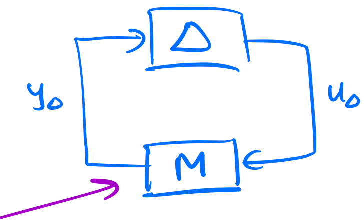
ROBUST STABILITY ANALYSIS FOR MIMO SYSTEMS

GOAL: ROBUST STABILITY TESTS FOR STRUCTURED OR UNSTRUCTURED UNCERTAINTY MODELS (MIMO)

APPROACH: PUT BLOCK DIAGRAM INTO "$M\Delta$" FORM AND APPLY THE SMALL GAIN THEOREM



STABLE!

TYPES OF UNCERTAINTY (Δ)

(i) UNSTRUCTURED: Δ CAN BE ANY TF MATRIX AS LONG AS IT IS PROPER AND STABLE, $\|\Delta\|_\infty \leq 1$ (FULL COMPLEX PERTURBATIONS) — MOST GENERAL FORM

(ii) STRUCTURED:

(a) BLOCK COMPLEX: $\Delta(s) = \begin{pmatrix} \Delta_1(s) & & 0 \\ & \ddots & \\ 0 & & \Delta_n(s) \end{pmatrix}$ WHERE THE Δ_i ARE PROPER, STABLE $\|\Delta_i\|_\infty \leq 1$.

(b) BLOCK REAL: $\Delta(s) = \begin{pmatrix} \delta_i & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{pmatrix}$ WHERE THE $\delta_i \in \mathbb{R}^{i \times j}$ ARE REAL # MATRICES

- THIS IS THE MOST RESTRICTIVE FORM
- PARAMETRIC (CHANGES IN PARAMETERS)

(c) MIXED: COMBO OF COMPLEX/REAL BLOCKS

MOST DIFFICULT TO ANALYZE

NOTE: THE VERSION OF THE SMALL GAIN THEOREM DEPENDS ON THE TYPE OF UNCERTAINTY USED.

DETERMINANT STABILITY CONDITION

APPLIES TO REAL OR COMPLEX PERTURBATIONS :

LET $M(s)$ AND $\Delta(s)$ BE STABLE. THEN THE $M-\Delta$ SYSTEM IS ROBUSTLY STABLE (RS)

$\iff$ NYQUIST PLOT OF $\det(I - M\Delta)$ DOES NOT ENCIRCLE THE ORIGIN $\forall \Delta$ (SINCE M, Δ STABLE)

$\iff$ $\det(I - M\Delta) \neq 0 \quad \forall \omega, \Delta$ (ZEROS ARE CLOSED LOOP POLES)

PROOF : APPLY MULTIVARIABLE NYQUIST THEOREM WITH $p = 0 \Rightarrow z = n = 0$ FOR STABILITY

SPECTRAL RADIUS CONDITION (COMPLEX PERTURBATIONS ONLY)

LET $M(s)$ AND $\Delta(s)$ BE STABLE. THEN THE $M-\Delta$ SYSTEM IS ROBUSTLY STABLE (RS) $\iff \rho(M\Delta) < 1 \quad \forall \omega, \Delta$

PROOF

NECESSITY ($\Leftarrow$)

RECALL SPECTRAL RADIUS $\rho(M\Delta) = \max_i |\lambda_i(M\Delta)|$

ASSUME $\rho(M\Delta) < 1 \quad \forall \omega, \Delta$

$$\det(I - M\Delta) = \prod_i \lambda_i(I - M\Delta) = 0 \quad \text{(PRODUCT OF } \lambda\text{'s)}$$

$$= \prod_i [1 - \lambda_i(M\Delta)] = 0$$

EVALS OF $A + cI$ $= \lambda_i(A) + c$

$\rho(M\Delta) < 1 \Rightarrow \det(I - M\Delta) \neq 0$ ∎

SUFFICENCY

($\Rightarrow$) PROVE USING THE CONTRAPOSITIVE :

$A \Rightarrow B$ IS EQUIVALENT TO NOT$(B) \Rightarrow$ NOT(A)

NOT $\rho(M\Delta) < 1 \Rightarrow$ NOT $\det(I - M\Delta) \neq 0$

ASSUME $\rho(M\Delta) = 1 \Rightarrow \max_i |\lambda_i(M\Delta)| = 1$

PICK A Δ' SUCH THAT $\lambda_i(M\Delta') = +1$ (CAN DO THIS ONLY IF Δ IS ALLOWED TO BE COMPLEX)

$\Rightarrow \det(I - M\Delta) = \prod_i [1 - \lambda_i(M\Delta)] = 0$ ∎

NOTE : THE ($\Leftarrow$) DIRECTION HOLDS FOR BOTH REAL AND COMPLEX PERTURBATIONS

PROPOSITION FOR ANY MATRIX NORM, $\rho(A) \leq \|A\|$

PROOF : LET $\lambda_i(A)$ BE ANY EIGENVALUE OF A.

THEN $A v_i = \lambda_i v_i$, AND

$$|\lambda_i| \, \|v_i\| = \|\lambda_i v_i\| = \|A v_i\| \leq \|A\| \, \|v_i\|$$

(FOR ANY VECTOR / MATRIX NORMS THAT ARE COMPATIBLE)

THUS $|\lambda_i(A)| \leq \|A\|$ FOR ANY i

$\Rightarrow \rho(A) \leq \|A\|$ ∎

EQUIVALENCE OF $\max \rho(\cdot)$ AND $\|\cdot\|_\infty$

LET Δ BE THE SET OF ALL COMPLEX MATRICES SUCH THAT $\bar{\sigma}(\Delta) = \|\Delta\|_\infty \leq 1$ (FULL COMPLEX PERTURBATION). THEN

$$\max_{\Delta} \rho(M\Delta) = \max_{\Delta} \bar{\sigma}(M\Delta) = \max_{\Delta} \bar{\sigma}(M)\bar{\sigma}(\Delta) = \bar{\sigma}(M)$$

PROOF:

$$\max_{\Delta} \rho(M\Delta) \underset{\text{PROPOSITION}}{\leq} \max_{\Delta} \bar{\sigma}(M\Delta) \underset{\bar{\sigma}(AB) \leq \bar{\sigma}(A)\bar{\sigma}(B)}{\leq} \max_{\Delta} \bar{\sigma}(M)\bar{\sigma}(\Delta) \underset{\bar{\sigma}(\Delta) \leq 1}{\leq} \bar{\sigma}(M)$$

TO SHOW EQUALITY, CONSTRUCT A Δ' SUCH THAT

$$\rho(M\Delta') = \bar{\sigma}(M\Delta')$$

Δ' WILL EXIST IF WE ALLOW Δ TO BE A FULL COMPLEX PERTURBATION, Δ STABLE, $\|\Delta\|_\infty \leq 1$.

TAKE SVD $M = U\Sigma V^*$ (FOR EACH ω) AND CHOOSE $\Delta' = VU^*$ (U, V ORTHOGONAL, SVs ALL 1)

REQUIRES FULL COMPLEX

THEN $\bar{\sigma}(\Delta') = 1$ AND

$$\rho(M\Delta') = \rho(U\Sigma V^* V U^*) = \underbrace{\rho(U\Sigma U^*)}_{\text{SIMILARITY TF}} = \rho(\Sigma) = \bar{\sigma}(M) \quad \blacksquare$$

RECALL $U^* = U^{-1}$ AND THE EIGENVALUES OF A MATRIX ARE INVARIANT UNDER SIMILARITY TRANSFORMATIONS.

BACK TO THE SPECTRAL RADIUS CONDITION, ASSUME Δ FULL COMPLEX:

NOW, RS $\iff \rho\,[M(j\omega)\,\Delta(j\omega)] < 1 \quad \forall \omega, \Delta$ (SPECTRAL RADIUS CONDITION)

$\iff \bar{\sigma}\,[M(j\omega)\,\Delta(j\omega)] < 1 \quad \forall \omega, \Delta$ $\quad \Delta$ FULL, COMPLEX

$\iff \bar{\sigma}\,[M(j\omega)] < 1 \quad \forall \omega$ $\quad \|\Delta\|_\infty \leq 1$

$\iff \|M\|_\infty < 1$

WE HAVE PROVEN THE FOLLOWING:

SMALL GAIN THEOREM (H_∞ NORM CASE)

ASSUME $\Delta(s)$ IS STABLE, $\|\Delta\|_\infty \leq 1$, Δ FULL COMPLEX.

THEN WE HAVE ROBUST STABILITY (RS) $\iff$

$$\bar{\sigma}\,[M(j\omega)]\;\bar{\sigma}\,[\Delta(j\omega)] < 1 \quad \forall\, \omega, \Delta$$

OR EQUIVALENTLY

$$\bar{\sigma}\,[M(j\omega)] < 1 \quad \forall \omega \qquad (\;\|M\|_\infty < 1\;)$$

NOTES

(1) SGT HOLDS FOR ANY INDUCED SYSTEM NORM SATISFYING $\|AB\| \leq \|A\|\,\|B\|$ (SUB MULTIPLICATIVE) BUT IS ONLY A SUFFICIENT CONDITION UNLESS WE USE $\|\cdot\|_\infty$ AND HAVE FULL COMPLEX Δ.

(2) IN GENERAL, RESULT IS CONSERVATIVE SINCE PHASE INFO NOT TAKEN INTO ACCOUNT

(3) TO GUARANTEE INTERNAL STABILITY, NEED TO VERIFY THERE ARE NO UNSTABLE HIDDEN MODES (P/Z CANCELLATIONS IN RHP IN $L = M\Delta$).