SIGNAL SPACES & NORMS

WE DEFINE A SIGNAL AS AN ELEMENT OF AN APPROPRIATELY DEFINED SET $S = \{ f : \mathbb{R} \to \mathbb{R}^n , \; P \}$ WHERE "P" ARE ADDITIONAL PROPERTIES, SPECIFICATIONS OR CONSTRAINTS

TIME DOMAIN SIGNAL SPACES

EX VECTOR-VALUED LEBESGUE 2-SPACE ON $\mathbb{R} = (-\infty, \infty)$

$$L_2^n(-\infty, \infty) = \Big\{ f : (-\infty, \infty) \to \mathbb{R}^n , \; \underbrace{\int_{-\infty}^{\infty} \|f(t)\|^2 \, dt}_{\text{SIGNAL ENERGY}} < \infty \Big\}$$

$$f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

$\| \cdot \|$ IS THE EUCLIDEAN NORM,

$$\|f(t)\| = (f^T f)^{1/2} = (\langle f, f \rangle)^{1/2}$$

(HOLD t CONSTANT)

INTERPRETATION:

SQUARE OF A SIGNAL $\longrightarrow$ INSTANTANEOUS POWER $\|f(t)\|^2$

INTEGRAL OF POWER $\longrightarrow$ ENERGY

L_2^n ARE SIGNALS WITH FINITE / BOUNDED ENERGY

ALSO, L_2^n IS AN INNER PRODUCT SPACE,

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f^T(t)\, g(t)\, dt$$

INDUCED NORM: $\|f(t)\|_{L_2} = \left(\langle f(t), f(t) \rangle \right)^{1/2}$

USE "2" INSTEAD

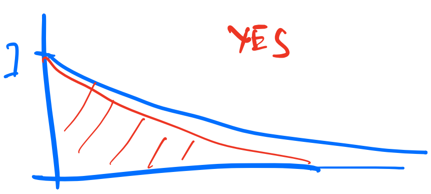
$$= \left(\int_{-\infty}^{\infty} f^T(t)\, f(t)\, dt \right)^{1/2}$$

$$= \left(\int_{-\infty}^{\infty} \|f(t)\|^2\, dt \right)^{1/2}$$

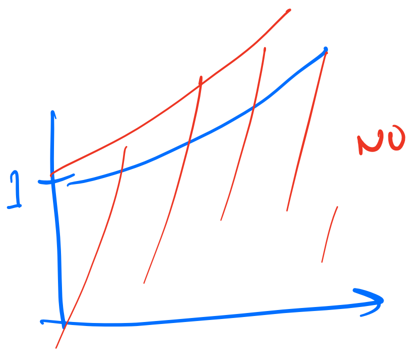
HENCE THE L_2 NORM IS THE SQUARE ROOT OF THE SIGNAL'S ENERGY

EXAMPLES

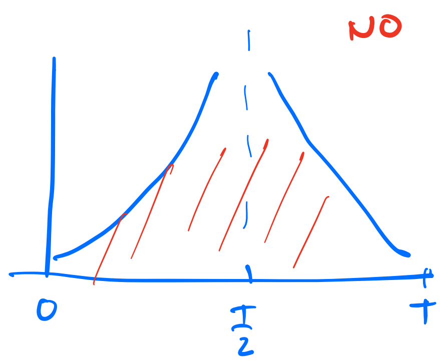
$f(t) = e^{-t}$



$f(t) = e^{t}$



$f(t) = \dfrac{1}{|2T-1|}$



CAN ALSO DEFINE L_2 SPACES WITH A FINITE TIME HORIZON

$$L_2^n[0,T] = \left\{ f : [0,T] \to \mathbb{R}^n,\ \int_0^T \|f(t)\|^2\, dt < \infty \right\}$$

SIGNALS THAT ARE BOUNDED ON FINITE TIME INTERVALS

OTHER TIME DOMAIN SIGNAL NORMS

	2 - NORM	∞ - NORM
CONSTANT VECTOR $z \in \mathbb{R}^n$	$\|z\| = \sqrt{z^T z} = \sqrt{\sum_{i=1}^{n} \|z_i\|^2}$	$\|z\|_\infty = \max_i \|z_i\|$
SCALAR SIGNAL $f \in L_2(-\infty,\infty)$	$\|f(t)\|_2 = \left(\int_{-\infty}^{\infty} \|f(t)\|^2 dt \right)^{1/2}$	$\|f(t)\|_\infty = \max_t \|f(t)\|$
VECTOR SIGNAL $f \in L_2^n(-\infty,\infty)$	$\|f(t)\|_2 = \left(\int_{-\infty}^{\infty} \|f(t)\|^2 dt \right)^{1/2} = \left(\int_{-\infty}^{\infty} f(t)^T f(t)\, dt \right)^{1/2}$	$\|f(t)\|_\infty = \left\| \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} \right\|_\infty = \max_i \|f_i(t)\|_\infty$
	AVERAGE	PEAK

FREQUENCY DOMAIN SIGNAL SPACES

A FREQUENCY DOMAIN SIGNAL IS A FUNCTION

$$\hat{f}(j\omega) : \underbrace{(-j\infty, j\infty)}_{j\mathbb{R}} \rightarrow \mathbb{C}^n$$

$j\mathbb{R} \leftarrow$ IMAG AXIS

WITH THE PROPERTY

$$\hat{f}^*(j\omega) = \hat{f}^T(-j\omega)$$

ω: REAL FREQ (RAD/S)

$(\cdot)^*$: HERMITIAN (COMPLEX CONJUGATE TRANSPOSE)

EX:

$$\hat{f}(j\omega) = \begin{pmatrix} \dfrac{1}{j\omega + a} \\ \dfrac{j\omega}{(j\omega)^2 + b^2} \end{pmatrix}$$

ONE FREQ DOMAIN SIGNAL SPACE OF INTEREST IS

$$\hat{L}_2^n(j\mathbb{R}) = \left\{ \hat{f} : j\mathbb{R} \rightarrow \mathbb{C}^n, \ \|\hat{f}\|_2 < \infty \right\}$$

FREQ DOMAIN SIGNALS WITH FINITE 2-NORM;

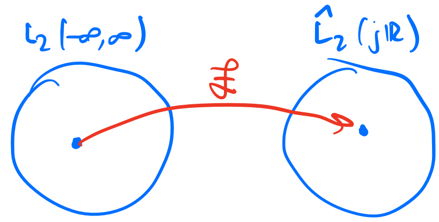
$$\|\hat{f}\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}^*(j\omega)\, \hat{f}(j\omega)\, d\omega \right)^{1/2}$$

THM PARSEVAL'S IDENTITY $\|f(t)\|_2 = \|\hat{f}(j\omega)\|_2$

2-NORM IN TIME OR FREQ DOMAIN IS PRESERVED

THIS IS BECAUSE $L_2^n(-\infty,\infty)$ AND $\hat{L}_2^n(j\mathbb{R})$ ARE ISOMORPHIC (1-1, ONTO) UNDER THE FOURIER TRANSFORM:

TAKE $n=1$:



$$\mathcal{F} : L_2(-\infty,\infty) \rightarrow \hat{L}_2(j\mathbb{R})$$

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \ \in \hat{L}_2(j\mathbb{R})$$

ANOTHER FREQ DOMAIN SPACE OF INTEREST IS H_2 (HARDY 2-SPACE): FUNCTIONS OF A COMPLEX VARIABLE THAT ARE ANALYTIC AND BOUNDED IN THE OPEN RIGHT HALF OF THE COMPLEX PLANE

$$H_2 = \Big\{ \hat{f} : \mathbb{C} \to \mathbb{C}^n,\ \hat{f}(s) \text{ ANALYTIC IN } \mathrm{Re}\{s\} > 0,\ \underbrace{\|\hat{f}\|_2}_{H_2 \text{ NORM}} < \infty \Big\}$$

$$\hat{f}(s) : \underbrace{\mathbb{C}}_{\text{COMPLEX PLANE}} \to \mathbb{C}^n, \qquad \hat{f}^*(s) = \hat{f}^T(\bar{s})$$

$s = \alpha + j\omega$

$\bar{s} = \alpha - j\omega$

EX:
$$\hat{f}(s) = \begin{pmatrix} \frac{1}{s+4} \\ \frac{1}{2s+1} \end{pmatrix}$$

NOT EX: $\hat{f}(s) = \frac{s^2}{s+4}$ (NOT STRICTLY PROPER)

OR $\frac{1}{s-4}$ (RHP POLE)

ANALYTIC SMOOTH (INFINITELY COMPLEX DIFF'BLE ON DOMAIN)

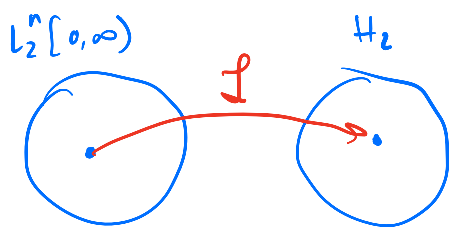
BOUNDED
$$\|\hat{f}\|_2 = \left(\sup_{\alpha > 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}^*(\alpha + j\omega)\, \hat{f}(\alpha + j\omega)\, d\omega \right)^{1/2} < \infty$$

H_2 NORM

SIMILARLY, $\|f(t)\|_2 = \|\hat{f}(s)\|_2$ (PARSEVAL'S IDENTITY)

SINCE $L_2^n[0, \infty)$ AND H_2 ARE ISOMORPHIC UNDER THE LAPLACE TRANSFORM:

$$\mathcal{L} : L_2^n[0, \infty) \longrightarrow H_2$$



$$\mathcal{L}[f(t)] = \hat{f}(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

ADDITIONAL INSIGHTS

RATIONAL FREQ DOMAIN FUNCTIONS

$$\hat{f}(s) = \frac{p(s)}{q(s)} \quad \text{OR} \quad \hat{f}(j\omega) = \frac{p(j\omega)}{q(j\omega)}$$ RATIO OF POLYNOMIALS

DENOTE $R\hat{L}_2$, RH_2 AS THE SETS OF RATIONAL FUNCTIONS IN $\hat{L}_2$, H_2 RESPECTIVELY

(THESE ARE ALSO SIGNAL SPACES)

ALSO NOTE STRICTLY PROPER $\Rightarrow$ DEG(p) < DEG(q)

PROPER $\Rightarrow$ DEG(p) = DEG(q)

IMPROPER $\Rightarrow$ DEG(p) > DEG(q)

THEN, A RATIONAL FREQ DOMAIN FUNCTION IS

(a) IN $R\hat{L}_2$ IF IT IS STRICTLY PROPER AND HAS NO POLES ON THE Im AXIS

EX: $\frac{1}{j\omega+a}$, $a \neq 0$ NO: $\frac{1}{j\omega}$, $\frac{j\omega}{j\omega+2}$

(b) IN RH_2 IF IT IS STRICTLY PROPER AND HAS NO POLES IN THE CLOSED RHP (STABLE)

EX: $\frac{1}{s+a}$, $a>0$ NO: $\frac{1}{s+a}$, $a<0$, $\frac{s}{s+a}$

MORE: SEE DULLERUD, PAGANINI (TEXT)