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A Karhunen–Loeve decomposition of a Gaussian process generated by independent pairs of exponential random variables

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We dedicate this article to Paul Malliavin

Abstract

We obtain the explicit Karhunen–Loeve decomposition of a Gaussian process generated as the limit of an empirical process based upon independent pairs of exponential random variables. The orthogonal eigenfunctions of the covariance kernel have simple expressions in terms of Jacobi polynomials. Statistical applications, in extreme value and reliability theory, include a Cramér–von Mises test of bivariate independence, whose null distribution and critical values are tabulated.

Keywords: Nonparametric tests; Tests of independence; Cramér–von Mises-type tests; Extreme values; Reliability; Lifetime analysis; Gaussian processes; Empirical processes; Karhunen–Loeve expansions; Weak laws; Jacobi polynomials; Orthogonal polynomials; Fredholm integral equations

1. A probabilist introduction and statement of the main result

In this paper, we give the *Karhunen–Loeve* (KL) decomposition of a special case of centered Gaussian process $\{Z_0(t):\ 0 \leqslant t \leqslant 1\}$, which will be shown to have some useful statistical applications.

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To motivate the present study, we start by a discussion of the probabilistic properties of the KL decomposition of a general centered Gaussian process $\{Z(t)\colon\ 0 \leqslant t \leqslant 1\}$. This is followed by the statement of our main result (in Theorem 1.1 below) for the particular case of Gaussian process we are considering.

In Section 2, we continue by the exposition of a more statistical viewpoint. We begin by describing an empirical process based upon independent pairs of exponential random variables which we show to converge weakly to $\{Z_0(t)\colon\ 0 \leqslant t \leqslant 1\}$. We then combine this result with that of Section 1 to provide a numerical evaluation of the limiting distribution of a Cramér–von Mises-type test of independence, whose critical levels are tabulated. We conclude by a discussion of the relevance of this methodology in the framework of reliability and extreme values theory. Section 3 collects the proofs of these results, together with additional details of interest, concerning the structure of $\{Z_0(t)\colon\ 0 \leqslant t \leqslant 1\}$.

We recall the following useful facts. Let $\{Z(u)\colon\ 0 \leqslant u \leqslant 1\}$ denote a centered Gaussian process with continuous covariance function

$$R(u, v) = \mathbb{E}\big(Z(u)Z(v)\big) \quad \text{for } 0 \leqslant u, v \leqslant 1,$$

and almost surely continuous sample paths. In this case, it is well known (see, e.g. [33], [48, pp. 206–218], [1, pp. 66–79]) that there exist constants $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$, together with continuous functions $e_1(t), e_2(t), \ldots,$ on $[0, 1]$ (the *eigenfunctions* of the *covariance kernel* $R(u, v)$), such that the following properties are fulfilled:

(K.1) The $\{e_k\colon\ k \geqslant 1\}$ are orthonormal in $L^2[0, 1]$, i.e.

$$\int_0^1 e_i(t)e_j(t)\,dt = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.1}$$

(K.2) The $\{(\lambda_k, e_k)\colon\ k \geqslant 1\}$ form a complete set of solutions of the Fredholm-type equation in (λ, e),

$$\lambda e(u) = \int_0^1 R(u, v)e(v)\,dv \quad \text{for } 0 \leqslant u \leqslant 1, \quad \text{and} \quad \int_0^1 e^2(u)\,du = 1. \tag{1.2}$$

(K.3) We have

$$R(u, v) = \sum_{k=1}^{\infty} \lambda_k e_k(u)e_k(v), \tag{1.3}$$

where the series on the right-hand side of (1.3) is absolutely and uniformly convergent on $[0, 1]^2$.

(K.4) There exists a sequence $\{\omega_k\colon\ k \geqslant 1\}$ of independent $N(0, 1)$ random variables such that the following *Karhunen–Loeve* (KL) expansion holds. For all $0 \leqslant u \leqslant 1$

$$Z(u) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\omega_k e_k(u), \tag{1.4}$$

where the series is convergent in integrated mean square, and uniformly on [0, 1] with probability 1.

The KL expansion induced by (K.1)–(K.4) is of great interest for several reasons. Below, we discuss two of the most important of its applications. We limit ourselves to the non-trivial case where $\lambda_k > 0$ for all $k \geqslant 1$, and assume implicitly that this condition is fulfilled in the examples we consider.

The first major consequence of the KL expansion (1.3), (1.4) is that it yields an explicit description of the *reproducing kernel Hilbert space* (RKHS) of Z (see, e.g., [34], and [1, Theorem 3.16]). When Z is considered as a random variable with values in the Banach space (denoted hereafter by $(C[0,1],\mathcal{U})$) $C[0,1]$ of continuous functions on [0, 1] endowed by the uniform topology $\mathcal{U}$, defined by the sup-norm $\|f\| = \sup_{0\leqslant u\leqslant 1}|f(u)|$, the RKHS $\mathbb{H}$ of Z is the Hilbert subspace of $C[0,1]$ given by

$$\mathbb{H} = \left\{f:\ f(u) = \sum_{k=1}^{\infty} a_k\sqrt{\lambda_k}e_k(u),\ 0\leqslant u\leqslant 1,\ \sum_{k=1}^{\infty} a_k^2 < \infty\right\}, \tag{1.5}$$

with inner product (assuming $\lambda_k > 0$ for all $k \geqslant 1$)

$$\langle f, g\rangle_{\mathbb{H}} = \sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty}\frac{1}{\lambda_k}\left(\int_0^1 g(u)e_k(u)\,du\right)\left(\int_0^1 f(u)e_k(u)\,du\right), \tag{1.6}$$

where

$$f(u) = \sum_{k=1}^{\infty} a_k\sqrt{\lambda_k}e_k(u) \quad \text{and} \quad g(u) = \sum_{k=1}^{\infty} b_k\sqrt{\lambda_k}e_k(u) \quad \text{for } 0\leqslant u\leqslant 1. \tag{1.7}$$

As follows from (1.5)–(1.6), the sequence $\{\sqrt{\lambda_k}e_k:\ k\geqslant 1\}$ yields a *convergent orthonormal sequence* (CONS) in $\mathbb{H}$, the latter providing an essential tool to describe the probabilistic structure of Z (refer to [36,37]).

A second, more statistically oriented, consequence of interest of (K.1)–(K.4), it that it provides an indirect description of the distribution function of

$$\mathcal{J}^2 = \int_0^1 Z^2(u)\,du = \sum_{k=1}^{\infty}\lambda_k\omega_k^2, \tag{1.8}$$

via its characteristic function, given by

$$\mathbb{E}\big(\exp\big(iu\mathcal{J}^2\big)\big) = \prod_{k=1}^{\infty}(1-2iu\lambda_k)^{-1/2} \quad \text{for } u\in\mathbb{R}. \tag{1.9}$$

The relations (1.8)–(1.9) have obvious statistical applications when $\{Z(u):\ 0 \leqslant u \leqslant 1\}$ is the weak limit (in an appropriate functional space) of a sequence $\{\zeta_{n,*}(u):\ 0 \leqslant u \leqslant 1\}$ of empirical processes. In such a setting, the statistic

$$\mathcal{J}_{n,*}^2 = \int_0^1 \zeta_{n,*}^2(u)\,du, \tag{1.10}$$

is typically of interest for tests of goodness of fit, since its distribution can be approximated for large values of n by that of $\mathcal{J}^2$. This, however, necessitates in practice a numerical evaluation of the various quantiles of interest of the latter distribution. It is not too difficult (see, e.g. [32, Section 18-8, pp. 444–450]) to invert numerically a finite product approximation of the right-hand side of (1.9) leading to the desired values of $\mathbb{P}(\mathcal{J}^2 \leqslant x)$, for specified choices of x. For related methods of the kind, refer to [30,38,39] (see Remark 2.1 in the sequel). This, in turn, requires a prior *explicit* knowledge of the eigenvalues $\{\lambda_k:\ k \geqslant 1\}$, which are *implicit* in terms of $R(u,v)$ via (1.2). In most examples of interest, one knows only $R(u,v)$ and the needed numerical evaluation of the λ_k's can only be made by tedious recursions, which do not allow to achieve any reasonable precision for higher order terms (see, e.g., [5]). Therefore, the only case where (1.11) is useful for such applications is when there exist sufficiently simple closed-form expressions for the λ_k's. If such is not the case, one must use different techniques (see, e.g., [18,40]), which, besides being more time-consuming for the computer than a direct approach, do not yield more than a very superficial insight concerning the specific form of the KL decomposition.

Unfortunately, for most Gaussian processes of interest with respect to statistics, the values of the λ_k's are unknown, even though their existence remains guaranteed through the knowledge of $R(u,v)$ (see, e.g., [1, p. 76]). The practical application of this theory to statistics is therefore limited to a small number of particular cases. Below, we review some examples of Gaussian processes on [0, 1] for which the constants $\{\lambda_k:\ k \geqslant 1\}$ in the KL expansion are known (refer to [11,12], for results of the kind for processes indexed on $[0,1]^d$ with $d \geqslant 2$).

- The (restriction on [0,1] of the) Wiener process $\{W(t):\ t \geqslant 0\}$, with $Z = W$ and $\lambda_k = 1/((k-\frac{1}{2})\pi)^2$ for $k \geqslant 1$ (see, e.g. [1, p. 77]);
- The Brownian bridge $\{B(t):\ 0 \leqslant t \leqslant 1\}$, with $Z = B$ and $\lambda_k = 1/(k\pi)^2$ for $k \geqslant 1$. With respect to our discussion relative to $\mathcal{J}^2$ and $\mathcal{J}_{n,*}^2$, we obtain in this case the celebrated Cramér–von Mises statistic when $\zeta_{n,*}$ is the uniform empirical process on [0, 1] (see, e.g., [48, Proposition 1 and Theorem 1, pp. 213–217], [20, p. 32], and [13, p. 15]). We note here that explicit formulas for $\mathbb{P}(\mathcal{J}^2 \leqslant x)$ are given both by [2,51] (see also [14]).
- The limiting process of the Anderson–Darling statistic, with $Z(t) = B(t)/\sqrt{t(1-t)}$, and $B(t)$ denoting again a Brownian bridge (see, e.g. [3,58,59], [48, pp. 148, 224–227]). In this case, $\lambda_k = 1/(k(k+1))$ for $k \geqslant 1$.

The discussion above motivates clearly the usefulness of deriving KL expansions for all possible Gaussian processes of interest for which an explicit computation of the λ_k's is possible.

It is the purpose of this paper to exhibit a new non-trivial example of KL expansion with statistical applications in the framework of extreme values and reliability theory. The centered Gaussian process $\{Z_0(t):\ 0 \leqslant t \leqslant 1\}$ on which this expansion is based, is defined by its covariance function $R_0(u,v) = \mathbb{E}(Z_0(u)Z_0(v))$, which is such that

$$R_0(u,v) = R_0(v,u) = \frac{2v-u^2-v^2}{(1-u)v} - 1 - (1-u)(1-v) - uv \quad \text{for } 0 \leqslant u \leqslant v \leqslant 1. \tag{1.11}$$

In our main theorem, stated below, we show the unexpected result that the eigenvalues $\{\lambda_{k,0}\colon k \geqslant 1\}$ and eigenfunctions $\{e_{k,0}\colon k \geqslant 1\}$ pertaining to $R(u,v) = R_0(u,v)$ have relatively simple expressions. The following notation and facts from the theory of orthogonal polynomials will be needed.

The *Jacobi polynomials* (see, e.g., [56, pp. 160–177], [53]) are usually denoted by $P_n^{\alpha,\beta}(x)$ for $n \geqslant 0$ and $\alpha, \beta > -1$, with $x \in [-1,1]$. We will need here to modify this definition by the change of variable $u = (x+1)/2$ and define the *modified Jacobi polynomials* via the formula (see, e.g., [10, (2.1) p. 143]), for $n \geqslant 0$,

$$Q_n^{\alpha,\beta}(u) = P_n^{\alpha,\beta}(2u-1) = \frac{(-1)^n}{n!}\frac{1}{u^\beta(1-u)^\alpha}\frac{d^n}{du^n}\left\{u^{\beta+n}(1-u)^{\alpha+n}\right\} \quad \text{for } 0 \leqslant u \leqslant 1. \tag{1.12}$$

We will make use of the fact that the modified Jacobi polynomials $\{Q_n^{\alpha,\beta}\colon n \geqslant 0\}$ fulfill the orthogonality relations (see, e.g., [10, (2.18), p. 148]), for $m, n \geqslant 0$,

$$\int_0^1 Q_m^{\alpha,\beta}(u) Q_n^{\alpha,\beta}(u) u^\beta (1-u)^\alpha \, du = \begin{cases} 0 & \text{when } m \neq n, \\ \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\, n!} & \text{when } m = n. \end{cases} \tag{1.13}$$

We will specialize in the case where $\alpha = \beta = 2$, and consider the sequence of polynomials defined for $n \geqslant 0$ by

$$P_n(u) = Q_n^{2,2}(u) = \frac{(-1)^n}{n!}\frac{1}{u^2(1-u)^2}\frac{d^n}{du^n}\left\{u^{n+2}(1-u)^{n+2}\right\} \quad \text{for } 0 \leqslant u \leqslant 1, \tag{1.14}$$

and fulfilling the relations, for $m, n \geqslant 0$,

$$\int_0^1 u^2(1-u)^2 P_m(u) P_n(u)\, du = \begin{cases} 0 & \text{when } m \neq n, \\ \Delta_n := \frac{(n+2)!(n+2)!}{(2n+5)(n+4)!n!} = \frac{1}{2n+5} \cdot \frac{(n+1)(n+2)}{(n+3)(n+4)} & \text{when } m = n. \end{cases} \tag{1.15}$$

Theorem 1.1. *Let $Z = Z_0$ and $R(u,v) = R_0(u,v)$ be as in* (1.11). *Then,* (K.1)–(K.4) *hold with*

$$\lambda_k = \lambda_{k,0} = \frac{6}{k(k+1)(k+2)(k+3)} \quad \textit{for } k \geqslant 1, \tag{1.16}$$

and, for $k \geqslant 1$,

$$e_k(u) = e_{k,0}(u) = \frac{u(1-u)}{\sqrt{\Delta_{k-1}}} P_{k-1}(u) = \left\{(2k+3) \cdot \frac{(k+2)(k+3)}{k(k+1)}\right\}^{1/2}$$
$$\times \frac{(-1)^{k-1}}{(k-1)!}\frac{1}{u(1-u)}\frac{d^{k-1}}{du^{k-1}}\left\{u^{k+1}(1-u)^{k+1}\right\} \quad \textit{for } 0 \leqslant u \leqslant 1. \tag{1.17}$$

Remark 1.1. For explicit computation of the eigenfunctions $\{e_{k,0}\colon\ k \geqslant 1\}$ in (1.17), one may either use a binomial expansion of $(u-1)^{j+1}$ in the formula (see, e.g., [10, (2.60), p. 144])

$$e_k(u) = -\left\{(2k+3)\cdot\frac{(k+2)(k+3)}{k(k+1)}\right\}^{1/2}\sum_{j=0}^{k-1}\binom{k+1}{k-1-j}\binom{k+1}{j}u^{k-j}(u-1)^{j+1}, \tag{1.18}$$

or make use of the following relations whose validity will be established in Section 3. First we evaluate $\{\theta_{j,k}\colon\ 1 \leqslant j \leqslant k\}$ through the recursions

$$\theta_{j,k} = \left\{\frac{\lambda_{i,0}}{\lambda_{k,0}} - 1\right\}\frac{1}{\theta_{j+1,k} - 2} \quad \text{for } 1 \leqslant j \leqslant k-1, \qquad \theta_{k,k} = 0, \tag{1.19}$$

then, we compute the constants $\{a_{j,k}\colon\ 0 \leqslant j \leqslant k\}$ and c_k by setting

$$c_k = \left\{\frac{3(2k+3)}{2}\lambda_{k,0}\right\}^{1/2} = \left\{\frac{3(2k+3)}{2}\cdot\frac{k(k+1)(k+2)(k+3)}{6}\right\}^{1/2},$$

$$a_{0,k} = 1, \qquad a_{k,k} = 0, \qquad a_{j,k} = \prod_{\ell=1}^{j}\theta_{\ell,k} \quad \text{for } 1 \leqslant \ell \leqslant k-1. \tag{1.20}$$

Finally, we write, for $k \geqslant 1$,

$$\begin{aligned} e_k(u) = e_{k,0}(u) &= (-1)^{k-1}c_k u(1-u)\sum_{j=0}^{k-1}a_{j,k}u^j \\ &= (-1)^{k-1}c_k u(1-u)\left\{1 + \sum_{j=1}^{k-1}a_{j,k}u^j\right\}. \end{aligned} \tag{1.21}$$

The first of these eigenfunctions are given by

$$\begin{aligned} e_{1,0}(u) &= \sqrt{30}u(1-u), \\ e_{2,0}(u) &= -\sqrt{210}u(1-u)(2u-1), \\ e_{3,0}(u) &= 3\sqrt{10}u(1-u)\big(14u^2 - 14u + 3\big), \\ e_{4,0}(u) &= -3\sqrt{2310}u(1-u)\big(12u^3 - 18u^2 + 8u - 1\big), \\ e_{5,0}(u) &= 2\sqrt{1365}u(1-u)\big(33u^4 - 66u^3 + 45u^2 - 12u + 1\big). \end{aligned} \tag{1.22}$$

We note that Theorem 1.1 remains valid if one replaces $e_{k,0}(u)$ in (1.17) by $-e_{k,0}(u)$. The choice of sign which was used here ensures that $a_{0,k} = 1$ in (1.21), or equivalently that $e'_{k,0}(0) > 0$.

Remark 1.2. 1° It follows from (1.17) that $e_{k,0}(1-u) = (-1)^{k+1}e_{k,0}(u)$ for $k \geqslant 1$. This, when combined with the version of (1.4) holding for Z_0, shows that

$$Z_0(1-u) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,(-1)^{k+1}\omega_k e_{k,0}(u) =_d Z_0(u) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\,\omega_k e_{k,0}(u), \tag{1.23}$$

where "$=_d$" denotes equality in probability. That $\{Z_0(1-u)\colon 0 \leqslant u \leqslant 1\} =_d \{Z_0(u)\colon 0 \leqslant u \leqslant 1\}$ may be checked directly from the equality $R_0(u,v) = R(1-u, 1-v)$ for all $0 \leqslant u, v \leqslant 1$.

2° The equality (1.23) implies that the processes

$$Z_0^S(u) = \frac{1}{2}\{Z_0(u) + Z_0(1-u)\} = \sum_{\ell=1}^{\infty} \sqrt{\lambda_{2\ell+1}}\,\omega_{2\ell+1} e_{2\ell+1}(u), \tag{1.24}$$

and

$$Z_0^A(u) = \frac{1}{2}\{Z_0(u) - Z_0(1-u)\} = \sum_{\ell=1}^{\infty} \sqrt{\lambda_{2\ell}}\,\omega_{2\ell} e_{2\ell}(u), \tag{1.25}$$

are independent, with KL expansions given as above.

The proof of Theorem 1.1 is given in Section 3, together with other results of interest. In particular, we show in this section that the process $\{Z_0(u)\colon 0 \leqslant u \leqslant 1\}$ is closely related to the bivariate Brownian bridge. Section 2 below gives some insight on how the process $\{Z_0(u)\colon 0 \leqslant u \leqslant 1\}$ may be generated as the limiting form of simple empirical processes. The statistical applications which will follow from these theorems are likely to provide an additional motivation to the present study.

2. Statistical motivation and applications

In this section, we will show that the centered Gaussian process $\{Z_0(t)\colon 0 \leqslant t \leqslant 1\}$ with covariance function $R_0(u,v)$ given by (1.11) arises naturally as the weak limit of empirical processes with important statistical applications. We start by a description of the statistical models which turns out to lead ultimately to this limiting process.

Consider a sequence $\{(X_n, Y_n)\colon n \geqslant 1\}$ of independent and identically distributed [i.i.d.] bivariate random vectors, and assume that the distribution of $(X,Y) = (X_1, Y_1)$, denoted hereafter by $(X,Y) \equiv E_A(\gamma,\nu)$, is such that, for constants $\gamma > 0$ and $\nu > 0$

$$\mathbb{P}(X \geqslant \gamma x, Y \geqslant \nu y) = \exp\left(-(x+y)A\left(\frac{x}{x+y}\right)\right) \quad \text{for } x > 0 \text{ and } y > 0, \tag{2.1}$$

where the so-called *dependence function* $\{A(u)\colon 0 \leqslant u \leqslant 1\}$ fulfills the assumptions

(A.1) $\max\{u, 1-u\} \leqslant A(u) \leqslant 1$ for $0 \leqslant u \leqslant 1$;
(A.2) A is convex on [0, 1].

We refer to [22, Section 4.2, pp. 111–118] (see also [47, Chapter 5]) for a discussion of this model. It is noteworthy (see, e.g., [24,43,44]) that (A.1), (A.2) are necessary and sufficient conditions for $\mathbb{P}(X \geqslant x, Y \geqslant y)$ to define, via (2.1) the *survivor function* of a proper bivariate probability distribution. The latter has the following characteristic property. Whenever $(X,Y) \equiv E_A(\gamma,\nu)$, then, for any pair of constants $c > 0$ and $d > 0$, the random variable $\min(cX, dY)$ follows an exponential distribution. This property holds, in particular, for X (with

$c = 1$, $d = \infty$) and Y (with $c = \infty$, $d = 1$), which follow exponential distributions with means given by $\mathbb{E}(X) = 1/\gamma$ and $\mathbb{E}(Y) = 1/\nu$.

Below, we seek to derive some appropriate tests of the null hypothesis, denoted hereafter by (H.0), that X and Y are *independent*. Under the $E_A(\gamma, \nu)$ model, this property is equivalent to

(H.0) $A(u) = 1$ for $0 \leqslant u \leqslant 1$.

Towards the aim of testing (H.0) against the general alternative (H.1) that $A(u) \neq 1$ for some $0 < u < 1$, we introduce the empirical process

$$\zeta_{n,0}(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \min\left(\frac{X_i/\overline{X}_n}{u}, \frac{Y_i/\overline{Y}_n}{1-u} \right) - 1 \right\} \quad \text{for } 0 \leqslant u \leqslant 1, \tag{2.2}$$

where we use the convention that $x/0 = \infty$ when $x > 0$. We note that $\zeta_{n,0}$ defines a random variable with values in $(C[0,1], \mathcal{U})$ and distribution independent of $\gamma > 0$ and $\nu > 0$. Given this notation, the main result of the present section may now be stated as follows. Recall the definition of $\{Z_0(u)\colon\ 0 \leqslant u \leqslant 1\}$ via its covariance function (1.11).

Theorem 2.1. *Under* (H.0)*, the empirical process* $\{\zeta_{n,0}(u)\colon\ 0 \leqslant u \leqslant 1\}$ *in* (2.2) *converges weakly in* $(C[0,1], \mathcal{U})$ *to* $\{Z_0(u)\colon\ 0 \leqslant u \leqslant 1\}$ *as* $n \to \infty$.

The proof of Theorem 2.1 is postponed until the end of this section. An immediate corollary of Theorems 1.1 and 2.1 is given below in terms of the statistic $\mathcal{J}_{n,0}^2$ and random variable $\mathcal{J}_0^2$ defined respectively by

$$\mathcal{J}_{n,0}^2 = \int_0^1 \zeta_{n,0}^2(u)\,du \quad \text{and} \quad \mathcal{J}_0^2 = \int_0^1 Z_0^2(u)\,du. \tag{2.3}$$

Let $\{\lambda_{k,0}\colon\ k \geqslant 1\}$ be as in (1.16).

Corollary 2.1. *Under* (H.0)*, we have*

$$\lim_{n\to\infty} \mathbb{E}\big(\exp\big(iu\mathcal{J}_{n,0}^2\big)\big) = \mathbb{E}\big(\exp\big(iu\mathcal{J}_0^2\big)\big) = \prod_{k=1}^{\infty} (1 - 2iu\lambda_{k,0})^{-1/2} \quad \textit{for } u \in \mathbb{R}. \tag{2.4}$$

Proof. Combine Theorems 1.1 and 2.1 with (1.8), (1.9) and (2.3). □

A logical use of the statistic $\mathcal{J}_{n,0}^2$ in (2.3) is to test (H.0) against (H.1) by rejecting the null hypothesis when $\mathcal{J}_{n,0}^2$ exceeds a high critical level $c_{n,\alpha}$, chosen in such a way that, for a specified $0 < \alpha < 1$, $\mathbb{P}(\mathcal{J}_{n,0}^2 \geqslant c_{n,\alpha} \mid (\text{H.0})) = \alpha$. The evaluation of the exact values of $c_{n,\alpha}$ for the various possible choices of the risk level $\alpha \in (0,1)$, and the sample size $n \geqslant 1$, is beyond the scope of the present paper. Below, we limit ourselves to a minimal tabulation of limiting constants c_α such that

$$\lim_{n\to\infty} \mathbb{P}\big(\mathcal{J}_{n,0}^2 \geqslant c_\alpha \bigm| (\text{H.0})\big) = \mathbb{P}\big(\mathcal{J}_0^2 \geqslant c_\alpha\big) = \alpha. \tag{2.5}$$

Table 1
Critical points of $\mathcal{J}_0^2$

α (%)	c_α
20	0.507
10	0.770
5	1.053
1	1.750

Table 2
The distribution function of $\mathcal{J}_0^2$

$y \backslash z$	0.0	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
0.0	00.000	08.721	24.493	37.689	48.128	56.390	63.018	68.399	72.829	76.515
0.5	79.612	82.236	84.474	86.395	88.051	89.484	90.730	91.817	92.766	93.598
1.0	94.328	94.970	95.535	96.034	96.475	96.864	97.209	97.514	97.785	98.026
1.5	98.239	98.429	98.598	98.748	98.881	99.000	99.107	99.201	99.286	99.361
2.0	99.428	99.488	99.542	99.589	99.632	99.671	99.705	99.735	99.763	99.787
2.5	99.809	99.829	99.847	99.862	99.877	99.889	99.901	99.911	99.920	99.928
3.0	99.935	99.942	99.948	99.953	99.958	99.962	99.966	99.969	99.973	99.975
3.5	99.978	99.980	99.982	99.984	99.986	99.987	99.988	99.989	99.991	99.991
4.0	99.992	99.993	99.994	99.994	99.995	99.995	99.996	99.996	99.997	99.997
4.5	99.997	99.998	99.998	99.998	99.998	99.998	99.999	99.999	99.999	99.999

In Table 1, we give c_α with a precision of 10^{-3} for $\alpha = 20\%, 10\%, 5\%, 1\%$.

Table 2 gives the percentage points $100 \times \mathbb{P}(\mathcal{J}_0^2 \leqslant y + z)$ of the distribution function of $\mathcal{J}_0^2$ with error not exceeding 0.001%.

Remark 2.1. A general description of the numerical methods which may be used to evaluate the above quantities, is to be found in [38–40] and [18]. The approach which has been followed here for the computation of the constants in Table 1 is based on the Smirnov formula [50,51]

$$\mathbb{P}\big(\mathcal{J}^2 \leqslant x\big) = 1 + \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k \int_{\gamma_{2k-1}}^{\gamma_{2k}} \frac{e^{xg/2}\, dg}{g\sqrt{|D(g)|}} \quad \text{for } x \geqslant 0, \tag{2.6}$$

where $D(g)$ denotes the *Fredholm determinant*

$$D(g) = \prod_{k=1}^{\infty} \left(1 - \frac{g}{\gamma_k}\right). \tag{2.7}$$

We assume here that $\{\gamma_k\colon k \geqslant 1\}$ and $\{\lambda_k\colon k \geqslant 1\}$ are sequences of positive constants, related via

$$\gamma_k = \frac{1}{\lambda_k} \quad \text{for } k \geqslant 1, \quad \text{with } \sum_{k=1}^{\infty} \lambda_k = \sum_{k=1}^{\infty} \frac{1}{\gamma_k} < \infty \text{ and } \lambda_1 > \lambda_2 > \cdots > 0, \tag{2.8}$$

and let $\mathcal{J}^2$ be defined, as in (1.8), by

$$\mathcal{J}^2 = \sum_{k=1}^{\infty} \lambda_k \, \omega_k^2 = \sum_{k=1}^{\infty} \frac{\omega_k^2}{\gamma_k}, \tag{2.9}$$

where $\{\omega_k \colon k \geqslant 1\}$ is an i.i.d. sequence of $N(0,1)$ random variables. We note that in the special case where $\lambda_k = \lambda_{k,0}$ the convergence of the series in (2.6) is guaranteed (see, e.g., [38] for characterizations of this property). It is convenient, for the numerical computation of the integral

$$I_k = \int_{\gamma_{2k-1}}^{\gamma_{2k}} \frac{e^{xg/2} \, dg}{g\sqrt{|D(g)|}},$$

in (2.6), to make a change of variable by writing (refer to [26])

$$I_k = \int_{-1}^{1} \frac{(p_k(z) - \gamma_{2k-1})(\gamma_{2k} - p_k(z)) \exp(-p_k(z)/2) \, dz}{p_k(z)\sqrt{1-z^2} \cdot \sqrt{|D(p_k(z))|}},$$

where $p_k(z) = [(\gamma_{2k} - \gamma_{2k-1})z + \gamma_{2k} - \gamma_{2k-1}]/2$, then, to evaluate numerically the latter integral by the quadrature formula

$$\int_{-1}^{1} \frac{f(z)\, dz}{\sqrt{1-z^2}} \approx \frac{\pi}{m} \sum_{\ell=1}^{m} f\left(\cos\left\{\frac{2k-1}{2m}\right\}\right),$$

which is accurate for sufficiently large values of m.

Remark 2.2. Since the present section is primarily focused on the derivations of statistical applications to illustrate the KL expansion in Theorem 1.1, we will not provide here any details concerning the speed of convergence in (2.5), nor a discussion of the efficiency of the test $\mathcal{J}_{n,0}^2$, with respect to alternative methods. It would require some extensive studies to compare the performances of the above test methodology with that of the numerous other tests available in the literature, and this would necessitate a considerable extension of the contents of our paper with respect to its present volume. These problems are therefore left open for future research. Along this line, the reviews of tests of independence between exponential pairs in [6–8,19,45] should be of interest.

Remark 2.3. 1° Among the many possible statistics which (in addition to $\mathcal{J}_{n,0}$) may be used to test (H.0) against (H.1), one should mention the *principal component test statistics* defined by

$$T_{n,k} = \sqrt{\lambda_k} \int_0^1 \zeta_{n,0}(u) e_{k,0}(u) \, du \quad \text{for } k \geqslant 1. \tag{2.10}$$

A direct consequence of Theorems 1.1 and 2.1 is that, under (H.0), for each $k \geqslant 1$,

$$T_{n,k} \to_d N(0,1), \tag{2.11}$$

and we may use this property to reject (H.0) when $\pm T_{n,k}$ or $|T_{n,k}|$ exceeds the appropriate quantile of the $N(0,1)$ law. The fact that the $e_{k,0}$ have explicit expressions allows a simple use of this methodology. For example, making use of (1.22), we obtain readily that, under (H.0),

$$\begin{aligned}T_{n,1} &= 2\sqrt{30}\int_0^1 \zeta_{n,0}(u)u(1-u)\,du\\ &= \frac{\sqrt{30}}{\sqrt{n}}\sum_{i=1}^n\left\{\frac{X_iY_i}{X_i\overline{Y}_n+Y_i\overline{X}_n}-\frac{1}{3}\right\}\to_d N(0,1)\quad\text{as } n\to\infty.\end{aligned}\tag{2.12}$$

It is noteworthy that, under (H.1), we have

$$T_{n,1}=\big(1+o(1)\big)\sqrt{n}\cdot 2\sqrt{30}\int_0^1\left\{\frac{1}{A(u)}-1\right\}u(1-u)\,du\to\infty\quad\text{a.s.},$$

so that the test of (H.0) based upon $T_{n,1}$ is consistent. This property is not shared by $T_{n,k}$ for higher values of k. An example is given for $k=2$ and when $A(u)=A(1-u)$ (see, e.g., (2.20) in the sequel), in which case we infer from (1.22) that

$$\int_0^1\left\{\frac{1}{A(u)}-1\right\}e_2(u)\,du=\int_0^1\left\{\frac{1}{A(u)}-1\right\}\sqrt{210}u(1-u)(1-2u)\,du=0.$$

2° Making use of (1.24) and (1.25), we infer readily from Theorems 1.1 and 2.1 the limiting distributions under (H.0) of the statistics

$$\mathcal{J}_{n,0}^S=\frac{1}{4}\int_0^1\big\{\zeta_{n,0}(u)+\zeta_{n,0}(1-u)\big\}^2\,du,\tag{2.13}$$

and

$$\mathcal{J}_{n,0}^A=\frac{1}{4}\int_0^1\big\{\zeta_{n,0}(u)-\zeta_{n,0}(1-u)\big\}^2\,du.\tag{2.14}$$

We have namely, under (H.0), for $u\in\mathbb{R}$,

$$\lim_{n\to\infty}\mathbb{E}\big(\exp\big(iu\mathcal{J}_{n,0}^S\big)\big)=\prod_{\ell=1}^\infty(1-2iu\lambda_{2\ell+1,0})^{-1/2}\tag{2.15}$$

and

$$\lim_{n\to\infty}\mathbb{E}\big(\exp\big(iu\mathcal{J}_{n,0}^A\big)\big)=\prod_{\ell=1}^\infty(1-2iu\lambda_{2\ell,0})^{-1/2}.\tag{2.16}$$

We will not discuss here the use of these statistics to investigate various possible assumptions on A in (2.1).

In the remainder of this section, we discuss some related problems and give some further motivation for the use of the statistic $\{A_{n,0}(u)\colon\ 0 \leqslant u \leqslant 1\}$ defined by

$$\frac{1}{A_{n,0}(u)} = \frac{1}{n}\sum_{i=1}^{n}\min\biggl(\frac{X_i/\overline{X}_n}{u}, \frac{Y_i/\overline{Y}_n}{1-u}\biggr), \tag{2.17}$$

as an estimator of $\{A(u)\colon\ 0 \leqslant u \leqslant 1\}$. We note that, when combined with the SLLN for $\overline{X}_n$ and $\overline{Y}_n$, the results of [17] imply readily that, for an arbitrary dependence function A fulfilling (A.1), (A.2), we have $\|A_{n,0} - A\| \to 0$ a.s. This, in turn, shows that $\mathcal{J}_{n,0}^2 \to \infty$ a.s. under (H.1), which establishes the consistency of the just-given test of (H.0) based upon $\mathcal{J}_{n,0}^2$. Below, we will be primarily concerned with the weak limiting behavior of $A_{n,0}$ under (H.0). First, we review some relevant results obtained on the $E_A(\gamma, \nu)$ model, making use, at times, of the following notation, to work in the *reduced case* where $\gamma = \nu = 1$. We set namely, for convenience $(U_n, V_n) = (X_n/\gamma, Y_n/\nu)$ for $n \geqslant 1$, and $(U, V) = (U_1, V_1) = (X/\gamma, Y/\nu) \equiv E_A(1, 1)$. This notation allows us to write

$$\frac{1}{A_{n,0}(u)} = \frac{1}{n}\sum_{i=1}^{n}\min\biggl(\frac{U_i\,/\,\overline{U}_n}{u}, \frac{V_i\,/\,\overline{V}_n}{1-u}\biggr), \quad \text{with } \overline{U}_n = \frac{1}{n}\sum_{i=1}^{n} U_i,\ \overline{V}_n = \frac{1}{n}\sum_{i=1}^{n} V_i. \tag{2.18}$$

Keeping in mind that X and Y (respectively U and V) follow exponential distributions, with respective means $\mathbb{E}(X) = \gamma$ and $\mathbb{E}(Y) = \nu$ (respectively $\mathbb{E}(U) = \mathbb{E}(V) = 1$), we continue our exposition by a discussion of the relevance of $E_A(\gamma, \nu)$ model to describe the dependence structure between exponential marginals.

There is a huge scientific literature on the applications of exponential laws (see, e.g., [6], and the references therein) and we will consider only the classical setting where X and Y are used to model the lifetimes of two components from the same equipment. There is no need to stress the importance of finding a proper model for the dependence relationships between X and Y. In particular, it is useful to know if the fact that one component has failed conditions the lifetime of the other component or not.

For theoretical reasons which will be explicited below, the dependence relationships induced on X and Y by the $E_A(\lambda, \nu)$ model is very natural. Below, we consider two particular cases of interest.

A first example is the popular Marshall and Olkin [41] model, which is readily verified to fall in the class of $E_A(\lambda, \nu)$ bivariate distributions, with

$$A(u) = 1 - \max\bigl\{ru, s(1-u)\bigr\} \quad \text{for } 0 \leqslant u \leqslant 1, \tag{2.19}$$

where $0 \leqslant r, s \leqslant 1$ are arbitrary parameters.

A second example is Gumbel's third model (see, e.g., Gumbel [27], Balakrishnan and Basu [6, p. 328]) which describes a class of $E_A(\lambda, \nu)$ distributions such that, for some $m \geqslant 1$,

$$A(u) = \bigl\{u^m + (1-u)^m\bigr\}^{1/m} \quad \text{for } 0 \leqslant u \leqslant 1. \tag{2.20}$$

We limit our survey to these examples which are sufficient to show the importance of the $E_A(\gamma,\nu)$ class of distributions. The task of reviewing all models of the kind would need considerable space and efforts. Moreover and in spite of the fact that the models considered are essentially identical, there does not seem to be much cross-referencing between applied work in extreme value theory and reliability, so that a large number of relevant papers need to be mentioned in these two, at times overlapping, domains. We may cite, among others, the writings of Gumbel [28], Freund [23], Downton [19], Proschan and Sullo [45], Arnold [4], Johnson and Kotz [31], Marshall and Olkin [42], Raftery [46], Tiago de Oliveira [55], Barnett [7], Galambos [24], Basu [8], Balakrishnan and Basu [6], where additional discussions and details may be found.

The main reason why the distributions $E_A(\gamma,\nu)$ are so appropriate to model lifetimes originates from extreme value theory. Grossly speaking, a lifetime X of a component can be considered (by the *weakest-link principle*, see, e.g., [35, Chapter 14]) as the minimum of the lifetimes of each of its sub-components. Thus, it is natural to chose its distribution within the class of all nondegenerate limit laws generated by minima of a large number of i.i.d. random variables. At this point we recall the following basic facts from extreme value theory (refer to [24]). Let $\{(\xi_n',\xi_n''):\ n\geqslant 1\}$ denote an i.i.d. sequence of $\mathbb{R}^2$-valued random vectors, and set, for $n\geqslant 1$,

$$\eta_n' = \min_{1\leqslant i\leqslant n} \xi_i' \quad \text{and} \quad \eta_n'' = \min_{1\leqslant i\leqslant n} \xi_i''.$$

Following earlier work of Geffroy [25], Sibuya [49], Tiago de Oliveira [54], and de Haan and Resnick [29], Pickands [43] gave the following characterization for limit laws of bivariate extremes (see, e.g. the review in [16]). Assume that there exist norming constants $a_n'>0$, $a_n''>0$, b_n' and b_n'' such that the weak convergence of distributions in $\mathbb{R}^2$

$$\big(a_n'\big(\eta_n'-b_n'\big), a_n''\big(\eta_n''-b_n''\big)\big) \to_d (\mathcal{W}',\mathcal{W}''), \tag{2.21}$$

holds as $n\to\infty$. If $\mathcal{W}'$ and $\mathcal{W}''$ are nondegenerate, then there exist norming constants $c'>0$, $c''>0$, $d'\in\mathbb{R}$, $d''\in\mathbb{R}$, together with indexes $r'\in\mathbb{R}$, $r''\in\mathbb{R}$ such that the distributional identity

$$(\mathcal{W}',\mathcal{W}'') =_d \big(c'(1+r'U)^{1/r'}-d', c''(1+r''V)^{1/r''}-d''\big), \tag{2.22}$$

holds with $(U,V)\equiv E_A(1,1)$ and for a suitable choice of $\{A(u):\ 0\leqslant u\leqslant 1\}$ fulfilling (A.1), (A.2). In (2.22), we use the convention that $(1+rx)^{1/r}=e^x$ for $r=0$. In the particular case of (2.22) when $r'=r''=1$, $c'=d'=\gamma$ and $c''=d''=\nu$, we obtain $(\mathcal{W}',\mathcal{W}'')=_d (X,Y)$ with distribution $E_A(\gamma,\nu)$ as in (2.1). The case where r' or $r''\neq 1$ corresponds to Weibull or Gumbel margins and will not be discussed here (see also [21,52]).

By all this, when one considers joint distributions with exponential marginals to model equipment lifetimes, as above, it is very natural to assume that (X,Y) follows an $E_A(\lambda,\nu)$ distribution. To achieve the statistical analysis of data of this type, one may distinguish two possible main cases of interest:

Case (i). The marginals laws are known, but not the dependence between margins;
Case (ii). Both the marginal laws and dependence structure are unknown.

We consider first the (often unrealistic) Case (i) where the marginal distributions of X and Y are known, or equivalently, when $\gamma>0$ and $\nu>0$ are specified and $A(u)$ unknown. In this case, we may work, without loss of generality on $(U,V)=(X/\gamma,Y/\mu)$ and $\{(U_i,V_i):\ n\geqslant 1\}$.

We assume therefore that the data available is $\{(U_i, V_i)\colon\ 1 \leqslant i \leqslant n\}$. In this framework, the maximum likelihood estimator of $A(u)$ based upon $\{(U_i, V_i)\colon\ 1 \leqslant i \leqslant n\}$ is given by $A_{n,1}(u)$, the latter being defined via the relation ([43,44], see, e.g., [17])

$$\frac{1}{A_{n,1}(u)} = \frac{1}{n}\sum_{i=1}^{n} \min\left(\frac{U_i}{u}, \frac{V_i}{1-u}\right) \quad \text{for } 0 \leqslant u \leqslant 1. \tag{2.23}$$

In (2.23), we use the convention that $1/0 = \infty$. In view of this relation, it is natural to introduce the *empirical dependence processes* (see, e.g., [17]) defined for an arbitrary $A(u)$, by

$$\begin{aligned}\zeta_{n,1;A}(u) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\min\left(\frac{U_i}{u}, \frac{V_i}{1-u}\right) - \frac{1}{A(u)}\right\} \\ &= n^{1/2}\left\{\frac{1}{A_{n,1}(u)} - \frac{1}{A(u)}\right\} \quad \text{for } 0 \leqslant u \leqslant 1,\end{aligned} \tag{2.24}$$

and, under (H.0), with $A(u) = 1$ for $0 \leqslant u \leqslant 1$, by

$$\begin{aligned}\zeta_{n,1}(u) = \zeta_{n,1;1}(u) &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{\min\left(\frac{U_i}{u}, \frac{V_i}{1-u}\right) - 1\right\} \\ &= n^{1/2}\left\{\frac{1}{A_{n,1}(u)} - 1\right\} \quad \text{for } 0 \leqslant u \leqslant 1.\end{aligned} \tag{2.25}$$

Recall the notation $(C[0,1], \mathcal{U})$, to denote the space $C[0,1]$ of continuous functions on $[0,1]$, endowed with the uniform topology $\mathcal{U}$, defined by the sup-norm $\|f\| = \sup_{0\leqslant t\leqslant 1}|f(t)|$. Let $R_{1;A}(u,v)$ be the covariance function defined, for $0 \leqslant u, v \leqslant 1$, by

$$R_{1;A}(u,v) = \mathbb{E}\left(\left\{\min\left(\frac{U}{u}, \frac{V}{1-u}\right) - \frac{1}{A(u)}\right\}\left\{\min\left(\frac{U}{v}, \frac{V}{1-v}\right) - \frac{1}{A(v)}\right\}\right). \tag{2.26}$$

Deheuvels [17] established the following theorem.

Theorem 2.2. *Whenever A fulfills* (A.1), (A.2)*, the empirical dependence process $\{\zeta_{n,1;A}(u)\colon 0 \leqslant u \leqslant 1\}$ converges weakly as $n \to \infty$, in $(C[0,1], \mathcal{U})$, to a centered Gaussian process $\{Z_{1;A}(u)\colon\ 0 \leqslant u \leqslant 1\}$ with almost surely continuous sample paths and covariance function $\{R_{1;A}(u,v)\colon\ 0 \leqslant u, v \leqslant 1\}$.*

For an arbitrary $A(u)$ fulfilling (A.1), (A.2), the expression of $R_{1;A}$ following from (2.26), after integration with respect to the survivor function (2.1) turns out, in general, to be quite involved (see, e.g., [17]). However, it simplifies to a great extent under the assumption (H.0) of *independent margins*, i.e., when $A(u) = 1$ for $0 \leqslant u \leqslant 1$. Setting for convenience $Z_1 = Z_{1;1}$ and $\Gamma_1 = \Gamma_{1;1}$ in this case, we infer readily from (2.26) the following expression for the covariance function $R_1(u,v) = R_{1;1}(u,v) = E(Z_1(u)Z_1(v))$ (see, e.g., [17, (2.19), p. 434]). We have

$$R_1(u,v) = R_1(v,u) = \frac{2v - u^2 - v^2}{(1-u)v} - 1 \quad \text{for } 0 \leqslant u \leqslant v \leqslant 1. \tag{2.27}$$

In the sequel, we shall refer to the limiting centered Gaussian process $\{Z_1(u):\ 0 \leqslant u \leqslant 1\}$, with covariance function $R_1(u,v)$ given by (2.27), as the *Pickands process*. We refer to [17] and [18] for the description of statistics based upon $\{\zeta_{n,1}(u):\ 0 \leqslant u \leqslant 1\}$ to test the assumption (H.0) of independent margins, against the alternative. In particular, the latter reference provides a tabulation of the limiting critical levels of the Cramér–von Mises-type statistic

$$\mathcal{J}_{n,1}^2 = \int_0^1 \zeta_{n,1}(u)^2\,du, \tag{2.28}$$

which, by Theorem 2.2 (see, e.g., [17, Theorem 2.1]), converges weakly as $n \to \infty$ under (H.0) to

$$\mathcal{J}_1^2 = \int_0^1 Z_1(u)^2\,du. \tag{2.29}$$

In spite of the fact that the covariance function $R_1(u,v)$ in (2.27) is much simpler that $R_0(u,v)$ in (1.11) we have not been able to find any closed form expression for the corresponding KL coefficients $\{\lambda_k = \lambda_{k,1}:\ k \geqslant 1\}$ (see, e.g., Proposition 3.2 in the sequel). In [18], a different method was therefore used to tabulate $\mathbb{P}(\mathcal{J}_1^2 \leqslant x)$, as to render possible the use of $\mathcal{J}_{n,1}^2$ for testing (H.0) in this setting.

A close look at the empirical dependence process $\zeta_{n,1}$ under (H.0) shows that it shares with Z_1 the property of having a covariance function $R_1(u,v)$ (see (2.27)) fulfilling

$$\mathbb{E}\big(\zeta_{n,1}(u)^2\big) = \mathbb{E}\big(Z_1(u)^2\big) = R_1(u,u) = 1 \quad \text{for } 0 \leqslant u \leqslant 1.$$

In particular, the empirical dependence process at the endpoints of its definition interval is such that

$$\zeta_{n,1}(0) = n^{1/2}(\overline{U}_n - 1) \to_d N(0,1) \quad \text{and} \quad \zeta_{n,1}(1) = n^{1/2}(\overline{V}_n - 1) \to_d N(0,1), \tag{2.30}$$

where we set, in agreement with the notation (2.18),

$$\overline{U}_n = \frac{1}{n}\sum_{i=1}^n U_i \quad \text{and} \quad \overline{V}_n = \frac{1}{n}\sum_{i=1}^n V_i.$$

It is obvious from (2.30) that each of the statistics $\zeta_{n,1}(0)$ and $\zeta_{n,1}(1)$, depending only upon one of the marginal samples $\{U_i:\ 1 \leqslant i \leqslant n\}$ and $\{V_i:\ 1 \leqslant i \leqslant n\}$, does not carry information on A under the general assumption that $(U,V) \equiv E_A(1,1)$. It is therefore logical to substract from $\zeta_{n,1}$ these non-informative sources of variation, by introducing the *modified dependence empirical process* (see, e.g. [17, (3.1), (3.2)]), defined by

$$\zeta_{n,0}^*(u) = \zeta_{n,0}(u) - u\zeta_{n,0}(1) - (1-u)\zeta_{n,0}(1) \quad \text{for } 0 \leqslant u \leqslant 1. \tag{2.31}$$

In the next theorem, we show that, under (H.0), the Gaussian process $\{Z_0(u)\colon 0 \leqslant u \leqslant 1\}$ in Section 1 is the weak limit of the modified empirical dependence process $\{\zeta^*_{n,0}(u)\colon 0 \leqslant u \leqslant 1\}$ as $n \to \infty$.

Theorem 2.3. *Under* (H.0)*, the modified dependence empirical process* $\{\zeta^*_{n,0}(u)\colon 0 \leqslant u \leqslant 1\}$ *converges weakly as* $n \to \infty$*, in* $(C[0,1], \mathcal{U})$*, to the centered Gaussian process defined, in terms of* $\{Z_1(u)\colon 0 \leqslant u \leqslant 1\}$*, by*

$$Z_0(u) = Z_1(u) - uZ_1(1) - (1-u)Z_1(0) \quad \text{for } 0 \leqslant u \leqslant 1, \tag{2.32}$$

and with covariance function fulfilling $\mathbb{E}(Z_0(u)Z_0(v)) = R_0(u,v)$ *for* $0 \leqslant u, v \leqslant 1$*, where*

$$R_0(u,v) = R_0(v,u) = \frac{2v - u^2 - v^2}{(1-u)v} - 1 - (1-u)(1-v) - uv \quad \text{for } 0 \leqslant u \leqslant v \leqslant 1. \tag{2.33}$$

Proof. The first part of the theorem is straightforward by combining (2.31) with Theorem 2.2. To establish (2.33), we let $R_1(u,v)$ be as in (2.27), and observe that, under (H.0), for $0 \leqslant u \leqslant v \leqslant 1$,

$$\begin{aligned} &R_1(u,1) = u, \qquad R_1(0,v) = 1-v, \qquad R_1(0,0) = 1, \\ &R_1(1,1) = 1, \qquad R_1(0,1) = 0. \end{aligned} \tag{2.34}$$

Thus, we have, for all $0 \leqslant u \leqslant v \leqslant 1$,

$$\begin{aligned} \mathbb{E}\big(Z_0(u)Z_0(v)\big) &= R_1(u,v) - vR_1(u,1) - (1-v)R_1(0,u) - uR_1(v,1) \\ &\quad + uvR_1(1,1) + u(1-v)R_1(0,1) - (1-u)R_1(0,v) \\ &\quad + v(1-u)R_1(0,1) + (1-u)(1-v)R(0,0) \\ &= \frac{2v - u^2 - v^2}{(1-u)v} - 1 - vu - (1-u)(1-v) \\ &= R_0(u,v). \end{aligned} \tag{2.35}$$

Making use of a similar argument when $0 \leqslant v \leqslant u \leqslant 1$, we so obtain (2.33). □

Remark 2.4. The notation introduced in (2.32), (2.33) is in agreement with our original definition of $\{Z_0(u)\colon 0 \leqslant u \leqslant 1\}$ via (1.11). The reciprocal relation induced by the next theorem allows to express the Pickands process $\{Z_1(u)\colon 0 \leqslant u \leqslant 1\}$ as a linear combination of the process $\{Z_0(u)\colon 0 \leqslant u \leqslant 1\}$ and linear functions weighted by independent $N(0,1)$ random variables.

Theorem 2.4. *Let* Z_0 *and* Z_1 *be as in* (2.32)*. Then,* $\{Z_0(u)\colon 0 \leqslant u \leqslant 1\}$*,* $Z_1(0)$ *and* $Z_1(1)$ *are independent.*

Proof. It follows from (2.32) that $\{Z_0(u)\colon 0 \leqslant u \leqslant 1\}$, $Z_1(0)$ and $Z_1(1)$ follow a joint centered Gaussian distribution. Therefore, all we need is to show that

$$\mathbb{E}\big(Z_0(u)Z_1(0)\big) = R_1(0,u) - uR_1(0,1) - (1-u)R_1(0,0) = 0,$$
$$\mathbb{E}\big(Z_0(u)Z_1(1)\big) = R_1(u,1) - uR_1(1,1) - (1-u)R_1(0,1) = 0,$$

which is straightforward by (2.34). □

Remark 2.5. Theorems 1.1 and 2.3, yield jointly the following representation of the Pickands process. There exists an i.i.d. sequence $\{\omega_0', \omega_0'', \omega_k\colon\ k \geqslant 1\}$ of $N(0,1)$ random variables such that

$$Z_1(u) = \omega_0' u + \omega_0''(1-u) + Z_0(u) = \omega_0' u + \omega_0''(1-u) + \sum_{k=1}^{\infty} \sqrt{\lambda_{k,0}}\,\omega_k e_{k,0}(u). \tag{2.36}$$

This, however, does not constitute a KL expansion for $\{Z_1(u)\colon\ 0 \leqslant u \leqslant 1\}$, since the functions u, $1-u$ and $\{e_k(u)\colon\ k \geqslant 1\}$ are not orthogonal in $L^2[0,1]$ (which is needed for (K.1)).

As follows from Theorems 2.1 and 2.3, under (H.0), the empirical processes $\{\zeta_{n,1}(u)\colon 0 \leqslant u \leqslant 1\}$ and $\{\zeta_{n,1}^*(u)\colon\ 0 \leqslant u \leqslant 1\}$ converge weakly to the same limiting process $\{Z_0(u)\colon 0 \leqslant u \leqslant 1\}$. The following theorem gives an explanation of the fact that the limits in both cases coincide.

Theorem 2.5. *Under* (H.0)*, we have*

$$\big\|\zeta_{n,1} - \zeta_{n,1}^*\big\| = o_P(1) \quad \text{as } n \to \infty. \tag{2.37}$$

The proof of Theorem 2.5 is postponed until Section 3. Given this theorem, we obtain a one line proof of Theorem 2.1 as follows.

Proof of Theorem 2.1. Combine Theorems 2.3 and 2.5. □

We conclude by the observation that, as far as practical applications are concerned, there is no need to distinguish between Cases (i) and (ii). In both situations, one may, just as well, base the statistical decision procedure aiming to decide whether X and Y are independent on not, directly on appropriate functionals of $\{\zeta_{n,0}(u)\colon\ 0 \leqslant u \leqslant 1\}$. The use of $\{\zeta_{n,1}(u)\colon\ 0 \leqslant u \leqslant 1\}$, restricted to the case where both margins are known, appears therefore as limited in practice to very few examples of interest.

3. Proofs and complementary results

3.1. Introduction

Besides the proofs of Theorems 1.1 and 2.5, we will give in this section the following representation of the Pickands process $\{Z_1(u)\colon\ 0 \leqslant u \leqslant 1\}$. We will show namely that, on an appropriate probability space, there exists a bivariate Brownian bridge $\{B(s,t)\colon\ 0 \leqslant s,t \leqslant 1\}$ such that

$$Z_1(u) = \int_0^u B\big(e^{-zu}, e^{-z(1-u)}\big)\,dz \quad \text{for } 0 \leqslant u \leqslant 1. \tag{3.1}$$

By a *bivariate Brownian bridge* is meant here a centered Gaussian process with covariance function

$$\mathbb{E}\big(B(s',t')B(s'',t'')\big) = (s' \wedge s'')(t' \wedge t'') - s's''t't'' \quad \text{for } 0 \leqslant s', s'', t', t'' \leqslant 1. \tag{3.2}$$

This process verifies $B(0,t) = B(s,0) = 0$ for $0 \leqslant s, t \leqslant 1$ and $B(1,1) = 0$. On the other hand, $B(1,t)$ and $B(s,1)$ do not vanish identically, and constitute, as functions of $s, t \in [0,1]$, univariate Brownian bridges in the usual sense. The process

$$B^*(s,t) = B(s,t) - tB(s,1) - sB(1,t) \quad \text{for } 0 \leqslant s, t \leqslant 1, \tag{3.3}$$

called a *tied-down bivariate Brownian bridge*, fulfills $B^*(0,t) = B^*(s,0) = B^*(1,t) = B^*(s,1) = 0$ for all $0 \leqslant s, t \leqslant 1$, and defines a centered Gaussian process with covariance function

$$\mathbb{E}\big(B^*(s',t')B^*(s'',t'')\big) = (s' \wedge s'' - s's'')(t' \wedge t'' - t't'') \quad \text{for } 0 \leqslant s', s'', t', t'' \leqslant 1. \tag{3.4}$$

We mention that the tied-down bivariate Brownian bridge has been shown to be the limiting process of the *Hoeffding, Blum, Kiefer, Rosenblatt multivariate empirical process* (see, e.g., [11, 12,15]).

We will show in the forthcoming Section 3.3 that the following representation of $\{Z_0(u)\colon 0 \leqslant u \leqslant 1\}$ holds. On an appropriate probability space, there exists a tied-down bivariate Brownian bridge $\{B^*(s,t)\colon 0 \leqslant s, t \leqslant 1\}$ such that

$$Z_0(u) = \int_0^u B^*\big(e^{-zu}, e^{-z(1-u)}\big)\,dz \quad \text{for } 0 \leqslant u \leqslant 1. \tag{3.5}$$

The remainder of this section is organized as follows. In Section 3.2, we give details on the purely analytic proof of Theorem 1.1, which consists, namely in sorting out the solutions of the Fredholm-type equation induced by (K.2) for $R(u,v) = R_0(u,v)$. Section 3.3 establishes the representations (3.1)–(3.5) and apply the latter to prove Theorem 2.5.

3.2. *Proof of Theorem 1.1*

To avoid any ambiguous statement, we will make use below of the following notation. Letting $R_0(u,v)$ be as in (1.11), we will denote by $\lambda_{1,0} > \lambda_{2,0} > \cdots$ the complete set of eigenvalues $\lambda > 0$ of the Fredholm-type equation

$$\lambda\mathcal{Y}(u) = \int_0^1 R_0(u,v)\mathcal{Y}(v)\,dv, \tag{3.6}$$

holding for some $\mathcal{Y} \neq 0$. Our aim is to show that these eigenvalues coincide with the coefficients

$$\lambda^*_{k,n} = \frac{6}{k(k+1)(k+2)(k+3)} \quad \text{for } k \geqslant 1. \tag{3.7}$$

We will use therefore a different notation for $\{\lambda_{k,n}\colon k \geqslant 1\}$ and $\{\lambda^*_{k,n}\colon k \geqslant 1\}$ until we have established the equalities of these sequences, allowing us afterwards to drop the "$*$."

Towards this aim, we start by the following proposition which will play an essential role in the forthcoming proof of Theorem 1.1.

Proposition 3.1. *A function $\mathcal{Y} \in C[0,1]$ is such that there exists a $\lambda \in \mathbb{R}$ fulfilling the identity*

$$\lambda \mathcal{Y}(u) = \int_0^1 R_0(u,v)\mathcal{Y}(v)\,dv \quad \textit{for } 0 \leqslant u \leqslant 1, \tag{3.8}$$

if and only if the function $y(u) = u(1-u)\mathcal{Y}(u)$ for $0 \leqslant u \leqslant 1$, is solution of the differential equation with limit conditions

$$\lambda u^2(1-u)^2 y^{(4)} - 6y = 0, \qquad y(0) = y(1) = y'(0) = y'(1) = 0. \tag{3.9}$$

Proof. Consider the Fredholm-type equation, for $0 \leqslant u \leqslant 1$

$$\begin{aligned}\lambda \mathcal{Y}(u) &= \int_0^1 R_0(u,v)\mathcal{Y}(v)\,dv \\ &= \int_0^u R_0(u,v)\mathcal{Y}(v)\,dv + \int_u^1 R_0(u,v)\mathcal{Y}(v)\,dv,\end{aligned} \tag{3.10}$$

where, in view of (1.11),

$$\begin{aligned}R_0(u,v) = R_0(v,u) &= \frac{2v - u^2 - v^2}{(1-u)v} - 1 - (1-u)(1-v) - uv \\ &= \frac{-u^2 + 3uv - u^2v - 3uv^2 + 2u^2v^2}{(1-u)v} \quad \text{for } 0 \leqslant u \leqslant v \leqslant 1.\end{aligned} \tag{3.11}$$

It is obvious from (1.11) and (3.10), (3.11) that $\mathcal{Y}$ as above is necessarily continuous on $[0,1]$. By a straightforward recursion, we obtain likewise that $\mathcal{Y}$ is C^∞ on $(0,1)$. There is, therefore, no loss of generality to make the change of variable

$$y(u) = u(1-u)\mathcal{Y}(u) \quad \text{for } 0 \leqslant u \leqslant 1, \tag{3.12}$$

with y assumed to be C^∞ on $(0,1)$. We so obtain the equations

$$\lambda y(u) = \int_0^u \frac{(-v^2 + 3uv - uv^2 - 3u^2v + 2u^2v^2)(1-u)}{v(1-v)^2} y(v)\,dv$$

$$+\int_u^1 \frac{(-u^2+3uv-u^2v-3uv^2+2u^2v^2)u}{v^2(1-v)}y(v)\,dv$$

$$=: \int_0^u Q(u,v)y(v)\,dv + \int_u^1 T(u,v)y(v)\,dv. \tag{3.13}$$

Note that $Q(u,u)=T(u,u)$ and $T(0,v)=Q(1,v)=0$ for all $0\leqslant u,v\leqslant 1$. We may therefore check from the above relations the equalities (rendered necessary by (3.12))

$$y(0)=y(1)=0. \tag{3.14}$$

Moreover, by differentiating both sides of (3.13), we obtain that

$$\begin{aligned}
\lambda y'(u) &= Q(u,u)y(u)-T(u,u)y(u)+\int_0^u \frac{\partial}{\partial u}Q(u,v)y(v)\,dv+\int_u^1 \frac{\partial}{\partial u}T(u,v)y(v)\,dv \\
&= \int_0^u \frac{\partial}{\partial u}Q(u,v)y(v)\,dv+\int_u^1 \frac{\partial}{\partial u}R(u,v)y(v)\,dv \\
&= 3\int_0^u \frac{1-4u+2uv+3u^2-2u^2v}{(1-v)^2}y(v)\,dv+3\int_u^1 \frac{-u^2+2uv-2u^2v}{v^2}y(v)\,dv \\
&=: 3\int_0^u Q_1(u,v)y(v)\,dv+3\int_u^1 T_1(u,v)y(v)\,dv.
\end{aligned} \tag{3.15}$$

Observe that $Q_1(u,u)=T_1(u,u)=1-2u$ and $T_1(0,v)=Q_1(1,v)=0$ for all $0\leqslant u,v\leqslant 1$. We may therefore infer from the above relations that

$$y'(0)=y'(1)=0. \tag{3.16}$$

By differentiating both sides of (3.15), we obtain in turn that

$$\begin{aligned}
\lambda y''(u) &= 3Q_1(u,u)y(u)-3T_1(u,u)y(u) \\
&\quad +3\int_0^u \frac{\partial}{\partial u}Q_1(u,v)y(v)\,dv+3\int_u^1 \frac{\partial}{\partial u}T_1(u,v)y(v)\,dv \\
&= 6\int_0^u \frac{-2+v+3u-2uv}{(1-v)^2}y(v)\,dv+6\int_u^1 \frac{-u+v-2uv}{v^2}y(v)\,dv
\end{aligned}$$

$$=: 6\int_0^u Q_2(u,v)y(v)\,dv + 6\int_u^1 T_2(u,v)y(v)\,dv. \tag{3.17}$$

Observe that $Q_2(u,u) = T_2(u,u) = -2$. It follows that

$$\begin{aligned}\lambda y^{(3)}(u) &= 6Q_2(u,u)y(u) - 6T_2(u,u)y(u)\\ &\quad + 6\int_0^u \frac{\partial}{\partial u} Q_2(u,v)y(v)\,dv + 6\int_u^1 \frac{\partial}{\partial u} T_2(u,v)y(v)\,dv\\ &= 6\int_0^u \frac{3-2v}{(1-v)^2} y(v)\,dv + 6\int_u^1 \frac{-1-2v}{v^2} y(v)\,dv. \end{aligned}\tag{3.18}$$

Finally, by derivating both sides of (3.18), we are led to the differential equation with limit conditions

$$\lambda u^2(1-u)^2 y^{(4)}(u) = 6y(u) \quad \text{with } y(0) = y(1) = y'(0) = y'(1) = 0 \tag{3.19}$$

(recall (3.14)–(3.16)) which completes our proof. □

Lemma 3.1. *We have the equality*

$$\sum_{k=1}^{\infty} \frac{6}{k(k+1)(k+2)(k+3)} = \frac{1}{3}. \tag{3.20}$$

Proof. It is readily checked that, for all $0 \leqslant u < 1$,

$$\sum_{k=1}^{\infty} \frac{2u^{k+2}}{k(k+1)(k+2)} = -(1-u)^2\log(1-u) - u + \frac{3}{2}u^2,$$

whence, by integrating both sides on $(0, 1)$, and setting for convenience $v = 1 - u$,

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)} &= -\frac{1}{2}\int_0^1 \left\{(1-u)^2\log(1-u) + u - \frac{3}{2}u^2\right\} du\\ &= -\frac{1}{2}\int_0^1 \left\{v^2\log v\right\} dv - \frac{1}{2}\left[\frac{u^2}{2} - \frac{u^3}{2}\right]_0^1 = -\frac{1}{2}\int_0^1 \left\{v^2\log v\right\} dv\\ &= -\frac{1}{2}\left[\frac{v^3}{3}\log v - \frac{v^3}{9}\right]_0^1 = \frac{1}{18}.\end{aligned}$$

This, in turn, readily implies (3.20). □

Lemma 3.2. *We have*

$$\mathbb{E}\left(\mathcal{J}_0^2\right)=\mathbb{E}\left\{\int_0^1 Z_0(u)^2\,du\right\}=\sum_{k=1}^{\infty}\lambda_{k,0}^2=\frac{1}{3}. \tag{3.21}$$

Proof. It follows from (1.8), taken with $\mathcal{J}^2=\mathcal{J}_0^2$ and $\lambda_k=\lambda_{k,0}$ for $k\geqslant 1$, that

$$\mathbb{E}\left\{\int_0^1 Z_0(u)^2\,du\right\}=\mathbb{E}\left\{\sum_{k=1}^{\infty}\lambda_{k,0}\omega_k^2\right\}=\sum_{k=1}^{\infty}\lambda_{k,0}. \tag{3.22}$$

On the other hand, an application of Fubini's theorem in combination with (1.11) shows that

$$\begin{aligned}\mathbb{E}\left\{\int_0^1 Z_0(u)^2\,du\right\}&=\int_0^1 \mathbb{E}Z_0(u)^2\,du=\int_0^1 R_0(u,u)^2\,du=\int_0^1\left\{2u-2u^2\right\}du\\&=1-\frac{2}{3}=\frac{1}{3}.\end{aligned} \tag{3.23}$$

We conclude (3.21) by combining (3.22) with (3.23). □

Proof of Theorem 1.1. We have now in hand all the necessary ingredients for the proof of Theorem 1.1. In view of Proposition 3.1 and Lemmas 3.1, 3.2, in order to show that

$$\lambda_{k,0}=\lambda_{k,0}^{*}\quad\text{for } k\geqslant 1, \tag{3.24}$$

it suffices to show that each of the $\lambda_{k,0}^{*}$'s is an admissible value of $\lambda>0$ with respect to (3.9), namely, such that there exists a function y, non-vanishing on [0, 1], and fulfilling

$$\lambda u^2(1-u)^2y^{(4)}(u)=6y(u)\quad\text{with } y(0)=y(1)=y'(0)=y'(1)=0. \tag{3.25}$$

To establish this property, we will show, for each $k\geqslant 1$, the existence of polynomial solutions of (3.25) taken with $\lambda=\lambda_{k,0}^{*}$, of the form

$$\begin{aligned}y(u)=y_k(u)&=u^2(1-u)^2\left\{1+a_1u+a_2u^2+\cdots+a_{k-1}u^{k-1}\right\}=u^2(1-u)^2\sum_{j=-\infty}^{\infty}a_ju^j\\&=\sum_{j=0}^{\infty}u^{j+2}\{a_j-2a_{j-1}+a_{j-2}\}=\sum_{j=0}^{\infty}u^{j+2}b_j,\end{aligned} \tag{3.26}$$

where we set for convenience $a_0=1$, $a_j=0$ for $j\geqslant k$ or $j\leqslant -1$, and $b_j=a_j-2a_{j-1}+a_{j-2}$ for $j\in\mathbb{Z}$. Given this notation, we see that (3.25) taken with $\lambda=\lambda_{k,0}^{*}$ and $y(u)$ as in (3.8) reduces to

$$\frac{u^2(1-u)^2y^{(4)}(u)}{6} = u^2(1-u)^2\sum_{j=2}^{\infty}u^{j-2}\gamma_{j-1}b_j = \sum_{j=0}^{\infty}u^{j+2}\{\gamma_{j-1}b_j - 2\gamma_j b_{j+1} + \gamma_{j+1}b_{j+2}\}$$
$$= \gamma_k y(u) = \sum_{j=0}^{\infty}u^{j+2}\gamma_k b_j, \tag{3.27}$$

where we set $\gamma_j = \{j(j+1)(j+2)(j+3)\}/6 = 1/\lambda^*_{j,0}$ for $j \in \mathbb{Z}$. Given the fact that we must have $b_j = 0$ for either $j < 0$ or $j \geqslant k+2$, we see that (3.27) is equivalent to

$$\begin{aligned}
\gamma_k b_0 &= \gamma_1 b_2,\\
\gamma_k b_1 &= -2\gamma_1 b_2 + \gamma_2 b_3,\\
\gamma_k b_2 &= \gamma_1 b_2 - 2\gamma_2 b_3 + \gamma_3 b_4,\\
&\ \ \vdots\\
\gamma_k b_k &= \gamma_{k-1} b_k - 2\gamma_k b_{k+1},\\
\gamma_k b_{k+1} &= \gamma_k b_{k+1}.
\end{aligned} \tag{3.28}$$

It is readily checked that (3.28) is, in turn, equivalent to

$$\begin{aligned}
\gamma_1(a_2 - 2a_1 + a_0) &= \gamma_1 b_2 = \gamma_k b_0 = \gamma_k a_0,\\
\gamma_2(a_3 - 2a_2 + a_1) &= \gamma_2 b_3 = \gamma_k(2b_0 + b_1) = \gamma_k a_1,\\
\gamma_3(a_4 - 2a_3 + a_2) &= \gamma_3 b_4 = \gamma_k(3b_0 + 2b_1 + b_2) = \gamma_k a_2,\\
&\ \ \vdots\\
\gamma_{k-2}(a_{k-1} - 2a_{k-2} + a_{k-3}) &= \gamma_{k-2} b_{k-1} = \gamma_k\big((k-2)b_0 + \cdots + b_{k-3}\big) = \gamma_k a_{k-3},\\
\gamma_{k-1}(-2a_{k-1} + a_{k-2}) &= \gamma_{k-1} b_k = \gamma_k\big((k-1)b_0 + \cdots + b_{k-2}\big) = \gamma_k a_{k-2},\\
\gamma_k a_{k-1} &= \gamma_k b_{k+1} = \gamma_k(kb_0 + \cdots + b_{k-1}) = \gamma_k a_{k-1}.
\end{aligned} \tag{3.29}$$

Thus, $y(u)$ in (3.26) fulfills (3.25) with $\lambda = \lambda^*_{k,0} = 1/\gamma_k$ if and only if the coefficients $\{a_j\colon\ j \geqslant 0\}$ fulfill the set of relations

$$\gamma_j(a_{j+1} - 2a_j + a_{j-1}) = \gamma_k a_{j-1} \quad \text{for } 1 \leqslant j \leqslant k. \tag{3.30}$$

Given $a_0 = a_{0,k} = 1$, the linear system (3.30) has a unique set of solutions $\{a_j = a_{j,k}\colon j \geqslant 0\}$ with $a_j = 0$ for $j \geqslant k$, which are readily verified to fulfill (1.19), (1.20). Making use of Proposition 3.1, we see that the function

$$\mathcal{Y}(u) = \mathcal{Y}_k(u) = \frac{y_k(u)}{u(1-u)} = u(1-u)\sum_{j=0}^{k-1}a_{j,k}u^j \quad \text{for } 0 \leqslant u \leqslant 1, \tag{3.31}$$

satisfies (3.8) with $\lambda = \lambda_{k,0}$. We have just shown that the $e_{k,0}(u)$ are of the form

$$e_{k,0}(u) = u(1-u)Q_{k-1}(u),$$

with Q_n being, for each $n \geqslant 0$ a polynomial of degree n, we infer from the orthogonality relationships in (K.1) that we must have

$$\int_0^1 u^2(1-u)^2 Q_m(u)Q_n(u)\,du = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

These relations, however, are sufficient to show that the sequence $\{\pm\Delta_n^{1/2} Q_n\colon n \geqslant 0\}$ coincides with the sequence of modified Jacobi polynomials $\{Q_n^{2,2}\colon n \geqslant 1\}$. The proof of Theorem 1.1 is therefore completed. □

Remark 3.1. Let $R_1(u,v)$ be as in (2.27), so that

$$R_1(u,v) = R_0(u,v) + uv + (1-u)(1-v) \quad \text{for } 0 \leqslant u, v \leqslant 1. \tag{3.32}$$

In the following proposition, we show that the solutions of the Fredholm-type equation induced by (K.2) for $R(u,v) = R_1(u,v)$ fulfill the same differential equation as that corresponding to $R(u,v) = R_0(u,v)$, but with different limit conditions. Unfortunately, we have not been able to find simple closed-form expressions for the corresponding KL coefficients $\{\lambda_k\colon k \geqslant 1\}$.

Proposition 3.2. *A function $\mathcal{Y} \in C[0,1]$ is such that there exists a $\lambda \in \mathbb{R}$ fulfilling the identity*

$$\lambda\mathcal{Y}(u) = \int_0^1 R_1(u,v)\mathcal{Y}(v)\,dv \quad \textit{for } 0 \leqslant u \leqslant 1, \tag{3.33}$$

if and only if the function $y(u) = u(1-u)\mathcal{Y}(u)$ for $0 \leqslant u \leqslant 1$ is solution of the differential equation with limit conditions

$$\begin{gathered} \lambda u^2(1-u)^2 y^{(4)} - 6y = 0, \qquad y(0) = y(1) = 0, \\ y'(0) = \lambda \int_0^1 \frac{y(u)}{u}\,du, \qquad y'(1) = \lambda \int_0^1 \frac{y(u)}{1-u}\,du. \end{gathered} \tag{3.34}$$

Proof. The proof being very similar to the proof of Proposition 3.1, we omit details. □

3.3. *Proof of Theorem 2.5*

In the sequel, we let $\{(U_n, V_n)\colon n \geqslant 1\}$ be as in Section 2, and assume throughout that the assumption (H.0) is fulfilled. We so assume that $\{U_m, V_n\colon m, n \geqslant 1\}$ defines a doubly indexed array of independent exponential random variables with mean 1. We denote the empirical survivor function based upon the first $n \geqslant 1$ of these observations by

$$S_n(x,y) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{U_i \geqslant x, V_i \geqslant y\}} \quad \text{for } u, v \in \mathbb{R}, \tag{3.35}$$

where $\mathbb{1}_E$ stands for the indicator of E. The corresponding empirical process is given by

$$a_n(x,y) = n^{1/2}\big(S_n(x,y) - e^{-(x+y)}\big) \quad \text{for } x, y \geqslant 0. \tag{3.36}$$

It will be convenient to introduce the i.i.d. sequence $\{(\mathcal{U}_n, \mathcal{V}_n) = (\exp(-U_n), \exp(-V_n))\colon\ n \geqslant 1\}$ of bivariate uniform random variables on $[0,1]^2$, with empirical distribution function

$$F_n(s,t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{\mathcal{U}_i \leqslant s, \mathcal{V}_i \leqslant t\}} \quad \text{for } s, t \in \mathbb{R}, \tag{3.37}$$

and empirical process

$$\alpha_n(s,t) = n^{1/2}\big(F_n(s,t) - st\big). \tag{3.38}$$

Note for further use that we have the identity, for all $x, y \geqslant 0$,

$$a_n(x,y) = \alpha_n\big(e^{-x}, e^{-y}\big). \tag{3.39}$$

The following fact, due to [57] (see also [9, Theorem 2.3, p. 429]), will be useful.

Fact 3.1. *On an appropriate probability space $(\Omega, \mathcal{A}, \mathbb{P})$, it is possible to define $\{(\mathcal{U}_n, \mathcal{V}_n)\colon\ n \geqslant 1\}$ jointly with a sequence $\{B_n(s,t)\colon\ n \geqslant 1\}$ of bivariate Brownian bridges, with*

$$\mathbb{E}\big(B_n(s',t')B_n(s'',t'')\big) = (s' \wedge s'')(t' \wedge t'') - s's''t't'' \quad \textit{for } 0 \leqslant s', s'', t', t'' \leqslant 1, \tag{3.40}$$

in such a way that

$$\sup_{0 \leqslant s,t \leqslant 1} \big|\alpha_n(s,t) + B_n(s,t)\big| = O_P\left(\frac{\log^2 n}{\sqrt{n}}\right) \quad \textit{as } n \to \infty. \tag{3.41}$$

We next state a simple lemma.

Lemma 3.3. *There exists with probability* 1 *an $n_0 < \infty$ such that, for all $n \geqslant n_0$*

$$F_n(s,t) = 0 \quad \textit{for all } s \geqslant 0, t \geqslant 0 \textit{ such that } s \wedge t \leqslant 1/n^2. \tag{3.42}$$

Proof. By the definition (3.37) of $F_n(s,t)$, we see that $F_n(s,t) = 0$ whenever $s < \min\{\mathcal{U}_1, \ldots, \mathcal{U}_n\}$ or $t < \min\{\mathcal{V}_1, \ldots, \mathcal{V}_n\}$. Thus, to prove (3.42), we need only show that

$$\mathbb{P}\big(\min\{\mathcal{U}_1, \ldots, \mathcal{U}_n\} \leqslant 1/n^2 \text{ i.o.}\big) = \mathbb{P}\big(\min\{\mathcal{V}_1, \ldots, \mathcal{V}_n\} \leqslant 1/n^2 \text{ i.o.}\big) = 0.$$

This, however, is straightforward since these properties are equivalent to

$$\mathbb{P}\big(\mathcal{U}_n \leqslant 1/n^2 \text{ i.o.}\big) = \mathbb{P}\big(\mathcal{V}_n \leqslant 1/n^2 \text{ i.o.}\big) = 0,$$

and follow readily from an application of the Borel–Cantelli lemma. □

The next lemma gives only a rough upper bound which will turn out to be largely sufficient for our needs. Set, for convenience, $B(s,t) = B_1(s,t)$ for $0 \leqslant s, t \leqslant 1$.

Lemma 3.4. *For any* $0 < \varepsilon < 1/2$*, we have*

$$\lim_{s\wedge t\to 0} \frac{|B(s,t)|}{(s\wedge t)^{1/2-\varepsilon}} = 0 \quad a.s. \tag{3.43}$$

Proof. Following [9, Proposition 4.2, pp. 462, 463], we see that there exists positive constants $\mathcal{C}_1, \mathcal{C}_2$ and an $0 < \eta < 1$ such that, for all $0 < \theta \leqslant \eta$,

$$\mathbb{P}\Big(\sup_{s\wedge t\leqslant\theta} \big|B(s,t)\big| \geqslant x\theta^{1/2}\Big) \leqslant \mathcal{C}_1 \exp\big(-\mathcal{C}_2 x^2\big) \quad \text{for } x \geqslant 0. \tag{3.44}$$

Fix $0 < \varepsilon < 1$ and select an arbitrary $\varepsilon_1 > 0$. Set $\theta = \theta_n = e^{-n}$ and $x = \varepsilon_1\theta_n^{-\varepsilon} = \varepsilon_1 e^{n\varepsilon}$ in (3.44). We so obtain that, for all large n,

$$P_n(\varepsilon_1) := \mathbb{P}\Big(\sup_{s\wedge t\leqslant\theta_n} \big|B(s,t)\big| \geqslant \theta_n^{1/2-\varepsilon}\Big) \leqslant \mathcal{C}_1 \exp\big(-\mathcal{C}_2\varepsilon_1^2 e^{2n\varepsilon}\big). \tag{3.45}$$

Since this implies that $\sum_n P_n(\varepsilon_1) < \infty$ independently of $\varepsilon_1 > 0$, we obtain readily (3.43) by an application of the Borel–Cantelli lemma. □

Introduce next the empirical process

$$\begin{aligned}
\mathcal{E}_n(u,v) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \min\left(\frac{U_i}{u}, \frac{V_i}{v}\right) - \frac{1}{u+v} \right\} \\
&= \int_0^\infty\int_0^\infty \min\left(\frac{x}{u}, \frac{y}{v}\right) a_n(dx, dy) \\
&= -\int_0^\infty a_n(zu, zv)\, dz \\
&= -\int_0^\infty \alpha_n\big(e^{-zu}, e^{-zv}\big)\, dz \quad \text{for } u, v > 0.
\end{aligned} \tag{3.46}$$

Define likewise, in view of (3.40), for $n \geqslant 1$, and $u, v \geqslant 0$, with $u \wedge v > 0$, the centered Gaussian processes

$$\mathcal{G}_n(u,v)=\int_0^\infty B_n\left(e^{-zu},e^{-zv}\right)dz \quad \text{and} \quad \mathcal{G}(u,v)=\mathcal{G}_1(u,v)=\int_0^\infty B\left(e^{-zu},e^{-zv}\right)dz, \tag{3.47}$$

the existence of which being a straightforward consequence of (3.43). In view of this notation, we obtain the following approximation theorem.

Theorem 3.1. *Assume* (H.0). *Then, on* $(\Omega,\mathcal{A},\mathbb{P})$*, we have, for each* $0<\delta<1$*,*

$$\sup_{u\vee v\geqslant\delta}\left|\mathcal{E}_n(u,v)-\mathcal{G}_n(u,v)\right|=O_P\left(\frac{\log^3 n}{\sqrt{n}}\right) \quad \text{as } n\to\infty. \tag{3.48}$$

Moreover, $\{\mathcal{G}_1(u,v)\colon\ u\vee v\geqslant\delta\}$ *is continuous.*

Proof. Fix $\delta>0$ and set $c_n=(12\log n)/\delta$ for $n\geqslant 1$. It follows from (3.41), (3.46) and (3.47) that, on $(\Omega,\mathcal{A},\mathbb{P})$,

$$\begin{aligned}
\sup_{u\vee v\geqslant\delta}\left|\mathcal{E}_n(u,v)-\mathcal{G}_n(u,v)\right| &= \sup_{u\vee v\geqslant\delta}\left|\int_0^\infty\left\{\alpha_n\left(e^{-zu},e^{-zv}\right)+B_n\left(e^{-zu},e^{-zv}\right)\right\}dz\right| \\
&\leqslant \sup_{u\vee v\geqslant\delta}\left|\int_0^{c_n}\left\{\alpha_n\left(e^{-zu},e^{-zv}\right)+B_n\left(e^{-zu},e^{-zv}\right)\right\}dz\right| \\
&\quad + \sup_{u\vee v\geqslant\delta}\left|\int_{c_n}^\infty\left\{\alpha_n\left(e^{-zu},e^{-zv}\right)\right\}dz\right| \\
&\quad + \sup_{u\vee v\geqslant\delta}\left|\int_{c_n}^\infty\left\{B_n\left(e^{-zu},e^{-zv}\right)\right\}dz\right| \\
&=: K_{n,1}+K_{n,2}+K_{n,3}.
\end{aligned} \tag{3.49}$$

By (3.41) and the definition of c_n,

$$\begin{aligned}
K_{n,1} &= \sup_{u\vee v\geqslant\delta}\left|\int_0^{c_n}\left\{\alpha_n\left(e^{-zu},e^{-zv}\right)+B_n\left(e^{-zu},e^{-zv}\right)\right\}dz\right| \\
&\leqslant c_n\times\sup_{0\leqslant s,t\leqslant 1}\left|\alpha_n(s,t)+B_n(s,t)\right|=O_P\left(\frac{\log^3 n}{\sqrt{n}}\right) \quad \text{as } n\to\infty.
\end{aligned} \tag{3.50}$$

Next, we observe from the definition (3.38) of $\alpha_n(s,t)$, that

$$K_{n,2} = \sup_{u\vee v\geqslant\delta}\left|\int_{c_n}^\infty \alpha_n\left(e^{-zu},e^{-zv}\right)dz\right|$$

$$\leqslant \sup_{u\vee v\geqslant\delta}\left|\int_{c_n}^{\infty}\left\{n^{1/2}F_n\left(e^{-zu},e^{-zv}\right)\right\}dz\right| + \sup_{u\vee v\geqslant\delta}\left|\int_{c_n}^{\infty}\left\{n^{1/2}e^{-z(u+v)}\right\}dz\right|$$
$$=:\ K'_{n,2}+K''_{n,2}.$$

Now, (3.42) implies that $K'_{n,2}=0$ almost surely for all large n, whereas a direct integration yields

$$K''_{n,2}=\sup_{u\vee v\geqslant\delta}\frac{n^{1/2}e^{-c_n(u+v)}}{u+v}\leqslant\frac{n^{1/2}e^{-c_n\delta}}{\delta}\leqslant\frac{1}{\delta n^2}.$$

These two statements, when combined, are more than enough to show that

$$K_{n,2}=o_P\left(\frac{\log^3 n}{\sqrt{n}}\right)\quad\text{as } n\to\infty. \tag{3.51}$$

Finally, we consider

$$K_{n,3}=\sup_{u\vee v\geqslant\delta}\left|\int_{c_n}^{\infty}\left\{B_n\left(e^{-zu},e^{-zv}\right)\right\}dz\right| =_d K_{n,4}=\sup_{u\vee v\geqslant\delta}\left|\int_{c_n}^{\infty}\left\{B\left(e^{-zu},e^{-zv}\right)\right\}dz\right|,$$

where " $=_d$" denotes equality in distribution. By setting $\varepsilon=1/4$ in (3.43), we see that, almost surely for all large n, the following inequality holds, uniformly over all $z\geqslant c_n$ and $u>0$, $v>0$ with $u\vee v\geqslant\delta$,

$$\left|B\left(e^{-zu},e^{-zv}\right)\right|\leqslant\exp\left(-\frac{z}{4}(u\vee v)\right). \tag{3.52}$$

It follows that, almost surely for all large n,

$$K_{n,4}\leqslant\sup_{u\vee v\geqslant\delta}\left\{\frac{4e^{-c_n(u\vee v)/4}}{u\vee v}\right\}\leqslant\frac{4e^{-c_n\delta/4}}{\delta}\leqslant\frac{1}{\delta n^2}. \tag{3.53}$$

Since almost sure convergence implies convergence in probability and $K_{n,3}=_d K_{n,4}$, it follows readily from this last statement that

$$K_{n,3}=o_P\left(\frac{\log^3 n}{\sqrt{n}}\right)\quad\text{as } n\to\infty, \tag{3.54}$$

we conclude (3.48) by a joint application of (3.50), (3.51), (3.54) and the triangle inequality. The proof that $\mathcal{E}_1(u,v)$ is continuous on $\{(u,v)\colon\ u\vee v\geqslant\delta\}$ may be achieved along the same lines or by making use of the explicit form of the covariance function of $\mathcal{G}(u,v)$. We omit the details of this routine argument. $\square$

Proof of Theorem 2.5. Below, we will work on the probability space $(\Omega,\mathcal{A},\mathbb{P})$ of Fact 3.1. Letting $\{B_n(s,t)\colon\ 0\leqslant s,t\leqslant 1\}$ be as in Fact 3.1, we set, for $n\geqslant 1$,

$$B_n^*(s,t) = B_n(s,t) - sB_n(1,t) - tB_n(s,1) \quad \text{for } 0 \leqslant s,t \leqslant 1, \tag{3.55}$$

and observe that the so-defined $\{B_n^*(s,t)\colon\ 0 \leqslant s,t \leqslant 1\}$ constitute for $n \geqslant 1$ a sequence of tied-down bivariate Brownian bridges. In view of (3.47) and (3.55), set for $n \geqslant 1$ and $0 \leqslant u \leqslant 1$,

$$Z_1^{(n)}(u) = \mathcal{G}_n(u, 1-u) = \int_0^\infty B_n\big(e^{-zu}, e^{-zv}\big)\, dz, \tag{3.56}$$

and

$$\begin{aligned} Z_0^{(n)}(u) &= Z_1^{(n)}(u) - uZ_1^{(n)}(1) - vZ_1^{(n)}(0) \\ &= \mathcal{G}_n(u, 1-u) - u\mathcal{G}_n(1,0) - (1-u)\mathcal{G}_n(0,1) = \int_0^\infty B_n^*\big(e^{-zu}, e^{-zv}\big)\, dz. \end{aligned} \tag{3.57}$$

Recalling (2.25), (2.31) and (3.46), we see that, under (H.0),

$$\zeta_{n,1}(u) = \mathcal{E}_n(u, 1-u), \tag{3.58}$$

and

$$\zeta_{n,0}^*(u) = \mathcal{E}_n(u, 1-u) - u\mathcal{E}_n(1,0) - (1-u)\mathcal{E}_n(0,1). \tag{3.59}$$

Moreover, by (2.30) and (3.58), $\mathcal{E}_n(1,0) = n^{1/2}(\overline{U}_n - 1)$ and $\mathcal{E}_n(0,1) = n^{1/2}(\overline{V}_n - 1)$. This, in combination with (2.2), (3.46) and (3.48), entails readily that, as $n \to \infty$, the following relations hold uniformly in $0 \leqslant u \leqslant 1$,

$$\begin{aligned} \zeta_{n,0}(u) &= \mathcal{E}_n\big(u\overline{U}_n, (1-u)\overline{V}_n\big) + n^{1/2}\left\{\frac{1}{\overline{U}_n u + \overline{V}_n(1-u)} - 1\right\} \\ &= \mathcal{E}_n\big(u\overline{U}_n, (1-u)\overline{V}_n\big) - \frac{u\mathcal{E}_n(1,0) + (1-u)\mathcal{E}_n(0,1)}{1 + n^{-1/2}(u\mathcal{E}_n(1,0) + (1-u)\mathcal{E}_n(0,1))} \\ &= \mathcal{E}_n\big(u\overline{U}_n, (1-u)\overline{V}_n\big) - u\mathcal{E}_n(1,0) - (1-u)\mathcal{E}_n(0,1) + O_P\left(\frac{1}{\sqrt{n}}\right) \\ &= \mathcal{G}_n\big(u\overline{U}_n, (1-u)\overline{V}_n\big) - u\mathcal{G}_n(1,0) - (1-u)\mathcal{G}_n(0,1) + O_P\left(\frac{\log^3 n}{\sqrt{n}}\right) \\ &= \mathcal{G}_n\big(u\overline{U}_n, (1-u)\overline{V}_n\big) - \mathcal{G}_n(u, 1-u) + Z_0^{(n)}(u) + O_P\left(\frac{\log^3 n}{\sqrt{n}}\right). \end{aligned} \tag{3.60}$$

Denote by $\mathbf{I}(u) = u$ the identity. We infer readily from (3.48), (3.56) and (3.58), in combination with the triangle inequality, that, under (H.0), as $n \to \infty$,

$$\big\|\zeta_{n,1} - Z_1^{(n)}\big\| = \big\|\mathcal{E}_n(\mathbf{I}, 1-\mathbf{I}) - \mathcal{G}_n(\mathbf{I}, 1-\mathbf{I})\big\| = O_P\left(\frac{\log^3 n}{\sqrt{n}}\right). \tag{3.61}$$

Likewise, by (3.48), (3.57) and (3.59), we obtain that, under (H.0), as $n \to \infty$,

$$\begin{aligned}\left\| \zeta_{n,0}^{*} - Z_0^{(n)} \right\| &= \left\| \left\{ \mathcal{E}_n(\mathbf{I}, 1-\mathbf{I}) - \mathbf{I}\mathcal{E}_n(1,0) - (1-\mathbf{I})\mathcal{E}_n(0,1) \right\} \right. \\ &\quad \left. - \left\{ \mathcal{G}_n(\mathbf{I}, 1-\mathbf{I}) - \mathbf{I}\mathcal{G}_n(1,0) - (1-\mathbf{I})\mathcal{G}_n(0,1) \right\} \right\| \\ &= O_P\left(\frac{\log^3 n}{\sqrt{n}} \right). \end{aligned} \tag{3.62}$$

We note here that (3.61), (3.62), when combined with Theorems 2.1 and 2.3, imply (3.1) and (3.5).

Next, making use of the continuity of $\mathcal{G}_n(u,v) =_d \mathcal{G}_1(u,v)$ on $\{(u,v)\colon u \vee v \geqslant \delta\}$ for each specified $0 < \delta < 1/2$, the latter following from Theorem 3.1, we infer from (3.60) and the LLN for $\overline{U}_n$ and $\overline{V}_n$ that, under (H.0), as $n \to \infty$,

$$\left\| \zeta_{n,0} - Z_0^{(n)} \right\| = \left\| \mathcal{G}_n\big(\mathbf{I}\overline{U}_n, (1-\mathbf{I})\overline{V}_n\big) - \mathcal{G}_n(\mathbf{I}, 1-\mathbf{I}) \right\| + O_P\left(\frac{\log^3 n}{\sqrt{n}} \right) = o_P(1). \tag{3.63}$$

We conclude (2.37) by combining (3.62) and (3.63) with the triangle inequality. □

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