MATH 6644
Project 2

Stephan Boettcher

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Fixed-point and Newton's Method for Nonlinear systems

Consider the discrete Chandrasekhar H-equation:

$$F_i(\vec{x}) = x_i - (1 - \frac{c}{2N}\sum_{j=1}^{N}\frac{\mu_i x_j}{\mu_i + \mu_j})^{-1} = 0$$

where $c \in (0,1)$ is a given constant, $\mu_i = \frac{(i-\frac{1}{2}}{N}$ for $1 \leq i \leq N$ and N is the dimension of the unknown vector $\vec{x}$. Write your own code and compute the solution of the equation for $N = 200$ and $c = 0.9$ by using:

1. *Fixed-point method,*
2. *Chord method,*
3. *Newton method,*
4. *Shamanskii method with $m = 2$.*

In all of the computations, the initial guess is taken as $\vec{x} = [1, 1, \ldots, 1]^T$ and the stopping condition is given.

The Chandrasekhar H-equation was introduced by astrophysicist Subrahmanyan Chandrasekhar is used to solve exit distribution problems in radiative transfer. In the world of Numerical Methods, this equation serves as a well-understood problem that nonlinear solvers can tackle. In this project, the Fixed point and variations on Newton's methods were used to solve the discrete Chandrasekhar H-equation with a constant of $c = 0.9$ and $N = 200$. These four methods are all able to solve nonlinear systems of equations with various rates of convergence.

Fixed-Point Method

The Fixed point method is a nonlinear solver that relies on contraction mapping to find a solution. The Fixed point iteration is given by:

$$\vec{x_{n+1}} = \vec{x_n} - F(\vec{x_n})$$

By inducing a contraction mapping, given by $\vec{x} = k(\vec{x})$, the method will converge, so long as the following conditions are met:

$$||k(\vec{x}) - k(\vec{y})|| \leq \gamma||\vec{x} - \vec{y}|| \quad 0 < \gamma < 1$$

where $k(\vec{x}) = \vec{x}_{n+1}$, $k(\vec{y}), \vec{y} = \vec{x}^*$ and γ is a constant. The Fixed-point method may not converge as quickly as other methods, but it is a relatively straight forward method and does not need to calculate a Jacobian matrix to converge. This can be a huge advantage over the Newton's family of methods, should finding the Jacobian prove to be difficult or costly.

Newton Method

Newton's method is a numerical method that iteratively finds successively better approximations to the zeros of a real function. The Newton iteration is defined as:

$$\vec{x}_{n+1} = \vec{x}_n - \frac{F(\vec{x}_n)}{F'(\vec{x}_n)}$$

Chord Method

The chord

Shamanskii Method with $m = 2$

Results

n	Serial (sec)	Parallel For(sec)	DST (sec)
258	0.0194	0.0494	0.5190
514	0.0111	0.0564	0.0196
1026	0.0423	0.0771	0.0706
2050	0.1704	0.1439	0.2003
4098	0.5087	0.4694	1.1024
8194	8.1068	1.9299	3.9583
16386	37.3268	7.9876	29.3784

Figure 1: Runtime for generating the S

The second symmetric Toeplitz system is defined by:

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

where $f(\theta) = \theta^4 + 1$ for $-\pi \leq \theta \leq \pi$. This means that a_k is the Fourier coefficients of $f(\theta)$ and were computed via FFT. For the second system, the PCG and CG method were each run for the 6 different values of n only. Once again, the CG method was compared to the PCG method with Chan's preconditioner and Strang's preconditioner. For this new A matrix based upon the Fourier coefficients of $f(\theta)$. These coefficients were generated by first creating an array of $f(\theta)$ values, and then performing a FFT on the values. The result was an array of complex values that were then used to generate the A matrix. Once the A matrix had been formed, Chan and Strang's circulant matrices were generated. In order to achieve fast convergence rates, the A matrix was preconditioned with each of these circulant matrices. The final preconditioned matrix passed to the PCG method was $C_n A C_n'$ and the matrix used in the PCG calculation was $(C_n A C_n')^{-1}$. Figures 2 and 3 show the final runtimes and number of iterations used by each method.

pcsb.png

Figure 2: The final runtimes of the CG iterative method and the PCG iterative method with a Strang circulant preconditioner.

As can be seen in Figures 2 and 3, the PCG method greatly outpaces the CG method in both runtime and number of iterations required to converge. However, once again the Chan and Strang circulant preconditioners are roughly equivalent. Both methods are able to converge within one iteration and both methods are able to handle the large matrices without issue. A brief analysis of the runtimes show that the $\approx \mathcal{O}(n\,log\,n)$ growth rate discussed above is maintained with the second symmetric Toeplitz system analyzed here.

pccb.png

Figure 3: The final runtimes of the CG iterative method and the PCG iterative method with a Chan circulant preconditioner.