Determinant

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \rightsquigarrow \det(A) = a_{11}\cdot a_{22} - a_{12}\cdot a_{21}.$$

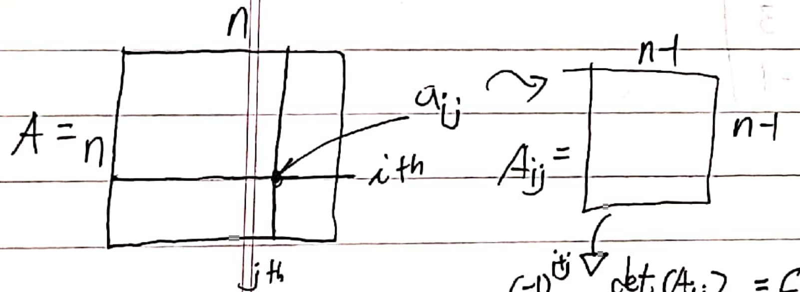
$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightsquigarrow \det(A) =$$

Def

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\boxed{\det(A) = \underbrace{\sum_{\sigma}}_{\substack{\text{permutation} \\ n!\text{개}}} \overset{(-1)^{\#\text{ of transposition for } \sigma}}{(\pm)}\, a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} \cdots a_{n\sigma(n)}}$$

Thm. (Cofactor Expansion thm)



$(-1)^{i+j} \det(A_{ij}) = C_{ij}$ (the cofactor of a_{ij})

Fix j th column

$$a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \det(A) \text{ for any } j.$$

Fix i th row.

$$a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \det(A) \text{ for any } i \quad //$$

Thm.

$A \xrightarrow{\forall} B$

$\det(A) \overset{?}{\longleftrightarrow} \det(B)$

① ~~multiply a~~ add a multiple of a row to another row.

$\Rightarrow \det(A) = \det(B)$

② multiply a row by non zero constant λ

$\Rightarrow \lambda \det(A) = \det(B)$

③ interchange two rows.

$\Rightarrow -\det(A) = \det(B)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \end{bmatrix}$$

$(a_{11} + 3a_{21})\square + \cdots + (a_{1n} + 3a_{2n})\square$

$= \det(A) + 3(a_{21}\square + \cdots)$

$\overset{?}{=} 0$

$$\hookrightarrow A' = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$ 같으면 0000 이상X.

$\det(A') = (a_{21}\square \cdots) = 0$

↪ 0000 말고 중요한 것 강의 보고 이해하기.

① interchange

$\det(A') = -\det(A')$

$\therefore \det(A') = 0.$

< Cramer's Rule.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \qquad \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{b}$$

$$A_1(b) = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$$

$$A_2(b) = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$

assume this has an unique solution.

$\hookrightarrow a_{11} : a_{12} \neq a_{21} : a_{22}$

$$\overset{0}{\neq}$$

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = b_1 a_{22} - b_2 a_{12}.$$

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\det(A_1(b))}{\det(A)}$$

$$\Rightarrow x_2 = \frac{-b_1 a_{21} + b_2 a_{11}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\det(A_2(b))}{\det(A)}.$$

Thm (Cramer's Rule)

$Ax = b \quad A : n \times n,\ inv,$

$\hookrightarrow x = A^{-1} \cdot b$ is a unique solu.

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad x_i = \frac{\det(A_i(b))}{\det(A)}.$$

<pf>

column vector

$A = [a_1 \cdots a_n]$, $I = [e_1, e_2 \cdots e_n]$

$$I_i(x) = [e_1 \cdots \underset{\downarrow x}{\cancel{e_i}} \cdots e_n]$$

$$A \cdot I_i(x) = [Ae_1 \cdots Ax \cdots Ae_n] = [a_1 \cdots b \cdots a_n] = A_i(b)$$

$$\det(A_i(b)) = \det(A) \cdot \det(I_i(x))$$

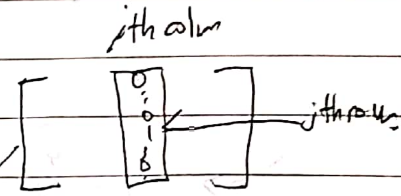
$$\Rightarrow \det(I_i(x)) = \frac{\det(A_i(b))}{\det(A)}$$

CamScanner로 스

Thm

$$A = \begin{bmatrix} & & \\ & & \end{bmatrix} \rightsquigarrow C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & \cdots & & C_{2n} \\ \vdots & & & \\ C_{n1} & \cdots & & C_{nn} \end{bmatrix}$$

$$\mathrm{adj}(A) = C^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ & \vdots & & \\ C_{1n} & \cdots & & C_{nn} \end{bmatrix}$$



$$A^{-1} = \frac{1}{\det(A)} \cdot \mathrm{adj}(A)$$

<pf>

$$A^{-1} = \begin{array}{c} \text{jth} \\ \begin{bmatrix} & x_j & \end{bmatrix} \end{array}$$

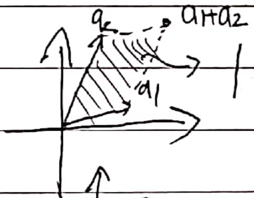
$A \cdot A^{-1} = I$

by Cramer's Rule

$Ax = e_j$, Consider ith entry of x, $= x_i = \dfrac{\det(A_i(e_j)) = C_{ji}}{\det(A)}$

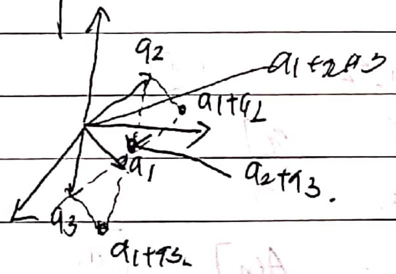
Thm

① $A = 2\times 2 = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$



$\rightarrow |\det(A)|$

② $A = 3\times 3 \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$



Volume $= |\det(A)|$

$\det(A) = 0 \Leftrightarrow a_1, a_2, a_3$ are linearly depen.

Thm

① $T: \mathbb{R}^2 \to \mathbb{R}^2$: a lin. transform

$\cup$
S
$\shortparallel$
parallelogram

$T(S) \subset \mathbb{R}^2$

$\therefore$ Area of $T(S) = |\det([T])|\,\mathrm{area}(S)$.

lin. transform T 에 대해 자세하게 이해하기

~~[illegible]~~ 자코비안

→ 미적 2 책에서 찾아보기.

CH 4. Vector Spaces.

Def

$V \underset{TS}{\neq} \emptyset$ a nonempty set w/ $f : V \times V \to V$ (addition operation)

$g : \mathbb{R} \times V \to V$ (scalar multiplication)

satisfying.

① $\underset{u+v}{f(u,v)} = \underset{v+u}{f(v,u)}$ $\forall u, v \in V$.

② $\underset{u+(v+w)}{f(u, f(v,w))} = \underset{(u+v)+w}{f(f(u,v), w)}$ $\forall u, v, w \in V$. → associative (결합)

③ $\exists\, \vec{0} \in V$ s.t $f(u, \vec{0}) = f(\vec{0}, u) = u$ $\forall u \in V$. → $\vec{0}$ is identity $(=0)$

④ $\forall u \in V$, $\exists\, \Box \in V$ s.t $f(u, \Box) = f(\Box, u) = \vec{0}$ → $\Box$ is $-u$ (is inverse)

⑤ $\underset{c(u+v)}{g(c, f(u,v))} = \underset{= cu + cv.}{f(g(c,u), g(c,v))}$ $\forall c \in \mathbb{R}$, $u, v \in V$

⑥ $g(c+d, u) = f(g(c,u), g(d,u))$ $\forall c, d \in \mathbb{R}$, $u \in V$.

⑦ $g(c, g(d,u)) = g(c \cdot d, u)$, $c, d \in \mathbb{R}$, $u \in V$

⑧ $g(1, u) = u$ $\forall u \in V$

⟹ V is a vector space over $\mathbb{R}$ $V \ni u$: a vector (element of vector space)

Rmk

① $\underset{\mathbb{R}}{0} \cdot u = \underset{V}{\mathbf{0}}$ (zero vector).

<pf>

$$0u = 0 \cdot u + \mathbf{0} \quad (③)$$
$$= 0 \cdot u + (0 \cdot u + (-0 \cdot u)) \quad (④)$$
$$= (0 \cdot u + 0 \cdot u) + (-(0 \cdot u)) \quad (②)$$
$$= ((0+0)u + (-(0 \cdot u)) \quad (⑥)$$
$$= 0 \cdot u + (-(0 \cdot u)) = \mathbf{0} \quad (④)$$

② $c \in \mathbb{R}$ $c \cdot \mathbf{0} = \mathbf{0}$.

③ $\underset{\mathbb{R}}{(-1)} \cdot u = -u$

ch5 Eigenvalue, Eigenvectors.

$$T_A : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \to A \cdot x$$

"fixed point?" $x \in \mathbb{R}^n$ s.t $T_A(x) = x = I \cdot x$.

$$A \cdot x = x.$$

$$(A - I)x = 0.$$

T_A has a nontrivial fixed pt

$$\Longleftrightarrow \det(A-I) = 0.$$

$\boxed{Q}$ Can we have a line $\overset{0}{\underset{\cap}{L}} \subset \mathbb{R}^n$, which is fixed by $T_A : \mathbb{R}^n \to \mathbb{R}^n$

Or can we find L s.t $T_A(L) \subset L$?

$\underset{x \neq 0}{\overset{\cup}{}}$, $T_A(x) \in L$.

$\qquad\qquad \overset{\shortparallel}{\lambda \cdot x}$.

생각해보기.

Def.

$\exists x \neq 0$, $\lambda \in \mathbb{R}$ s.t $Ax = \lambda x$ (eigenvalue를 찾는 이유가 뭘까?)

λ eigenvalue for A. ↳ (추) 혹시 간단하게 표현하기 위해서?

x ↳ eigenvector assoc. to λ

Ex).

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}.$$

λ : an eigenvalue for A

$\leadsto \exists x \neq 0$ s.t $Ax = \lambda x = \lambda \cdot Ix$

$(A - \lambda I)x = 0$ $\quad (A - \lambda I) = \begin{bmatrix} 7-\lambda & 2 \\ -4 & 1-\lambda \end{bmatrix}$

$\Longleftrightarrow \det(A - \lambda I) = 0.$ — a polynomial in λ of degree n.

$\to (7-\lambda)(1-\lambda) + 8 = 0$ $\quad$ → the characteristic poly. of A.

$\to \lambda^2 - 8\lambda + 15 = 0$

$(\lambda - 3)(\lambda - 5) = 0 \Rightarrow \lambda = 3$ or $\lambda = 5$

① $\lambda = 3$.

$(A - 3I)x = 0$.

$$\overset{\shortparallel}{\begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix}} \leadsto \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \quad x_1 + \frac{1}{2}x_2 = 0 \quad x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \Rightarrow x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

② $\lambda = 5$.

$(A - 5I)x = 0$

$$\overset{\shortparallel}{\begin{bmatrix} 2 & 2 \\ -4 & -4 \end{bmatrix}} \leadsto \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \to x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Cor

if A has n distinct eigenvalues $\Rightarrow$ then A is diagonalizable
$\Leftarrow$ not true.

Ex)

$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ $\rightsquigarrow$ diagonalizable $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -2 & & 0 \\ & -2 & \\ 0 & & 1 \end{bmatrix}$

$P^{-1}AP = D$

$\times 1$ 해도 det 변화없다.

$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 3 & 3 \\ 0 & -2-\lambda & -2-\lambda \\ 3 & 3 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 3 & 0 \\ 0 & -2-\lambda & 0 \\ 3 & 3 & -2-\lambda \end{vmatrix}$

$\det(A-\lambda I) = -(\lambda+2)^2(\lambda-1)$

$\lambda = 1$ or $\lambda = -2$

" $=1$ algebraic multiplicity $= 2$

1 $\qquad 2$

$\lambda = -2.$

$(A+2I) = \begin{vmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{vmatrix} \Rightarrow x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ geo. mult (x_2, x_3 둘다 0만 아니면 된다)

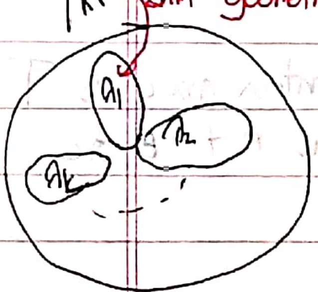
$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$

$\lambda = 1$

$(A-I) = \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} \phi & \phi & \phi \\ 0 & 1 & \phi \\ 0 & 0 & \end{pmatrix}$ $\quad x_1 = x_3$, $x_2 = -x_3$ $\quad x_3\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$

A
$\lambda_1 \cdots \lambda_k$

$\mathbb{R}^n$ dim = geometric multiplicity.



$a_i = g_i$
$\Updownarrow$
n $\rightarrow$ diagonalizable
$\Uparrow$
can select at most $(g_1+g_2+\cdots+g_k)$ lin. indep. eigenvectors

Def.

$S \subset \mathbb{R}^n$ a subset.

$S^\perp = \{ x \in \mathbb{R}^n \mid x \cdot s = 0 \ \forall s \in S \}$

↳ orthogonal complement of S

$\boxed{\begin{array}{l} T \subset \mathbb{R}^n \text{ another subset w/ } S \subset T. \\ \Rightarrow S^\perp \supset T^\perp \end{array}}$

span(S)

$S \subset \text{span}(S)$ 둘의 차이가 뭔지 생각해봐 $\boxed{\text{중요내용}}$

$\Rightarrow S^\perp \supset (\text{span}(S))^\perp$.

In fact, have "="

$S^\perp = \text{span}(S)^\perp$

$S^\perp \ni x \Rightarrow x \in \text{span}(S)^\perp$

i.e. $x \cdot v = 0 \ \forall v \in \text{span}(S)$

$v = c_1 s_1 + \cdots + c_k s_k$

$\rightarrow c_1 \underbrace{x \cdot s_1}_{=0} + \cdots + c_k \underbrace{x \cdot s_k}_{=0} = 0.$

$Null(A) = Row(A)^\perp$

$Null(A^T) = Col(A)^\perp$

A. $m \times n$

$\mathbb{R}^n \supset Null(A) = Row(A)^\perp$

$\mathbb{R}^m \supset Col(A)$

$\mathbb{R}^n \supset row(A)$

$Null(A^T) = Col(A)^\perp$

$Col(A^T)$

$Row(A^T)$

$Null(A) \ni x$

↳ $A \cdot x = 0 = \begin{bmatrix} r_1 x \\ r_2 x \\ \vdots \\ r_m x \end{bmatrix}$, $r_1 \cdot x$ $\Longleftrightarrow x \in \{\text{row vectors of } A\}^\perp$

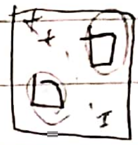
row vectors $A = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}$

$= \text{span}(\cdots)^\perp$

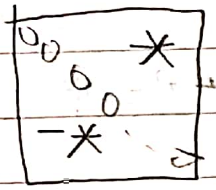
$= Row(A)^\perp$

Def.

$A = A^T \Rightarrow A$ is a symmetric $\rightarrow$



$A = -A^T \rightarrow$ skew-symmetric $\rightarrow$



$A = n \times n$, general

$= \boxed{\frac{1}{2}(A+A^T)} + \boxed{\frac{1}{2}(A-A^T)}$

symmetric.

$\rightarrow$ skew-symmetric

$\frac{1}{2}(A^T + A)$

$A = A^T$

$A = -A^T$

lemma

A: symm.

$\lambda_1 \neq \lambda_2$: eigenvalues of A

v_1, v_2 : eigenvectors

$\lambda_1(v_1 \cdot v_2) = (Av_1)^T \times v_2$

$= v_1^T A^T v_2$

$= v_1^T \lambda_2 v_2$

$= \lambda_2 v_1 \cdot v_2$

$\lambda_1(v_1 \cdot v_2) = \lambda_1(v_1^T v_2) = (\lambda_1 v_1)^T \cdot v_2 = (Av_1)^T \cdot v_2$

$\lambda_1 v_1 = A \cdot v_1$

$= v_1^T A^T v_2$

$= v_1^T(A^T v_2) = v_1^T(\lambda_2 v_2)$

$= \lambda_2(v_1 \cdot v_2)$

$\therefore v_1 \cdot v_2 = 0.$

Thm

A = symm. $\rightsquigarrow Q_A(x) = x^TAx$ a quadratic form

$Q_A : \mathbb{R}^n \to \mathbb{R}$

$x \mapsto x^TAx$

$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$: maximum, minimum value 존재

$n=2 : S^1 \leftarrow$ unit circle

$n=3 : S^2 \leftarrow$ unit sphere

$M = \max\{x^TAx \mid \|x\| = 1\}$

$m = \min\{x^TAx \mid \|x\| = 1\}$

$\Rightarrow P^TAP = D \quad x = Py \quad \lambda_1 > \lambda_2 > \cdots > \lambda_n$

$\|x\| = \|Py\| = \|y\|$

$x^TAx = y^TDy$

$= \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$

$\le \lambda_1 y_1^2 + \lambda_1 y_2^2 + \cdots + \lambda_1 y_n^2$

$= \lambda_1 \|y\|^2 = \lambda_1$

$\Rightarrow$ ① M is the greatest eigenvalue of A.

$x^TAx = M$ for x an unit eigenvector corres to greatest eigenvalue

② m is the smallest eigenvale of A

$x^TAx = m$ smallest eigenvalue

$A = PDP^T$ w/ $P = [v_1 \cdots v_n]$ $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ w/ $\lambda_1 \ge \cdots \ge \lambda_n$

$\max\{x^TAx \mid \|x\| = 1, x^Tv_1 = x^Tv_2 = \cdots = x^Tv_{k-1} = 0\}$

$= \lambda_k$ acheived at $x = v_k$

$\begin{matrix} 5 & -2 \\ -2 & 5 \end{matrix}$ $\quad 5-\lambda$

$\lambda^2 - 10\lambda + 21 = 0$

$\lambda = 3 \quad \lambda = 7$

$\begin{matrix} 2 & -2 \\ 1 & -1 \end{matrix}$ $\quad \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$