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Course: Algorithmic Motion Planning – ASEN 52540-001 – Fall 2024
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Date: September 13, 2024

Homework 3 Assignment Submission

Exercise 1.

Consider the following Euler representation of rotation:
$R(\alpha, \beta, \gamma) = R_z(\gamma)R_y(\beta)R_z(\alpha)$.
(a) Determine matrix $R(\alpha, \beta, \gamma)$.
(b) Show that $R(\alpha, \beta, \gamma) = R(\alpha - \pi, -\beta, \gamma - \pi)$.
(c) Given a rotation matrix R′, determine α, β, and γ in terms of elements of R′. (Hint: denote the element of R′ in the ith row and jth column by R'_{ij} , and write your solutions in terms of these elements.)

Matrix R(α, β, γ):

To find the rotation matrix R(α, β, γ), we have to find the rotation matrices for each individual rotation and then multiply them together.

The first rotation $R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$

$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$R_z(\gamma)R_y(\beta)R_z(\alpha) = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix}$

$= R(\alpha, \beta, \gamma)$

R(α, β, γ) = R(α − π, −β, γ − π):

We can show that R(α, β, γ) = R(α − π, −β, γ − π) by substituting those new values (α − π, −β, γ − π) into the matrices we already derived and show that they have the same value.

$$R_z(\alpha-\pi)= \begin{pmatrix} \cos(\alpha-\pi) & -\sin(\alpha-\pi) & 0 \\ \sin(\alpha-\pi) & \cos(\alpha-\pi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & -\cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y(-\beta) = \begin{pmatrix} \cos(-\beta) & 0 & \sin(-\beta) \\ 0 & 1 & 0 \\ -\sin(-\beta) & 0 & \cos(-\beta) \end{pmatrix} = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$R_z(\gamma-\pi)= \begin{pmatrix} \cos(\gamma-\pi) & -\sin(\gamma-\pi) & 0 \\ \sin(\gamma-\pi) & \cos(\gamma-\pi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & -\cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R(\alpha-\pi,-\beta,\gamma-\pi)= \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma-\sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma+\cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{pmatrix}$$

= R(α, β, γ)

Given Rotation Matrix R’ Determine α, β, γ in Terms of R’:

Let the elements of R′ be denoted as:

$$R' = \begin{pmatrix} R'11 & R'12 & R'13 \\ R'21 & R'22 & R'23 \\ R'31 & R'32 & R'33 \end{pmatrix}$$

From the structure of R(α,β,γ), we can extract:

- β can be determined from the third row, third column element:

  cosβ=R’33⇒β=arccos(R’33)
- α can be determined from the first and second rows of the third column:

  sinα=R’23/sinβ, cosα=R’13/sinβ

  Thus:

  α=arctan2(R’23/sinβ,R’13/sinβ)
- γ can be determined from the first and second rows of the first column:

  sinγ=R’21/sinβ, cosγ=R’11/sinβ

  Thus:

  γ=arctan2(R21′/sinβ, R11’/sinβ)

Exercise 2.

Consider the 3-link manipulator in Figure 1. The links A1, A2, and A3 are identical.

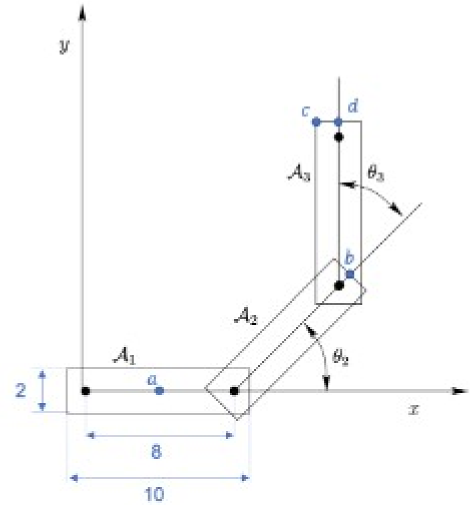

Figure 1: 3-link manipulator.

(a) For the configuration (θ1, θ2, θ3) = (π/4, π/2, −π/6), determine the locations of points a, b, and c.
(b) Find the configuration(s) of the robot when point d is at (0, 4).

Locations For Points a, b, c at (θ1, θ2, θ3) = (π/4, π/2, −π/6):

The manipulator's configuration involves calculating the positions of points a, b, and c based on the joint angles and link lengths.

Point a: is located halfway up the first link and rotated up by pi/4 so is located at the point (4cos(pi/4), 4sin(pi/4)).

Point b: Point b is 9 units away from point a along the second link, and the second link is rotated by another 2π relative to point a. The total rotation at point b is the sum of the rotations from the base and the second joint: Total rotation at b=π/4+π/2=3π/4.

The change in coordinates from a to b is given by:

$\Delta x=9\cos(3\pi/4)$

$\Delta y=9\sin(3\pi/4)$

Since $\cos(3\pi/4)= \dfrac{-\sqrt{2}}{2}$ and $\sin(3\pi/4)= \dfrac{\sqrt{2}}{2}$

$\Delta x=9\times \dfrac{-\sqrt{2}}{2} \approx 9\times -0.707\approx -6.363$

$\Delta y=9\times \dfrac{\sqrt{2}}{2} \approx 9\times 0.707\approx 6.363$

Adding these changes to the coordinates of point a (which are (2.828,2.828)):

$b_x=2.828-6.363\approx -3.535$ $b_y=2.828+6.363\approx 9.191$

Therefore, the coordinates of point b are approximately:

$b=(-3.535,9.191)$

This places point b 9 units away from point a, rotated by a total of 3π/4.

Point c:

Position the end of the third link relative to point b:

We already calculated the position of point b as (−3.535,9.191).

The third link is 9 units long, and it connects 1 unit back from point b in the direction of the previous rotation, which is at 3π/4.

So, we first move 1 unit back from point b along this direction.

The change in position for this movement is:

$\Delta x=1\cos(3\pi/4)=1\times \dfrac{-\sqrt{2}}{2} \approx -0.707$ $\Delta y=1\sin(3\pi/4)=1\times \dfrac{\sqrt{2}}{2} \approx 0.707$

So, the coordinates of the point where the third link starts, call it b′, are:

$bx'=bx+(-0.707)=-3.535-0.707=-4.242$ $by'=by+0.707=9.191+0.707=9.898$

So, $b'\approx(-4.242, 9.898)$.

Find the end of the third link (before accounting for the side shift):

From b′, the third link extends 9 units in the direction of the **current rotation**. Since the second link rotated by π/2, the new rotation for the third link = 3π/4-π/6 = 5π/12

The change in position for the end of the third link (let’s call it c′) is:

$\Delta x=9\cos(5\pi/12)$ $\Delta y=9\sin(5\pi/12)$

Numerically:

$\cos(5\pi/12)\approx 0.2588$ $\sin(5\pi/12)\approx 0.9659$

Therefore, the change in coordinates is:

$\Delta x=9\times 0.2588\approx 2.33$ $\Delta y=9\times 0.9659\approx 8.693$

Adding these to the coordinates of b′:

$c_x' = b_x' + 2.33 = -4.242 + 2.33 = -1.912$ $c_y' = b_y' + 8.693 = 9.898 + 8.693 = 18.591$

So, the coordinates of $c' \approx (-1.912, 18.591)$.

Shift point c by 1 unit counterclockwise:

Now, we need to apply the final shift: point c is 1 unit **counterclockwise** from the end of the third link. A counterclockwise shift by 90 degrees from $-\pi/6$ (the rotation at c') brings the direction to $\pi/3$.

The change in position due to this shift is:

$\Delta x = 1\cos(\pi/3) = 1 \times \frac{1}{2} = 0.5$ $\Delta y = 1\sin(\pi/3) = 1 \times \frac{\sqrt{3}}{2} \approx 0.866$

So, the final coordinates of point c are:

$c_x = c_x' + 0.5 = -1.912 + 0.5 = -1.412$ $c_y = c_y' + 0.866 = 18.591 + 0.866 = 19.457$

Therefore, the corrected coordinates of point c are approximately:

$c = (-1.412, 19.457)$

Configurations When d is at (0,4):

Set up the kinematic equations: The end position of the manipulator's point d is determined by the joint angles and the lengths of the links.

The forward kinematics in terms of θ_1, θ_2, and θ_3 for the position of point d can be written as:

$x_d = 8\cos(\theta_1) + 8\cos(\theta_1+\theta_2) + 9\cos(\theta_1+\theta_2+\theta_3)$ $y_d = 8\sin(\theta_1) + 8\sin(\theta_1+\theta_2) + 9\sin(\theta_1+\theta_2+\theta_3)$

Since point d is at (0,4), we have:

$0 = 8\cos(\theta_1) + 8\cos(\theta_1+\theta_2) + 9\cos(\theta_1+\theta_2+\theta_3)$ $4 = 8\sin(\theta_1 + 8\sin(\theta_1+\theta_2) + 9\sin(\theta_1+\theta_2+\theta_3)$

Solve for θ_1: The fact that $x_d = 0$ suggests that the manipulator must align symmetrically about the y-axis. A natural assumption here is that $\theta_1 = 2\pi$, which would align the first link vertically along the y-axis.

Substituting $\theta_1 = 2\pi$ simplifies the equations:

$0 = 8\cos(2\pi) + 8\cos(2\pi+\theta_2) + 9\cos(2\pi+\theta_2+\theta_3)$ $4 = 8\sin(2\pi) + 8\sin(2\pi+\theta_2) + 9\sin(2\pi+\theta_2+\theta_3)$

These simplify further to:

$0 = -8\sin(\theta_2) - 9\sin(\theta_2+\theta_3)$ $4 = 8 + 8\cos(\theta_2) + 9\cos(\theta_2+\theta_3)$

Solve for θ_2 and θ_3: From the first equation:

$\sin(\theta_2) + 9/8\sin(\theta_2+\theta_3) = 0$

This implies:

$\sin(\theta_2+\theta_3) = -8/9\sin(\theta_2)$

Now, use the second equation:

$4 = 8 + 8\cos(\theta_2) + 9\cos(\theta_2+\theta_3)$

Rearrange this to:

$-4=8\cos(\theta_2)+9\cos(\theta_2+\theta_3)$

Solving these two equations simultaneously you get:

$\theta_2 \approx 1.586$ radians $\theta_3 \approx -2.681$ radians

Exercise 3.

Express the configuration spaces of the following systems in terms of a Cartesian product

of simpler spaces (such as Rn, Sn, etc.) and determine their dimensions. Justify your answer.

(a) Two trains on two train tracks.

(b) A spacecraft that can translate and rotate in 2D.

(c) Two mobile robots rotating and translating in the plane.

(d) Two translating and rotating planar mobile robots connected rigidly by a bar.

(e) A cylindrical rod that can translate and rotate in 3D. (Hint: if the rod is rotated about its central

axis, it is assumed that the rod's position and orientation are not changed in any detectable way.)

(f) A spacecraft that can translate and rotate in 3D and is equipped with a 3-link robot arm

(revolute joints only).

(g) The manipulator in Figure 2.

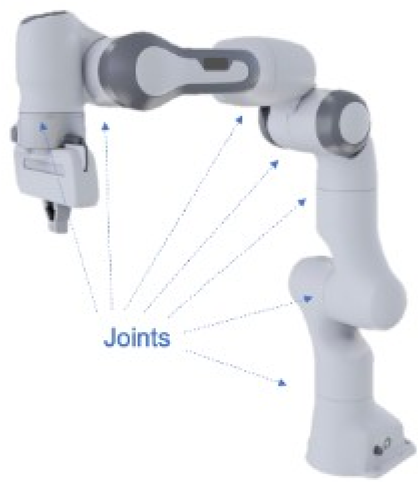

Figure 2: Robotic manipulator with revolute joints

(a) Two trains on two train tracks

Each train can be described by its position along a track and its orientation (direction). Assuming the tracks are straight and infinite:

- Each train's position can be represented by R (a real line).
- Orientation can be represented by S1 (a circle, since it can face any direction).

Thus, the configuration space for one train is R×S1. For two trains:

C=(R×S1)×(R×S1)=R2×(S1)2.

Dimension: 2+2=4.

(b) A spacecraft that can translate and rotate in 2D

The spacecraft can translate in 2D space and can rotate about a point:

- Translational degrees of freedom: R2.
- Rotational degree of freedom: S1.

The configuration space is:

C=R2×S1.

Dimension: 2+1=3.

(c) Two mobile robots rotating and translating in the plane

Each robot can translate in 2D and rotate:

- For one robot: R2×S1.
- For two robots:

C=(R2×S1)×(R2×S1)=R4×(S1)2.

Dimension: 4+2=6.

(d) Two translating and rotating planar mobile robots connected rigidly by a bar

The two robots have the same translational and rotational coordinates. However, since they are connected by a rigid bar, they can't move independently:

- One robot's position and orientation: R2×S1.

The configuration space can be described by the position of one robot and the angle between the bar and the reference direction:

- Position: R2.
- Angle: S1.

Thus:

C=R2×S1.

Dimension: 2+1=3.

(e) A cylindrical rod that can translate and rotate in 3D

The rod can translate in 3D and can rotate about its central axis:

- Translational degrees of freedom: R3.
- Rotational degree of freedom (around the central axis, which can be represented as a circle): S1.

The configuration space is:

C=R3×S1.

Dimension: 3+1=4.

(f) A spacecraft that can translate and rotate in 3D and is equipped with a 3-link robot arm (revolute joints only)

The spacecraft can translate and rotate in 3D:

- Translational degrees of freedom: R3.
- Rotational degrees of freedom: S3 (for full orientation in 3D).

The 3-link robot arm has 3 revolute joints:

- Each joint adds one rotational degree of freedom: S1 for each joint.

Thus, the configuration space for the arm is (S1)3.

Combining these:

C=R3×S3×(S1)3.

Dimension: 3+3+3=9.

Configuration Space for a 7-Joint Revolute Manipulator Arm

Each revolute joint contributes one rotational degree of freedom, represented as S1. Therefore, for 7 revolute joints, the configuration space can be expressed as:

C=(S1)7.

Dimension

The dimension of this configuration space is simply the number of joints, which is:

Dimension: 7.

Exercise 4.

Consider workspace W ⊆ Rn with convex obstacles. Show that the C-space obstacles

are also convex for a convex robot with transitional motion in W.

Select Two Configurations: Let q_1 and q_2 be two configurations in the C-space such that both configurations lead to the robot intersecting the obstacle O. Each configuration q_i consists of a position in the workspace and a configuration of the robot.

Translate the Robot: The C-space obstacle corresponding to O can be defined as:

$O_C=\{q\in C \mid R(q)\cap O\neq\varnothing\}$

where R(q) is the position and orientation of the robot based on the configuration q.

Consider a Line Segment: For $\lambda\in[0,1]$, define a point on the line segment connecting q_1 and q_2 as:

$q_\lambda=(1-\lambda)q_1+\lambda q_2$.
Check Intersections: To show O_C is convex, we need to show that $R(q_\lambda)\cap O\neq\varnothing$:

Since O is convex, for any points x_1 and x_2 in O, the line segment connecting them lies entirely within O.
If both configurations q_1 and q_2 lead to intersections with O, then the positions $R(q_1)$ and $R(q_2)$ must both intersect the obstacle.
Therefore, for any point along the line segment connecting the configurations, the robot can be placed in a way that it still intersects the obstacle.

Conclusion: Since any linear combination of configurations q_λ results in the robot intersecting the convex obstacle O, it follows that the C-space obstacle O_C is also convex.