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Homework 2 Assignment Submission

Question 1:

Show whether or not each of the following sets constitutes a valid group or field, under the ordinary addition and multiplication rules for the set elements.
a. S = { Strictly proper real-rational functions (ratios of polynomials with real coefficients, with numerator polynomial degree less than the denominator degree) }
b. S = { Rational numbers (ratios of integers) }
c. S = { the set of points in space lying on the surface and exterior of a sphere}.

Strictly Proper Real-Rational Functions:

For Addition:

- **Closure**: The sum of two strictly proper rational functions may not be strictly proper. For example, $\frac{x^2}{x^2+1}+\frac{1}{x}=\frac{x^3+x^2+1}{x^3+x}$ is a rational function be it's degree is not strictly proper (both the numerator and denominator are degree 3). Thus, the set is not closed under addition.

For Multiplication:

- **Closure**: The product of two strictly proper real-rational functions will always be strictly proper, so the set is closed under multiplication. This is because whenever you multiply two ratios together you add the degrees of the numerators together and you add the degrees of the denominators to each other so if you have more in the initial denominators you will always have more in the resulting denominator.

Conclusion: This set is not a valid group under addition, and it is not closed under addition, so it is neither a group nor a field.

Rational Numbers:

For Addition:

- **Closure**: The sum of two rational numbers is a rational number. $\frac{m}{n}+\frac{p}{q}=\frac{(mq+np)}{nq}$.

- **Associativity**: Addition of rational numbers is associative. $(\frac{a}{b}+\frac{c}{d})+\frac{e}{f}=\frac{(ad+bc)f+ebd}{bdf}$ and $\frac{a}{b}+(\frac{c}{d}+\frac{e}{f})=\frac{(ad+bc)f+ebd}{bdf}$
- **Identity element**: The additive identity is 0, which is a rational number (0 can be written as $\frac{0}{1}$).
- **Inverse element**: For every rational number $\frac{a}{b}$, its additive inverse is $\frac{-a}{b}$ which is also a rational number.

For Multiplication:

- **Closure**: The product of two rational numbers is a rational number. For example, $\frac{m}{n}*\frac{p}{q}=\frac{mp}{nq}$.
- **Associativity**: Multiplication of rational numbers is associative. $(\frac{a}{b}*\frac{c}{d})*\frac{e}{f}=\frac{ace}{bdf}$ $\frac{a}{b}*(\frac{c}{d}*\frac{e}{f})=\frac{ace}{bdf}$
- **Identity element**: The multiplicative identity is 1, which is a rational number (1 can be written as $\frac{1}{1}$).
- **Inverse element**: For every non-zero rational number $\frac{a}{b}$, its multiplicative inverse is $\frac{b}{a}$, which is also a rational number.

Conclusion: The set of rational numbers is both a group and a field under ordinary addition and multiplication.

Points on Surface/Exterior of Sphere:

For Addition:

- **Closure**: Points in 3D space can be added together but the resulting point may not be on the surface or exterior of the sphere.

For Multiplication:

- **Closure**: Multiplication has a similar issue where multiplication first of all is not very clearly defined in this situation and if you multiplied points together component by component or by dot product you might not get a point on the surface or exterior.

Conclusion: The set of points on the surface and exterior of a sphere is neither a group nor a field under addition and multiplication, as these operations are not properly defined for points in this set.

Question 2:

Show whether or not each of the following sets V = (G, F) are valid vector spaces.
Except in part e), the group operator is parallelogram "vector" addition.
a. G is the set of points (x,y) in the Cartesian plane such that y = 3x + 2. F is the field of real numbers.
b. G is the set of points in the Cartesian plane containing only the zero element (x=0,y=0), F is again the reals.
c. G is the set of points in the Cartesian plane (x,y) such that y = x3 - 2x. F is the reals.
d. G is the set of all points (x,y) in the Cartesian plane. F is the reals.
e. G is the set of ordered pairs (p,q) where p and q each belong to corresponding vector spaces P and Q, which are both vector spaces over the same field F. For each p1, p2 in P and each q_1, q_2 in Q, define the sum ("□" operation) of two elements in G by $(p_1, q_1) + (p_2, q_2) = (p_1 + p_2, q_1 + q_2)$.
Scalar multiplication is defined by $\alpha(p_1, q_1) = (\alpha p_1, \alpha q_1)$, for any α in F.

G Set of Points such that y=3x+2, F Field of Real Numbers:

Vector Addition: Consider two points on the line: $(x_1, 3x_1+2)$ and $(x_2, 3x_2+2)$. Their sum should be $(x_1+x_2, 3(x_1+x_2)+2)$, but when you add $(x_1,3x_1+2)+(x_2,3x_2+2)$, the result is $(x_1+x_2,3x_1+3x_2+4)$. This does **not satisfy** the equation y=3x+2, so it's not closed under addition.
Conclusion: Not a valid vector space.

G Set of Points Cartesian Plane Containing Only (x=0, y=0), F Field of Real Numbers:

Group Validation: Adding two zero vectors and multiplying by a scalar both give (0,0). Associativity and commutativity: adding any amount of zero vectors in any order results in a zero vector. Identity is a zero vector. Inverse is a zero vector.

Field Validation: Identity is 1, inverse is reciprocal.

Conclusion: Valid vector space.

G Set of Points such that $y=x_3$-2x, F Field of Real Numbers:

Group Validation: Consider two points $(x_1, x_1^3-2x_1)$ and $(x_1, x_2^3-2x_2)$. Adding these gives $(x_1+x_2, (x_1^3-2x_1)+(x_2^3-2x_2))$, which does not in general yield a point of the form (x, x_3-2x).

Conclusion: Not a valid vector space.

G Set of All Points in Cartesian Plane, F Field of Real Numbers:

Group Validation: This seems like a bad joke, is a Cartesian Plane a valid vector space? But we're gonna do our due diligence. Adding any two points in R^2 remains in R^2 and multiplying any point by a scalar remains in R^2 . Associativity and Commutativity: ((a,b) + (c,d)) + (e,f) = (a+c+e, b+d+f) = (a,b) + ((c,d) + (e,f)). Identity: (0,0). Inverse is the point multiplied by -1.

Conclusion: Valid vector space.

G = {(p,q) | p ϵ P, q ϵ Q}, F Field of Real Numbers:

Vector addition and scalar multiplication: Defined as $(p_1,q_1)+(p_2,q_2)=(p_1+p_2,q_1+q_2)$ and $\alpha(p_1, q_1)=(\alpha p_1,\alpha q_1)$. Since both P and Q are vector spaces, the operations in each component satisfy the vector space axioms. Hence, the operations on pairs of elements from P and Q also satisfy the axioms.
Conclusion: Valid vector space (direct product of two vector spaces).

Question 3:

Show that if $\{\boldsymbol{v}i\}$ is a set of elements from a vector space Vo, then $span\{\boldsymbol{v}_i\}$ is a vector space V_1. Does $Vo = V_1$?

Span{v_i} is a Vector Space:

The **span** of a set of vectors $\{v_i\}$ is the set of all linear combinations of the vectors in $\{v_i\}$. Formally:

$\text{span}\{v_i\}=\{\sum\alpha_i v_i \mid \alpha_i \in F\}$

where F is the underlying field, and α_i are scalars from F.

To show that span$\{v_i\}$ is a vector space, we need to verify that it satisfies the following properties (the vector space axioms):

1. Closed under addition:

Let $u=\sum\alpha_i v_i$ and $w=\sum\beta_i v_i$ be two elements in span$\{v_i\}$. Their sum is:

$u+w=\sum\alpha_i v_i+\sum\beta_i v_i=\sum(\alpha_i+\beta_i)v_i$

Since $\alpha_i+\beta_i \in F$, the sum is still a linear combination of the v_i, meaning $u+w \in \text{span}\{v_i\}$. Therefore, span$\{v_i\}$ is closed under addition.

2. Closed under scalar multiplication:

Let $u=\sum\alpha_i v_i \in \text{span}\{v_i\}$ and let $\lambda \in F$. The scalar multiplication is:

$\lambda u=\lambda\sum\alpha_i v_i=\sum(\lambda\alpha_i)v_i$

Since $\lambda\alpha_i \in F$, λu is still a linear combination of the v_i, meaning $\lambda u \in \text{span}\{v_i\}$. Therefore, span$\{v_i\}$ is closed under scalar multiplication.

3. Contains the zero vector:

The zero vector can be written as the linear combination:

$0=\sum 0 \cdot v_i$

Thus, the zero vector is in span$\{v_i\}$.

4. Associativity, commutativity of addition, distributive properties:

These properties naturally hold because they are inherited from the properties of the underlying vector space V_0, where the vectors v_i come from, and F, the field over which the vector space is defined.

Since span$\{v_i\}$ satisfies all the properties of a vector space, it is indeed a vector space. Let's call this vector space V_1.

Determine whether $V_0=V_1$:

- **Case 1**: If $\{v_i\}$ is a set of vectors that spans all of V_0, meaning that any vector in V_0 can be written as a linear combination of the vectors in $\{v_i\}$, then span$\{v_i\}=V_0$. In this case, $V_0=V_1$.
- **Case 2**: If $\{v_i\}$ is a set of vectors that spans only a **subset** of V_0, then span$\{v_i\}=V_1$ is a proper subspace of V_0, and thus $V_0 \neq V_1$.

Conclusion:

span$\{v_i\}$, denoted V_1, is always a vector space. Whether $V_0=V_1$ depends on whether the set $\{v_i\}$ spans all of V_0. If it does, then $V_0=V_1$. Otherwise, V_1 is a subspace of V_0, and $V_0 \neq V_1$.