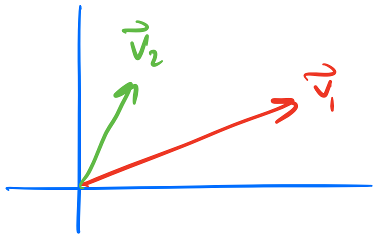
INTRODUCTION

QUESTIONS LIKE THE FOLLOWING OFTEN ARISE IN THIS COURSE:

- HOW LARGE OF A DISTURBANCE CAN THE SYSTEM STAND?
- HOW SMALL IS THE ERROR IN THE FEEDBACK LOOP FOR 2 DIFFERENT CONTROLLERS?
- HOW MUCH UNCERTAINTY CAN THE SYSTEM TOLERATE BEFORE BECOMING UNSTABLE?

EX: VECTORS IN $\mathbb{R}^2$

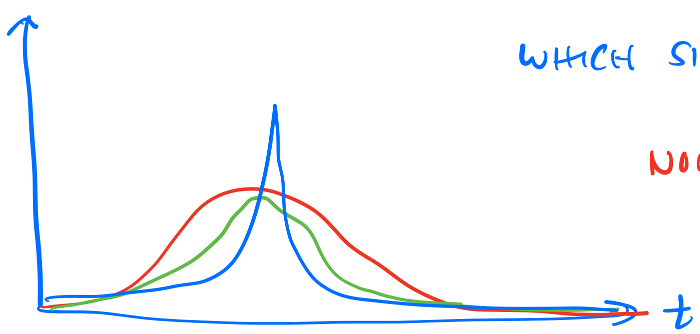


CLEARLY $\vec{v}_1$ IS LARGER THAN $\vec{v}_2$

INTRODUCE A FUNCTION $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ CALLED A NORM (SIZE OF THE VECTOR)

$$\|\vec{v}\| = \left(\sum_{i=1}^{2} |v_i|^2 \right)^{1/2}$$ (EUCLIDEAN NORM)

EX SCALAR-VALUED SIGNALS



WHICH SIGNAL IS THE LARGEST?

NORM: PEAK?

AREA UNDER CURVE?

EXAMPLES OF MATHEMATICAL FORMALISM

LEVELS OF MATHEMATICAL FORMALISM

DEF'N: A **SET** IS ANY COLLECTION OF WELL-DEFINED MATHEMATICAL OBJECTS

$$\{-2, -1, 0, 1, 2\} \qquad \{A \in \mathbb{R}^{n\times n} \mid \overbrace{A^T = -A}^{\text{PROPERTIES}}\}$$

ELEMENTS

BRACES

INTRODUCING LAYERS OF STRUCTURE ON SETS PROVIDES A FRAMEWORK FOR MATHEMATICAL ARGUMENTS AND CALCULATIONS (CONVERGENCE, ORTHOGONALITY, CONTINUITY)

(I) ALGEBRAIC STRUCTURE (LINEAR / VECTOR SPACES)

EQUIP SETS WITH
- ZERO VECTOR
- SCALAR MULTIPLICATION
- VECTOR ADDITION
- ⋮

} AXIOMS

$\Rightarrow$ LINEAR INDEPENDENCE
BASIS / DIMENSION
INVERTABILITY (1-1, ONTO)

(II) TOPOLOGICAL STRUCTURE

(A) NORMED SPACES (BANACH SPACE IF COMPLETE)

EQUIP A LINEAR SPACE V WITH A NORM

$$\|\cdot\| : \underbrace{V}_{\text{DOMAIN}} \to \underbrace{\mathbb{R}}_{\text{CODOMAIN}}$$ (SIZE OF AN ELEMENT)

$\Rightarrow$ BOUNDEDNESS OF FUNCTIONS / MAPS / TRANSFORMATIONS / OPERATORS

EX EUCLIDEAN NORM ON $\mathbb{R}^n$ $\quad \|\vec{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$

(B) INNER PRODUCT SPACES (HILBERT SPACE IF COMPLETE)

EQUIP A LINEAR SPACE V WITH AN INNER PRODUCT

$\langle \cdot , \cdot \rangle : \underbrace{V \times V}_{\text{CARTESIAN PRODUCT}} \rightarrow \mathbb{R}$ (ANGLE B/T ELEMENTS)

GEOMETRY OF SPACE

$\Rightarrow$ ORTHOGONALITY, PROJECTIONS, FOURIER SERIES

EX EUCLIDEAN INNER PRODUCT ON $\mathbb{R}^n$ (DOT PRODUCT)

$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i y_i = \vec{x}^T \vec{y}$ (OR $\vec{x} \cdot \vec{y}$)

THM EVERY INNER PRODUCT SPACE IS A NORMED SPACE UNDER THE INDUCED NORM $\|\vec{x}\| = (\langle \vec{x}, \vec{x} \rangle)^{1/2}$

EX EUCLIDEAN INNER PRODUCT ON $\mathbb{R}^n$ $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n} x_i y_i$ INDUCES THE EUCLIDEAN NORM $\|\vec{x}\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$

EX DEFINE $C[a,b] = \{ f : [a,b] \rightarrow \mathbb{R} \, , \, f \text{ CONTINUOUS} \}$

INNER PRODUCT ON $C[a,b]$: $\langle f, g \rangle = \int_a^b f(t) g(t) \, dt$

INDUCES THE SIGNAL 2-NORM: $\|f\|_2 = \left(\int_a^b |f(t)|^2 dt \right)^{1/2}$

HENCE, INNER PRODUCT SPACE $\Rightarrow$ NORMED SPACE $\Rightarrow$ METRIC SPACE (DISTANCE FUNCTION)

$\xleftrightarrow{\text{STRUCTURE}}$

MOST ... LEAST

$\boxed{\text{SIGNAL AND SYSTEM NORMS}}$

BIG PICTURE: FRAMEWORK FOR QUANTIFYING THE PERFORMANCE OF A DYNAMIC SYSTEM

DEF'N: A SIGNAL IS ANY MATHEMATICAL OBJECT THAT CONTAINS INFORMATION ABOUT A SYSTEM

SIGNAL NORMS: QUANTITATIVE MEASURE OF SIGNAL SIZE OR STRENGTH. WANT THESE TO CAPTURE THE PERFORMANCE OBJECTIVE (SMALL TRACKING ERROR, ETC.) AND BE EASILY COMPUTABLE.

DEF'N: A SYSTEM IS A MAP FROM ONE SIGNAL SPACE TO ANOTHER

(MAP / TRANSFORMATION / OPERATOR / ...)

SYSTEM NORMS: A MEASURE OF SIGNAL GAIN / NORM (SIZE, STRENGTH) AS IT PASSES THROUGH THE SYSTEM. WANT THESE TO CAPTURE THE EFFECT OF THE SYSTEM ON THE SIGNAL.

SIGNAL NORMS

WE DEFINE A SIGNAL AS AN ELEMENT OF AN APPROPRIATELY DEFINED SET $S = \{ f : \mathbb{R} \to \mathbb{R}^n ,\ P \}$ WHERE "P" ARE ADDITIONAL PROPERTIES, SPECIFICATIONS OR CONSTRAINTS

TIME DOMAIN SIGNAL SPACES

EX VECTOR-VALUED LEBESGUE 2-SPACE ON $\mathbb{R} = (-\infty, \infty)$

$$L_2^n(-\infty, \infty) = \left\{ f : (-\infty, \infty) \to \mathbb{R}^n ,\ \int_{-\infty}^{\infty} \| f(t) \|^2 \, dt < \infty \right\}$$

$$f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

$\| \cdot \|$ IS THE EUCLIDEAN NORM,

$$\| f(t) \| = (f^T f)^{1/2} = (\langle f, f \rangle)^{1/2}$$

(HOLD t CONSTANT)

INTERPRETATION:

SQUARE OF A SIGNAL $\longrightarrow$ INSTANTANEOUS POWER $\| f(t) \|^2$

INTEGRAL OF POWER $\longrightarrow$ ENERGY

INFORMALLY, L_2^n ARE SIGNALS WITH FINITE / BOUNDED ENERGY

ALSO, L_2^n IS AN INNER PRODUCT SPACE,

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f^T(t)\, g(t)\, dt$$

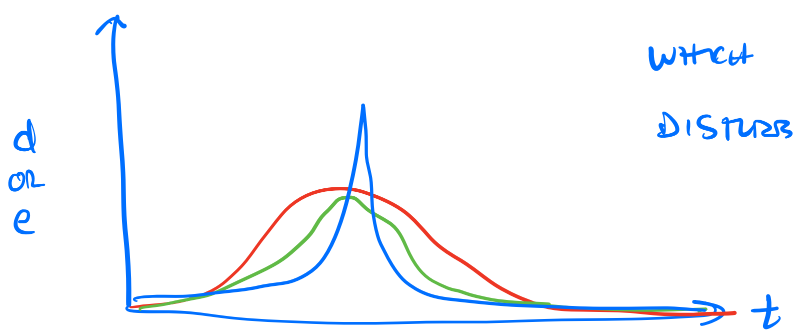
INDUCED NORM: $\|f(t)\|_{L_2} = \left(\langle f(t), f(t) \rangle \right)^{1/2}$

USE "2" INSTEAD

$$= \left(\int_{-\infty}^{\infty} f^T(t)\, f(t)\, dt \right)^{1/2}$$

$$= \left(\int_{-\infty}^{\infty} \|f(t)\|^2\, dt \right)^{1/2}$$

HENCE THE L_2 NORM IS THE SQUARE ROOT OF THE SIGNAL'S ENERGY



WHICH ONE IS THE LARGEST DISTURBANCE OR ERROR

CAN ALSO DEFINE L_2 SPACES WITH A FINITE TIME HORIZON.

$$L_2^n[0,T] = \left\{ f : [0,T] \to \mathbb{R}^n,\ \int_0^T \|f(t)\|^2\, dt < \infty \right\}$$

SIGNALS THAT ARE BOUNDED ON FINITE TIME INTERVALS

EX SIGNAL NOT IN L_2:

$$f(t) = \frac{1}{|2T - 1|}$$