# Homework 4

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Course: Linear Control Design – ASEN 5014-001 – Fall 2024
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# 1 Question 1

Express the vector $\mathbf{z} = \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}$ in terms of the orthonormal basis set $\hat{\mathbf{v}}_i$ of Problem 5.39.

$\hat{\mathbf{v}}_i =$

$$\hat{\mathbf{q}}_1 = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{14}} \end{bmatrix}$$

$$\hat{\mathbf{q}}_2 = \begin{bmatrix} \frac{1}{\sqrt{35}} \\ \sqrt{\frac{5}{7}} \\ \frac{3}{\sqrt{35}} \end{bmatrix}$$

$$\hat{\mathbf{q}}_3 = \begin{bmatrix} \frac{-31}{\sqrt{1010}} \\ 0 \\ \frac{7}{\sqrt{1010}} \end{bmatrix}$$

## 1.1 Vector z in Terms of Orthonormal Basis Set

To express the vector $\mathbf{z} = \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}$ in terms of the orthonormal basis set $\{\hat{\mathbf{q}}_1, \hat{\mathbf{q}}_2, \hat{\mathbf{q}}_3\}$, we represent $\mathbf{z}$ as:

$$\mathbf{z} = c_1\hat{\mathbf{q}}_1 + c_2\hat{\mathbf{q}}_2 + c_3\hat{\mathbf{q}}_3$$

where the coefficients $c_1, c_2, c_3$ are given by:

$$c_i = \hat{\mathbf{q}}_i^T \mathbf{z}, \quad \text{for } i = 1, 2, 3$$

**Step 1: Calculate $c_1$**

$$c_1 = \hat{\mathbf{q}}_1^T \mathbf{z} = \begin{bmatrix} \frac{1}{\sqrt{14}} & \sqrt{\frac{2}{7}} & \frac{3}{\sqrt{14}} \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}$$

$$c_1 = \frac{1}{\sqrt{14}} \cdot 6 + \sqrt{\frac{2}{7}} \cdot 4 + \frac{3}{\sqrt{14}} \cdot (-3)$$

$$c_1 = \frac{6}{\sqrt{14}} + \frac{4\sqrt{2}}{\sqrt{7}} - \frac{9}{\sqrt{14}}$$

$$c_1 = \frac{-3}{\sqrt{14}} + \frac{4\sqrt{2}}{\sqrt{7}}$$

**Step 2: Calculate $c_2$**

$$c_2 = \hat{\mathbf{q}}_2^T \mathbf{z} = \begin{bmatrix} \frac{1}{\sqrt{35}} & \sqrt{\frac{5}{7}} & \frac{3}{\sqrt{35}} \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}$$

$$c_2 = \frac{1}{\sqrt{35}} \cdot 6 + \sqrt{\frac{5}{7}} \cdot 4 + \frac{3}{\sqrt{35}} \cdot (-3)$$

$$c_2 = \frac{6}{\sqrt{35}} + \frac{4\sqrt{5}}{\sqrt{7}} - \frac{9}{\sqrt{35}}$$

$$c_2 = \frac{-3}{\sqrt{35}} + \frac{4\sqrt{5}}{\sqrt{7}}$$

**Step 3: Calculate $c_3$**

$$c_3 = \hat{\mathbf{q}}_3^T \mathbf{z} = \begin{bmatrix} \frac{-31}{\sqrt{1010}} & 0 & \frac{7}{\sqrt{1010}} \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}$$

$$c_3 = \frac{-31}{\sqrt{1010}} \cdot 6 + 0 \cdot 4 + \frac{7}{\sqrt{1010}} \cdot (-3)$$

$$c_3 = \frac{-186}{\sqrt{1010}} - \frac{21}{\sqrt{1010}}$$

$$c_3 = \frac{-207}{\sqrt{1010}}$$

**Final Representation of z**

$$\mathbf{z} = c_1\hat{\mathbf{q}}_1 + c_2\hat{\mathbf{q}}_2 + c_3\hat{\mathbf{q}}_3$$

$$\mathbf{z} = \left( \frac{-3}{\sqrt{14}} + \frac{4\sqrt{2}}{\sqrt{7}} \right) \hat{\mathbf{q}}_1 + \left( \frac{-3}{\sqrt{35}} + \frac{4\sqrt{5}}{\sqrt{7}} \right) \hat{\mathbf{q}}_2 + \left( \frac{-207}{\sqrt{1010}} \right) \hat{\mathbf{q}}_3$$

# 2 Question 2

Find orthogonal basis sets for the row space, right null space, column space, and left null space of the following matrix of a linear mapping A.

$$M = \begin{bmatrix} 3 & -1 & 4 & 0 & 7 \\ 3 & 7 & 11 & -9 & 8 \\ 1 & -3 & -1 & 3 & 2 \\ -10 & 6 & -11 & -3 & -23 \end{bmatrix}$$

## 2.1 Find an Orthogonal Basis for the Row Space

Let the rows of matrix $M$ be:

$$\begin{aligned} \mathbf{r}_1 &= \begin{bmatrix} 3 & -1 & 4 & 0 & 7 \end{bmatrix}, \\ \mathbf{r}_2 &= \begin{bmatrix} 3 & 7 & 11 & -9 & 8 \end{bmatrix}, \\ \mathbf{r}_3 &= \begin{bmatrix} 1 & -3 & -1 & 3 & 2 \end{bmatrix}, \\ \mathbf{r}_4 &= \begin{bmatrix} -10 & 6 & -11 & -3 & -23 \end{bmatrix} \end{aligned}$$

$$\mathbf{q}_1 = \mathbf{r}_1 = \begin{bmatrix} 3 & -1 & 4 & 0 & 7 \end{bmatrix}$$

$$\mathbf{q}_2 = \mathbf{r}_2 - \frac{\mathbf{r}_2 \cdot \mathbf{q}_1^T}{\mathbf{q}_1 \cdot \mathbf{q}_1^T}\mathbf{q}_1$$

$$\mathbf{q}_2 = \begin{bmatrix}3 & 7 & 11 & -9 & 8\end{bmatrix} - \frac{102}{75}\begin{bmatrix}3 & -1 & 4 & 0 & 7\end{bmatrix}$$

$$\mathbf{q}_2 = \begin{bmatrix}3 & 7 & 11 & -9 & 8\end{bmatrix} - 1.36\begin{bmatrix}3 & -1 & 4 & 0 & 7\end{bmatrix}$$

$$\mathbf{q}_2 = \begin{bmatrix}-1.08 & 8.36 & 5.56 & -9 & -1.52\end{bmatrix}$$

$$\mathbf{q}_3 = \mathbf{r}_3 - \frac{\mathbf{r}_3 \cdot \mathbf{q}_1^T}{\mathbf{q}_1 \cdot \mathbf{q}_1^T}\mathbf{q}_1 - \frac{\mathbf{r}_3 \cdot \mathbf{q}_2^T}{\mathbf{q}_2 \cdot \mathbf{q}_2^T}\mathbf{q}_2$$

$$\mathbf{q}_3 = \begin{bmatrix}1 & -3 & -1 & 3 & 2\end{bmatrix} - \frac{16}{75}\begin{bmatrix}3 & -1 & 4 & 0 & 7\end{bmatrix} - \frac{-61.76}{185.28}\begin{bmatrix}-0.72 & 8.24 & 6.04 & -9 & -0.68\end{bmatrix}$$

$$\mathbf{q}_3 = \begin{bmatrix}.36 & -2.7867 & -1.853 & 3 & 0.5067\end{bmatrix} - \frac{-61.76}{185.28}\begin{bmatrix}-0.72 & 8.24 & 6.04 & -9 & -0.68\end{bmatrix}$$

$$\mathbf{q}_3 = \begin{bmatrix}0 & -4.44x10^{-16} & -4.44x10^{-16} & 0 & -4.44x10^{-16}\end{bmatrix}$$

$$\mathbf{q}_4 = \mathbf{r}_4 - \frac{\mathbf{r}_4 \cdot \mathbf{q}_1}{\mathbf{q}_1 \cdot \mathbf{q}_1}\mathbf{q}_1 - \frac{\mathbf{r}_4 \cdot \mathbf{q}_2}{\mathbf{q}_2 \cdot \mathbf{q}_2}\mathbf{q}_2 - \frac{\mathbf{r}_4 \cdot \mathbf{q}_3}{\mathbf{q}_3 \cdot \mathbf{q}_3}\mathbf{q}_3$$

$$\mathbf{q}_4 = \begin{bmatrix}6.66x10^{-16} & 9.33 & 9.33 & 8.88x10^{-16} & 9.33\end{bmatrix}$$

The vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ form an orthogonal basis for the row space of $M$.

## 2.2 Find an Orthogonal Basis for the Right Nullspace

Doing QR Decomposition on the matrix $M^T$ we get

$$M^T = Q_T R_T$$

Matrix $Q_T$

$$Q_T = \begin{bmatrix}-0.3464 & 0.0793 & 0.2089 & 0.8944 \\ 0.1155 & -0.6142 & 0.7232 & -0.1216 \\ -0.4619 & -0.4085 & 0.0191 & 0.0054 \\ 0 & 0.6612 & 0.6580 & -0.1488 \\ -0.8083 & 0.1117 & 0.0029 & -0.4038\end{bmatrix}$$

Matrix $R_T$

$$R_T = \begin{bmatrix}-8.6603 & -11.7780 & -1.8475 & 27.8283 \\ 0 & -13.6118 & 4.5373 & -4.5373 \\ 0 & 0 & -7.8274 \times 10^{-16} & 4.5169 \times 10^{-16} \\ 0 & 0 & 0 & 2.4636 \times 10^{-15}\end{bmatrix}$$

Given that the last two rows of the R matrix are effectively 0 vectors, that makes the last 2 columns of the $Q_T$ matrix the orthogonal basis for the Right Nullspace.

$$RN(M^T) = \begin{bmatrix}0.2089 & 0.8944 \\ 0.7232 & -0.1216 \\ 0.0191 & 0.0054 \\ 0.6580 & -0.1488 \\ 0.0029 & -0.4038\end{bmatrix}$$

### 2.3 Find an Orthogonal Basis for the Column Space

Let the columns of matrix $M$ be:

$$\mathbf{c}_1 = \begin{bmatrix} 3 \\ 3 \\ 1 \\ -10 \end{bmatrix}$$

$$\mathbf{c}_2 = \begin{bmatrix} -1 \\ 7 \\ -3 \\ 6 \end{bmatrix}$$

$$\mathbf{c}_3 = \begin{bmatrix} 4 \\ 11 \\ -1 \\ -11 \end{bmatrix}$$

$$\mathbf{c}_4 = \begin{bmatrix} 0 \\ -9 \\ 3 \\ -3 \end{bmatrix}$$

$$\mathbf{c}_5 = \begin{bmatrix} 7 \\ 8 \\ 2 \\ -23 \end{bmatrix}$$

We apply the Gram-Schmidt process to obtain orthogonal vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5$:

$$\mathbf{p}_1 = \mathbf{c}_1 = \begin{bmatrix} 3 \\ 3 \\ 1 \\ -10 \end{bmatrix}$$

$$\mathbf{p}_2 = \mathbf{c}_2 - \frac{\mathbf{c}_2 \cdot \mathbf{p}_1}{\mathbf{p}_1 \cdot \mathbf{p}_1}\mathbf{p}_1$$

$$\mathbf{p}_2 = \begin{bmatrix} 0.1345 \\ 8.1345 \\ -2.6218 \\ 2.2185 \end{bmatrix}$$

$$\mathbf{p}_3 = \mathbf{c}_3 - \frac{\mathbf{c}_3 \cdot \mathbf{p}_1}{\mathbf{p}_1 \cdot \mathbf{p}_1}\mathbf{p}_1 - \frac{\mathbf{c}_3 \cdot \mathbf{p}_2}{\mathbf{p}_2 \cdot \mathbf{p}_2}\mathbf{p}_2$$

$$\mathbf{p}_3 = \begin{bmatrix} -4.16x10^{-16} \\ -1.78x10^{-15} \\ 8.88x10^{-16} \\ -4.44x10^{-16} \end{bmatrix}$$

$$\mathbf{p}_4 = \mathbf{c}_4 - \frac{\mathbf{c}_4 \cdot \mathbf{p}_1}{\mathbf{p}_1 \cdot \mathbf{p}_1}\mathbf{p}_1 - \frac{\mathbf{c}_4 \cdot \mathbf{p}_2}{\mathbf{p}_2 \cdot \mathbf{p}_2}\mathbf{p}_2 - \frac{\mathbf{c}_4 \cdot \mathbf{p}_3}{\mathbf{p}_3 \cdot \mathbf{p}_3}\mathbf{p}_3$$

$$\mathbf{p}_4 = \begin{bmatrix} 1.9282 \\ 8.2271 \\ -4.1136 \\ 2.0568 \end{bmatrix}$$

$$\mathbf{p}_5 = \mathbf{c}_5 - \frac{\mathbf{c}_5 \cdot \mathbf{p}_1}{\mathbf{p}_1 \cdot \mathbf{p}_1}\mathbf{p}_1 - \frac{\mathbf{c}_5 \cdot \mathbf{p}_2}{\mathbf{p}_2 \cdot \mathbf{p}_2}\mathbf{p}_2 - \frac{\mathbf{c}_5 \cdot \mathbf{p}_3}{\mathbf{p}_3 \cdot \mathbf{p}_3}\mathbf{p}_3 - \frac{\mathbf{c}_5 \cdot \mathbf{p}_4}{\mathbf{p}_4 \cdot \mathbf{p}_4}\mathbf{p}_4$$

$$\mathbf{p}_5 = \begin{bmatrix} -0.9909 \\ -4.2278 \\ 2.1139 \\ -1.0570 \end{bmatrix}$$

The vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5$ form an orthogonal basis for the column space of

### 2.4 Find an Orthogonal Basis for the Left Nullspace

Doing QR Decomposition on the matrix $M$ we get

$$M = QR$$

Matrix $Q$

$$Q = \begin{bmatrix} -0.2750 & -0.0152 & 0.9604 & 0.0412 \\ -0.2750 & -0.9211 & -0.1043 & 0.2549 \\ -0.0917 & 0.2969 & -0.0623 & 0.9485 \\ 0.9167 & -0.2512 & 0.2506 & 0.1837 \end{bmatrix}$$

Matrix $R$

$$R = \begin{bmatrix} -10.9087 & 4.1251 & -14.1172 & -0.5500 & -25.3925 \\ 0 & -8.8308 & -7.7270 & 9.9347 & -1.1039 \\ 0 & 0 & -5.4792 \times 10^{-15} & 2.0652 \times 10^{-15} & -6.2765 \times 10^{-15} \\ 0 & 0 & 0 & -2.4241 \times 10^{-17} & -5.1599 \times 10^{-16} \end{bmatrix}$$

Given that the last two rows of the R matrix are effectively 0 vectors, that makes the last 2 columns of the Q matrix the orthogonal basis for the Left Nullspace.

$$LN(M) = \begin{bmatrix} 0.9604 & 0.0412 \\ -0.1043 & 0.2549 \\ -0.0623 & 0.9485 \\ 0.2506 & 0.1837 \end{bmatrix}$$

## 3 Question 3

Let $S$ be the subspace of $\mathbf{R}^5$ spanned by the vectors $\{v_i\}$ below. Find a basis for the orthogonal complement $C$ of $S$ in $\mathbf{R}^5$.

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

### 3.1 QR Decomposition

The orthogonal complement $C$ of the subspace $S$ in $\mathbf{R}^5$ is found by determining the null space of the matrix formed by $v_1$ and $v_2$ as rows.

Given the matrix $A$, which is the matrix whose columns are the vectors $v_1$ and $v_2$:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 1 \\ 2 & 2 \\ 2 & 1 \end{bmatrix}$$

The QR decomposition of $A$ is:

$$A = Q_A R_A$$

$$Q_A = \begin{bmatrix} -0.5547 & -0.2892 \\ 0 & -0.7518 \\ -0.2774 & 0.2313 \\ -0.5547 & 0.4627 \\ -0.5547 & -0.2892 \end{bmatrix}$$

$$R_A = \begin{bmatrix} -3.6056 & -2.4962 \\ 0 & 1.3301 \end{bmatrix}$$

### 3.2 Parametric Form

From the QR Decomposition, we can express the leading variables ($x_1$ and $x_2$) in terms of the free variables ($x_3, x_4, x_5$):

$$x_1 = \frac{-0.2774x_3 - 0.5547x_4 - 0.5547x_5}{0.5547}$$

$$x_1 = -0.5x_3 - x_4 - x_5$$

$$x_2 = \frac{-0.2892(-0.5x_3 - x_4 - x_5) + 0.2313x_3 + 0.4627x_4 - 0.2892x_5}{0.7518}$$

$$x_2 = 0.5x_3 + x_4$$

Let $t_1, t_2, t_3$ be free variables representing $x_3, x_4, x_5$, respectively. Then:

$$\mathbf{x} = t_1 \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

### 3.3 Basis for Orthogonal Complement

The basis for the orthogonal complement $C$ is:

$$\left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

## 4 Question 4

Prove that the right null space of a linear mapping A is a subspace of the domain.

### 4.1 Proof: The Right Null Space of a Linear Mapping $A$ is a Subspace of the Domain

Let $A : \mathbf{R}^n \to \mathbf{R}^m$ be a linear mapping represented by an $m \times n$ matrix. The right null space of $A$ is defined as:

$$\mathrm{RN}(A) = \{\mathbf{x} \in \mathbf{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

We want to prove that $\mathrm{RN}(A)$ is a subspace of the domain, which is $\mathbf{R}^n$.
To prove that $\mathrm{RN}(A)$ is a subspace, we must show that:
1. $\mathrm{RN}(A)$ contains the zero vector.
2. $\mathrm{RN}(A)$ is closed under vector addition.
3. $\mathrm{RN}(A)$ is closed under scalar multiplication.

### 4.2 The Zero Vector is in RN($A$)

Consider the zero vector $\mathbf{0} \in \mathbf{R}^n$. By the definition of the linear mapping $A$:

$$A\mathbf{0} = \mathbf{0}$$

Thus, $\mathbf{0} \in \text{RN}(A)$. Therefore, $\text{RN}(A)$ contains the zero vector.

### 4.3 Step 2: Closure Under Vector Addition

Let $\mathbf{u}, \mathbf{v} \in \text{RN}(A)$. By definition, this means:

$$A\mathbf{u} = \mathbf{0} \quad \text{and} \quad A\mathbf{v} = \mathbf{0}$$

We need to show that $\mathbf{u} + \mathbf{v} \in \text{RN}(A)$. Consider:

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

Since $A(\mathbf{u} + \mathbf{v}) = \mathbf{0}$, it follows that $\mathbf{u} + \mathbf{v} \in \text{RN}(A)$. Thus, $\text{RN}(A)$ is closed under vector addition.

### 4.4 Closure Under Scalar Multiplication

Let $\mathbf{u} \in \text{RN}(A)$ and let $c \in \mathbf{R}$ be any scalar. By definition:

$$A\mathbf{u} = \mathbf{0}$$

We need to show that $c\mathbf{u} \in \text{RN}(A)$. Consider:

$$A(c\mathbf{u}) = cA\mathbf{u} = c \cdot \mathbf{0} = \mathbf{0}$$

Since $A(c\mathbf{u}) = \mathbf{0}$, it follows that $c\mathbf{u} \in \text{RN}(A)$. Thus, $\text{RN}(A)$ is closed under scalar multiplication.

### 4.5 Conclusion

Since $\text{RN}(A)$ contains the zero vector, is closed under vector addition, and is closed under scalar multiplication, it follows that $\text{RN}(A)$ is a subspace of the domain $\mathbf{R}^n$.

## 5 Question 5

Prove that the column space of a linear mapping A is a subspace of the co-domain.

### 5.1 Proof: The Column Space of a Linear Mapping $A$ is a Subspace of the Codomain

Let $A : \mathbf{R}^n \to \mathbf{R}^m$ be a linear mapping represented by an $m \times n$ matrix. The column space of $A$ is defined as:

$$\text{CS}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbf{R}^n\}$$

In other words, the column space consists of all possible linear combinations of the columns of $A$. We want to prove that the column space $\text{CS}(A)$ is a subspace of the codomain, which is $\mathbf{R}^m$.

To prove that $\text{CS}(A)$ is a subspace, we must show that:
1. $\text{CS}(A)$ contains the zero vector.
2. $\text{CS}(A)$ is closed under vector addition.
3. $\text{CS}(A)$ is closed under scalar multiplication.

### 5.2 Step 1: The Zero Vector is in CS($A$)

Consider the zero vector $\mathbf{0} \in \mathbf{R}^n$. By the definition of the linear mapping $A$:

$$A\mathbf{0} = \mathbf{0}$$

Thus, $\mathbf{0} \in \text{CS}(A)$. Therefore, $\text{CS}(A)$ contains the zero vector.

### 5.3 Step 2: Closure Under Vector Addition

Let $\mathbf{y}_1, \mathbf{y}_2 \in \mathrm{CS}(A)$. By definition, this means there exist vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^n$ such that:

$$\mathbf{y}_1 = A\mathbf{x}_1, \quad \mathbf{y}_2 = A\mathbf{x}_2$$

We need to show that $\mathbf{y}_1 + \mathbf{y}_2 \in \mathrm{CS}(A)$. Consider:

$$\mathbf{y}_1 + \mathbf{y}_2 = A\mathbf{x}_1 + A\mathbf{x}_2 = A(\mathbf{x}_1 + \mathbf{x}_2)$$

Since $\mathbf{x}_1 + \mathbf{x}_2 \in \mathbf{R}^n$, it follows that $\mathbf{y}_1 + \mathbf{y}_2 \in \mathrm{CS}(A)$. Thus, $\mathrm{CS}(A)$ is closed under vector addition.

### 5.4 Step 3: Closure Under Scalar Multiplication

Let $\mathbf{y} \in \mathrm{CS}(A)$ and let $c \in \mathbf{R}$ be any scalar. By definition, there exists a vector $\mathbf{x} \in \mathbf{R}^n$ such that:

$$\mathbf{y} = A\mathbf{x}$$

We need to show that $c\mathbf{y} \in \mathrm{CS}(A)$. Consider:

$$c\mathbf{y} = c(A\mathbf{x}) = A(c\mathbf{x})$$

Since $c\mathbf{x} \in \mathbf{R}^n$, it follows that $c\mathbf{y} \in \mathrm{C}(A)$. Thus, $\mathrm{C}(A)$ is closed under scalar multiplication.

### 5.5 Conclusion

Since $\mathrm{C}(A)$ contains the zero vector, is closed under vector addition, and is closed under scalar multiplication, it follows that $\mathrm{C}(A)$ is a subspace of the codomain $\mathbf{R}^m$.