Class: Robust Multivariate Control
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Assignment: Homework 6

1.

In this problem you will use H_∞ synthesis techniques to design a roll angle hold control system for a fixed wing UAS with actuator dynamics. The roll dynamics are given by

$$\dot{p} = L_p p + L_{\delta_a}\delta_a,$$
$$\dot{\phi} = p,$$

where p is the roll rate, ϕ is the roll angle and δ_a is the aileron deflection. The stability and control derivatives are $L_p = -1$ and $L_{\delta_a} = 30$. We will assume first order actuator dynamics for the aileron servomotors, hence

$$\dot{\delta}_a = -\frac{1}{\tau}\delta_a + \frac{1}{\tau}V$$

where V is the input voltage to the servomotor and $\tau = 0.1$ is the motor time constant.

(a)

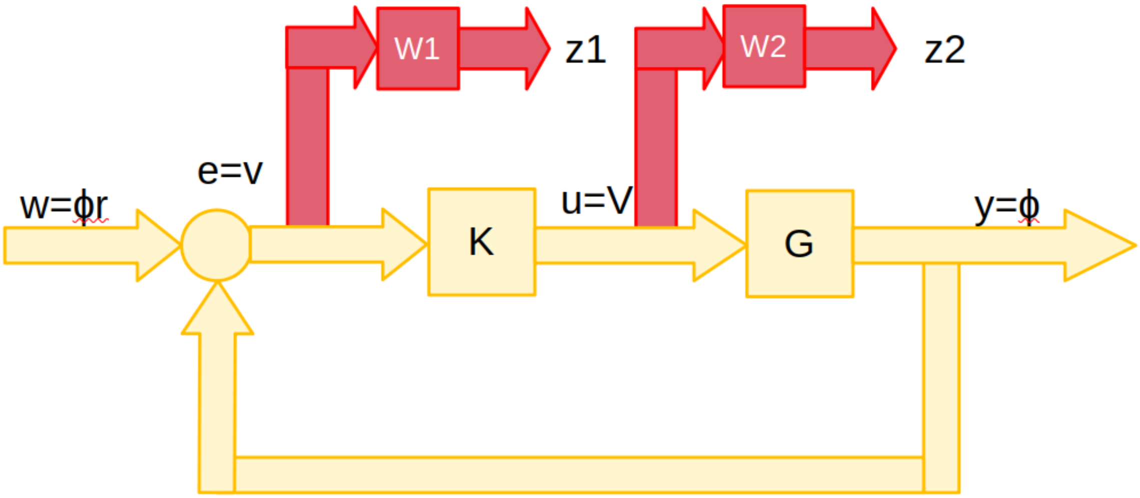
Augment the roll dynamics with the actuator dynamics to generate a new state space model for the combined plant G with state $x = [p\ \phi\ \delta_a]^T$ and input V.

The roll dynamics with actuator dynamics are given by:

$$\begin{bmatrix} \dot{p} \\ \dot{\phi} \\ \dot{\delta}_a \end{bmatrix} = \begin{bmatrix} L_p & 0 & L_{\delta_a} \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{\tau} \end{bmatrix} \begin{bmatrix} p \\ \phi \\ \delta_a \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\tau} \end{bmatrix} V$$

(b)

Generate a block diagram of the closed loop system assuming our measurement is the roll angle ϕ_r. We want to have the closed loop system track the roll angle reference ϕ_r with actuator limits, so assume performance outputs $z_1 = W_1 e$ and $z_2 = W_2 u$. Be sure to label all the important signal quantities (z_1, z_2, v, w, u) on the block diagram.



(c)

Compute the generalized plant P and close the lower LFT to compute the corresponding $w \rightarrow z$ transfer function and test for H_∞ nominal performance.

The performance outputs and signals are defined as:

$$\begin{aligned} z_1 &= W_1(w - Gu), \\ z_2 &= W_2 u, \\ v &= e = w - Gu. \end{aligned}$$

The generalized plant P is constructed as:

$$\begin{bmatrix} z_1 \\ z_2 \\ v \end{bmatrix} = \begin{bmatrix} W_1 & -W_1 G \\ 0 & W_2 \\ I & -G \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

The closed-loop transfer function $F_\ell(P, K)$ is derived as:

$$\begin{aligned} F_\ell(P,K) &= P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}, \\ F_\ell(P,K) &= \begin{bmatrix} W_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -W_1 G \\ W_2 \end{bmatrix} K(I + GK)^{-1} I, \\ &= \begin{bmatrix} W_1 - W_1 GK(I+GK)^{-1} \\ W_2 K(I+GK)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} W_1(I - T_O) \\ W_2 K S_O \end{bmatrix} \\ &= \begin{bmatrix} W_1 S_O \\ W_2 K S_O \end{bmatrix} \end{aligned}$$

The H_∞ performance criteria are:

$$\| \begin{bmatrix} W_1 S_O \\ W_2 K S_O \end{bmatrix} \|_\infty < 1$$

where $S_O = (I + GK)^{-1}$ is the sensitivity function.

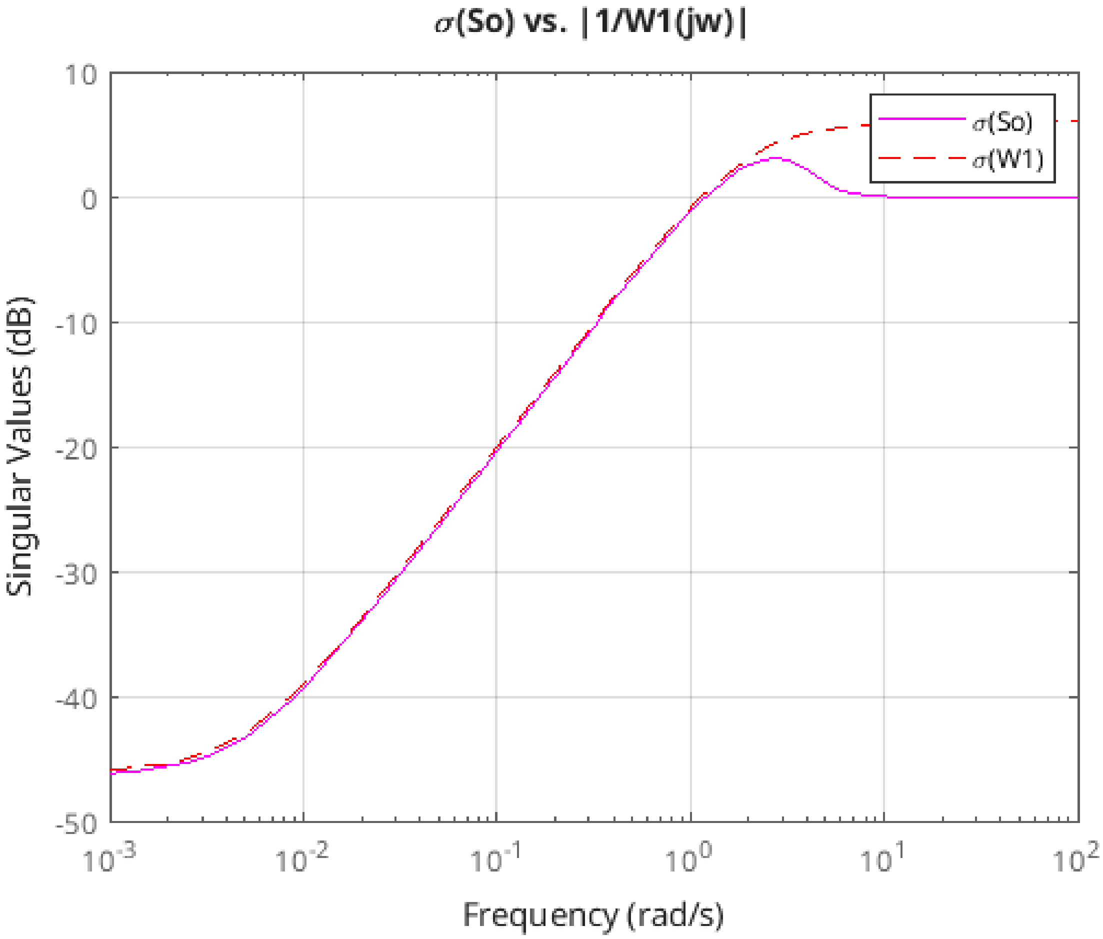
(d)

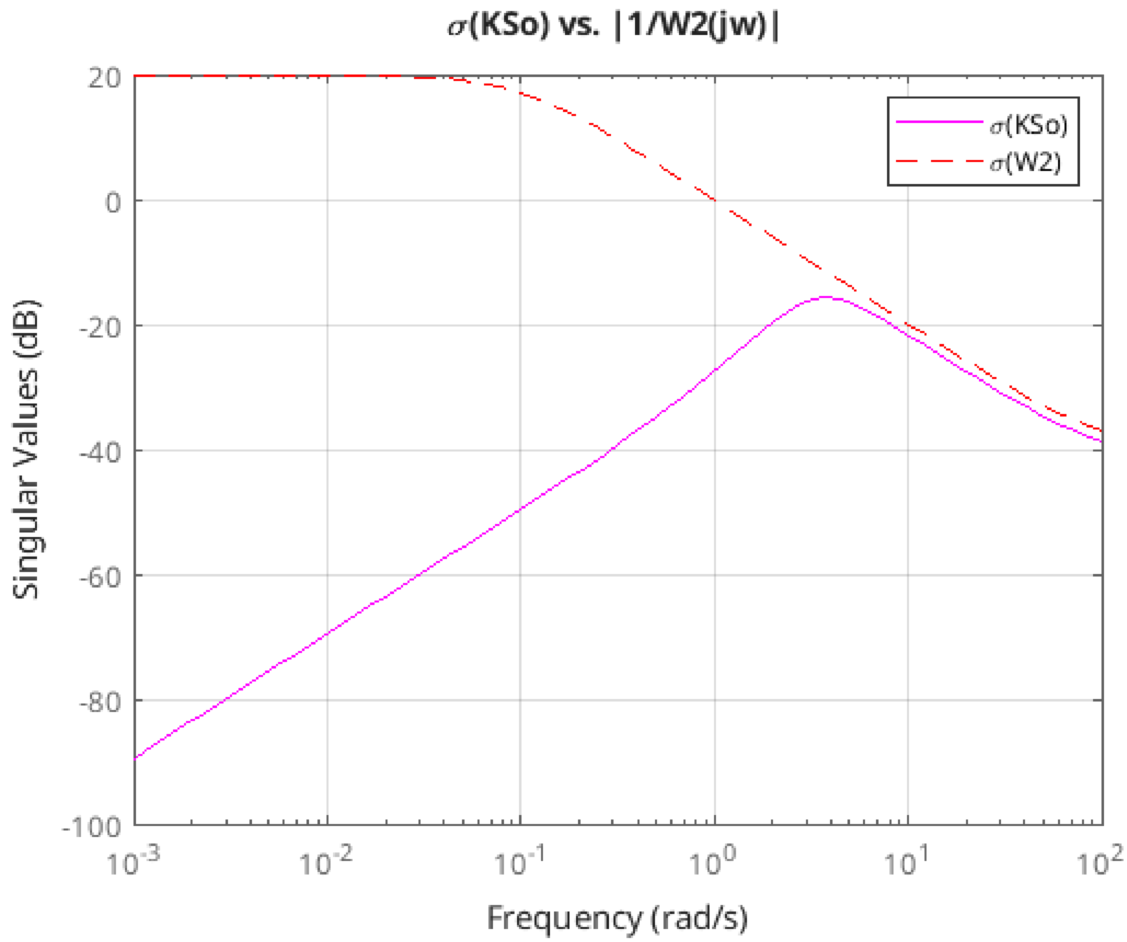
*Assume a tracking performance weighting function to be $W_1(s) = (s/M + \omega_B^*)/(s + \omega_B^* A)$ and an actuator usage weighting function (predetermined by the given actuator performance) to be $W_2(s) = 100(s+0.1)/(s+100)$. Implement MATLAB code to synthesize a H_∞ controller K using **hinfyn** and **sysic**. Your design parameters for tracking are A, M and ω_B^*. Ideally the rise time for the closed loop would be less than 4 seconds with minimal steady-state error.*

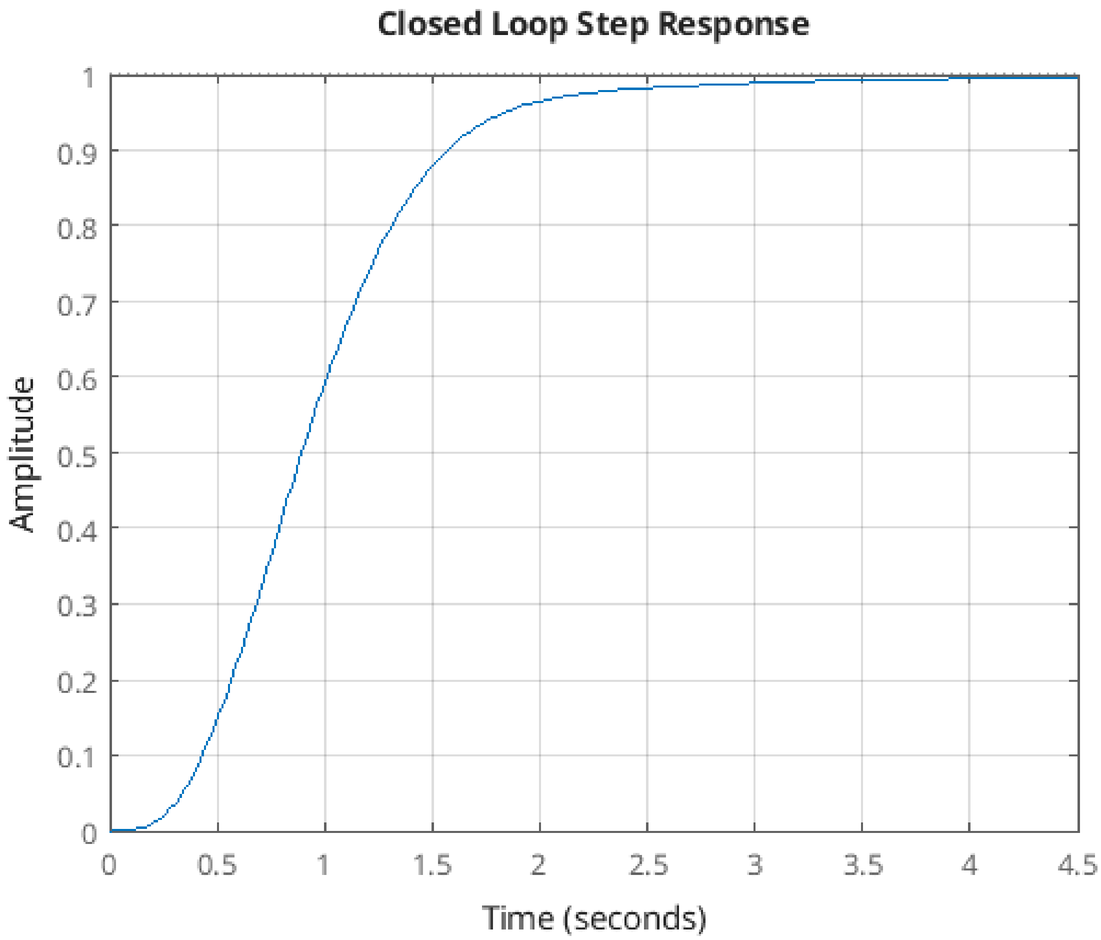
```
Lp = -1;
Lda = 30;
tau = 0.1;

A = [Lp 0 Lda; 1 0 0; 0 0 -1/tau];
B = [0; 0; 1/tau];
C = [0 1 0];
D = 0;
G = ss(A,B,C,D);

M = 2;
A = 0.005;
omegaB = 1;

W1 = tf([1/M omegaB], [1 omegaB*A]);
W2 = tf([100 10], [1 100]);

systemnames = 'G W1 W2';
inputvar = '[w; u]';
outputvar = '[W1; W2; w-G]';
input_to_G = '[u]';
input_to_W1 = '[w-G]';
input_to_W2 = '[u]';
sysoutname = 'P';
sysic;
P = minreal(ss(P));

m = size(B,2);
p = size(C,1);
[K, CL, gamma, info] = hinfsyn(P,p,m,'method','ric','Tolgam',1e-3,'DISPLAY','on');
```

(e)

When you have settled on a sufficient design, verify that your system satisfies nominal performance by plotting the $|1/W_1(j\omega)|$ and $\hat{\sigma}(S_0)$ curves, and the $|1/W_2(j\omega)|$ and $\hat{\sigma}(KS_0)$ curves. Also plot the step response for a unit step input in ϕ_r.


σ(So) vs. |1/W1(jw)|
10
0
-10
-20
-30
-40
-50
Singular Values (dB)
10^-3
10^-2
10^-1
10^0
10^1
10^2
Frequency (rad/s)
σ(So)
σ(W1)


σ(KSo) vs. |1/W2(jw)|
σ(KSo)
σ(W2)
Singular Values (dB)
Frequency (rad/s)
20
0
-20
-40
-60
-80
-100
10^-3
10^-2
10^-1
10^0
10^1
10^2

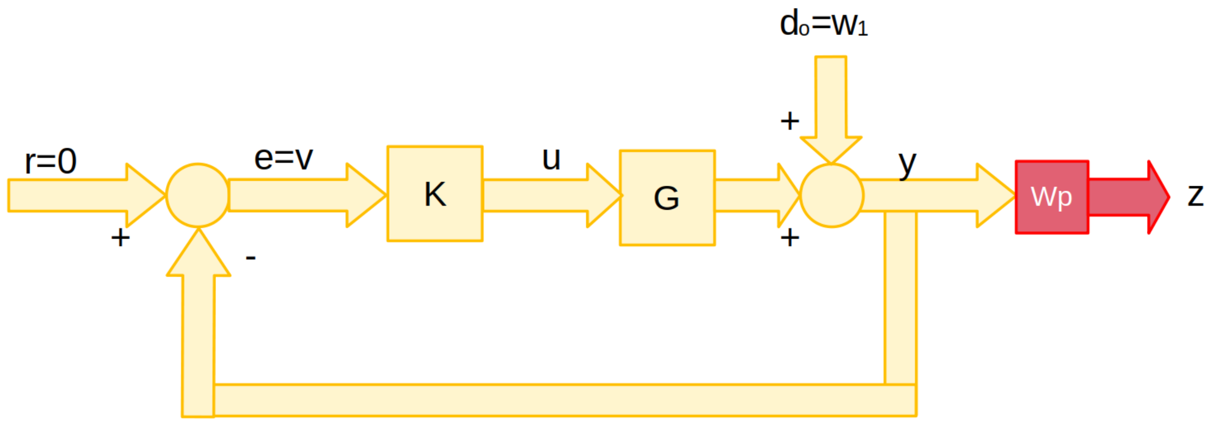


2.

In this problem you will gain some experience with a MIMO plant by synthesizing a disturbance rejection controller for the lateral dynamics of an A-4 aircraft. The plant has two states, β (sideslip angle) and r (yaw rate), along with two control inputs δ_a (aileron angle) and δ_r (rudder angle). Assume both states are measurable, i.e., $C = I$. The dynamics are available to download in an m-file from Canvas.

(a)

Generate a block diagram of the closed loop system assuming an output disturbance $w_1 = d_o \in \mathbb{R}^2$, with a performance output $z = W_p y \in \mathbb{R}^2$. Be sure to label all the important signal quantities (z, v, w, u) on the block diagram.



(b)

Compute the generalized plant P and close the lower LFT to compute the corresponding $w \to z$ transfer function and test for H_∞ nominal performance.

The performance output and signals are defined as:

$$
\begin{aligned}
z &= W_p y = W_p(w + Gu), \\
v &= e = -y = -(w + Gu).
\end{aligned}
$$

The generalized plant P is constructed as:

$$
\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} W_p & W_p G \\ -I & -G \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}
$$

The closed-loop transfer function $F_\ell(P, K)$ is derived as:

$$
\begin{aligned}
F_\ell(P, K) &= P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}, \\
&= W_p + W_p G K (I + GK)^{-1}(-I), \\
&= W_p (I - T_O)
\end{aligned}
$$

The H_∞ performance criterion is:

$$
\|W_p S_0\|_\infty < 1,
$$

where $S_0 = (I + GK)^{-1}$ is the sensitivity function, and the transfer function from $w \to z$ is T_O.

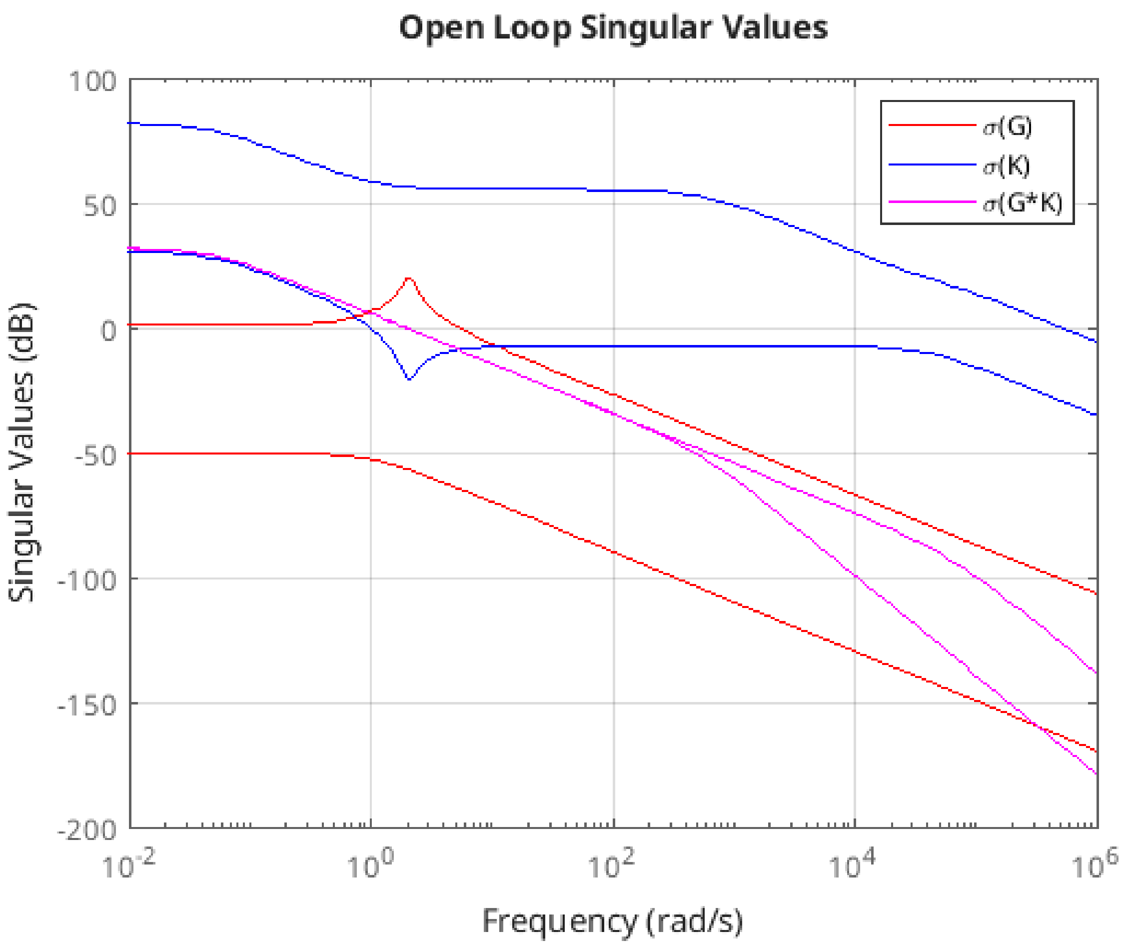
(c)

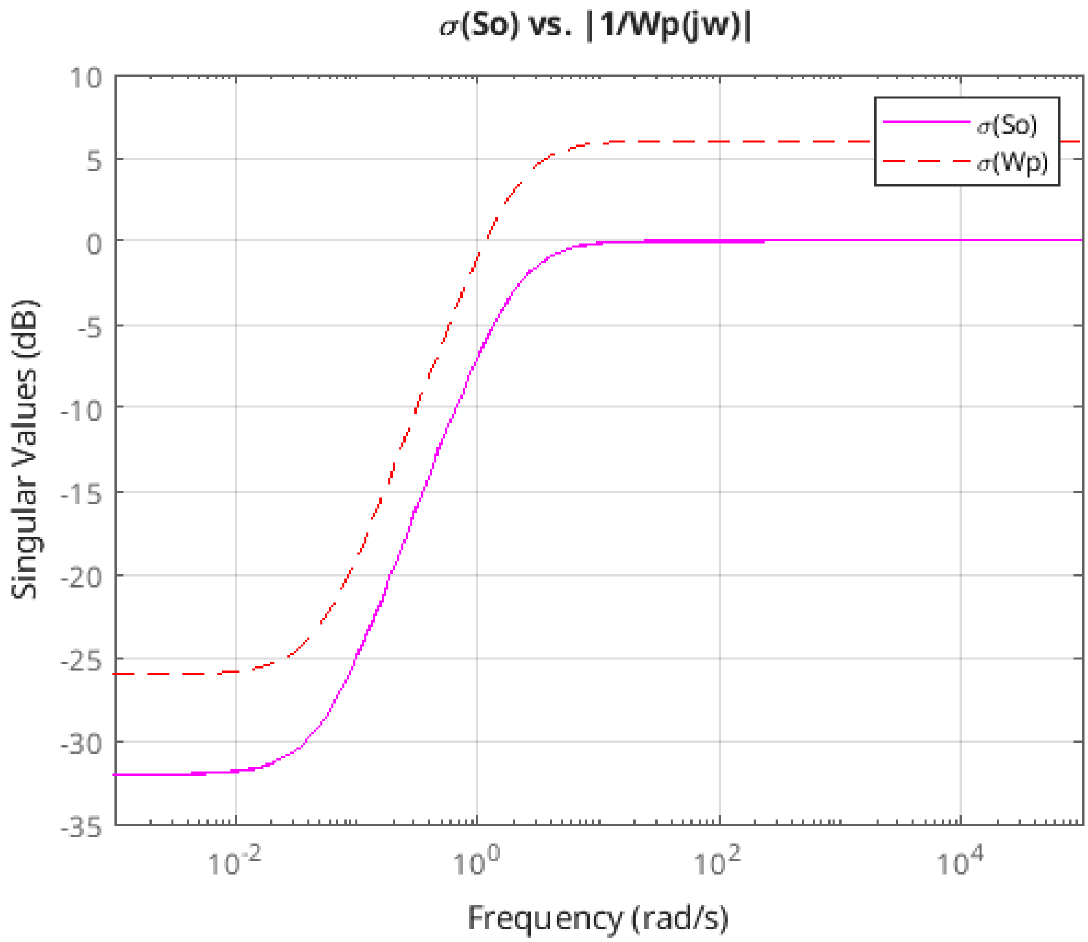
*Assume a tracking performance weighting function of the form $W_p(s) = (s/M + \omega_B^*)/(s + \omega_B^* A)I$. Implement MATLAB code to synthesize a H_∞ controller K using either the **mixsyn** or **hinfyn** functions. Select design parameters A, M and ω_B^* so that the resulting closed loop has a disturbance rejection capability of 10X and a closed loop bandwidth that is less than a 3 second rise time in each of the states.*

```
A=0.05;
omegaB=1;
M=2;
s = tf('s');
Wp = (s/M+omegaB)/(s+omegaB*A)*eye(2);
W2=[];
W3=[];

[K,CL,GAM,INFO] = mixsyn(G,Wp,W2,W3);
```

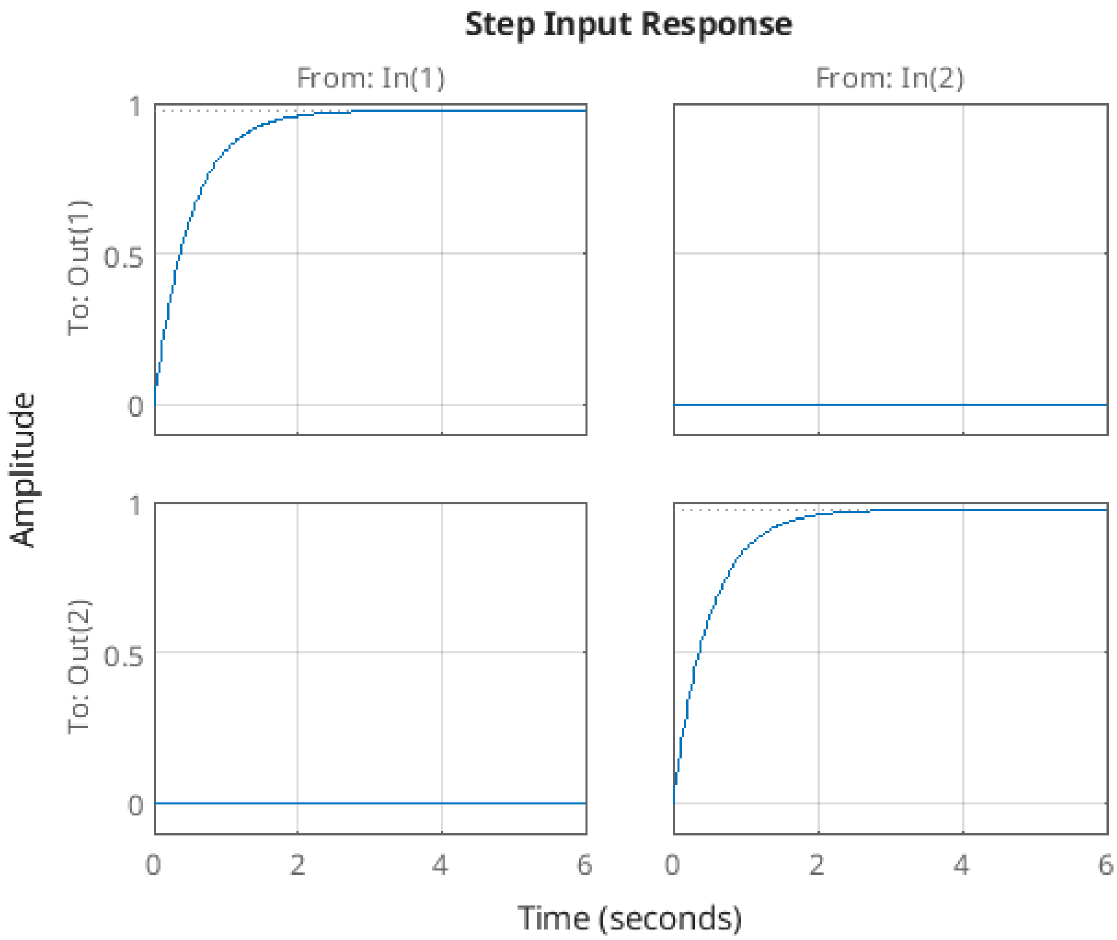
(d)

When you have settled on a sufficient design, generate the following plots: (i) a comparison of the singular values of the plant G, the controller K and the open loop transfer function matrix GK, (ii) verify that your system satisfies nominal performance by plotting the $|1/W_1(j\omega)|$ and $\hat{\sigma}(S_0)$ curves, (iii) the step response of the state β for a unit step input in the δ_a and the step response of the state r for a unit step input in the δ_r, and (iv) the response of the β state to a step in the first disturbance component d_1, and the response of the second state r to a step in the second disturbance component d_2, where $d_o = [d_1\ d_2]^T$.

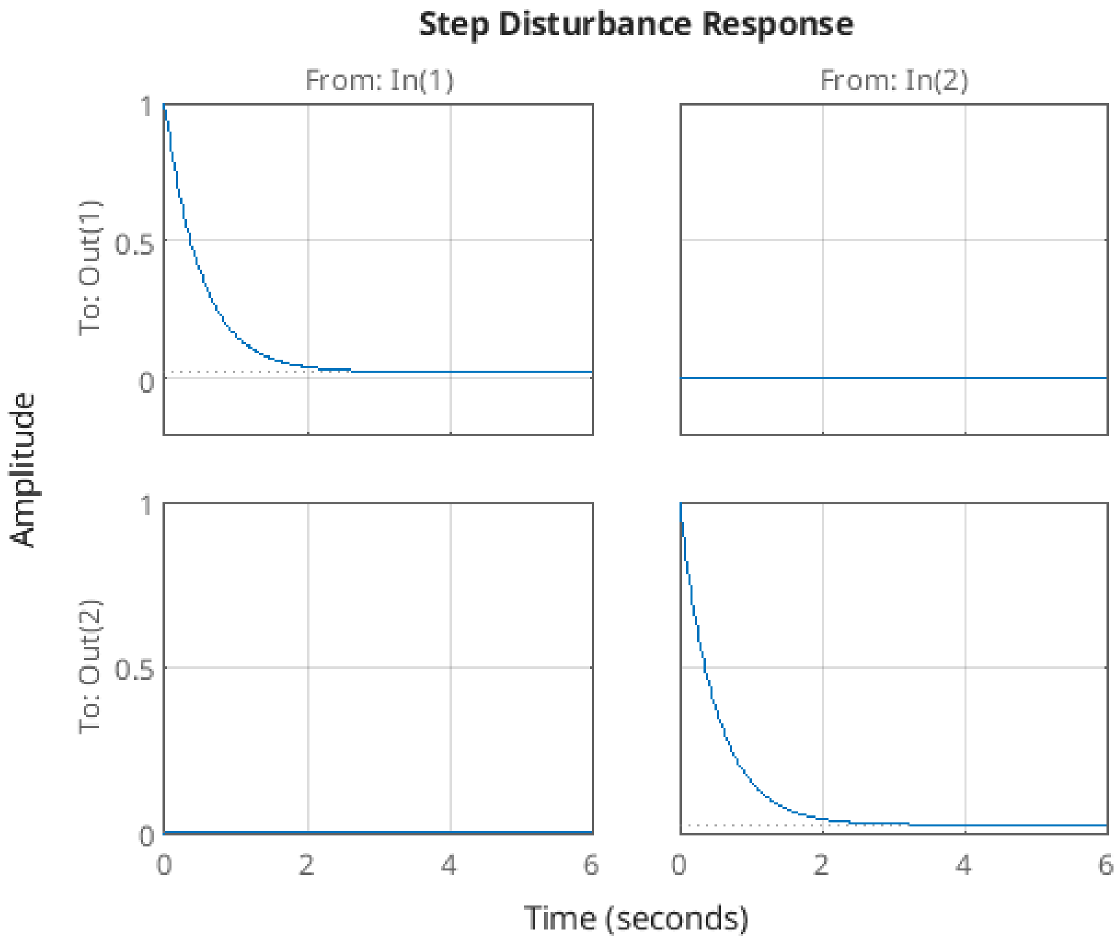




σ(So) vs. |1/Wp(jw)|
σ(So)
σ(Wp)
Singular Values (dB)
Frequency (rad/s)
10
5
0
-5
-10
-15
-20
-25
-30
-35
10^{-2}
10^{0}
10^{2}
10^{4}



Step Input Response
From: In(1)
From: In(2)
Amplitude
To: Out(1)
To: Out(2)
1
0.5
0
1
0.5
0
0
2
4
6
0
2
4
6
Time (seconds)


Step Disturbance Response
From: In(1)
From: In(2)
To: Out(1)
To: Out(2)
Amplitude
Time (seconds)
1
0.5
0
1
0.5
0
0
2
4
6
0
2
4
6