Class: Robust Multivariate Control
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Assignment: Homework 4

1. H_∞ State Feedback Control

Consider a general LTI system of the form $\dot{x} = Ax + B_2u + B_1w$, $z = C_1x$ where

$$A = \begin{pmatrix} -5 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.5 \\ 0 \\ 0.3 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

(a)

Implement the H_∞ LMI synthesis feasibility problem for a given attenuation level γ as a function and design a full state ($y = x$) feedback control law $u = Kx$ such that the closed loop system is stable and the transfer function matrix satisfies

$$\|G_{zw}\|_\infty < 0.5$$

I used the following LMI we were given in class derived from the Bounded Real Lemma:

$$\begin{bmatrix} (AX + B_2W)^T + AX + B_2W & B_1 & (C_1X + D_{12}W)^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1X + D_{12}W & D_11 & -\gamma I \end{bmatrix} < 0$$

D being 0 in this case I put together the LMI in Matlab like so:

```
A = [-5   1   0;     % 3x3 matrix A
      0   1   1;
      1   1   1];

B2 = [0   0;         % 3x2 matrix B2
      0   1;
      1   0];
```

```
B1 = [0.5;          % 3x1 matrix B1
        0;
      0.3];

C1 = [1  0  0;     % 2x3 matrix C1
      0  2  1];

nStates = size(A, 1);
nInputs = size(B2, 2);
nOutputs = size(C1, 1);

setlmis([])

X = lmivar(1, [nStates 1]);
W = lmivar(2, [nInputs nStates]);
gamma = 0.5;

lmiterm([-1 1 1 X],1,1);
lmiterm([2 1 1 X],A,1,'s');
lmiterm([2 1 1 W],B2,1,'s');
lmiterm([2 1 2 0],B1);
lmiterm([2 1 3 X],1,C1');
lmiterm([2 2 2 0],-gamma);
lmiterm([2 2 3 0],0);
lmiterm([2 3 3 0],-gamma);

lmiDstab = getlmis;

[tmin, xfeas] = feasp(lmiDstab);
X = dec2mat(lmiDstab,xfeas,X);
W = dec2mat(lmiDstab,xfeas,W);
```

(b)

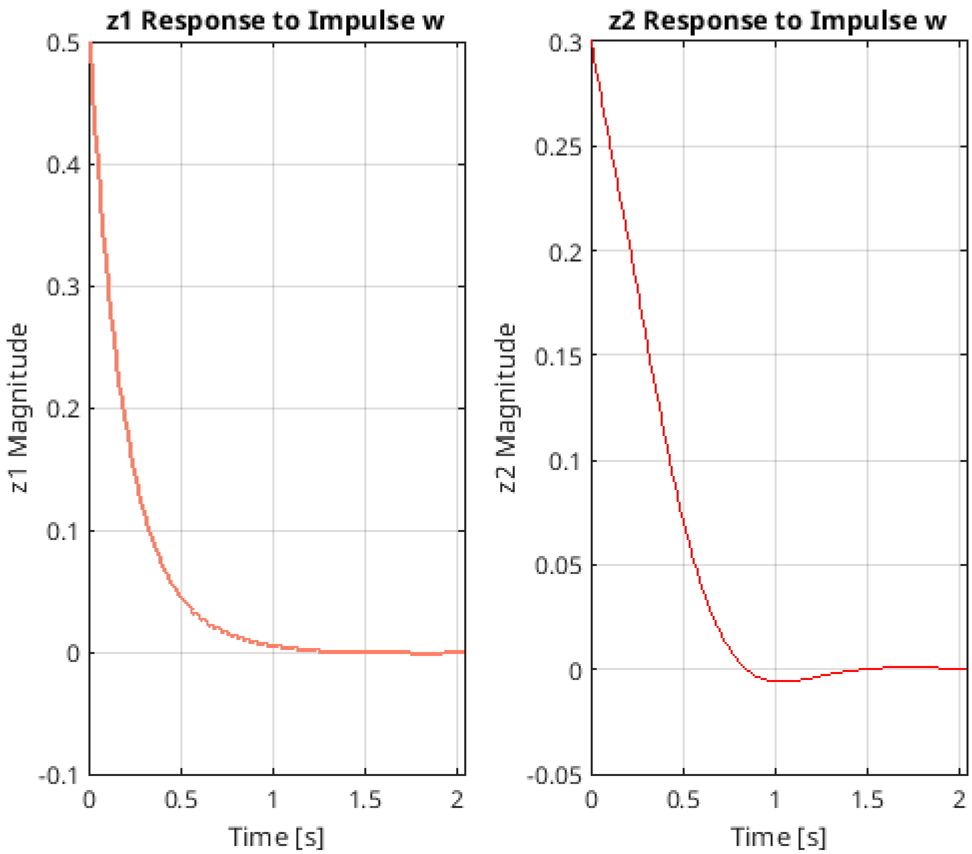
Plot the regulated output responses $z(t)$ to an impulse in w (use the ***subfigure*** *command and include one signal per plot).*

I derived K from the resulting W and X matrices and then used A+BK to create the closed loop state space model with the 'ss' function. Then used the 'impulse' function to generate and plot the output response.

```
K = W*inv(X);

hinfSys = ss(A+B2*K, B1, C1, 0);

[z, tOut, x] = impulse(hinfSys);

figure;

subplot(1,2,1);
plot(tOut, z(:,1), 'LineWidth', 1.5, 'Color', [255/255 127/255 102/255]);
grid on;
```

```
title('z1 Response to Impulse w');
xlabel('Time [s]');
ylabel('z1 Magnitude');

subplot(1,2,2);
plot(tOut, z(:,2), 'r-');
grid on;
title('z2 Response to Impulse w');
xlabel('Time [s]');
ylabel('z2 Magnitude');
```

The results are plotted below:



(c)

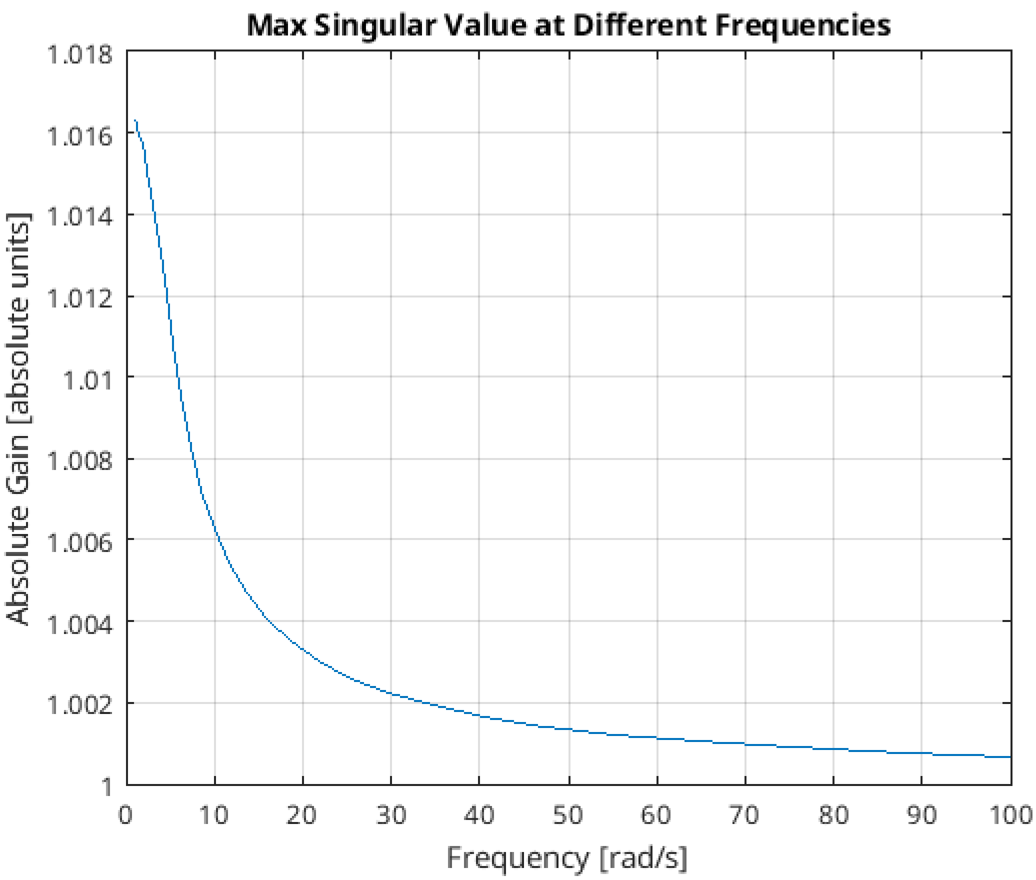
Generate a max singular value plot of the closed loop system $G_{zw}(s)$ using the command **`sigma`**. *Does this system amplify (absolute gain > 1) or attenuate (absolute gain < 1) disturbances? Note that the* **`sigma`** *plot generates the gain in dB.*

Here is my code following the procedure detailed above, I converted from dB to absolute units in the plot function.

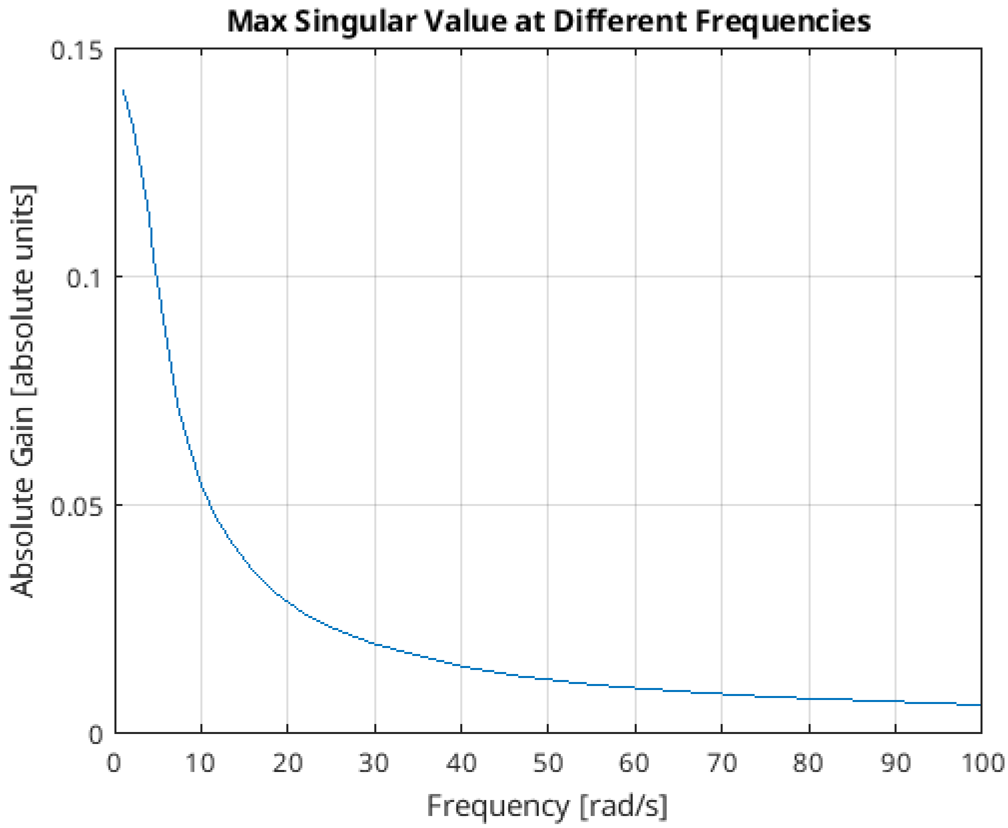
```
[sv,wout] = sigma(hinfSys);

```

```
figure;
plot(wout, 10.^(sv/20));
grid on;
title('Max Singular Value at Different Frequencies');
xlabel('Frequency [rad/s]');
ylabel('Absolute Gain [absolute units]');
```

These were the results and as you can see the absolute gain is slightly higher than 1 meaning the system amplifies disturbances which does not make a lot of sense to me given the previous results.



I wonder if 'sigma' is printing the singular values in decibels because when I use 'hinfnorm' on the system I get 0.1431 and when I plot the singular values without converting I get:



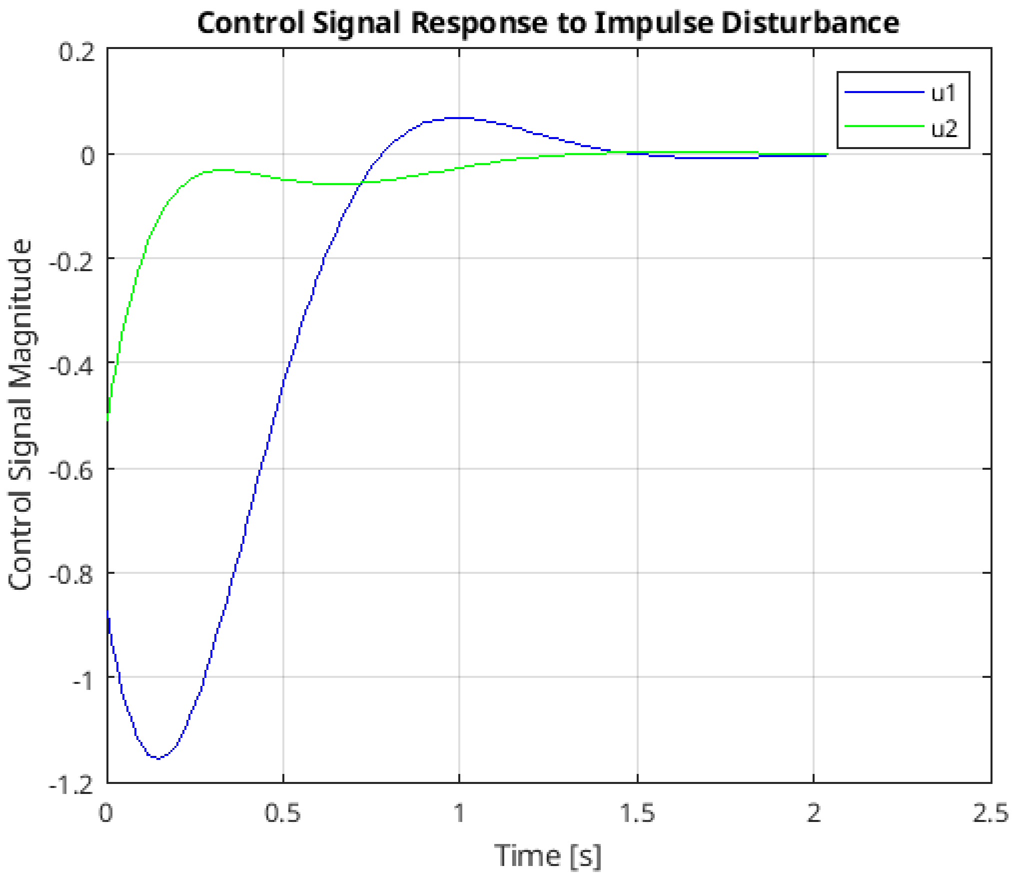
(d)

Plot the two control signals $u_1(t)$ and $u_2(t)$ and compute their L_2 norms. Which channel requires more input energy?

I calculated u using K and the x output from the 'impulse' function:

```
u = K*x';

figure;
plot(tOut, u(1,:), 'r-', tOut, u(2,:), 'y-');
legend('u1', 'u2');
grid on;
title('Control Signal Response to Impulse Disturbance');
xlabel('Time [s]');
ylabel('Control Signal Magnitude');

disp("u1 L2 Norm: ")
disp(norm(u(1,:),2));

disp("u2 L2 Norm: ")
disp(norm(u(2,:),2));
```

The plot looks like this:



I computed the L_2 norm using Matlab's 'norm' function and got 5.6361 for u1 and 1.0919 for u2 which makes sense looking at the plot there's definitely more area under the curve for u1 and I think this shows that this input channel requires a stronger response to deal with that disturbance.

2. H_2 State Feedback Control

In this problem, you will implement the H_2 LMI synthesis approach and design a feedback controller for the DC-8 example that was presented in class the previous week in the H_∞ state feedback example. The MATLAB code to set up the system matrices will be provided.

(a)

Implement the H_2 LMI conditions derived in lecture for the feasibility problem given a prescribed attenuation level $\|G(s)\|_2 < \gamma$ as a function using the LMI Toolbox in MATLAB. Design γ for the H_2 controller to match the control energy (L_2 norm) for an equivalent H_∞ controller with attenuation level $\gamma = 2.5$.

I first put the system into the previous Matlab code for the H_∞ controller using attenuation level 2.5 and got 6.8823 for the control signal L2 norm (0.1416 H_∞ Norm). Then

I created the following H_2 synthesis in Matlab to generate the H_2 controller and get the control energy for that.

```
A = [-0.0869 0 0.039 -1 ; % x1 = delta beta (rad)
      -4.424 -1.184 0 0.335; % x2 = delta p (rad/s)
       0 1 0 0 ; % x3 = delta phi (rad)
       2.148 -0.021 0 -0.228]; % x4 = delta r (rad/s)
B1 = [0 0 0 0.288]'; % w = r_g (yaw rate gust, rad/s)
B2 = [0.0223 0.547 0 -1.169]'; % u1 = delta_r
C1 = eye(4); % C = I
D11 = zeros(4,1);
D12 = [0 0 0 0]';

nStates = size(A, 1);
nInputs = size(B2, 2);
nOutputs = size(C1, 1);

setlmis([])

X = lmivar(1, [nStates 1]);
W = lmivar(2, [nInputs nStates]);
Z = lmivar(1, [nStates 1]);
gamma = 1.14;

lmiterm([-1 1 1 X],1,1);

lmiterm([2 1 1 X],A,1,'s');
lmiterm([2 1 1 W],B2,1,'s');
lmiterm([2 1 1 0],B1*B1');

lmiterm([3 1 1 Z],-1,1);
lmiterm([3 1 2 X],C1,1);
lmiterm([3 1 2 W],D12,1);
lmiterm([3 2 2 X],-1,1);

lmiterm([4 1 1 Z],[1 0 0 0], [1 0 0 0]');
lmiterm([4 1 1 Z],[0 1 0 0], [0 1 0 0]');
lmiterm([4 1 1 Z],[0 0 1 0], [0 0 1 0]');
lmiterm([4 1 1 Z],[0 0 0 1], [0 0 0 1]');
lmiterm([-4 1 1 0],gamma);

h2lmi = getlmis;

[tmin, xfeas] = feasp(h2lmi);
X = dec2mat(h2lmi,xfeas,X);
W = dec2mat(h2lmi,xfeas,W);
Z = dec2mat(h2lmi,xfeas,Z);

K = W/X;

h2sys = ss(A+B2*K,B1,C1,D11);

[z, tOut, x] = impulse(h2sys);

```

```
u = K*x';

disp("u1 L2 Norm: ")
disp(norm(u(1,:),2));
```

I then played with the gamma level starting at 2.5 and going up and down until the L2 norm energy was about the same as it was for H_∞ which ended up having an attenuation level of 1.14 with a resulting 6.8938 control energy and 0.1644 H_2 norm.

(b)

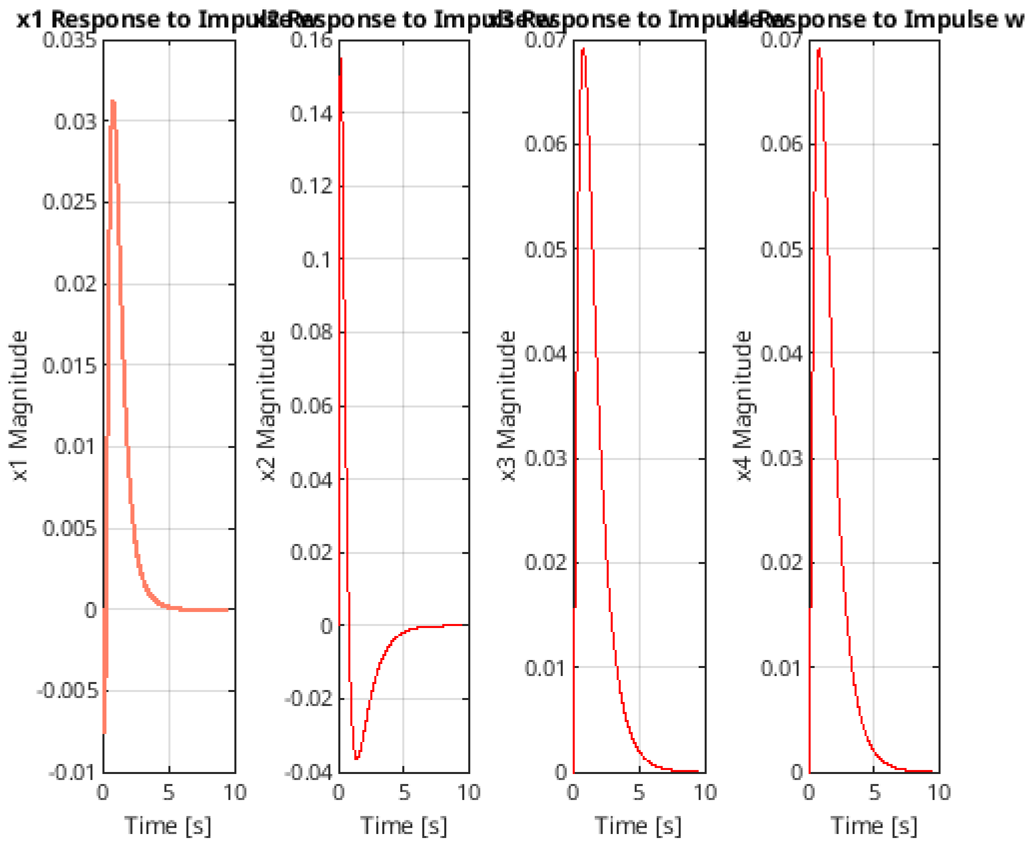
Generate the resulting singular value bode plots of the closed loop systems and the responses of the state x due to an impulse in the yaw disturbance r_g for both the H_2 and H_∞ controllers and compare both sets of plots.

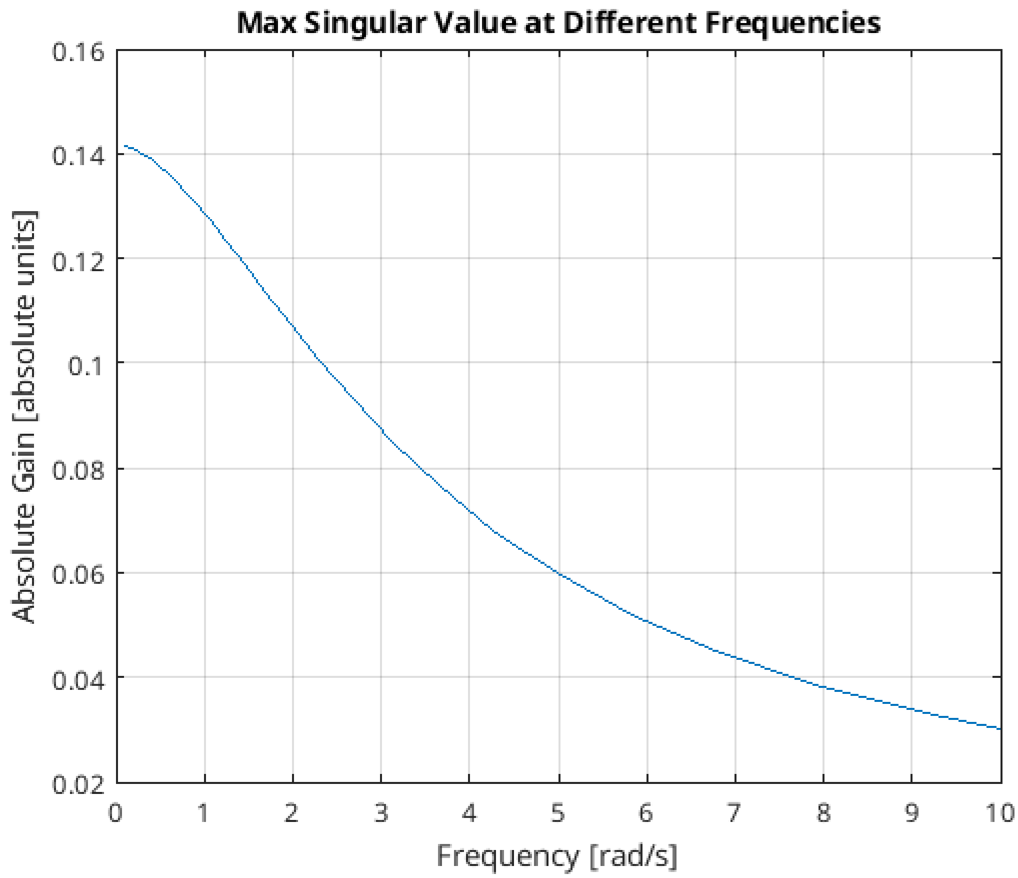
I took that x impulse function output at plotted the different state responses.

```
[z, tOut, x] = impulse(h2sys);

subplot(1,4,1);
plot(tOut, x(:,1), 'LineWidth', 1.5, 'Color', [255/255 127/255 102/255]);
grid on;
title('x1 Response to Impulse w');
xlabel('Time [s]');
ylabel('x1 Magnitude');
```

And then used the same sigma function and plotting as from before and the results follow.

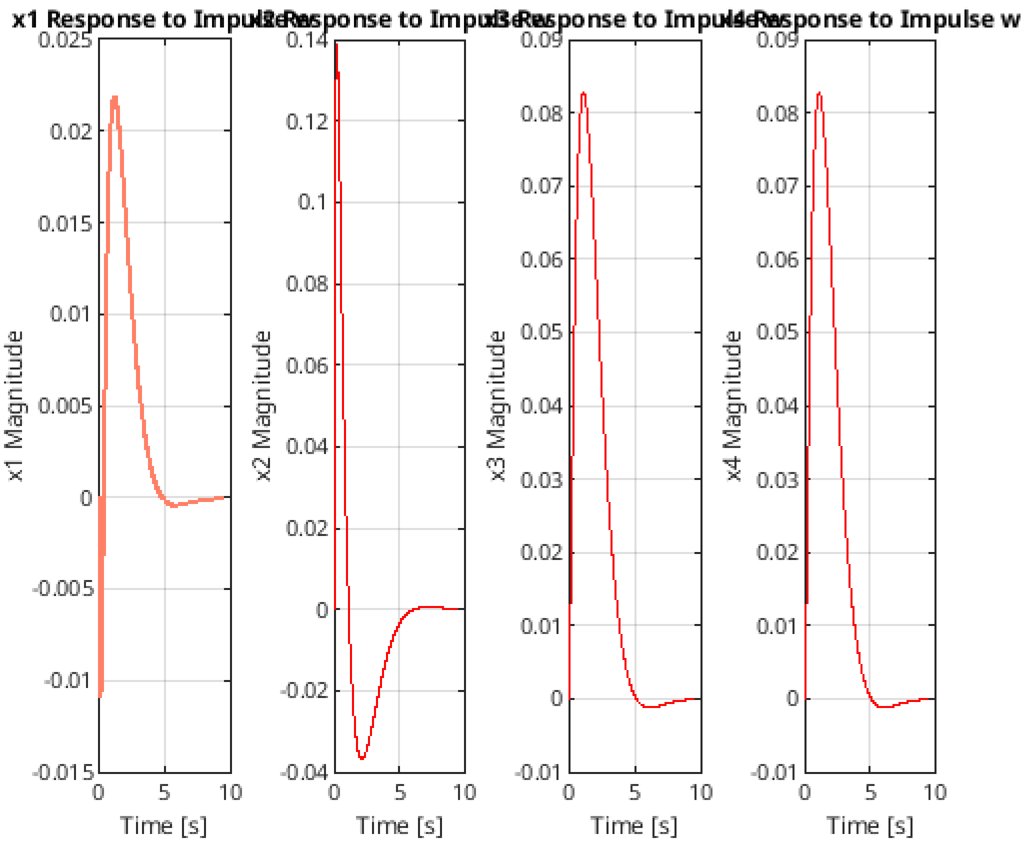
For H_∞:

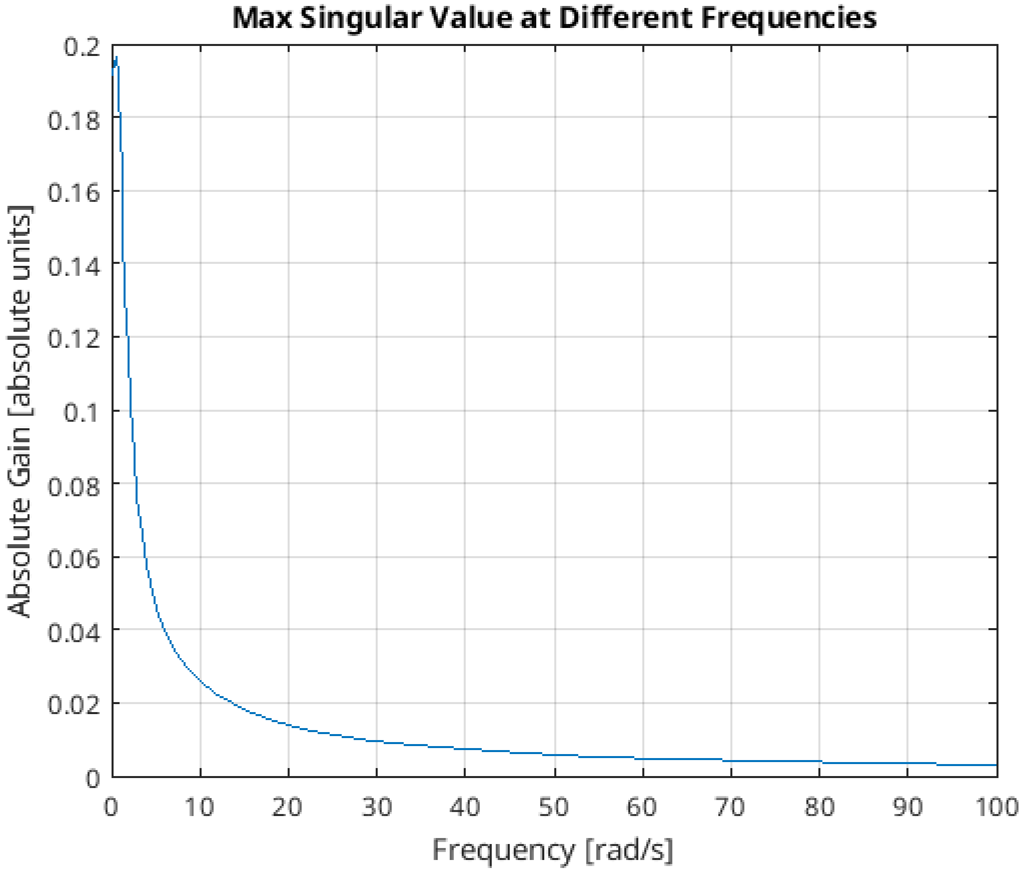

x1 Response to Impulse w
x2 Response to Impulse w
x3 Response to Impulse w
x4 Response to Impulse w
x1 Magnitude
x2 Magnitude
x3 Magnitude
x4 Magnitude
Time [s]
Time [s]
Time [s]
Time [s]


Max Singular Value at Different Frequencies
Absolute Gain [absolute units]
Frequency [rad/s]
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
1
2
3
4
5
6
7
8
9
10

.

And for H_2:


x1 Response to Impulse w
x2 Response to Impulse w
x3 Response to Impulse w
x4 Response to Impulse w
x1 Magnitude
x2 Magnitude
x3 Magnitude
x4 Magnitude
Time [s]
Time [s]
Time [s]
Time [s]



You can see the max singular values are a bit higher for the H_2 which makes sense given the explicit purpose of H_∞ is to minimize that value. It also seems that the state responses are a sharper I would say for H_∞, they seem to happen faster.

(c)

For a white noise yaw disturbance r_g of unit intensity, compute the average power in the states for each of the closed loop systems and compare.

I used the 'lyap' function to derive the covariance matrix for each system using $B_1 B_1'$ for unity intensity white noise and the trace of P to get the energy.

```
disp("Average Power in States: ");
Ph2 = lyap(A+B2*K, B1 * B1');
averagePower = trace(Ph2);
disp(averagePower);
```

The result for H_2 was 0.0270 and the result for H_∞ was 0.0224 which is close but does seem to verify the earlier observation about the state change.

3. H_∞ Observer Design

Consider the standard general LTI system with $D_{11} = D_{12} = 0$,

$$\dot{x} = Ax + B_2u + B_1w$$

$$z = C_1x$$

$$y = C_2x + D_{21}w + D_{22}u$$

with state x, measured output y, control input u, disturbance w, and the regulated outputs z. For this system, we introduce a full-order state observer in the following form:

$$\dot{\hat{x}} = A\hat{x} + B_2u + L(C_2\hat{x} + D_{22}u - y)$$

where $\hat{x}$ is the state observation and L is the observer gain. The estimate of the regulated outputs is given by $\hat{z} = C_1\hat{x}$, which is desired to have as small as possible influence from the disturbance w.

(a)

Show that for $e(t) = x(t) - \hat{x}(t)$ and $\tilde{z}(t) = z(t) - \hat{z}(t)$, the resulting observation error equation is given by

$$\dot{e} = (A + LC_2)e + (B_1 + LD_{21})w$$

$$\tilde{z} = C_1e$$

and has closed loop transfer function $G_{\tilde{z}w} = C_1[sI - (A + LC_2)]^{-1}(B_1 + LD_{21})$ from $w \to \tilde{z}$.

Substitute $y = C_2x + D_{21}w$ into the observer equation:

$$\dot{\hat{x}} = A\hat{x} + B_2u + L(C_2\hat{x} + D_{21}w - C_2x - D_{21}w) = A\hat{x} + B_2u + L(C_2\hat{x} - C_2x).$$

Since $D_{21} = 0$ (as specified), this simplifies to:

$$\dot{\hat{x}} = A\hat{x} + B_2u + LC_2(\hat{x} - x).$$

The error dynamics are:

$$\dot{e} = \dot{x} - \dot{\hat{x}}.$$

Substitute $\dot{x} = Ax + B_2u + B_1w$ and the observer equation:

$$\dot{e} = (Ax + B_2u + B_1w) - (A\hat{x} + B_2u + LC_2(\hat{x} - x)).$$

$$\dot{e} = Ax + B_1w - A\hat{x} - LC_2(\hat{x} - x).$$

$$\dot{e} = A(x - \hat{x}) + B_1w - LC_2(x - \hat{x}).$$

$$\dot{e} = (A - LC_2)e + B_1w.$$

Given $D_{21} = 0$, the term $LD_{21}w = 0$, so:

$$\dot{e} = (A + LC_2)e + B_1w.$$

Regulated Output Error $\tilde{z}$

$$z = C_1 x, \quad \hat{z} = C_1 \hat{x}.$$

$$\tilde{z} = z - \hat{z} = C_1 x - C_1 \hat{x} = C_1(x - \hat{x}) = C_1 e.$$

Thus, $\tilde{z} = C_1 e$.

Closed-Loop Transfer Function $G_{\tilde{z}w}$

The error dynamics in the Laplace domain (with zero initial conditions) are:

$$sE(s) = (A + LC_2)E(s) + B_1 W(s).$$

$$E(s) = [sI - (A + LC_2)]^{-1} B_1 W(s).$$

Then:

$$\tilde{Z}(s) = C_1 E(s) = C_1 [sI - (A + LC_2)]^{-1} B_1 W(s).$$

Thus, the transfer function from w to $\tilde{z}$ is:

$$G_{\tilde{z}w}(s) = C_1 [sI - (A + LC_2)]^{-1} B_1.$$

With $D_{21} = 0$, this matches $G_{\tilde{z}w} = C_1[sI - (A + LC_2)]^{-1}(B_1 + LD_{21})$.

(b)

Consider the following problem: Given the system in (a) and a positive scalar γ, find a matrix L such that $\|G_{\tilde{z}w}\|_\infty < \gamma$. Prove that this problem has a solution if and only if there exists a matrix W and a positive definite matrix $P > 0$ and the constraint

$$\begin{pmatrix} A^T P + PA + C_2^T W C_2 & PB_1 + WD_{21} & C_1^T \\ (PB_1 + WD_{21})^T & -\gamma I & 0 \\ C_1 & 0 & -\gamma I \end{pmatrix} < 0$$

holds. When such a pair of matrices W and P are found, the solution to the problem is given as $L = P^{-1} W$.

Proof

Problem Statement

We need to design L such that the H_∞ norm of the transfer function $G_{\tilde{z}w}(s) = C_1[sI - (A + LC_2)]^{-1}(B_1 + LD_{21})$ is less than γ, i.e., $||G_{\tilde{z}w}||_\infty < \gamma$. This means the worst-case gain from disturbance w to estimation error output $\tilde{z}$ is bounded by γ.

H_∞ Norm Condition

For the LTI system $\dot{e} = (A + LC_2)e + (B_1 + LD_{21})w$, $\tilde{z} = C_1 e$, the H_∞ norm condition $||G_{\tilde{z}w}||_\infty < \gamma$ is equivalent to the existence of a positive definite matrix $P > 0$ satisfying the Bounded Real Lemma:

$$(A + LC_2)^T P + P(A + LC_2) + C_1^T C_1 + \frac{1}{\gamma^2} P(B_1 + LD_{21})(B_1 + LD_{21})^T P < 0.$$

This ensures stability and the H_∞ performance bound.

LMI Formulation

To solve for L, we use the given LMI constraint:

$$\begin{pmatrix} A^T P + PA + C_1^T C_1 + WC_2 + C_2^T W^T & PB_1 + WD_{21} & C_1^T \\ (B_1^T P + D_{21}^T W^T) & -\gamma I & 0 \\ C_1 & 0 & -\gamma I \end{pmatrix} < 0,$$

where $W = PL$ (since $L = P^{-1}W$), and $P > 0$ is positive definite. This LMI ensures that the closed-loop system $A + LC_2$ is stable and $||G_{\tilde{z}w}||_\infty < \gamma$.

Sufficiency

If there exist $P > 0$ and W satisfying the LMI, then $L = P^{-1}W$ ensures the closed-loop matrix $A+LC_2$ is Hurwitz (stable) and the H_∞ norm condition holds. Substituting $W = PL$ into the LMI reduces it to the standard H_∞ performance condition for the error system, proving that $||G_{\tilde{z}w}||_\infty < \gamma$.

Necessity

If $||G_{\tilde{z}w}||_\infty < \gamma$ for some L, then there exists a $P > 0$ satisfying the Bounded Real Lemma for the error system. By defining $W = PL$, the LMI must hold, as it is a reformulation of the H_∞ condition in terms of P and W. Thus, the LMI is both necessary and sufficient for the existence of L satisfying the H_∞ bound.

Solution

When $P > 0$ and W are found by solving the LMI (e.g., using MATLAB's LMI Toolbox), the observer gain is:

$$L = P^{-1}W.$$

This L ensures the observer achieves the desired H_∞ performance.

(c)

Design an H_∞ observer according to the above approach such that $\|G_{\tilde{z}w}\|_\infty < 0.1$ for the system in Problem 1. Assume that

$$C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D_{21} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

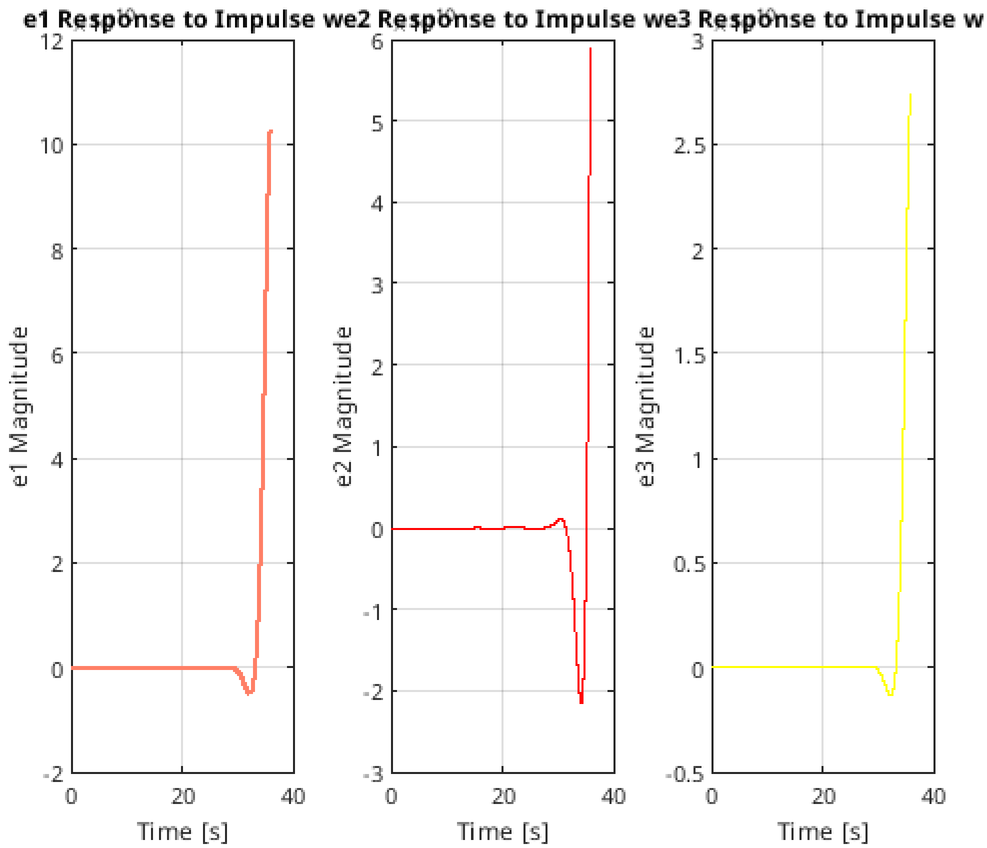
Provide plots of the error states $e_1(t)$, $e_2(t)$, and $e_3(t)$ for 2 seconds for an impulse input in w.

This is the code I used to generate the H_∞ Observer system and to plot the impulse response.

```
A = [-5   1   0;     % 3x3 matrix A
        0   1   1;
        1   1   1];

B2 = [0   0;         % 3x2 matrix B2
        0   1;
        1   0];

B1 = [0.5;            % 3x1 matrix B1
        0;
        0.3];

C1 = [1   0   0;     % 2x3 matrix C1
        0   2   1];

C2 = [1 0 0;
        0 0 1];

D21 = [0; 0];

nStates = size(A, 1);
nInputs = size(B2, 2);
nOutputs = size(C1, 1);

setlmis([])

P = lmivar(2, [nStates nStates]);
W = lmivar(2, [nStates nInputs]);
gamma = 0.1;

lmiterm([-1 1 1 P],1,1);

lmiterm([2 1 1 P],1,A,'s');
lmiterm([2 1 1 W],1, C2, 's');
lmiterm([2 1 2 P],1,B1);
lmiterm([2 1 3 0],C1');
lmiterm([2 2 2 0],-gamma);
lmiterm([2 2 3 0],0);
lmiterm([2 3 3 0],-gamma);

lmiH2obs = getlmis;
```

```
[tmin, xfeas] = feasp(lmiH2obs);
P = dec2mat(lmiH2obs,xfeas,P);
W = dec2mat(lmiH2obs,xfeas,W);

L = inv(P)*W;

hinfObsSys = ss(A+L*C2, B1, C1, 0);

[z, tOut, x] = impulse(hinfObsSys);

figure;

subplot(1,3,1);
plot(tOut, x(:,1), 'LineWidth', 1.5, 'Color', [255/255 127/255 102/255]);
grid on;
title('x1 Response to Impulse w');
xlabel('Time [s]');
ylabel('x1 Magnitude');

subplot(1,3,2);
plot(tOut, x(:,2), 'r-');
grid on;
title('x2 Response to Impulse w');
xlabel('Time [s]');
ylabel('x2 Magnitude');

subplot(1,3,3);
plot(tOut, x(:,3), 'y-');
grid on;
title('x3 Response to Impulse w');
xlabel('Time [s]');
ylabel('x3 Magnitude');
```

And below are the results:


e1 Response to Impulse w
e2 Response to Impulse w
e3 Response to Impulse w
e1 Magnitude
e2 Magnitude
e3 Magnitude
Time [s]
Time [s]
Time [s]