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AN OPTIMAL CIRCULANT PRECONDITIONER FOR TOEPLITZ SYSTEMS*

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Abstract. Given a Toeplitz matrix A, we derive an optimal circulant preconditioner C in the sense of minimizing $\|C - A\|_F$. It is in general different from the one proposed earlier by Strang ["A proposal for Toeplitz matrix calculations," Stud. Appl. Math., 74(1986), pp. 171-176], except in the case when A is itself circulant. The new preconditioner is easy to compute and in preliminary numerical experiments performs better than Strang's preconditioner in terms of reducing the condition number of $C^{-1}A$ and comparably in terms of clustering the spectrum around unity.

Key words. Toeplitz systems, circulants, iterative methods, preconditioned conjugate-gradient method

AMS(MOS) subject classification. 65

1. Introduction. Linear systems of the form $Ax = b$, where A is an n-by-n Toeplitz matrix (the entries of A are the same along each diagonal) arise in many applications, such as signal processing and time-series analysis. Since A obviously possesses a lot of structure (it is completely specified by only $2n - 1$ numbers), it is desirable to derive algorithms for solving Toeplitz systems in less than the $O(n^3)$ complexity for Gaussian elimination for a general matrix. In fact, algorithms with $O(n^2)$ complexity have been known for some time and most are based on the Levinson recursion formula [5], [8]. More recently, even faster algorithms with $O(n \log^2 n)$ complexity have been proposed [1], [2] but their stability properties are not yet clearly understood [3].

An alternative is to use iterative methods, based on using matrix-vector products of the form Av, which can be computed in $O(n \log n)$ complexity via the Fast Fourier Transform (FFT). To have any chance of beating the direct methods, such iterations must converge very rapidly and this naturally leads to the search for good *preconditioners* for A. Strang [6] proposed using *circulant* preconditioners, motivated by the observation that circulant systems can be solved efficiently by FFTs in $O(n \log n)$ complexity. In particular, for symmetric positive definite A, he obtained a circulant preconditioner S by copying the central diagonals of A and "bringing them around" to complete the circulant. Specifically, if $n = 2m$ and

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_0 & a_1 & \vdots \\ \vdots & a_1 & \ddots & \\ \vdots & & & a_1 \\ a_{n-1} & \cdots & a_1 & a_0 \end{pmatrix},$$

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then

$$S = \begin{pmatrix} a_0 & a_1 & & \cdots & & a_m & a_{m-1} & \cdots & a_1 \\ a_1 & a_0 & & & & \cdots & & & a_2 \\ \vdots & a_1 & & & & & & & \\ a_m & & & \ddots & & & & & \vdots \\ a_{m-1} & & \ddots & & & & & & \\ \vdots & & & & & & & & \\ a_1 & \cdots & & a_{m-1} & a_m & \cdots & & a_1 & a_0 \end{pmatrix}.$$

In numerical experiments, Strang found that the eigenvalues of $S^{-1}A$ cluster around 1, except for the largest and smallest eigenvalues. This is a very amazing discovery, considering that half the information in A is lost in obtaining C. An example is the special case of $a_k = t^k$, for which it can be proven that there are only five distinct values for the eigenvalues of $S^{-1}A$: $1/(1+t)$, $1/(1-t)$ (the extremes), 1 (a double eigenvalue), and $1/(1+t^{n/2})$, $1/(1-t^{n/2})$ (each repeated $\frac{n}{2}-2$ times!). Therefore, for n large and $|t| < 1$, the eigenvalues cluster around 1, except for the two extreme ones. This implies that when iterative methods such as the conjugate-gradient method is applied to solve the preconditioned system, the convergence is extremely fast.

In general, the convergence rate of the conjugate gradient iteration depends on the spectrum of $S^{-1}A$ and this problem is studied in detail in more recent papers by Strang and Edelman [7] and Chan and Strang [4]. In this note, we take a different direction and look for other (and hopefully better) circulant preconditioners to A.

2. The optimal preconditioner.

A preconditioner C for A can be viewed as an approximation to A that is efficiently invertible and can be used to obtain the following iterative method for solving the system $Ax = b$:

$$Cx_{i+1} = (C - A)x_i + b.$$

The convergence of this iteration (or an acceleration of it by conjugate gradient or other methods) depends on the spectrum of $C^{-1}(C-A)$: the smaller $\rho(C^{-1}(C-A))$, the faster the convergence. It can be easily shown that, if $B \equiv C - A$, then

$$\rho(C^{-1}B) \le \|C^{-1}B\| = \|(I + A^{-1}B)^{-1}A^{-1}B\| \le \frac{\|A^{-1}B\|}{1 - \|A^{-1}B\|},$$

provided $\|A^{-1}B\| < 1$ in some consistent matrix norm. Since the upper bound is a strictly increasing function of $\|A^{-1}B\|$, it is natural to choose C such that $\|A^{-1}B\|$ is minimized. Since $\|A^{-1}B\| \le \|A^{-1}\|\|B\|$, we are led to the following formulation for the construction of C:

$$\min_{C \text{ circulant}} \|C - A\|.$$

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Perhaps surprisingly, this problem is trivially solvable in the Frobenius norm for any (not necessarily symmetric positive definite) A.

THEOREM. *Let*

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{-1} & & \ddots & \vdots \\ \vdots & & & a_1 \\ a_{-(n-1)} & \cdots & a_{-1} & a_0 \end{pmatrix}$$

and

$$C = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \vdots \\ \vdots & \ddots & \ddots & \\ c_1 & \cdots & c_{n-1} & c_0 \end{pmatrix}.$$

Then the matrix C, whose entries are given by

$$c_i = \frac{ia_{-(n-i)} + (n-i)a_i}{n}, \qquad i = -(n-1), \ldots 0, \ldots (n-1),$$

minimizes $\|C - A\|_F$. The minimum is given by

$$\min_C \|C - A\|_F^2 = \sum_{i=0}^{n-1} (a_i - a_{-(n-i)})^2 \left(\frac{i(n-i)}{n}\right) \leq \frac{n}{4} \sum_{i=0}^{n-1} (a_i - a_{-(n-i)})^2.$$

Proof. Forming the Frobenius norm elementwise, we find that

$$\|C - A\|_F^2 = \sum_{i=0}^{n-1} \left[i(c_i - a_{-(n-i)})^2 + (n-i)(c_i - a_i)^2 \right].$$

Minimizing this by setting to zero the partial derivatives of $\|C - A\|_F^2$ with respect to c_i yields the first result. Substituting the expression for c_i into the expression for $\|C - A\|_F^2$ above yields the second result. □

This optimal preconditioner has a simple graphical description: the value of c_i is obtained by averaging the corresponding diagonal of A (extended to length n by a wraparound.)[1] It is also easy to see that C is symmetric if and only if A is symmetric. Moreover, $C = A$ if and only if $a_i = a_{-(n-i)}$ for $i = 1, 2, \cdots, n-1$, i.e., if A is itself circulant.

3. Comparison with Strang's preconditioner. That the preconditioner C is different from Strang's preconditioner S in general can best be illustrated in the $n = 4$ case for a symmetric A. If

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_0 & a_1 & a_2 \\ a_2 & a_1 & a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix},$$

then

$$S = \begin{pmatrix} a_0 & a_1 & a_2 & a_1 \\ a_1 & a_0 & a_1 & a_2 \\ a_2 & a_1 & a_0 & a_1 \\ a_1 & a_2 & a_1 & a_0 \end{pmatrix}, \qquad C = \begin{pmatrix} a_0 & \alpha & a_2 & \alpha \\ \alpha & a_0 & \alpha & a_2 \\ a_2 & \alpha & a_0 & \alpha \\ \alpha & a_2 & \alpha & a_0 \end{pmatrix}$$

where $\alpha = (3a_1 + a_3)/4$. The main difference is that whereas the information in a_3 is left out in S, it is incorporated in C through the averaging used in defining α. Note that $A = S = C$ if $a_1 = a_3$ (or $a_i = a_{n-i}$ in the general case), but otherwise the three matrices are different.

The results of some numerical experiments for four different Toeplitz matrices (also used by Strang [6]) with $n = 15$ are shown in Figs. 1-4; where the results for both the optimal C and Strang's S are shown. In all four cases, the spectrum of $C^{-1}A$ completely lies within that of $S^{-1}A$ (sometimes by a large margin), and hence the condition number of $C^{-1}A$ is smaller. On the other hand, it is difficult to choose between the two preconditioners in terms of clustering the spectrum around unity. S seems to cluster really well when a_k decreases rapidly with k (Figs. 1-3), whereas C seems to be better in other cases (Fig. 4).

4. Concluding remarks. This is only a preliminary study and many open problems remain. The preconditioners should extend easily to the block Toeplitz case. However, the stability and invertibility of C must be established. Finally, more detailed theoretical study of the spectral properties of $C^{-1}A$ is needed.

[1] This, in fact, characterizes the optimal preconditioner for a general non-Toeplitz A.

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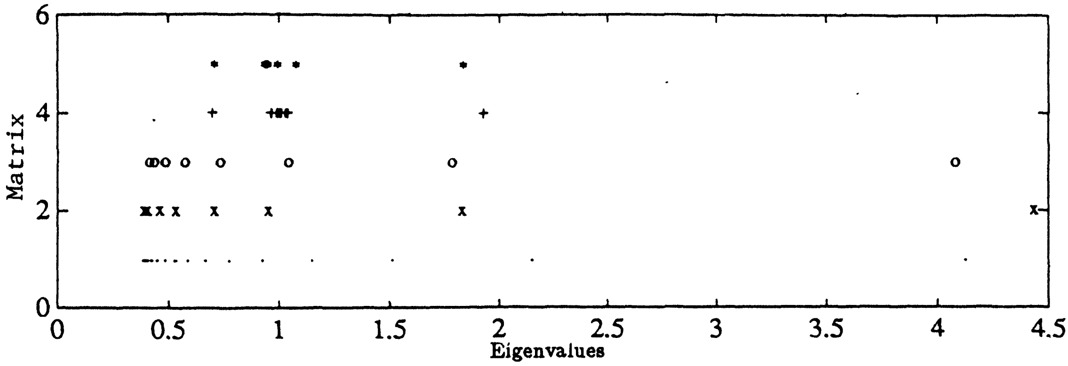

FIG. 1. *Eigenvalue distribution for* $a(k) = 1/(k+1)$.

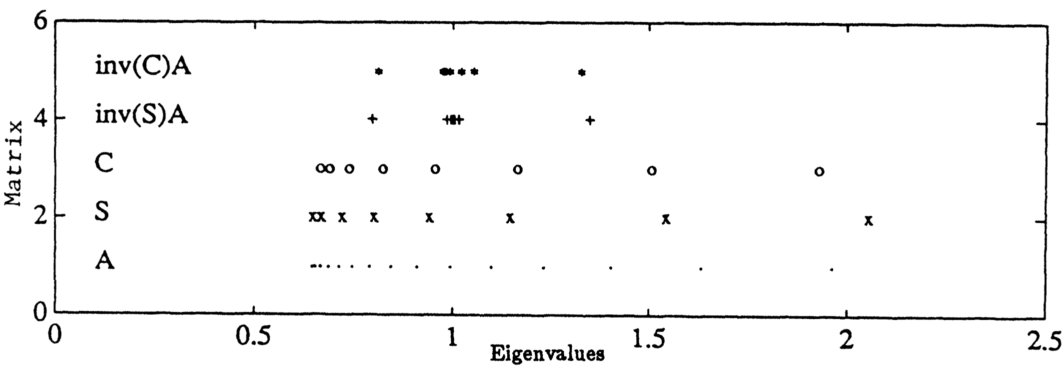

FIG. 2. *Eigenvalue distribution for* $a(k) = 1/(k+1) ** 2$.

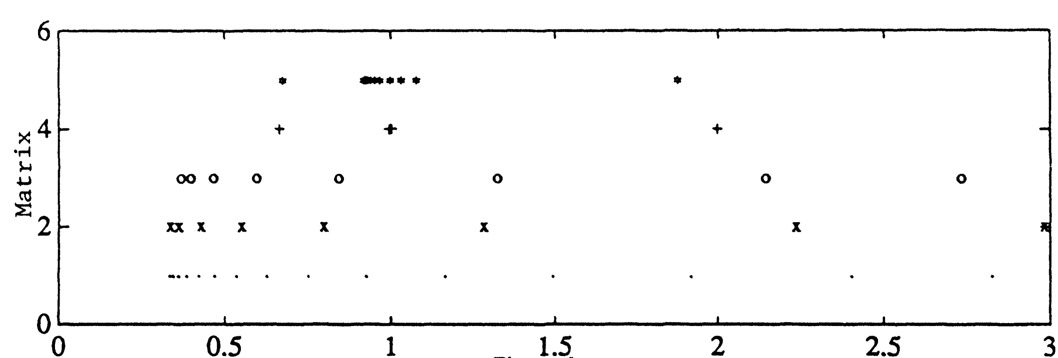

FIG. 3. *Eigenvalue distribution for* $a(k) = 2 ** (-k)$.

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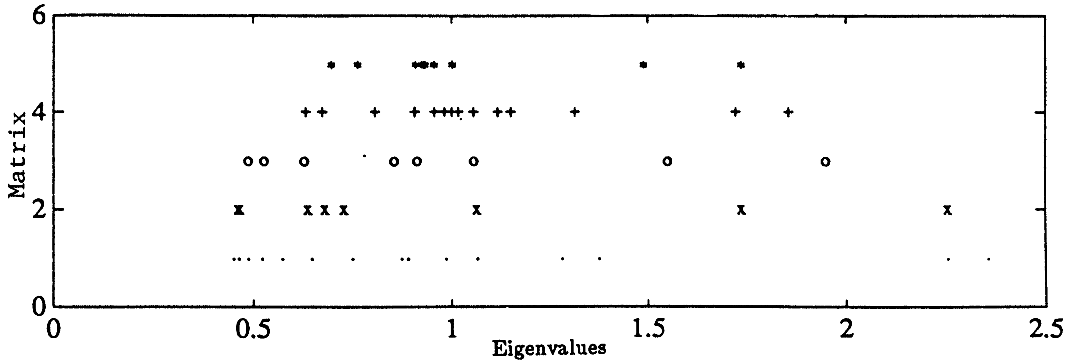

FIG. 4. *Eigenvalue distribution for* $a(k) = \cos(k)/(k+1)$.

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