Midterm
Course: Automatic Control Systems – ASEN 5114-001 – Spring 2025
Professor: Dale Lawrence
Teaching Assistant: Anantha Dhruva

Steve Gillet

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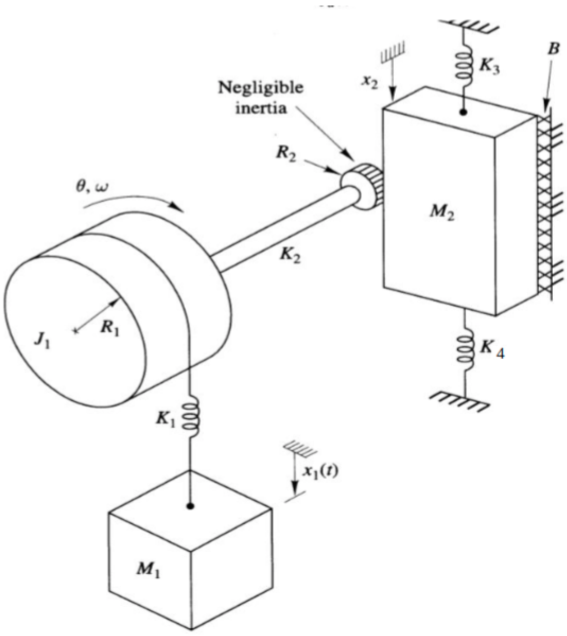
The following take-home midterm exam must be completed individually by each student. Students may use their notes, or the textbook. Any other resources used must be cited. Students are NOT allowed to communicate with any other person, except the instructor. Show your work and justify your answers for full credit. This exam is due at 11:59 PM on Monday, February 24 on Canvas. Students registered for ASEN 5114 do problems 2,3,4,5. Students registered for ASEN 4114 do problems 1,3,4,5.

2)

[40 pts] Find a direct state space model for the mechanical system at right, with an input force $f_a(t)$ oriented positive downward on M_2 and output $\omega(t)$. Neglect gravity and assume all positions are measured from the nominal position where the springs are at their rest length. The pinion gear with radius R_2 meshes with a rack gear on the side of M_2 without slipping. Model the sliding friction B with a linear damper. Note the rotating shaft has torsional stiffness K_2. Order the states in this model according to the element indexes, starting with inertial elements. Do not use the symbol "x" for state variables, since these are already used for displacements in the figure. θ and ω are inertially referenced.



1) Identify

- Input: $u = f$

- Output: $y = \omega$
- States:
  - Mass M_1: $w_1 = v_{M_1}$
  - Spring K_1: $w_2 = f_{K_1}$
  - Mass M_2: $w_3 = v_{M_2}$
  - Spring K_3: $w_4 = f_{K_3}$
  - Spring K_4: $w_5 = f_{K_4}$
  - Mass J_1: $w_6 = \omega = y$
  - Mass M_2: $w_7 = x_2$
  - Mass J_1: $w_8 = \Theta$

2) Complementary State Variables

$$w_1^* = f_{M_1}, \quad w_2^* = v_{K_1}, \quad w_3^* = f_{M_2}, \quad w_4^* = v_{K_3}, \quad w_5^* = v_{K_4}, \quad w_6^* = \tau_{J_1}$$

3) Topology Equations

$$\begin{aligned}
w_1^* &= f_{M_1} = -f_{K_1} = -w_2 \\
w_2^* &= v_{K_1} = v_{M_1} - \omega = w_1 - w_6 \\
w_3^* &= f_{M_2} = f - Bv_{M_2} + f_{K_4} - f_{K_3} - K_2\left(\Theta - \frac{x_2}{R_2}\right) \\
&= f - Bw_3 + w_5 - w_4 - K_2\left(w_8 - \frac{w_2}{R_2}\right) \\
w_4^* &= v_{K_3} = v_{M_2} = w_3 \\
w_5^* &= v_{K_4} = -v_{M_2} = -w_3 \\
w_6^* &= \tau_{J_1} = f_{K_1} - K_2\left(\theta - \frac{x_2}{R_2}\right) = w_2 - K_2\left(w_8 - \frac{w_7}{R_2}\right)
\end{aligned}$$

4) Energy Storage Element Equations

$$\begin{aligned}
w_1^* &= f_{M_1} = M_1\dot{v}_{M_1} = M_1\dot{w}_1 \implies \dot{w}_1 = -\frac{1}{M_1}w_2 \\
w_2^* &= v_{K_1} = \frac{1}{K_1}\dot{f}_{K_1} = \frac{1}{K_1}\dot{w}_2 \implies \dot{w}_2 = K_1(w_1 - w_6) \\
w_3^* &= f_{M_2} = M_2\dot{v}_{M_2} = M_2\dot{w}_3 \implies \dot{w}_3 = \frac{1}{M_2}(f - Bw_3 + w_5 - w_4 - K_2(w_8 - \frac{w_7}{R_2})) \\
w_4^* &= v_{K_3} = \frac{1}{K_3}\dot{f}_{K_3} = \frac{1}{K_3}\dot{w}_4 \implies \dot{w}_4 = K_3w_3 \\
w_5^* &= v_{K_4} = \frac{1}{K_4}\dot{f}_{K_4} = \frac{1}{K_4}\dot{w}_5 \implies \dot{w}_5 = -K_4w_3 \\
w_6^* &= \tau_{J_1} = J_1\dot{\omega} = J_1\dot{w}_6 \implies \dot{w}_6 = \frac{1}{J_1}w_2 - \frac{K_2}{J_1}w_8 + \frac{K_2}{J_1R_2}w_7
\end{aligned}$$

Note: $\dot{w}_7 = w_3$, $\dot{w}_8 = w_6$

5) Matrix Form

State vector: $\mathbf{w} = [w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8]^T$

State-space equations:

$$\dot{\mathbf{w}} = A\mathbf{w} + Bu, \quad y = C\mathbf{w} + Du$$

Where:

$$A = \begin{bmatrix} 0 & -\frac{1}{M_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ K_1 & 0 & 0 & 0 & 0 & -K_1 & 0 & 0 \\ 0 & 0 & -\frac{1}{M_2} & -\frac{1}{M_2} & \frac{1}{M_2} & 0 & \frac{K_2}{M_2} & -\frac{K_2}{M_2} \\ 0 & 0 & K_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -K_4 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{J_1} & 0 & 0 & 0 & 0 & \frac{K_2}{J_1 R_1} & -\frac{K_2}{J_1} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$