linear equation

definition => A linear equation in $x_1 \cdots x_n$ is an eqn

$$(*) \quad a_1x_1 + \cdots + a_nx_n = b \quad (a_i \in \mathbb{R}, b \in \mathbb{R})$$

A system of linear eqn is

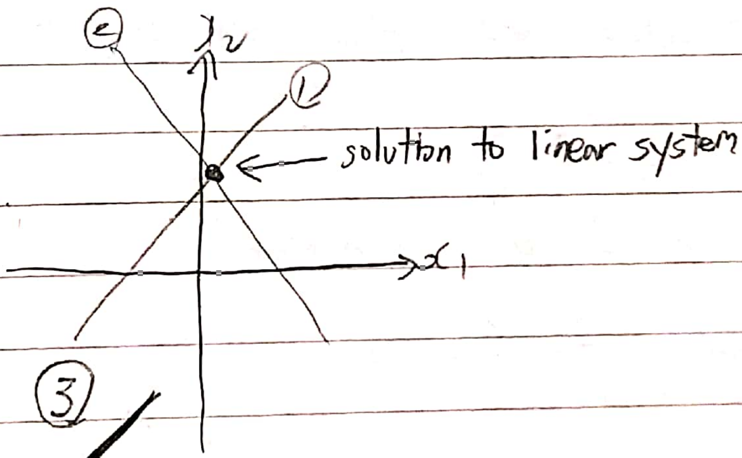
$$(**) \quad \begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n \end{matrix}$$

A point $|s_1 \cdots s_n| \in \mathbb{R}^n$ is a solution of $(*)$

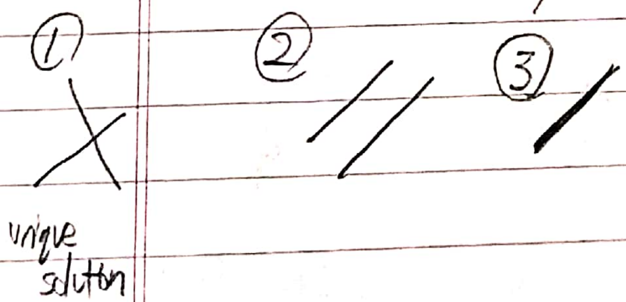
if $a_1s_1 + \cdots + a_ns_n = b$.

Ex) ① $a_{11}x_1 + a_{12}x_2 = b_1$ 적어도 (a_{11}, a_{12}) 중 하나는 0이 아니라 가정

② $a_{21}x_1 + a_{22}x_2 = b_2$



경우



<u>Thm</u>

The solution of a linear system is either unique, empty set or ∞ many pts

Row

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right] \leftarrow \text{matrix}$$

size (3×4)

augmented matrix

matrix of coefficient

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & \cdots ① \\ 2x_2 - 8x_3 = 8 & ② \\ 5x_1 - 5x_3 = 10 & ③ \end{cases}$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_1 - 7x_3 = +8 \\ 5x_1 - 5x_3 = 10. \end{cases}$$

$\times -5$
$+ ⑥$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_1 - 7x_3 = 8 \\ 30x_3 = \cancel{30} -1 \end{cases}$$

$$\downarrow$$

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ x_1 = 1 \\ x_3 = -1 \end{cases}$$

$$\downarrow$$

$$\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{cases}$$

이렇게 해도 되는 이유: 계속 system이 바뀌어도 system의 해가 같다고 믿기 때문

def of (elementary row operation) (row)

① add a multiple of 1 eqn. to another.

② Multiply an eqn. (row) by a nonzero const.

③ interchange two eqn. (rows)

→ The solution is invariant

Def row echelon form (REF)

① All zero rows are grouped together at the bottom

② The leading entry of a row is farther away to the right than the one in the above
(first non-zero entry)

③ The leading entry is 1 (← called the leading one).

④ The entries in the column with leading one should be 0 except leading one.

reduced row echelon form (RREF)

free variable

$$\longrightarrow \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$ RREF

pivot variable

column vector

① $x_1 = -6x_2 - 3x_4$
$x_3 = 4x_4 + 5$
$x_5 = 7$.

pivot — free variable + constant.

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -6x_2 - 3x_4 \\ x_2 \\ 4x_4 + 5 \\ x_4 \\ 7 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \\ 7 \end{bmatrix}$$

def.

A (column) vector in $\mathbb{R}^n$ is a matrix w/size $n \times 1$

$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \rightsquigarrow (u_1, \dots u_n) \in \mathbb{R}^n$

Can define (row) vector in $\mathbb{R}^m$

$u = [u_1 \cdots u_m] \rightsquigarrow (u_1, \cdots u_m) \in \mathbb{R}^m$

Remark

$u, v, w \in \mathbb{R}^n$, $c, d \in \mathbb{R}$

3 column vectors

① $u + v = v + u$ (commutative)

② $(u+v)+w = u+(v+w)$ (associative)

existence of identity ③ $u + 0 = 0 + u = u$. $(0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix})$

" of Inverse ④ $u + (-u) = (-u) + u = 0$.

⑤ $c(u+v) = cu + cv$.

⑥ $(c+d)u = cu + du$ — scalar multiplication

⑦ $c(du) = (cd)u$

⑧ $I \cdot u = u$

Def

$\mathbb{R}^n \ni v_1, \cdots, v_p$, $c_1, \cdots, c_p \in \mathbb{R}$ → weight

$c_1v_1 + c_2v_2 + \cdots + c_pv_p$ is called linear combination.

Ex)

$a_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, $b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

$x_1 \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ 5x_1 + 6x_2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ 5 & 6 & -3 \end{bmatrix}$

Thm

For any matrix, $\exists!$ (unique) RREF assoc. to it.

Def.

$\{c_1v_1 + \cdots + c_pv_p \mid c_i \in \mathbb{R}\} = \text{span}(v_1, \cdots, v_p)$

the space spanned by $v_1, \cdots, v_p$.

Def

$m \times n$ size A $[a_1\ a_2 \cdots a_n]$, a_i is a column vector in $\mathbb{R}^m$

$\mathbb{R}^n$ $X = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \rightsquigarrow \underset{(m\times n)(n\times 1)}{A \cdot X} = x_1a_1 + x_2a_2 + \cdots + x_na_n$ in $\mathbb{R}^m$

Thm.

$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$
$\vdots$
$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ \vdots \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = b$$

$$\Rightarrow A \cdot x = b$$

a lin. system w/ m eqns in n vars

a $m \times (n+1)$ matrix $[\underbrace{A}_{m\times n} | \underbrace{b}_{m\times 1}]$ ← augmented

the last column of $[A|b]$ is a pivot column $\Longleftrightarrow$ system is inconsistent.

Ex)

$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ ⇝ $Ax = b$ $\Rightarrow [A|b]$

⇒ system is inconsistent iff $2b_1 - b_2 + 2b_3 \neq 0$

원가가 일정할때 쓰임 (여기서는 항상 0이기 때문)

The case of $b = 0$, is called a homogeneous linear system

↳ $Ax = 0$

$\mathbb{R}^m$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}\right)$$

$x_i = 0$ 이면 답 (trivial solution)

$\mathbb{R}^n$

Thm

The homogeneous lin. system $Ax=0$ has a nontrivial soln

$\Updownarrow$ (if and only if)

$\exists$ at least 1 free variable

ex) $x_3 \begin{bmatrix} -\frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$ 이런게 general solutn

Thm

Assume $Ax=b$ is consistent (has at least 1 solutn)

say $p \in \mathbb{R}^n$ is a particular soln

Then, the general form of its solutn is $p + v_h$ where V_h is a soln to $(Ax=0)$

('span'이 뭔지 정확히 파악하기)

Def

$\{V_1 \cdots V_p\} \subset \mathbb{R}^n$ is linearly independent if

$(x_1V_1 + \cdots + x_pV_p = 0 \in \mathbb{R}^n \Rightarrow x_1 = \cdots = x_p = 0)$

Otherwise, it is linearly dependent

Ex) $\{V_1 \cdots V_p\}$ lin. dep.

$\Leftrightarrow x_1V_1 + \cdots + x_pV_p = 0$ w/ some $x_i \neq 0$

$x_iV_i = -x_1V_1 - x_2V_2 - \cdots - x_pV_p$

$V_i = -\frac{x_1}{x_i}V_1 - \cdots$

Ex) $V_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $V_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ $V_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$x_1V_1 + x_2V_2 + x_3V_3 = 0$

$\Vert$

$\underbrace{\begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = 0$ $\Rightarrow X=0$ (이면 linearly independent)

$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

free $\rightarrow$ linearly dependent

$\Rightarrow x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\therefore 2V_1 - V_2 + V_3 = 0$

Thm.

column vectors of A are linearly independent $\iff$ $Ax=0$ has only trivial solution

Thm

$\{v_1 \cdots v_p\} \subset \mathbb{R}^n$ w/ $p>n$

$\hookrightarrow$ always lin. dependent

\# of pivot variab.

$n \times p \to A = \begin{bmatrix} v_1 & \cdots & v_p \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & & \\ & 1 & \\ & & \\ \end{bmatrix}$ $n-a$, a (zero rows)

\# of free variables $\Rightarrow p-(n-a) = p-n+a \geq p-n \geq 1$

Def

$T: \mathbb{R}^n \to \mathbb{R}^m$ is transformation

domain, codomain

$T(x) =$ image of x via T

range of T $= \{T(x) \mid x \in \mathbb{R}^n\}$

$\parallel$

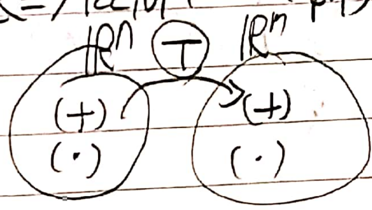
$\mathrm{Ran}(T)$

$T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation

if ① $T(u+v) = T(u)+T(v)$ $\to$ linearity

② $T(cu) = cT(u)$

$(\iff T(c_1v_1 + \cdots + c_pv_p) = c_1T(v_1) + \cdots + c_pT(v_p))$



Ex

T_A

$\parallel$

$A: (m \times n) \rightsquigarrow T: \mathbb{R}^n \to \mathbb{R}^m$

$x \to Ax$

$n \times 1$ $\quad (m \times n)(n \times 1) \Rightarrow m \times 1$

$T(u+v)$

$\parallel$

$A \cdot (u+v) = Au + Av = T(u) + T(v)$

$A \cdot (c \cdot u) = c(Au)$

Ex

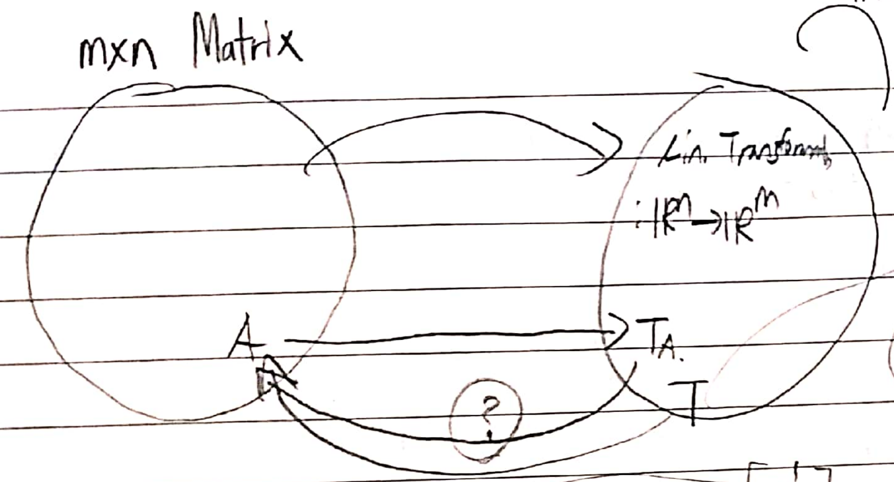
3×2 $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \rightsquigarrow T_A: \mathbb{R}^2 \to \mathbb{R}^3$

Is $C = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ in the Range of T_A? ($\mathrm{Ran}(T_A)$?

no!

$\iff \exists?\ x \in \mathbb{R}^2$ s.t. $Ax = C$?

$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} = [A \mid C] \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\checkmark$ inconsistency think about it



lin. transform 과 Matrix 의 information 은 같다

$T = T_A$

standard unit vector $e_1 \cdots e_n$ $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

Thm.

$T: \mathbb{R}^n \to \mathbb{R}^m$: a lin. transf.

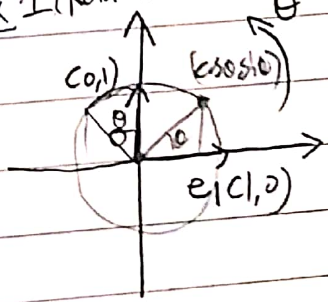
① $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ ② unique way

$\Rightarrow T(x) = x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n)$

$$\underset{n\times 1}{} = \underbrace{\begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}}_{\substack{\| \\ m\times n \quad A}} \times \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\substack{\| \\ X}} = Ax = T_A(x)$$

// $[T]$ — standard Matrix for T.

Ex 1 (Rotation)

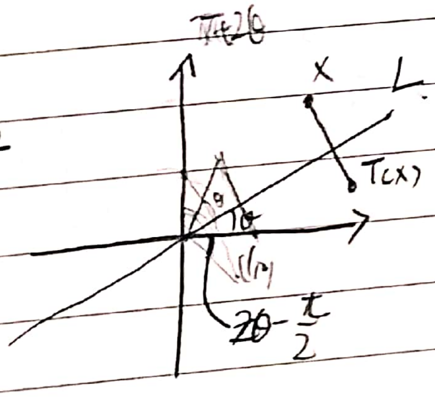


$T: \mathbb{R}^2 \to \mathbb{R}^2$

→ what's $[T]$? $= \underset{2\times 2}{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$

Ex2 (Reflection)

$T: \mathbb{R}^2 \to \mathbb{R}^2$



$[T] = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

$Ran(T) = \mathbb{R}^m$

1-1 $\Rightarrow$

Def

$T: \mathbb{R}^n \to \mathbb{R}^m$ a transformation

① T is called "onto" if $Ran(T) = \mathbb{R}^m$

② T is called "one to one" (1-1) if $x_1 \neq x_2$ in $\mathbb{R}^n \Rightarrow T(x_1) \neq T(x_2)$.

Thm.

if and only if $\Leftrightarrow$ $T(0) = 0$

~~$T: \mathbb{R}^n \to \mathbb{R}^m$ linear trans~~ $T(x)$

T is 1-1 iff $\underset{m \times n}{[T]}X = 0$ X has only trivial solution

1-1 $[T]X = 0$

if $T(x) = 0$ for some $x \neq 0$

$T(0) = 0$.

이거에서 only trivial solution이어야 가능.

$T(x_1) = T(x_2)$ for $x_1 \neq x_2$

$T(x_1 - x_2) = T(x_1) - T(x_2) = 0$

(Think about it)

$[T]$: $m \times n$ matrix colum

column vectors of $[T]$ are lin. indep

$[v_1 \cdots v_n]$

T is onto

$\Updownarrow$

$\forall y \in \mathbb{R}^m$ $Ran(T) = \{[T]X \mid x \in \mathbb{R}^n\}$ $\Longleftrightarrow$ The column vectors of $[T]$ span $\mathbb{R}^m$

$x_1 v_1 + \cdots + x_n v_n$

Matrix

$A=B$ iff ① they have the same size

② 〃 entries at each position

A, B w/ same size

$\hookrightarrow aA+bB$ for any constant a, b $(a, b \in \mathbb{R})$

$\rightarrow$ zero Matrix $O = O_{m,n} = O_{m \times n}$

$\underbrace{A+O}_{\text{same size 가정 필요}} = A$

multiplication 은 linear transformation

Def

$$\underset{m\times n}{A},\ \underset{n\times p}{B} \rightsquigarrow \underset{m\times p}{AB} := [Ab_1\ Ab_2 \cdots Ab_p] \quad \text{definition}$$

column vector: $B = [b_1 \cdots b_p]$; column vector: Ab_1

$(AB)_{ij}$ = the ith entry of $Ab_j = [a_1 \cdots a_n]\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$

$= b_{1j}a_1 + b_{2j}a_2 + \cdots + b_{nj}a_n$

$= b_{1j}a_{i1} + b_{2j}a_{i2} + \cdots + b_{nj}a_{in}$

$= (i\text{th row of } A) \cdot (j\text{th column of } B)$

In addition to this let $X = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} \in \mathbb{R}^p$

$\underset{m\times n}{A},\ \underset{n\times p}{B},\ \underset{p\times 1}{X}$

$$\underbrace{\underbrace{(AB)}_{m\times p} X}_{m\times 1} = \underbrace{A\underbrace{(BX)}_{n\times 1}}_{m\times 1}$$

$BX = x_1b_1 + \cdots + x_pb_p$

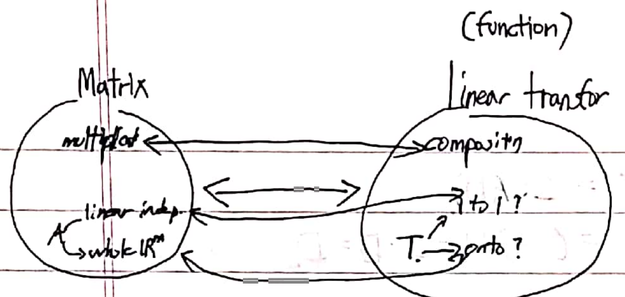
linear transformation 이므로

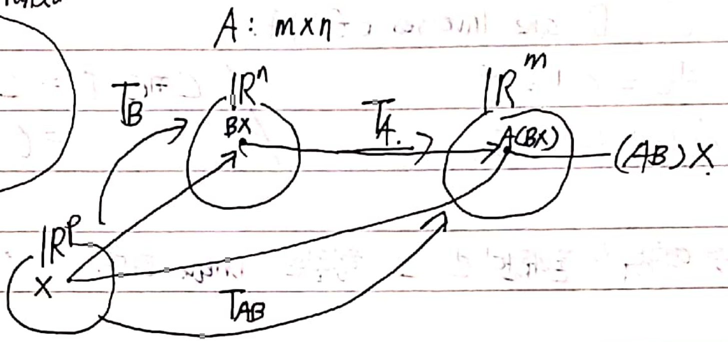
$= x_1(Ab_1) + \cdots + x_p(Ab_p)$

column vectors

$\|$

$[Ab_1 \cdots Ab_p]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$





$T_{AB} = T_A \circ T_B$

$$\underbrace{[c_1 \cdots c_q]}_{p\times 1 \text{ columns}} = \underset{p\times q}{C} \qquad \underset{m\times p}{AB}$$

$$\underset{m\times q}{(AB)C} = A\underset{n\times q}{(BC)} \quad (m\times q) \implies \text{associativity rule holds.}$$

$$= ABC$$

$$(AB)C = [(AB)c_1 \cdots (AB)c_q]$$

$$[A(Bc_1) \cdots A(Bc_q)]$$

$$= A\cdot[Bc_1 \cdots Bc_q] = A(BC)$$

A: $m\times n$, I_n : the identity matrix w/ size $n\times n$ = I

$\underset{m\times n}{A\cdot I_n} = A$. (check).

$\underset{m\times m}{I_m}\cdot A = A$.

$$A\cdot\underset{n\times ?}{\square} = \underset{?\times m}{\square}\cdot A = I. \quad \therefore n\times m$$

$m\times m \neq n\times n$ ($m\neq n$ 이라면)

$\implies m=n$ (A가 square matrix 이어야 함)

Def.

A: $n\times n$ square Matrix is invertible if $\exists$ square Matrix C in $n\times n$ st $AC=CA=I$.

C: an Inverse of A $= A^{-1}$

Otherwise, A is not invertible or singular.

Assume C, D are Inverses of A.

$\Rightarrow AC = CA = I$
$AD = DA = I$.

$C = C \cdot I = C(AD)$
$= (CA)D = I \cdot D = D$.

$\rightarrow$ 역행렬이 존재한다면 그 행렬은 unique 하다.

$A : n \times n = A^1$

$\underbrace{AA}_{n \times n} = A^2 \quad (AA) \cdot A = A \cdot A \cdot A = A^3 \quad \cdots \quad A^k$ for $k \geq 1$ 인 정수.

$A^0 = I. \quad A^r \cdot A^s = A^{r+s}.$

$A \cdot A^{-1} = A^{-1} \cdot A = I \; / \; A^1 \cdot A^{-1} = A^{-1} \cdot A^1 = A^0$

for $k \geq 1$ 인 정수라면.

$A^{-k} = (A^{-1})^k$

Ex)

$A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \quad C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \quad AC = CA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \therefore C = A^{-1}$

In general, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\Leftrightarrow ad - bc \neq 0$ — is called $\det(A)$

in which case $A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Thm

$Ax = b$ w/ A is invertible matrix.

always has a unique solution.

$$A^{-1}(Ax) = A^{-1} \cdot b$$
$$x = A^{-1} \cdot b$$

A가 invertible이라면 A^{-1} 도 invertible $(A^{-1})^{-1} = A$.

A, B가 같은 크기로 invertible 이라면

AB 도 invertible $(AB)^{-1} = B^{-1} \cdot A^{-1}$

<u>Def</u> /

An elementary matrix is a square matrix obtained by a single elementary row op. to an identity mat. (Identity matrix 에서 한번만 써서 만들 수 있는 Matrix)

Ex) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Thm

$A : m \times n \xrightarrow[\text{elem. row. op.}]{* \text{ operatn}} A'$

$I = I_m$

※ (곱을 걸 할 수 있는 이유 : row 개수가 같아서)

$I \rightarrow \underset{m \times m}{E}$

$$E \cdot A = A'$$
$m \times n$.

$E'E = I$

Inverse of ☆

E : elementary matrix ⇒ invertible

E' : elementary matrix ⇒ invertible

Thm

A : square of size $n \times n$.

: Invertible. ⇔ A is row equivalent to I_n

A is row equivalent to B if B is obtained by applying several elementary row op. to A.

$$A \xrightarrow{☆_1} \square \xrightarrow{☆_2} \cdots \xrightarrow{☆_k} I_n.$$

$$A \xrightarrow{\text{☆}1} \boxed{E_1A} \xrightarrow{\text{☆}2} \boxed{E_2E_1A} \longrightarrow \cdots\cdots \xrightarrow{\text{☆}k} \boxed{I_n}$$

$I \xrightarrow{\text{☆}1} E_1$

$$I_n = E_k^{-1} \times E_kE_{k-1}\cdots E_2E_1A = E_k^{-1}$$

각각은 elementary Matrix

$\therefore$ $A = \cancel{E_k^{-1}E_{k-1}^{-1}}\; E_1^{-1}\cdot E_2^{-1}\cdots E_k^{-1}$ 다 Invertible

$\therefore A$ is invertible.

A is not row equivalent to I $\Rightarrow A$ is not Invertible

$\boxed{\text{책에서 사진 찾아봐!}}$

$(b) \Rightarrow (a)$, $(a) \Rightarrow (j)$, $(j) \Rightarrow ③$, $③ \Rightarrow (d)$, $(d) \Rightarrow (c)$, $(c) \Rightarrow (b)$

$C = A^{-1}$

③ $Ax = 0 \Rightarrow x = 0$

$CAx = C\cdot 0 = 0$

$Ix = 0$

<u>Def</u>

$A = m\times n = (a_{ij})$

$\hookrightarrow \square : n\times m$, $i \times j$, $= (a_{ji})$ → called the transpose of A, denoted by A^T

$$(\overbrace{aA+bB}^{\text{size } m\times n})^T = aA^T + bB^T$$

$\underbrace{A \times B}_{m\times p}$ ($m\times n$, $n\times p$) → $p\times m$ Transpose

$\underbrace{B^T \cdot A^T}_{p\times m}$ ($p\times n$, $n\times m$)

$(AB)^T = B^T\cdot A^T$

$A\cdot A^{-1} = A^{-1}\cdot A = I$.

$(A\cdot A^{-1})^T = (A^{-1}\cdot A)^T = I^T = I$

$(A^{-1})^T (A^T) = A^T\cdot(A^{-1})^T = I$

역함수 존재.

$\Rightarrow A$ is invertible 하면 A^T: Invertible.

A^T 〃 $\rightarrow (A^T)^T$ 〃

$(A^T)^T = A$

$$X \underset{f^{-1}}{\overset{f,\ 1\text{-}1,\ \text{onto}}{\rightleftarrows}} Y$$

f^{-1} : Inverse function.

$f(x)=y \Leftrightarrow f^{-1}(y)=x$

def

lin. transf $T:\mathbb{R}^n \to \mathbb{R}^n$ is set to be invertible (as a functi) → $[T]$도 invertible

if there are som $\exists\ S:\mathbb{R}^n \to \mathbb{R}^n$ s.t $S(T(x))=T(S(x))=x \quad \forall x \in \mathbb{R}^n$

S ↳ T^{-1}

$[T]$ and $[T]^{-1}=[T^{-1}]=[S]$

Def

$H \subset \mathbb{R}^n$ $(H \neq \emptyset)$ is a subspace of $\mathbb{R}^n$ — addition, scalar multiplication 필요.

if ① $u+v \in H \quad \forall u,v \in H$

② $cu \in H \quad \forall u \in H,\ c \in \mathbb{R}$ (zero vector는 subspace에 항상 포함)

$\Rightarrow 0 \in H$.

* $H=\{0\} \Rightarrow$ trivial subspace.

if $H \ni u \neq 0$, then the line thru (0 and u) is contained in H.

($\mathbb{R}^2$를 예를 들면 ① trivial, ② a line thru u, ③ $\mathbb{R}^2$)

Ex)

$\mathbb{R}^n \ni v_1 \cdots v_p$

↳ $\text{span}\{v_1 \cdots v_p\} = \{c_1v_1 + \cdots + c_pv_p \mid c_i \in \mathbb{R}\}$

↳ a subspace of $\mathbb{R}^n$

$u = c_1v_1 + \cdots + c_pv_p$

$w = d_1v_1 + \cdots + d_pv_p$

$u+w = (c_1+d_1)v_1 + \cdots + (c_p+d_p)v_p$