Midterm
Course: Automatic Control Systems – ASEN 5114-001 – Spring 2025
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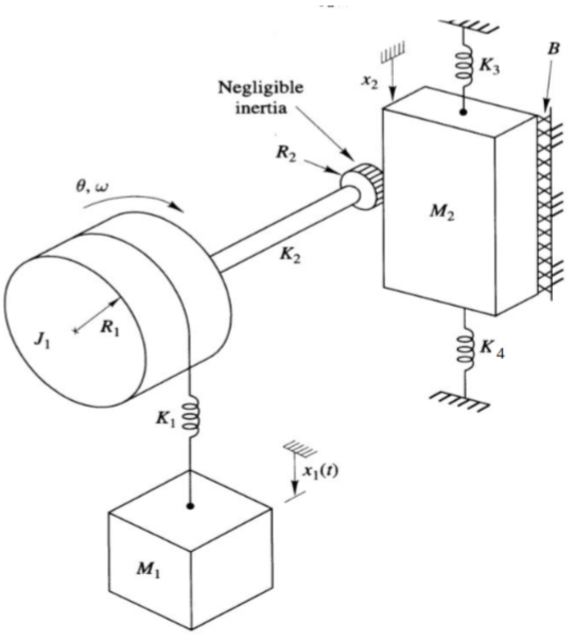
February 24, 2025

The following take-home midterm exam must be completed individually by each student. Students may use their notes, or the textbook. Any other resources used must be cited. Students are NOT allowed to communicate with any other person, except the instructor. Show your work and justify your answers for full credit. This exam is due at 11:59 PM on Monday, February 24 on Canvas. Students registered for ASEN 5114 do problems 2,3,4,5. Students registered for ASEN 4114 do problems 1,3,4,5.

2)

[40 pts] Find a direct state space model for the mechanical system at right, with an input force $f_a(t)$ oriented positive downward on M_2 and output $\omega(t)$. Neglect gravity and assume all positions are measured from the nominal position where the springs are at their rest length. The pinion gear with radius R_2 meshes with a rack gear on the side of M_2 without slipping. Model the sliding friction B with a linear damper. Note the rotating shaft has torsional stiffness K_2. Order the states in this model according to the element indexes, starting with inertial elements. Do not use the symbol "x" for state variables, since these are already used for displacements in the figure. θ and ω are inertially referenced.



1) Identify

- Input: $u = f$

- Output: $y = \omega$
- States:
  - Mass M_1: $w_1 = v_{M_1}$
  - Spring K_1: $w_2 = f_{K_1}$
  - Mass M_2: $w_3 = v_{M_2}$
  - Spring K_3: $w_4 = f_{K_3}$
  - Spring K_4: $w_5 = f_{K_4}$
  - Mass J_1: $w_6 = \omega = y$
  - Mass M_2: $w_7 = x_2$
  - Mass J_1: $w_8 = \Theta$

2) Complementary State Variables

$$w_1^* = f_{M_1}, \quad w_2^* = v_{K_1}, \quad w_3^* = f_{M_2}, \quad w_4^* = v_{K_3}, \quad w_5^* = v_{K_4}, \quad w_6^* = \tau_{J_1}$$

3) Topology Equations

$$\begin{aligned}
w_1^* &= f_{M_1} = -f_{K_1} = -w_2 \\
w_2^* &= v_{K_1} = v_{M_1} - \omega = w_1 - w_6 \\
w_3^* &= f_{M_2} = f - Bv_{M_2} + f_{K_4} - f_{K_3} - K_2\left(\Theta - \frac{x_2}{R_2}\right) \\
&= f - Bw_3 + w_5 - w_4 - K_2\left(w_8 - \frac{w_2}{R_2}\right) \\
w_4^* &= v_{K_3} = v_{M_2} = w_3 \\
w_5^* &= v_{K_4} = -v_{M_2} = -w_3 \\
w_6^* &= \tau_{J_1} = f_{K_1} - K_2\left(\theta - \frac{x_2}{R_2}\right) = w_2 - K_2\left(w_8 - \frac{w_7}{R_2}\right)
\end{aligned}$$

4) Energy Storage Element Equations

$$\begin{aligned}
w_1^* &= f_{M_1} = M_1\dot{v}_{M_1} = M_1\dot{w}_1 \implies \dot{w}_1 = -\frac{1}{M_1}w_2 \\
w_2^* &= v_{K_1} = \frac{1}{K_1}\dot{f}_{K_1} = \frac{1}{K_1}\dot{w}_2 \implies \dot{w}_2 = K_1(w_1 - w_6) \\
w_3^* &= f_{M_2} = M_2\dot{v}_{M_2} = M_2\dot{w}_3 \implies \dot{w}_3 = \frac{1}{M_2}(f - Bw_3 + w_5 - w_4 - K_2(w_8 - \frac{w_7}{R_2})) \\
w_4^* &= v_{K_3} = \frac{1}{K_3}\dot{f}_{K_3} = \frac{1}{K_3}\dot{w}_4 \implies \dot{w}_4 = K_3 w_3 \\
w_5^* &= v_{K_4} = \frac{1}{K_4}\dot{f}_{K_4} = \frac{1}{K_4}\dot{w}_5 \implies \dot{w}_5 = -K_4 w_3 \\
w_6^* &= \tau_{J_1} = J_1\dot{\omega} = J_1\dot{w}_6 \implies \dot{w}_6 = \frac{1}{J_1}w_2 - \frac{K_2}{J_1}w_8 + \frac{K_2}{J_1R_2}w_7
\end{aligned}$$

Note: $\dot{w}_7 = w_3$, $\dot{w}_8 = w_6$

5) Matrix Form

State vector: $\mathbf{w} = [w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8]^T$

State-space equations:

$$\dot{\mathbf{w}} = A\mathbf{w} + Bu, \quad y = C\mathbf{w} + Du$$

Where:

$$A = \begin{bmatrix} 0 & -\frac{1}{M_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ K_1 & 0 & 0 & 0 & 0 & -K_1 & 0 & 0 \\ 0 & 0 & -\frac{1}{M_2} & -\frac{1}{M_2} & \frac{1}{M_2} & 0 & \frac{K_2}{M_2} & -\frac{K_2}{M_2} \\ 0 & 0 & K_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -K_4 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{J_1} & 0 & 0 & 0 & 0 & \frac{K_2}{J_1 R_1} & -\frac{K_2}{J_1} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

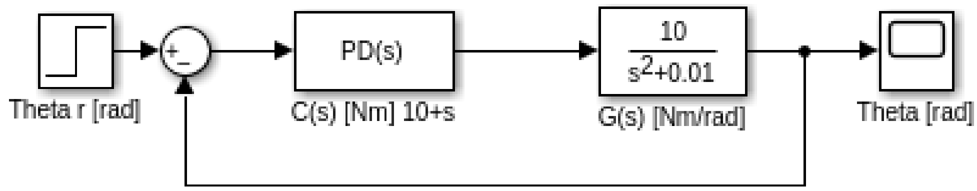
$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_2} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

3)

[20 pts] A rigid spacecraft stabilized by gravity gradient in low Earth orbit has the following transfer function relating applied reaction wheel torque τ to pitch angle θ relative to the local-vertical-local-horizontal frame. $G(s) = \frac{10}{s^2+0.01}$ [rad/Nm] Design a unity feedback control system to cause the pitch angle θ to track a pitch angle command θ_r such that the closed loop 5% settling time to a unit step command is less than 2 sec. Draw a block diagram and label all signals and units.

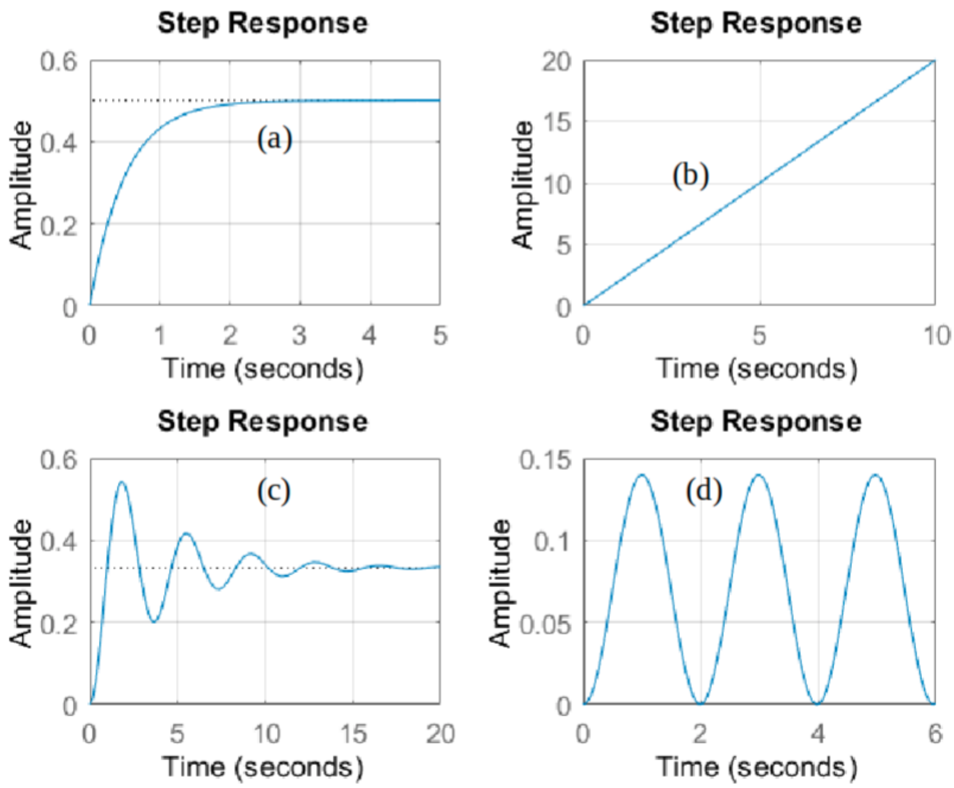
I started by using the closed loop transfer function $\frac{CG}{1+CG}$ and using a scalar value for C like a proportional gain control. I found that adjusting this proportional gain only shifted the poles up and down the imaginary axis so that wasn't going to give me the damping I needed to adjust the settling time. Then I added a derivative gain component so I started with $10 + 0.1s$ for the control law and then look at the denominator of the closed loop transfer function to get the roots. So the denominator became $1+(10+0.1s)\frac{10}{s^2+0.01}$ which equals $\frac{s^2+s+100.01}{s^2+0.01}$ and setting that equal to 0 and using the quadratic formula I got $-0.5 + j9.9875$ and the complex conjugate for the roots.

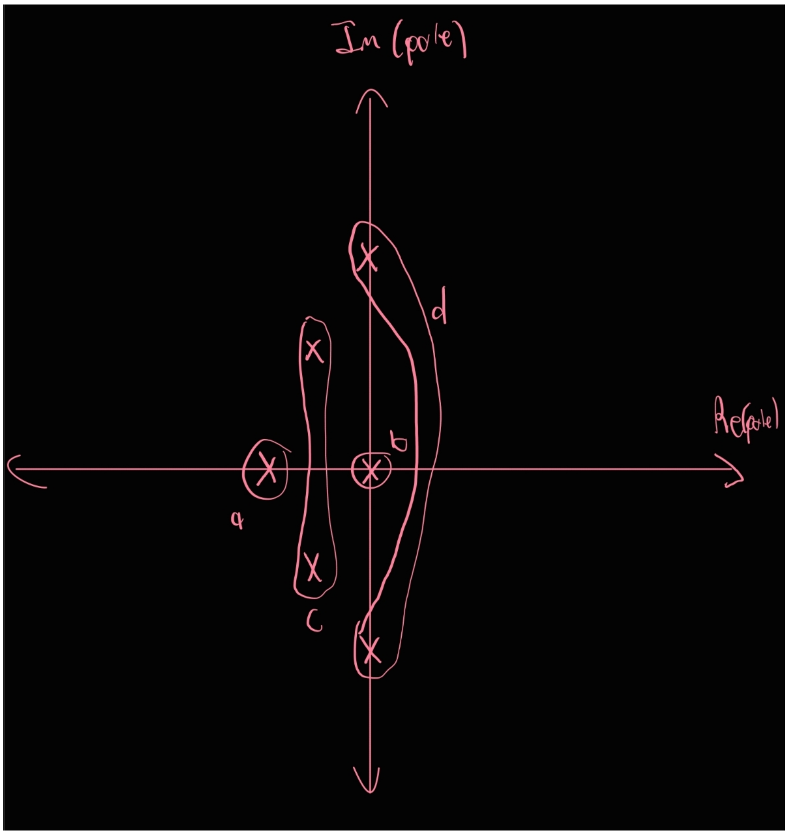
Then I used this formula for the 5% settling time and got 6 seconds. $t_s \approx \frac{-3}{-root} = 3\frac{1}{root} = 3\tau$ This told me that I needed to increase the gain in some way to bring the settling time down so I increased the proportional derivative to 1 and got $-5 + j9.9875$ and $-5 - j9.9875$ for the roots and putting that into the settling time formula I got 0.6 seconds.



4)

[20 pts] The following plots show unit step responses from different dynamical systems. In each case, quantitatively estimate the locations of the system poles in the complex plane.





(a) indicates a negative exponential as it settles to the final value of the transfer function somewhat quickly but not that quick indicates somewhere around -1. -2 (b) indicates a constant trajectory without any exponential or oscillatory effect which places it at the origin. (c) indicates a decaying exponential

Any estimation of the poles?

so the real component must be negative, the frequency and settling time are quite slow so I placed the poles near to the real and imaginary axes. (d) indicates a constant oscillation without growth or decay so the poles must lie on the imaginary axis, I put them out a bit further on the imaginary plane because the frequency seems a bit higher than it was for (c).

5)

[20 pts] A system with input u and output y is described by the following ordinary differential equation with initial conditions $y(0) = 0$, $\dot{y}(0) = 1$, $y(0) = 0$. $\ddot{y} + 12\ddot{y} + 22\dot{y} + 20y = 10u$ Assuming that the input $u(t)$ is a unit step function, use partial fractions to solve analytically for the following:

a. *The zero input response $y_{ZI}(t)$.*
b. *The zero initial condition response $y_{ZIC}(t)$.*
c. *The forced response $y_F(t)$.*
d. *The natural response $y_N(t)$.*

I started by doing a LaPlace Transform on the ODE to get the transfer function.

$s^3Y(s) - s^2y(0) - sy(0) - y(0)$ $+12$ $s^2Y(s) - 12sy(0) - 12y(0)$ $+22$ $sY(s) - 22y(0)$ $+20$ $Y(s) = \frac{10}{s}$

$s^3Y(s) - s + 12s^2Y(s) - 12 + 22sY(s) + 20Y(s) = \frac{10}{s}$

$s^3Y(s) + 12s^2Y(s) + 22sY(s) + 20Y(s) = s + 12 + \frac{10}{s}$

$Y(s)(s^3 + 12s^2 + 22s + 20) = \frac{10}{s} + s + 12$

$Y(s) = \frac{10/s+s+12}{s^3+12s^2+22s+20}$

$= \frac{A}{s+10} + \frac{B}{s+1-j} + \frac{C}{s+1+j}$

(a) is the transfer function without the input component so the partial fraction terms are the same as above with different values for A, B, and C Coefficients of each of the terms not provided

(b) does not have the initial conditions so the transfer function becomes

$Y(s) = \frac{10}{s}$ $s^3 + 12s^2 + 22s + 20$

which introduces another partial fraction term

$\frac{A}{s+10} + \frac{B}{s+1-j} + \frac{C}{s+1+j} + \frac{D}{s}$ Time domain signal was asked in the question

c) The forced response can be gotten from the final value theorem for step input $sY(s)$

$\lim_{s\to 0} sY(s) = s$ $\frac{10}{s(s^3+12s^2+22s+20)}$ $=$ $\frac{10}{20}$=0.5

d) The natural response would be the partial fractions from the zero input response

$\frac{A}{s+10} + \frac{B}{s+1-j} + \frac{C}{s+1+j}$

So it would have some negative decaying oscillatory natural response

$Ae^{-10t}e^{-t}(B\cos(t) + C\sin(t))$

Honor Code

Honor Code Pledge

On my honor, as a University of Colorado Boulder student, the solutions submitted are entirely my own, and I have neither given nor received unauthorized assistance on this exam.

-- Date: 2/29/25