Homework 5

Steve Gillet

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Course: Linear Control Design – ASEN 5014-001 – Fall 2024
Professor: Dale Lawrence
Teaching Assistant: Karan Muvvala

1 Question 1

If $A\mathbf{x} = 0$ has q linearly independent solutions $\mathbf{x}_i$ and $A\mathbf{x} = \mathbf{y}$ has $\mathbf{x}_0$ as a solution, show that
(a) $\mathbf{x}_c = \sum_{i=1}^{q} \alpha_i \mathbf{x}_i$ is also a solution of $A\mathbf{x} = 0$.
(b) $\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{q} \alpha_i \mathbf{x}_i$ is a solution of $A\mathbf{x} = \mathbf{y}$.

1.1 (a) $\mathbf{x}_c = \sum_{i=1}^{q} \alpha_i \mathbf{x}_i$ is a solution of $A\mathbf{x} = 0$:

Since $\mathbf{x}_i$ are linearly independent solutions of $A\mathbf{x} = 0$, we know that $A\mathbf{x}_i = 0$ for each i. Hence, for any linear combination $\mathbf{x}_c = \sum_{i=1}^{q} \alpha_i \mathbf{x}_i$, applying A to $\mathbf{x}_c$:

$$A\mathbf{x}_c = A\left(\sum_{i=1}^{q} \alpha_i \mathbf{x}_i\right) = \sum_{i=1}^{q} \alpha_i A\mathbf{x}_i = \sum_{i=1}^{q} \alpha_i \cdot 0 = 0.$$

Thus, $\mathbf{x}_c$ is indeed a solution of $A\mathbf{x} = 0$.

1.2 (b) $\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{q} \alpha_i \mathbf{x}_i$ is a solution of $A\mathbf{x} = \mathbf{y}$:

We are given that $\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{y}$, so:

$$A\mathbf{x}_0 = \mathbf{y}.$$

We need to show that $\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{q} \alpha_i \mathbf{x}_i$ is also a solution of $A\mathbf{x} = \mathbf{y}$. Applying A to this expression for $\mathbf{x}$:

$$A\mathbf{x} = A\left(\mathbf{x}_0 + \sum_{i=1}^{q} \alpha_i \mathbf{x}_i\right) = A\mathbf{x}_0 + A\left(\sum_{i=1}^{q} \alpha_i \mathbf{x}_i\right).$$

Using the fact that $A\mathbf{x}_i = 0$ for each i, this simplifies to:

$$A\mathbf{x} = A\mathbf{x}_0 + \sum_{i=1}^{q} \alpha_i A\mathbf{x}_i = \mathbf{y} + 0 = \mathbf{y}.$$

Thus, $\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{q} \alpha_i \mathbf{x}_i$ is indeed a solution of $A\mathbf{x} = \mathbf{y}$.

1.3 Dimension of the Right Null Space of A:

The right null space of A, also called the null space, consists of all vectors $\mathbf{x}$ such that $A\mathbf{x} = 0$. If there are q linearly independent solutions to $A\mathbf{x} = 0$, then the dimension of the null space is q.

Thus, the dimension of the right null space of A is q.

2 Question 2

Find all nontrivial solutions of $A\mathbf{x} = 0$, i.e., the null space of

$$A = \begin{bmatrix} 26 & 17 & 8 & 39 & 35 \\ 17 & 13 & 9 & 29 & 28 \\ 8 & 9 & 10 & 19 & 21 \\ 39 & 29 & 19 & 65 & 62 \\ 35 & 28 & 21 & 62 & 61 \end{bmatrix}$$

2.1 Null Space of A

2.1.1 QR Decomposition:

We perform the QR decomposition of A. QR decomposition breaks a matrix A into an orthogonal matrix Q and an upper triangular matrix R:

$$A = QR$$

- Q is an orthogonal matrix, meaning $Q^\top Q = I$. - R is an upper triangular matrix.

2.1.2 Find the Rank of A:

The rank of A is the number of non-zero rows in R. Let's denote the rank of A as r. This helps us determine how many linearly independent columns there are. The dimension of the null space is the number of columns of A minus the rank of A.

The number of null space vectors is determined by:

$$\text{nullity}(A) = \text{number of columns} - \text{rank}(A)$$

In this case, the nullity is 3, meaning there are 3 linearly independent vectors that form the null space.

2.1.3 Extract Null Space Vectors:

Once we have the QR decomposition, we can find the null space vectors by solving $A\mathbf{x} = 0$, or equivalently, $QR\mathbf{x} = 0$. Since Q is invertible, we have $R\mathbf{x} = 0$, which leads us to a system of linear equations from which we can solve for the free variables. This results in the null space vectors.

2.1.4 Result:

After performing the above operations, we find that the null space of A is spanned by the following three vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 0.3215 \\ -0.0843 \\ 0.6606 \\ 0.2484 \\ -0.6256 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0.1387 \\ -0.9490 \\ 0.0422 \\ 0.0691 \\ 0.2712 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0.5745 \\ 0.1178 \\ 0.2556 \\ -0.7224 \\ 0.2625 \end{bmatrix}$$

These vectors are the basis for the null space of A, meaning any vector in the null space is a linear combination of these three vectors.

3 Question 3

Given that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Measurements give $[y_1, y_2] = [3, 4]$. Find the least-squares estimate for x. Use a sketch in the y_1, y_2 plane to indicate the geometrical interpretation.

3.1 Least Squares Estimate for x

We are given the system of equations:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ represents the error terms.
The measurements are given as:

$$\mathbf{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

We can rewrite the system as:

$$\mathbf{y} = \mathbf{A}x + \mathbf{e}$$

where

$$\mathbf{A} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

To find the least-squares estimate for x, we minimize the sum of squared errors:

$$\min_x ||\mathbf{y} - \mathbf{A}x||^2.$$

The least-squares solution is given by the normal equation:

$$\mathbf{A}^\top \mathbf{A}x = \mathbf{A}^\top \mathbf{y}.$$

First, compute $\mathbf{A}^\top \mathbf{A}$:

$$\mathbf{A}^\top \mathbf{A} = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2^2 + 1^2 = 5.$$

Next, compute $\mathbf{A}^\top \mathbf{y}$:

$$\mathbf{A}^\top \mathbf{y} = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 2 \times 3 + 1 \times 4 = 6 + 4 = 10.$$

Now, solve for x:

$$x = \frac{\mathbf{A}^\top \mathbf{y}}{\mathbf{A}^\top \mathbf{A}} = \frac{10}{5} = 2.$$

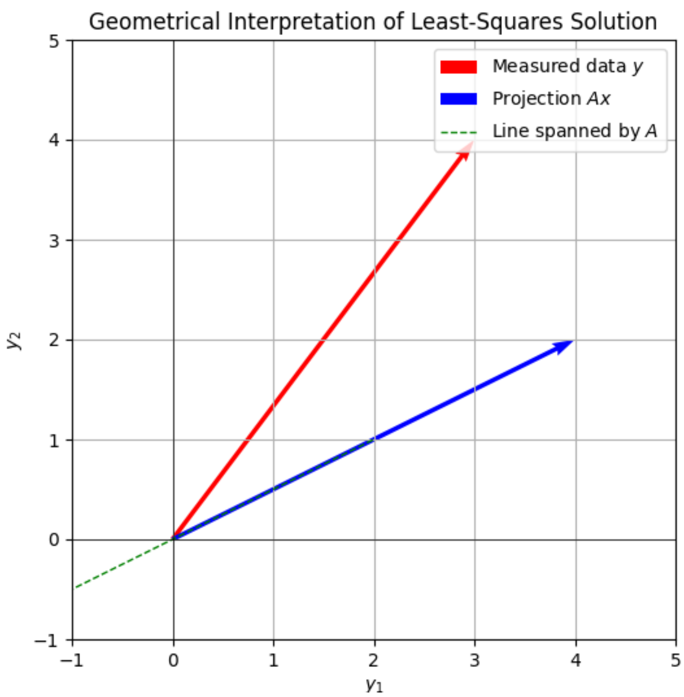
Thus, the least-squares estimate for x is:

$$x = 2.$$

3.2 Geometrical Interpretation:

The following plot shows the geometrical interpretation of the least-squares solution:

- The red vector represents the measured data $\mathbf{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.
- The blue vector is the projection of $\mathbf{y}$ onto the line spanned by $\mathbf{A} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, corresponding to the least-squares solution $x = 2$.
- The green dashed line is the line spanned by $\mathbf{A}$, representing all possible projections of $\mathbf{A}x$.



4 Question 4

The same device as in Problem 6.40 is considered. One more set of readings is taken as

$$u = 5, \quad y = 7$$

Find a least-squares estimate of a and b. Also, find the minimum mean-squared error in this straight line fit to the three points.

Problem 6.40:

6.40 A physical device is shown in Figure 6.10. It is believed that the output y is linearly related to the input u. That is, $y = au + b$. What are the values of a and b if the following data are taken?

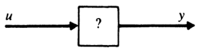


Figure 6.10

u	2	−2
y	5	1

4.1 Matrix Form:

We are given three data points:

$$u_1 = 2, \quad y_1 = 5$$
$$u_2 = -2, \quad y_2 = 1$$
$$u_3 = 5, \quad y_3 = 7$$

We are tasked with fitting a linear model $y = au + b$ to these points using least-squares estimation. We set up the system in matrix form:

$$\mathbf{Y} = \mathbf{U} \begin{bmatrix} a \\ b \end{bmatrix}$$

where

$$\mathbf{Y} = \begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 2 & 1 \\ -2 & 1 \\ 5 & 1 \end{bmatrix}$$

4.2 Least-Squares Solution:

The least-squares solution is given by:

$$\begin{bmatrix} a \\ b \end{bmatrix} = (\mathbf{U}^\top \mathbf{U})^{-1} \mathbf{U}^\top \mathbf{Y}$$

After computation, we find:

$$a = 0.865, \quad b = 2.892$$

4.3 Minimum Mean-Squared Error:

The minimum mean-squared error is calculated as:

$$\text{MSE} = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

where $\hat{y}_i$ are the predicted values from the least-squares line. The MSE is approximately:

$$\text{MSE} = 0.072$$

Thus, the least-squares line that best fits the data has a slope $a = 0.865$ and intercept $b = 2.892$, with a mean-squared error of 0.072.

5 Question 5

An empirical theory used by many distance runners states that the time T_i required to race a distance D_i can be expressed as $T_i = C(D_i)^\alpha$, where C and α are constants for a given person, determined by lung capacity, body build, etc. Obtain a least-squares fit to the following data for one middle-aged jogger. (Convert to a linear equation in the unknowns C and α by taking the logarithm of the above expression.) Predict the time for one mile.

Time (min)	Distance (mi)
185	26.2
79.6	12.4
60	9.5
37.9	6.2
11.5	2

5.1 Least Squares Fit

The given relation is $T_i = CD_i^\alpha$. Taking the natural logarithm of both sides:

$$\ln(T_i) = \ln(C) + \alpha \ln(D_i)$$

Let $\ln(T_i) = y_i$, $\ln(D_i) = x_i$, and $\ln(C) = b$. The equation becomes:

$$y_i = \alpha x_i + b$$

We can now apply a linear least-squares fit to the transformed data.

5.2 Transformed Data:

$\ln(T_i)$	$\ln(D_i)$
$\ln(185)$	$\ln(26.2)$
$\ln(79.6)$	$\ln(12.4)$
$\ln(60)$	$\ln(9.5)$
$\ln(37.9)$	$\ln(6.2)$
$\ln(11.5)$	$\ln(2)$

5.3 Matrix Form:

We write the system in matrix form as:

$$\mathbf{y} = \mathbf{X} \begin{bmatrix} \alpha \\ b \end{bmatrix}$$

where

$$\mathbf{y} = \begin{bmatrix} \ln(185) \\ \ln(79.6) \\ \ln(60) \\ \ln(37.9) \\ \ln(11.5) \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \ln(26.2) & 1 \\ \ln(12.4) & 1 \\ \ln(9.5) & 1 \\ \ln(6.2) & 1 \\ \ln(2) & 1 \end{bmatrix}$$

We solve the least-squares problem:

$$\begin{bmatrix} \alpha \\ b \end{bmatrix} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$$

After performing the least-squares calculation, we obtain the following values for α and C:

$$\alpha = 1.077, \quad C = 5.370$$

5.4 Prediction for One Mile:

Once we obtain α and $C = e^b$, we can predict the time for one mile by substituting $D = 1$ into the original equation:

$$T_{\text{one mile}} = C(1)^\alpha = C$$

Thus, the predicted time for one mile is:

$$T_{\text{one mile}} = 5.370 \text{ minutes.}$$