Class: Robust Multivariate Control
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Assignment: Homework 2

1. Find the singular value decompositions (SVDs) of the following:

(a) $A = \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}$

Find SVD

$$
\begin{aligned}
A &= \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \\
A^*A &= \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} \\
A^*A - \sigma^2 I &= \begin{bmatrix} 4-\lambda & -2 & 4 \\ -2 & 1-\lambda & -2 \\ 4 & -2 & 4-\lambda \end{bmatrix} \\
\det(A^*A - \sigma^2 I) &= 0 \\
-\lambda^3 + 9\lambda^2 &= 0 \\
-\lambda^2(\lambda - 9) &= 0 \\
\lambda_{2,3} &= 0, \lambda_1 = 9 \\
(A - \lambda_{2,3} I)v &= 0
\end{aligned}
$$

$$\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_{2,3} = c_1 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -5 & -2 & 4 \\ -2 & -8 & -2 \\ 4 & -2 & -5 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$\sigma_1 = 3, v_1 = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$\sigma_2, \sigma_3 = 0 \quad v_{2,3} = \frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

$$u_1 = \frac{Av_1}{\sigma_1}$$

$$u_1 = \frac{1}{3} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \frac{1}{9} \cdot 9$$

$$u_{2,3} = 0 \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = 0$$

$$A = [1] \begin{bmatrix} 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} & \frac{1}{\sqrt{18}} \\ \frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} & \frac{1}{\sqrt{18}} \end{bmatrix}$$

(b) The $m \times n$ matrix of zeros, $O_{m \times n}$.

For a 0 matrix you are going to get 0 for all the singular values because the determinant of that $A^*A - \lambda I$ is just going to give you n $\lambda = 0$. The V^T matrix will be $n \times n$ with 0s in the first row and then the other rows any vectors orthonormal to the 0 vector, same with the U matrix columns except the U matrix will be $m \times m$ and Σ will be $m \times n$ 0s.

$$O_{mxn} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{mxm} \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{mxn} \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{nxn}$$

Intuitively a 0 matrix scaled by 0 in every direction so all singular values will be 0.

(c) A general $x \neq 0 \in \mathbb{R}^n$ (in terms of x and $\|x\|$)

So the result of the SVD of any vector would be similar to part (a) where the Σ singular value diagonal matrix is just going to be a $1 \times n$ matrix with 1 singular value that will be the norm of the vector (since a vector only extends in one direction it makes sense that there's just one singular value that is the norm in that direction). Then U will just be a scalar 1 since m is just 1 and v_1 will be the normalized vector $\frac{x_i}{\|x\|}$ values transposed and $\frac{A}{\sigma_1}$ will be $\frac{x_i}{\|x\|}$ vector u_i will effectively be xx^T which will always come out to 1 in this case. The remaining V rows will be whatever vectors are orthonormal to v_1 so the whole thing will look something like:

$$x = [1] \begin{bmatrix} \|x\| & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \frac{x_1}{\|\vec{x}\|} & \cdots & \frac{x_{n-1}}{\|\vec{x}\|} & \frac{x_n}{\|\vec{x}\|} \\ v_2 \perp v_1 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ v_n \perp v_1 & \cdots & \cdots & \cdots \end{bmatrix}$$

2. The condition number of a matrix is defined by $cond(G) = \sigma(\bar{G})/\underline{\sigma}(G)$ The condition number of a matrix (or of a transfer function of a plant plotted as a function of frequency) is a measure of the difficulty in inverting the matrix (i.e., controlling the plant). If G and K are square, invertible complex matrices, prove the following using the submultiplicative property $\bar{\sigma}(GK) \leq \bar{\sigma}(G)\bar{\sigma}(K)$ of the matrix 2-norm, $\|\cdot\|_2 = \bar{\sigma}(\cdot)$ and the identity $\bar{\sigma}(G^{-1}) = \frac{1}{\underline{\sigma}(G)}$. (Hint: $GG^{-1} = I$).

(a) $\bar{\sigma}(GK) \leq \bar{\sigma}(KG)cond(G)$

Submultiplicative

$$\begin{aligned} \bar{\sigma}(GK) &\leq \bar{\sigma}(G)\bar{\sigma}(K) \\ \bar{\sigma}(GK) &\leq \bar{\sigma}(G)\bar{\sigma}(KGG^{-1}) \\ \bar{\sigma}(GK) &\leq \frac{\bar{\sigma}(G)\bar{\sigma}(KG)}{\underline{\sigma}(G)} \\ \bar{\sigma}(GK) &\leq \bar{\sigma}(KG)cond(G) \end{aligned}$$

(b) $\underline{\sigma}(GK) \geq \underline{\sigma}(KG)/cond(G)$

$$
\begin{aligned}
&\text{Submultiplicative} \\
\bar{\sigma}(G^{-1}K^{-1}) &\leq \bar{\sigma}(G^{-1})\bar{\sigma}(K^{-1}) \\
\frac{1}{\underline{\sigma}(GK)} &\leq \frac{1}{\underline{\sigma}(G)\underline{\sigma}(K)} \\
\underline{\sigma}((G)\underline{\sigma}(K)) &\leq \underline{\sigma}(GK) \\
\underline{\sigma}((G)\underline{\sigma}(KGG^{-1})) &\leq \underline{\sigma}(GK) \\
\frac{\underline{\sigma}(G)}{\bar{\sigma}(G)}\underline{\sigma}(KG) &\leq \underline{\sigma}(GK) \\
\frac{\underline{\sigma}(KG)}{cond(G)} &\leq \underline{\sigma}(GK)
\end{aligned}
$$