Class: Robust Multivariate Control
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TAs:
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Assignment: Homework 1

1. Analytically compute the 2-norms of the following time domain signals on the specified in- terval, if they exist.

The equation for the 2-norm of a continuous function over an interval is:
$\|f\|_2 = \sqrt{\int_a^b |f(t)|^2 \, dt}$

(a) f(t) = e^{-3t}, $[0, \infty)$

$$f(t) = e^{-3t}, \quad t \in [0, \infty)$$

$$\|f\|_2 = \sqrt{\int_0^\infty e^{-6t} \, dt}$$

$$\|f\|_2 = \sqrt{\left[-\frac{1}{6} e^{-6t} \right]_0^\infty}$$

$$\|f\|_2 = \sqrt{0 + \frac{1}{6}}$$

$$\|f\|_2 = \sqrt{\frac{1}{6}}$$

(b) f(t) = sin t, [0, ∞)

$$f(t) = \sin(t), \quad t \in [0, \infty)$$

$$\|f\|_2 = \sqrt{\int_0^\infty \sin^2(t)\, dt}$$

$$\|f\|_2 = \sqrt{\left[\frac{t}{2} - \frac{1}{4}\sin 2t\right]_0^\infty}$$

$\sin(\infty)$ does not converge so the 2-norm does not exist.

(c) f(t) = $\begin{bmatrix} e^{-t} \\ 1 \end{bmatrix}$, [0, T]

$$f(t) = \begin{bmatrix} e^{-t} \\ 1 \end{bmatrix}$$

$$\|f\|_2 = \begin{bmatrix} \sqrt{\int_0^T e^{-2t} dt} \\ \sqrt{\int_0^T 1 dt} \end{bmatrix}$$

$$\|f\|_2 = \begin{bmatrix} \sqrt{\left[-\frac{1}{2}e^{-2t}\right]_0^T} \\ \sqrt{[T]_0^T} \end{bmatrix}$$

$$\|f\|_2 = \begin{bmatrix} \sqrt{-\frac{1}{2}e^{-2t} - \frac{1}{2}} \\ \sqrt{T} \end{bmatrix}$$

$$\|f\|_2 = \sqrt{\frac{1}{2} - \frac{1}{2}e^{-2T} + T}$$

2. Analytically compute the 2-norms of the following frequency domain signals, if they exist. (Hint: Parseval's identity)

Parseval's identity:

$$\|\hat{f}\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^\infty \left|\hat{f}(j\omega)\right|^2 d\omega}$$

(a) $\hat{f}(j\omega) = \frac{1}{j\omega+a}, \quad a > 0$

$$\hat{f}(j\omega) = \frac{1}{j\omega + a}, \quad a > 0$$

$$\|\hat{f}(j\omega)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{1}{j\omega + a}\right|^2 d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\omega^2 + a^2}\, d\omega}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a}\arctan\left(\frac{\omega}{a}\right)\right]_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{\pi}{a}\right)$$

$$= \sqrt{\frac{1}{2a}}$$

(b) $\hat{f}(s) = \frac{1}{(s+a)^2}, \quad a > 0$

$$hatf(s) = \frac{1}{(s + a)^2}, \quad a > 0$$

$$\|\hat{f}(j\omega)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{1}{(j\omega + a)^2}\right|^2 d\omega}$$

$$= \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{(\omega^2 + a^2)^2}\, d\omega}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{\pi}{2a^3}\right]$$

$$= \sqrt{\frac{1}{4a^3}}$$

(c) $\hat{f}(s) = \begin{bmatrix} \frac{1}{s+a} \\ \frac{1}{s+b} \end{bmatrix}, \quad a > 0,\, b > 0$

$$\hat{f}(s) = \begin{bmatrix} \frac{1}{s+a} \\ \frac{1}{s+b} \end{bmatrix}$$

$$\|\hat{f}(j\omega)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\left|\frac{1}{j\omega + a}\right|^2 + \left|\frac{1}{j\omega + b}\right|^2\right) d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{\omega^2 + a^2} + \frac{1}{\omega^2 + b^2} \right) d\omega$$

$$= \frac{1}{2\pi} \left(\frac{\pi}{a} + \frac{\pi}{b} \right)$$

$$= \frac{1}{2a} + \frac{1}{2b}$$

3. If a scalar frequency domain signal $\hat{g}(s)$ is stable and strictly proper, its corresponding signal $g(t)$ in the time domain can be bounded by an exponential of the form ce^{-at}, $a > 0$ for all $t \geq 0$. Use this fact to show that its 1-norm in the time domain

$$\|g\|_1 = \int_{-\infty}^{\infty} |g(t)|\, dt$$

is bounded, i.e., $\|g\|_1 < \infty$

$$\|g\|_1 = \int_0^{\infty} |g(t)|\, dt$$

$$\leq \int_0^{\infty} ce^{-at}\, dt$$

$$= \left[-\frac{c}{a} e^{-at} \right]_0^{\infty}$$

$$= 0 + \frac{c}{a}$$

$$= \frac{c}{a}$$

$\frac{c}{a}$ is finite for any positive c and a.

$\therefore \|g\|_1 < \infty,$ and g can be bounded.