a cost of $c(e_h + e_i) = c(b)$. But there is one other assigned task that is left incomplete. Thus $e_i = b$ completes strictly more tasks with a weakly lesser cost for a higher net payoff. □

A.2 Characterization of Efficiency in Proposition 3

Proof. Repeating the statement of the claim: that the efficient effort at key task i solves $c'(e_i^*) = |S_i| \cdot c'(b - e_i^*)$ where S_i is its peripheral set. The efficient effort profile solves the following cost-minimization problem:

$$\min_{\{e_i\}_{i=1}^k} \sum_{i=1}^k c(e_i) \quad \text{s.t. } e_i + \sum_{\ell \in G_i} e_\ell \geq b \text{ and } e_i \geq 0 \text{ for all } i$$

Let $\{e_j^*\}_{j=1}^k$ denote the solution. The first-order conditions of the cost-minimization problem are:

$$c'(e_i^*) = \sum_{\ell \in G_i \cup \{i\}} \lambda_\ell + \mu_i \quad ; \quad c'(e_h^*) = \lambda_i + \lambda_h + \mu_h$$

where λ_ℓ is the Lagrange multiplier on the $e_\ell^* + \sum_{\ell' \in G_\ell} e_{\ell'} \geq b$ constraint and μ_i is the Lagrange multiplier on the nonnegativity constraint. Note that $e_i^* + e_h^* = b$ for all $h \in S_i$, otherwise the total cost is not minimized. Thereby $e_h^* = e_{h'}^*$ for all $h, h' \in S_i$. Before proceeding, I state a helpful Lemma.

Lemma: For all $h \in S_i$, $e_h^* > 0$.

Assume by contradiction that there exists $h \in S_i$ such that $e_h^* = 0$. This implies $e_i^* = b$ and subsequently $e_h^* = 0$ for all $h \in S_i$. Define $L := \{\ell \in G_i : e_\ell^* + \sum_{\ell' \in G_\ell \setminus \{i\}} e_{\ell'}^* = 0\}$. Note that $S_i \subseteq L$ by assumption, so L is non-empty. All of the tasks in L require $e_i^* = b$ so their respective effort thresholds are met. Consider the following ϵ-deviation effort profile: increase effort at all tasks in L by some $\epsilon > 0$ and decrease effort at task i by the same ϵ. This deviating effort profile completes the same number of tasks as before, but with a lower cost if ϵ is chosen to satisfy $|L| \cdot c(\epsilon) + c(b - \epsilon) < c(b)$.

Since c is C^1 and $c'(0) = 0$ by assumption, there exists $\epsilon_0 > 0$ such that for all $\epsilon_0' < \epsilon_0$, $\frac{c(\epsilon_0')}{\epsilon_0'} < \frac{c'(0.99b)}{|L|}$. There also exists $\epsilon_1 > 0$ such that for all $\epsilon_1' < \epsilon_1$, $\frac{c(b) - c(b - \epsilon_1')}{\epsilon_1'} > c'(0.99b)$ because c is strictly increasing and convex. Set $\epsilon = \min\{\epsilon_0, \epsilon_1\}$ so that $\frac{|L| \cdot c(\epsilon)}{\epsilon} < \frac{c(b) - c(b - \epsilon)}{\epsilon}$ which implies $|L| \cdot c(\epsilon) + c(b - \epsilon) < c(b)$ and proves the Lemma.

By the Lemma, $e_h^* > 0$ and so $\mu_h = 0$. Moreover, $e_i^* + |S_i| \cdot e_h^* > b$, which implies $\lambda_i = 0$. The first order condition can be simplified:

$$c'(e_h^*) = \lambda_i + \lambda_h + \mu_h \implies c'(e_h^*) = \lambda_h$$

$$c'(e_i^*) = \lambda_i + \sum_{h \in S_i} \lambda_h + \sum_{j \in G_i \setminus S_i} \lambda_j + \mu_i = |S_i| \cdot c'(e_h^*) + \sum_{j \in G_i \setminus S_i} \lambda_j$$

Since all other neighboring task nodes $j \in G_i \setminus S_i$ are also key tasks, then $\lambda_j = 0$. Then $\sum_{j \in G_i \setminus S_i} \lambda_j = 0$ and $c'(e_i^*) = |S_i| \cdot c'(e_h^*) = |S_i| \cdot c'(b - e_i^*)$. □

A.3 Proof of Theorem 5

Proof. Fix some network G with minimum specialization $\min_{i \in \mathcal{K}(G)} \sigma_i < 1$. By Theorem 4, the modular assignment is optimal for any cost function with $c(b) \leq \min_{i \in \mathcal{K}(G)} \sigma_i$.

Now I construct an alternate assignment and a suitable cost function that makes the modular assignment suboptimal. A cost function satisfying $c(b) > \min_{i \in \mathcal{K}(G)} \sigma_i$ is not enough as Example 2 will demonstrate; the shape of the cost function matters. Label the key task with lowest specialization as i and consider an alternate task assignment that uses i to support intermediate efforts at every neighboring key task: assign all tasks in $\{i\} \cup (G_i \setminus S_i)$ to one agent (call him agent a), each peripheral task $k \in S_j$ for key task $j \in G_i \setminus S_i$ to a different agent, and all other modules following the modular assignment. This assignment keeps the tasks in S_i unassigned.

Now consider the effort profile where agent a plays effort $\frac{b}{|G_i \setminus S_i|}$ at each task in $G_i \setminus S_i$ and zero effort at task i. Each agent assigned to a peripheral task of a key task residing in $G_i \setminus S_i$ plays effort $\frac{b(|G_i \setminus S_i| - 1)}{|G_i \setminus S_i|}$. Every other agent assigned to a module plays effort b at his respective key task and zero at his peripheral tasks.

Agent a does not have a profitable deviation in lowering his effort at any of his tasks lest he loses task i's completion payoff. Every agent assigned to a peripheral task of a key task in $G_i \setminus S_i$ is playing a best response. The alternate assignment implements said effort profile as a Nash equilibrium with intermediate effort at key tasks in $\{i\} \cup (G_i \setminus S_i)$ and replicates the modular payoff everywhere else. Thus, it is without loss to compare the payoffs between the alternate assignment and the modular assignment on the key tasks residing in $\{i\} \cup (G_i \setminus S_i)$ and their peripheral tasks. The payoff to the alternate task assignment is:

$$\underbrace{\sum_{j \in G_i \setminus S_i} |S_j| + |\{i\} \cup (G_i \setminus S_i)|}_{\text{total \# of tasks}} - \underbrace{c(0 + (|G_i \setminus S_i|) \cdot \tfrac{b}{|G_i \setminus S_i|})}_{=c(b)\ ;\ \text{agent } a\text{'s cost}} - \sum_{j \in G_i \setminus S_i} |S_j| \cdot \underbrace{c(\tfrac{b(|G_i \setminus S_i|-1)}{|G_i \setminus S_i|})}_{\text{each peripheral agent's cost}}$$

The payoff to the modular assignment is:

$$|S_i| + \sum_{j \in G_i \setminus S_i} |S_j| + |\{i\} \cup (G_i \setminus S_i)| - |\{i\} \cup (G_i \setminus S_i)| \cdot c(b)$$

The alternate assignment achieves a strictly higher payoff if:

$$|G_i \setminus S_i| \cdot c(b) > |S_i| + \sum_{j \in G_i \setminus S_i} |S_j| \cdot c(\tfrac{b(|G_i \setminus S_i|-1)}{|G_i \setminus S_i|})$$

The goal is to construct a cost function $c : \mathbb{R}^+ \to \mathbb{R}^+$ that is C^1, strictly increasing and strictly convex with $c(0) = 0$ and $c'(0) = 0$ that satisfies the above inequality. Set $c(b) = \frac{1}{2}(\frac{|S_i|}{|G_i \setminus S_i|}+1)$. By assumption, $\min_i \sigma_i = \frac{|S_i|}{|G_i \setminus S_i|} < 1$, so $\min_i \sigma_i < c(b) < 1$ and there exists some $\epsilon \in (0,1)$ such that $c(b) = \epsilon + \frac{|S_i|}{|G_i \setminus S_i|}$. Set $c(\frac{b(|G_i \setminus S_i|-1)}{|G_i \setminus S_i|}) = \frac{\epsilon \cdot |G_i \setminus S_i|}{2m \sum_{j \in G_i \setminus S_i} |S_j|}$ where m is a natural number chosen such that $\frac{\epsilon \cdot |G_i \setminus S_i|}{2m \sum_{j \in G_i \setminus S_i} |S_j|} < \frac{1}{2}(\frac{|S_i|}{|G_i \setminus S_i|} + 1)(\frac{(|G_i \setminus S_i|-1)}{|G_i \setminus S_i|})$, where the latter term is the secant line between $(0,0)$ and $(b, \frac{1}{2}(\frac{|S_i|}{|G_i \setminus S_i|}+1))$ evaluated at $\frac{b(|G_i \setminus S_i|-1)}{|G_i \setminus S_i|}$. This ensures convexity is satisfied and the alternate assignment achieves a strictly higher payoff than the modular assignment. We now have three points necessary to construct a polynomial: $(0,0)$, $(\frac{b(|G_i \setminus S_i|-1)}{|G_i \setminus S_i|}, \frac{\epsilon \cdot |G_i \setminus S_i|}{2m \sum_{j \in G_i \setminus S_i} |S_j|})$ and $(b, \frac{1}{2}(\frac{|S_i|}{|G_i \setminus S_i|} + 1))$. The desired C^1, strictly monotone and convex cost function with a specified endpoint derivative $c'(0) = 0$ can be generated by the Fritsch-Carlson method for shape-preserving piecewise cubic Hermite polynomial interpolation (see Fritsch and Carlson (1980) and Ferguson and Pruess (1991)). □

A.4 Proof of Theorem 6

Proof. Let G be a network composed of key, peripheral and intermediary tasks.

Cost Function for Optimality: Any assignment that achieves a strictly higher payoff than the modular must implement an equilibrium with intermediate effort $e_i \in (0, b)$ at some key task i and complete all of its peripheral tasks i.e. efforts $b - e_i$ played. For this to be an equilibrium, the agent assigned to key task i, call him agent a, must be assigned to a supporting neighboring task j. This supporting task can be key or intermediary.

Case #1: Task j is key. Then its peripheral tasks must be incomplete, and the modular assignment does better in this case for some cost function with $c(b) < \min_{i\in\mathcal{K}(G)} \sigma_i$. An example is cost function $c(e) = \left(\frac{e}{\beta b}\right)^\alpha$ where β is sufficiently large.

Case #2: Task j is intermediary. As an aside, first notice the first-best key task effort in a network that contains intermediary tasks is exactly the same as that of a standard key-peripheral network i.e. $c'(e_i) = |S_i|c'(b - e_i)$. Now we focus on the class of cost functions $c(e) = \left(\frac{e}{\beta b}\right)^\alpha$ with $\alpha > 1$ and β sufficiently large to ensure $c(b) < 1$. In the first-best:

$$e_i^* = b\left(\frac{\exp\left(\frac{\ln(|S_i|)}{\alpha-1}\right)}{1+\exp\left(\frac{\ln(|S_i|)}{\alpha-1}\right)}\right)$$

By L'Hôpital's rule, $\lim_{\alpha\to1^+} e_i^* = b$, and so for α sufficiently close to 1, the difference between the modular assignment and the first-best payoffs is negligible. Now returning to the original inquiry, if task j is intermediary, then there must be at least one key task in $k \in G_j$ different from i by definition. In equilibrium, it must be that $e_k < b$ lest $e_j + \sum_{\ell\in G_j} e_\ell > b$ and agent a will have a profitable deviation by reducing his effort at task i. The upper bound to the payoff of the alternate assignment on tasks in $\{i, j, k\} \cup S_i \cup S_k$ is to complete all tasks at a minimum cost:

$$\min_{e_i,e_j,e_k} c(e_i) + |S_i|c(b - e_i) + c(e_j) + c(e_k) + |S_k|c(b - e_k)$$

$$\text{s.t. } e_j + \sum_{\ell\in G_j} e_\ell = b \text{ ; } e_i, e_j, e_k \geq 0$$

Denote $\hat{e}_i$ and $\hat{e}_k$ to be the solutions to this cost minimization problem. For the class of cost functions $c(e) = e^\alpha$ with $\alpha > 1$, it cannot be that both $\lim_{\alpha\to1^+} \hat{e}_i = b$ *and* $\lim_{\alpha\to1^+} \hat{e}_k^* = b$ since the constraint $e_j + \sum_{\ell\in G_j} e_\ell = b$ must be obeyed so that $e_i \in (0, b)$ is an equilibrium effort profile. Thus there is always a nonnegligble wedge between the first-best payoff and that of this alternate assignment, and we can conclude that the modular assignment does better in this case for cost function $c(e) = \left(\frac{e}{\beta b}\right)^\alpha$ with α sufficiently close to 1.

Now to tie both cases together, the modular assignment is optimal for cost function $c(e) = \left(\frac{e}{\beta b}\right)^\alpha$ with sufficiently large β and α sufficiently close to 1.

Cost Function for Suboptimality: Let $\mathcal{I}(G)$ be the set of all intermediary tasks of network G. Choose some intermediary task j such that $G_j \cap D_{min} \not\supset G_k \cap D_{min}$ for all $k \in \mathcal{I}(G) \cap G_j$. Assign intermediary task j and all key tasks $G_j \cap D_{min}$ to some agent, call him agent a. Assign every peripheral task of every key task in $G_j \cap D_{min}$ to a different agent. Define the set $\mathcal{J} := \{k \mid G_k \cap D_{min} \subseteq (G_j \cap D_{min})\}$ and notice that $\mathcal{J} \cap G_j = \emptyset$ by property of task j. Assign the tasks in $\mathcal{J}$ according to Algorithm 4.

Algorithm 4

for $x \in \{1, 2, ..., |G_j \cap D_{min}|\}$ **do**
 for each $k \in \mathcal{J}$ with $|G_k \cap D_{min}| = x$ **do**
  if not linked to a task in $\mathcal{J}$ already assigned **then**
   Assign k to an unassigned agent
  else
   Remain unassigned
  end if
 end for
end for

For every other task $\ell \in D_{min}$, assign ℓ and two of its peripheral tasks to a different agent. This assignment implements the following effort profile as a Nash equilibrium. Agent a exerts effort $\frac{b}{|G_j|}$ at every task in G_j and zero effort at task j. An agent assigned to a task $k \in \mathcal{J}$ with $|G_k \cap D_{min}| = x$ best responds with effort $\frac{b(|G_j|-x)}{|G_j|}$. Every other agent plays effort b at his assigned task in D_{min} and zero elsewhere. The nontrivial best response to check is that of agent a. If he shirks at any of his tasks in G_j, then he loses the task completion payoff at task j. $\mathcal{J} \cap G_j = \emptyset$ ensures that any of the positive efforts played at intermediary tasks in $\mathcal{J}$ do not contribute to the total efforts at task j. Playing effort b at task j and zero at every other task in G_j is not profitable because every task in G_j is completed regardless of his effort provision. This is because the tasks in $G_j \cap D_{min}$ neighbor at least two peripheral tasks with efforts $\frac{b(|G_j|-1)}{|G_j|}$ with $|G_j| \geq 2$, so the total neighboring efforts total to at least the completion threshold b. Thereby agent a does not have a profitable deviation to playing effort b at task j.

The payoff to this assignment replicates the modular assignment everywhere besides the tasks in $\{j\} \cup G_j \cup \mathcal{J}$. It is easy to see that the completion threshold is met at tasks in $G_j \cup \{j\}$ and the assigned tasks in $\mathcal{J}$. For the unassigned tasks in $\mathcal{J}$, their neighboring efforts total to at least $\frac{b}{|G_j|} + \frac{b(|G_j|-1)}{|G_j|} = b$, so they are also completed. Thus the alternate assignment completes every task in $\{j\} \cup G_j \cup \mathcal{J}$ with an upper bound cost of:

$$c(|G_j| \cdot \tfrac{b}{|G_j|}) + |\mathcal{J}| \cdot c(\tfrac{b(|G_j|-1)}{|G_j|})$$

The modular assignment counts a cost of $|G_j \cap D_{min}| \cdot c(b)$ on the tasks in $\{j\} \cup G_j \cup \mathcal{J}$. Thus the assignment achieves a strictly higher payoff if:

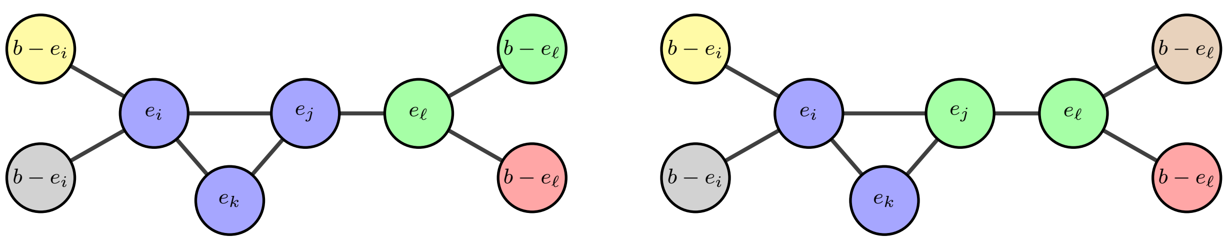
$$|\mathcal{J}| \cdot c(\frac{b(|G_j|-1)}{|G_j|}) < (|G_j \cap D_{min}| - 1)c(b) \iff c(\frac{b(|G_j|-1)}{|G_j|}) < \frac{|G_j \cap D_{min}|-1}{|\mathcal{J}|}c(b)$$

Now to construct the desired cost function, set $c(b)$ to some value $C \in (0,1)$. and set $c(\frac{b(|G_j|-1)}{|G_j|}) = \frac{|G_j \cap D_{min}|-1}{2m|\mathcal{J}|}$ where m is a natural number chosen such that $\frac{|G_j \cap D_{min}|-1}{2m|\mathcal{J}|} < C \cdot (\frac{b(|G_j|-1)}{|G_j|})$ to preserve convexity. Again, we appeal to the Fritsch-Carlson method for shape-preserving piecewise cubic Hermite polynomial interpolation to generate the desired cost function.

□

A.5 Example 4: Implementing $e_i \in (0, b)$

Proof. Implementing an intermediate effort $e_i \in (0, b)$ is also strictly worse than the modular assignment. The agent assigned to task i must be assigned to a neighboring task. If he is not assigned to task k, then it must remain incomplete and we return to a variation of the assignments shown in Figure 13. Consider the assignments where task k is assigned to that agent.



(a) Blue Task j (b) Green Task j

Figure 13: Two Possible Assignments

The left assignment implements an equilibrium only if $e_i + e_k + e_j = b$ for a total cost of at least $c(e_i + e_k + e_j) + c(e_\ell + b - e_\ell) = 2c(b)$, so it performs strictly worse than the modular assignment. Now let's focus on the right assignment. Since task k is assigned to the blue agent, it will be completed in equilibrium. By the previous analysis, there cannot be an intermediate effort at task ℓ in equilibrium. The right assignment thus accrues a total cost of at least $2c(b)$ in either case of $e_\ell = 0$ or $e_\ell = b$, making it strictly worse than the modular assignment. □

A.6 Proof of Theorem 7

Proof. The proof proceeds in two parts: showing that the modular assignment implements the minimum dominating effort profile under imperfect substitutability for a sufficiently large δ and that the modular assignment is optimal when $c(b) \leq \min_{i \in \mathcal{K}(G)} \sigma_i$ for a sufficiently large δ.

Implementability: The modular assignment assigns each agent to a module; an agent assigned to module $S_i \cup \{i\}$ receives payoff of $|S_i| + 1 - c(\frac{b}{\delta})$ if he plays according to the equilibrium effort profile.

We can inspect all possible deviations the agent has by holding his number of completed tasks in the module constant, which creates $|S_i|$ separate cost minimization problems and generates $|S_i|$ incentive-compatible (IC) constraints. We consider deviations that set $e_i \neq \frac{b}{\delta}$, lest they would not be profitable. The agent's best payoff from completing m tasks within the module $S_i \cup \{i\}$ with $e_i \neq \frac{b}{\delta}$ solves:

$$\min e_i + \sum_{\ell \in S_i} e_\ell$$

$$\text{s.t. } e_h + \delta \sum_{\ell \in G_h} e_\ell \geq b \text{ for exactly } m \text{ tasks and } e_h \geq 0 \text{ for all tasks } h \in S_i \cup \{i\}$$

Since all other agents are playing according to the minimum dominating effort profile, key task i can be completed by free-riding. The solution to the problem above for $m = 1$ is setting $e_i = e_l = 0$ for all $l \in S_i$, for a payoff of 1. To complete 2 tasks, the solution is to set one of the $e_l = b$ and the rest of the efforts equal to zero, for a payoff of $2 - c(b)$. To complete 3 tasks, the solution is to set $e_l = b$ for 2 tasks for a payoff of $3 - c(2b)$. Thereby, for playing $\frac{b}{\delta}$ at key task i to be a best response, the following constraints must hold:

$$|S_i| - c(\tfrac{b}{\delta}) \geq m - c((m-1)b) \text{ for all } m \in \{1, 2, ..., |S_i|\}$$

A sufficiently high δ that is less than 1 that satisfies all constraints exists because c is continuous and $|S_i| \geq 2$. Take the maximum over all δ values that make each agent's best response $e_i = \frac{b}{\delta}$ at their key task, this is the value of δ that makes the modular assignment implement the minimum dominating effort profile under imperfect substitutability.

Optimality: The payoff from the modular assignment is:

$$\sum_{i \in \mathcal{K}(G)} (|S_i| + 1) - |\mathcal{K}(G)| \cdot c(\tfrac{b}{\delta})$$

We break the analysis into cases for an arbitrary module with key task i:

Case #1: A deviation for the principal is to implement effort $e_i = 0$. Then the greatest payoff possible is if the principal assigns all other peripheral tasks to different agents. For modularization to be optimal on module $S_i \cup \{i\}$, the following inequality must hold:

$$|S_i| + 1 - c(b/\delta) \geq |S_i| + 1 - |S_i| \cdot c(b) \implies c(\tfrac{b}{\delta}) \leq |S_i| \cdot c(b)$$

Since the inequality holds when $\delta = 1$ and c is continuous, then there must be some $\delta_1 < 1$ such that the inequality holds for all $\delta > \delta_1$.

Case #2: Another deviation is to implement effort $e_i \in (0, \frac{b}{\delta})$.

Case #2.1: If all of the peripheral task nodes remain unassigned, then obviously none of them are completed and this is sub-optimal compared to the modular assignment. I split this into subcases.

Case #2.2: Assign $m \in \{1, ..., |S_i| - 1\}$ peripheral tasks to different agents. Each agent exerts effort level $e_h = b - \delta e_i$. The maximum payoff of $m + 1 - c(e_i) - m \cdot c(e_h)$ which is bounded from above by $m + 1$, which itself is bounded above by $|S_i|$. There exists δ_2 such that $|S_i| + 1 - c(\frac{b}{\delta_2}) > |S_i|$ again by continuity of c.

Case #2.3: Assign all tasks in S_i to different agents who each exert $e_h = b - \delta e_i$. The total effort level for an agent at task i is:

$$e_i + \delta|S_i|(b - \delta e_i) + \delta \sum_{\ell \in G_i \setminus S_i} e_\ell$$

It would be problematic if the effort threshold does not exceed b, as there would be no equilibrium trade-off for implementing an intermediate effort level: Observe that for any $e_i \in (0, \frac{b}{\delta})$:

$$\delta > \frac{1}{|S_i|} \implies e_i + \delta|S_i|(b - \delta e_i) > b$$

Let δ_4 be greater than $\frac{1}{|S_i|}$. Therefore implementing an intermediate effort level and completing every task within the module is by itself not an equilibrium.

Taking the maximum δ i.e. $\delta = \max\{\delta_1, \delta_2, \delta_3\}$ for the smallest module suffices to fend off profitable deviations in the form of Case #1-#2.3 for *all* modules within the project while guaranteeing that assigning agents as prescribed in Case #2.4 is not an equilibrium. The remainder of the proof of optimality now mirrors that of Theorem 4 assuming the δ is sufficiently high as previously shown except for the following observation.

Implement a key task effort of $e_i \in (0, \frac{b}{\delta})$ and assign all tasks in S_i to different agents. Since δ is sufficiently high, the principal needs to give the agent assigned to key task i a neighboring key task j that supports it in equilibrium. Just like in Proposition 2, the sum of total efforts at key task j must be exactly b i.e. $e_j + \delta \sum_{\ell \in G_j} e_\ell = b$ for this to be an equilibrium. If the principal assigns one of the peripheral tasks ℓ of key task j to an agent, then he will best respond by playing effort $e_\ell = b - \delta e_j$. Note that there exists a sufficiently large δ such that $e_j + \delta e_i + \delta(b - \delta e_j) > b$ since the inequality holds with $\delta = 1$ and the expression is continuous in δ. Thereby if δ is large enough, then none of the peripheral tasks of a supporting key task can be completed in equilibrium. Then all of the conclusions in Theorem 4 follow in the same fashion and modularization is optimal if $c(\frac{b}{\delta}) \leq \min_{i \in \mathcal{K}(G)} \sigma_i$. $\square$