Homework 3

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1 Question 1

5.39 Consider $x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ $x_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(a) Show that this set is linearly independent.
(b) Generate an orthonormal set using the Gram-Schmidt procedure.

1.1 Gram-Schmidt Orthonormal

I can answer part 'a' of the question in the process of doing part 'b' (specifically by showing that none of thee orthonormal vectors are zero vectors) and so I will start with part 'b' and the Gram-Schmidt procedure.

Finding Orthoganal Vectors:

$$q_1 = x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{1}$$

$$q_2 = x_2 - \left(\frac{x_2^T q_1}{q_1^T q_1} \right) q_1 \tag{2}$$

$$= \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - \left(\frac{\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \right) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} \frac{4}{7} \\ \frac{-20}{7} \\ \frac{12}{7} \end{bmatrix} \tag{4}$$

$$q_3 = x_3 - \left(\frac{x_3^T q_1}{q_1^T q_1} \right) q_1 - \left(\frac{x_3^T q_2}{q_2^T q_2} \right) q_2 \tag{5}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \right) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{7} \\ \frac{-20}{7} \\ \frac{12}{7} \end{bmatrix}}{\begin{bmatrix} \frac{4}{7} & \frac{-20}{7} & \frac{12}{7} \end{bmatrix} \begin{bmatrix} \frac{4}{7} \\ \frac{-20}{7} \\ \frac{12}{7} \end{bmatrix}} \right) \begin{bmatrix} \frac{4}{7} \\ \frac{-20}{7} \\ \frac{12}{7} \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} \frac{-31}{70} \\ 0 \\ \frac{1}{10} \end{bmatrix} \tag{7}$$

↑ None of these vectors are zero vectors answering 'part a' of the question. The set is linearly independent.

Normalizing Vectors to Get Final Orthonormal Vectors:

In order to normalize the orthogonal vectors we found we can divide them by their L2 norms.

$\hat{q}_i = \frac{q_i}{|q_i|}$

$$q_1 = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{14}} \end{bmatrix} \tag{8}$$

$$q_2 = \begin{bmatrix} \frac{1}{\sqrt{35}} \\ \sqrt{\frac{5}{7}} \\ \frac{3}{\sqrt{35}} \end{bmatrix} \tag{9}$$

$$q_3 = \begin{bmatrix} \frac{-31}{\sqrt{1010}} \\ 0 \\ \frac{7}{\sqrt{1010}} \end{bmatrix} \tag{10}$$

2 Question 2

Considering x_1, x_2, and x_3 of Problem 5.39 as a basis set, find the reciprocal basis set.

2.1 Reciprocal Basis Set

Given the vectors:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

we want to find the reciprocal basis set $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$.

The reciprocal basis vectors are computed using the formulas:

$$\mathbf{x}^1 = \frac{\mathbf{x}_2 \times \mathbf{x}_3}{\mathbf{x}_1 \cdot (\mathbf{x}_2 \times \mathbf{x}_3)}, \quad \mathbf{x}^2 = \frac{\mathbf{x}_3 \times \mathbf{x}_1}{\mathbf{x}_1 \cdot (\mathbf{x}_2 \times \mathbf{x}_3)}, \quad \mathbf{x}^3 = \frac{\mathbf{x}_1 \times \mathbf{x}_2}{\mathbf{x}_1 \cdot (\mathbf{x}_2 \times \mathbf{x}_3)}$$

First, we compute the cross products:

$$\mathbf{x}_2 \times \mathbf{x}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 0 & 1 & 1 \end{vmatrix} = \begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix}$$

Now, compute the scalar:

$$\mathbf{x}_1 \cdot (\mathbf{x}_2 \times \mathbf{x}_3) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix} = -4$$

Thus, we compute the reciprocal basis vectors:

$$\mathbf{x}^1 = \frac{1}{-4} \begin{bmatrix} -5 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}$$

Next:

$$\mathbf{x}_3 \times \mathbf{x}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{x}^2 = \frac{1}{-4} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Finally:

$$\mathbf{x}_1 \times \mathbf{x}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 1 & -2 & 3 \end{vmatrix} = \begin{bmatrix} 12 \\ 0 \\ -4 \end{bmatrix}$$

$$\mathbf{x}^3 = \frac{1}{-4} \begin{bmatrix} 12 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the reciprocal basis set is:

$$\mathbf{x}^1 = \begin{bmatrix} \frac{5}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}, \quad \mathbf{x}^2 = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix}, \quad \mathbf{x}^3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

3 Question 3

Express the vector $z = \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}$ in terms of the original basis set $\{x_i\}$ of the Problem 5.39 by using the reciprocal basis vectors $\{r_i\}$ found in Problem 5.40.

3.1 z in Terms of Basis Set Using Reciprocal

Given the vector:

$$\mathbf{z} = \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}$$

we want to express $\mathbf{z}$ in terms of the original basis vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ from the previous problem.

The relation is given by:

$$\mathbf{z} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$$

where the coefficients c_1, c_2, c_3 are found using the reciprocal basis vectors $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$:

$$c_1 = \mathbf{x}^1 \cdot \mathbf{z}, \quad c_2 = \mathbf{x}^2 \cdot \mathbf{z}, \quad c_3 = \mathbf{x}^3 \cdot \mathbf{z}$$

From the previous problem, the reciprocal basis vectors are:

$$\mathbf{x}^1 = \begin{bmatrix} \frac{5}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}, \quad \mathbf{x}^2 = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix}, \quad \mathbf{x}^3 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Now, we compute the coefficients:

$$c_1 = \mathbf{x}^1 \cdot \mathbf{z} = \begin{bmatrix} \frac{5}{4} \\ \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix} = \frac{5}{4}(6) + \frac{1}{4}(4) - \frac{1}{4}(-3) = 7.25$$

$$c_2 = \mathbf{x}^2 \cdot \mathbf{z} = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix} = -\frac{1}{4}(6) - \frac{1}{4}(4) + \frac{1}{4}(-3) = -3.25$$

$$c_3 = \mathbf{x}^3 \cdot \mathbf{z} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix} = -3(6) + 0(4) + 1(-3) = -21$$

Thus, the vector $\mathbf{z}$ in terms of the original basis vectors is:

$$\mathbf{z} = 7.25\mathbf{x}_1 - 3.25\mathbf{x}_2 - 21\mathbf{x}_3$$

4 Question 4

Compute the Grammian for the following vectors and draw conclusions about their linear independence.

$$y_1 = \begin{bmatrix} 1.0000000E+00 \\ 1.0000000E+00 \\ 1.0000000E+00 \\ 1.0000000E+00 \end{bmatrix} y_2 = \begin{bmatrix} 1.0001000E+00 \\ 9.9989998E-01 \\ 1.0000000E+00 \\ 1.0000000E+00 \end{bmatrix} y_3 = \begin{bmatrix} -2.0000000E+00 \\ -1.9999000E+00 \\ -2.0000000E+00 \\ -2.0000000E+00 \end{bmatrix}$$

4.1 Grammian

The Grammian matrix G is defined as:

$$G = \begin{bmatrix} \mathbf{y}_1^T\mathbf{y}_1 & \mathbf{y}_1^T\mathbf{y}_2 & \mathbf{y}_1^T\mathbf{y}_3 \\ \mathbf{y}_2^T\mathbf{y}_1 & \mathbf{y}_2^T\mathbf{y}_2 & \mathbf{y}_2^T\mathbf{y}_3 \\ \mathbf{y}_3^T\mathbf{y}_1 & \mathbf{y}_3^T\mathbf{y}_2 & \mathbf{y}_3^T\mathbf{y}_3 \end{bmatrix}$$

We calculate each element of the Grammian as follows:

$$\mathbf{y}_1^T\mathbf{y}_1 = 1 \times 1 + 1 \times 1 + 1 \times 1 + 1 \times 1 = 4$$

$$\mathbf{y}_1^T\mathbf{y}_2 = 1 \times 1.0001 + 1 \times 9.9989998E-01 + 1 \times 1 + 1 \times 1 = 4.0000000E+00$$

$$\mathbf{y}_1^T\mathbf{y}_3 = 1\times(-2)+1\times(-1.9999)+1\times(-2)+1\times(-2) = -7.9999$$

$$\mathbf{y}_2^T\mathbf{y}_2 = 1.0001\times 1.0001+9.9989998E{-}01\times 9.9989998E{-}01+1\times 1+1\times 1 = 4.0000002E{+}00$$

$$\mathbf{y}_2^T\mathbf{y}_3 = 1.0001\times(-2)+9.9989998E{-}01\times(-1.9999)+1\times(-2)+1\times(-2) = -7.9998998$$

$$\mathbf{y}_3^T\mathbf{y}_3 = (-2)\times(-2)+(-1.9999)\times(-1.9999)+(-2)\times(-2)+(-2)\times(-2) = 15.99960001$$

Thus, the Grammian matrix is:

$$G = \begin{bmatrix} 4 & 4.0000000E+00 & -7.9999 \\ 4.0000000E+00 & 4.0000002E+00 & -7.9998998 \\ -7.9999 & -7.9998998 & 15.99960001 \end{bmatrix}$$

4.2 Analyze the Grammian for Linear Independence

For the set of vectors to be linearly independent, the Grammian matrix must be invertible, meaning its determinant must be non-zero. If the determinant of the Grammian is zero, the vectors are linearly dependent.

We can calculate the determinant of the matrix G, and if $\det(G) \neq 0$, the vectors are linearly independent.

The determinant of the Grammian matrix is approximately $-1.54x10^{-13}$, which is very close to zero. This indicates that the vectors are linearly dependent.

5 Question 5

Determine the dimension of the vector space spanned by

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 3 \\ 4 \\ 4 \\ 3 \end{bmatrix}$$

5.1 Form the Matrix

We form the matrix A using the vectors as columns:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 4 \\ 2 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix}$$

5.2 Perform Row Reduction

To determine the dimension of the vector space spanned by the columns of A, we compute the rank of the matrix (the number of linearly independent columns).

1. Subtract 2 times the first row from the second and third rows and subtract the first row from the fourth row:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Add the second row to the third row:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

After performing row reduction, we find that the matrix has rank 2. Hence, the dimension of the vector space spanned by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ is 2.

$$\text{Dimension} = 2$$

Thus, the vectors span a 2-dimensional subspace of R^4.