Linear Programming Formulations for Small Examples of Discrete Optimization Problems

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Abstract

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1. Introduction

Polyhedral combinatorics studies discrete optimization problems using polyhedra. A discrete optimization problem is reduced to the optimization of a linear function over a finite set of vectors. The same result is obtained by optimizing over the convex hull of the set of vectors, which is a polyhedron. We have converted the original problem to a linear programming problem (Schrijver et al., 1987).

Describing problems as linear programming problems is useful, since linear programming has been extensively studied and good algorithms exist to solve them. However, these algorithms cannot be applied unless we know the matrix A and vector b that define the linear programming problem.

A standard approach is to guess the system $Ax \leq b$, followed by proving that the description is complete. One proof method was introduced by Lovász in "Graph Theory and Integer Programming" (1979). When using the method, one needs to show that the actual polytope P is a subset of $Ax \leq b$ and that every facet of P is defined by some inequality in $Ax \leq b$. When the matrix A has some special properties, for example when A is totally unimodular, simpler methods may be applicable (Schrijver et al., 1987).

When the full description has not been found, an approximation $Bx \leq c$ of the polytope can be used to give upper or lower bounds for the optimzation problem using linear programming (Schrijver et al., 1987). Often, an LP program defined using such an approximation can be solved in polynomial time, which makes them useful for dealing with NP hard problems (Carr & Konjevod, 2005).

There are many discrete optimization problems for which the complete systems $Ax \leq b$ are still unknown. Give some examples here.

In this paper we present a method that can be used to find A and b from the finite set of vectors corresponding to a sufficiently small example of a discrete optimization problem. We verify the correctness of our method using the Traveling Salesman problem. Here, we compare our system of inequalities with existing characterizations of the TSP polytope (Christof et al., 1991; Boyd & Cunningham, 1991). Then, we apply the method to add contributions here.

For both problems, we determine how the size of the system depends on the size of the optimization problem. Furthermore, we investigate how sparse the system of inequalities is and determine whether the inequalities can be grouped into sets of similar inequalities.

1.1. Organization

First, we describe the method used to determine the system of inequalities in section 2. Then, in section 3, we introduce the previous work done on the Traveling Salesman Problem, rederive the system of inequalities using our method and compare the results. In section 4, Add summary here. We state our conclusions in section 5.

2. Methodology

Suppose an instance of a discrete optimization problem is described by the finite set $S \subseteq \mathbb{R}^E$. We give a procedure using the software polymake (pol) to obtain a minimal description $Ax \leq b$ of the polyhedron conv.hull(S). We illustrate our method using the example $S = \{(1,0,1), (1,1,0), (0,1,1)\}$.

First, enumerate all points in S, representing them in homogeneous coordinates by adding an additional first dimension and setting it to 1. Note that polymake uses homogeneous coordinates to allow rays of a cone to be specified by points at infinity (**?**). Since $|S|$ is often exponential in the size of the problem (Schrijver et al., 1987), our method is only feasible for small examples of such problems. Define a `Polytope` using the resulting set of points, for example

```
$p = new Polytope(POINTS=>[[1,1,0,1],
[1,1,1,0], [1,0,1,1]]);
```

Figure 1 gives a visualization of this polytope. Now, `print_constraints($p)` gives us a minimal descrip-

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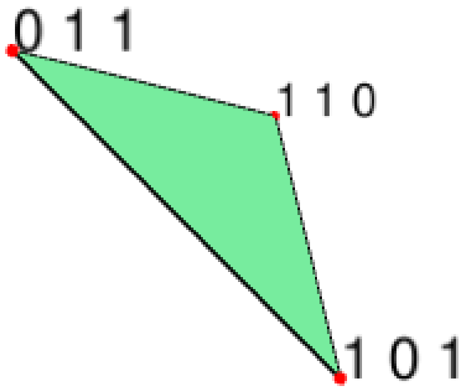


Figure 1. Visualization of the example polyhedron, made using polymake (pol)

tion of the polyhedron, seperated into facet-defining inequalities and implicit equalities. For the example, we get

```
Facets:
0:  -x3 >= -1
1:  x1 + x3 >= 1
2:  -x1 >= -1

Affine hull:
0:  x1 + x2 + x3 = 2
```

We get implicit equalities, which define the affine hull of the polyhedron, when the polyhedron is not full dimensional. All other inequalities are facet-definining inequalities and correspond to some facet of the polyhedron (Carr & Konjevod, 2005).

3. Traveling Salesman Problem

3.1. Problem Definition and Previous Work

Let $G(V, E)$ be the complete graph and define edge costs c_e for all $e \in E$. The Traveling Salesman problem (TSP) finds the minimum cost circuit in G that visits each node exactly once. Note that the set of feasible solutions is the set of all circuits in G, making the TSP polytope independent of the edge costs c_e.

A complete description of the TSP polytope has not been found. In 1991, Boyd and Cunningham published complete characterizations of the polytope for 6 and 7 nodes. Their paper identifies four classes of facet-defining inequalities (1991).

First, they introduce *nonnegativity inequalities* of the form

$$x_{ij} \geq 0 \quad \text{for} \quad (i, j) \in E$$

Next, let $\gamma(S) = \{(i, j) \in E : i \in S, j \in S\}$ and define $x(A) = \sum(x_{ij} : (i, j) \in A)$ for some $A \subseteq E$. The *subtour elimination inequalities* have the form

$$x(\gamma(S)) \leq |S| - 1 \text{ for } S \subset V,\ 2 \leq |S| \leq n - 2$$

Note that the subtour elimination inequalities for S and $V \backslash S$ are equivalent (Boyd & Cunningham, 1991).

Now, let $H \subset V$ and let $T_1, \ldots, T_{2k+1} \subset V$ for some integer k be mutually disjoint such that $T_j \cap H$ and $T_j \backslash H$ are nonempty for all j. The set H is called the handle and the sets T_j are called teeth. The *comb inequality* associated with these sets is

$$x(\gamma(H)) + \sum_{j=1}^{2k+1} x(\gamma(T_j)) \leq |H| + k + \sum_{j=1}^{2k+\text{‘}1} (|T_j| - 2)$$

The comb inequalities for $H, T_1, \ldots, T_{2k+1}$ and $V \backslash H, T_1, \ldots, T_{2k+1}$ are equivalent (Boyd & Cunningham, 1991).

Finally, *envelope inequalities* are defined specifically for the case $n = 7$. Label the nodes from 1 to 7 and define the sets $H_1 = \{1, 4, 6\}$, $H_2 = \{2, 5, 7\}$, $T_1 = \{1, 2, 3\}$, $T_2 = \{4, 5\}$ and $T_3 = \{6, 7\}$. Then the envelope inequality is

$$\sum_{i=1}^{2} x(\gamma(H_i)) + x(\gamma(T_1)) + 2\sum_{i=2}^{3} x(\gamma(T_i)) \leq 8$$

Note that different labelings of the vertices give different envelope inequalities. Boyd and Cunningham show that envelope inequalities are not facet-defining for $n = 8$ (1991).

For the $n = 8$ case, a variety of additional classes of inequalities are required, including bipartition, lifted bipartition, crown and chain inequalities (Boyd & Cunningham, 1991). The last of the required inequalities were found by Christof, Jünger and Reinelt, who proved that the resulting system of inequalities is complete (1991). For larger n, no complete description is known.

3.2. Obtaining the Inequalities with Polymake

To find the TSP polytope, we need to enumerate all Hamiltonian circuits in the graph G, given some number of nodes $n = |V|$. Label the nodes $1, \ldots, n$ and assume the starting vertex is n. Let $v_1, \ldots, v_{n-1}$ be some permutation of the remaining vertices $1, \ldots, n - 1$. Each permutation gives a circuit $n, v_1, \ldots, v_{n-1}, n$. Note that the procedure will list each circuit twice, since $n, v_1, \ldots, v_{n-1}, n$ is equivalent to $n, v_{n-1}, \ldots, v_1, n$. These duplicates do not cause any issues when defining a `Polytope` in polymake.

Now, we find a representation of each circuit C as a point $x \in \mathbb{R}^{|E|}$ where $x_e = 1$ if the edge $e \in C$ and $x_e = 0$ otherwise. We associate the first $n-1$ dimensions with the edges $(1,2), (1,3), \dots, (1,n)$, the next $n-2$ dimensions with the edges $(2,3), (2,4), \dots, (2,n)$, and so on. Now, we use a script to iterate through the circuits obtained above and transform each to a point $x \in \mathbb{R}^{|E|}$ using the described edge association.

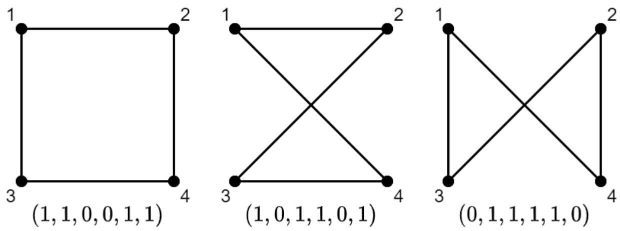


Figure 2. Possible circuits and corresponding points $x \in \mathbb{R}^6$ for $n = 8$

For $n = 4$, the unique circuits and corresponding vectors x are shown in Figure 2. Applying the method from section 2 to the resulting set S gives three inequalities and four implicit equalities. The following Lemma shows how to categorize the inequalities:

Lemma 3.1. *For $n = 4$, the TSP polytope is defined by one nonnegativity inequality, two subtour elimination inequalities and four implicit equalities.*

Proof. Polymake gives the following facet-defining inequalities:

$$x_{24} + x_{34} \geq 1 \tag{1}$$
$$x_{24} \leq 1 \tag{2}$$
$$x_{34} \leq 1 \tag{3}$$

Here, (2) and (3) are subtour elimination inequalities with $S = \{2,4\}$ and $S = \{3,4\}$ respectively. While equation (1) initially does not look like an inequality from the four classes defined in section 3.1, we can add $-1\cdot$ equation (2) to the equality to obtain $x_{34} \geq 0$, a nonnegativity inequality.

Note that the TSP polytope is not full dimension, since the set of Hamiltonian cycles is not a basis for the set of all possible edge combinations. For $n = 4$, the polytope has dimension 2. As the points x are in $\mathbb{R}^6$, we require 4 additional implicit equalities. □

Next, we use the same method to generate a description of the TSP polytope for $n = 5$. Lemma 3.2 categorizes a subset of the required inequalities.

Lemma 3.2. *For $n = 5$, the description of the TSP polytope constains at least the following: five nonnegativity inequalities, five subtour constraints, four comb inequalities and five implicit equalities.*

Proof. Polymake gives 20 facet-defining inequalities. Out of these, 5 have the form of nonnegativity constraints:

$$x_{12} \geq 0 \quad x_{13} \geq 0 \quad x_{24} \geq 0 \quad x_{34} \geq 0 \quad x_{45} \geq 0$$

A further five are subtour elimination inequalities for sets S with $|S| = 2$:

$$x_{12} \leq 1 \quad x_{13} \leq 1 \quad x_{24} \leq 1 \quad x_{34} \leq 1 \quad x_{45} \leq 1$$

Next, we have three inequalities of the form

$$x_{12} + x_{13} - x_{45} \leq 1$$

Adding one of the subtour elimination inequalities gives

$$x_{12} + x_{13} \leq 2$$

which is a comb inequality with $k = 0$, $H = \{1,2\}$ and $T_1 = \{1,3\}$. A similar modification can be made for the other two similar inequalities.

Additionally, we have the inequality

$$-x_{12} - x_{13} + x_{24} + x_{34} + x_{45} \leq 1$$

Ading two subtour elimination inequalities and subtracting one subtour elimination inequality gives

$$x_{24} + x_{34} \leq 2$$

which has the same form as the previous comb inequalities.

The polytope has dimension 5. Since the points $x \in \mathbb{R}^{10}$, 5 additional implicit equalities are required to fully determine the polytope. □

We were unable to classify the remaining six inequalities:

$$\begin{aligned} -x_{13} + x_{24} + x_{45} &\geq 0 \\ -x_{12} + x_{34} + x_{45} &\geq 0 \\ x_{12} + x_{13} - x_{45} &\geq 0 \\ x_{24} + x_{34} + x_{45} &\geq 1 \\ x_{24} + x_{34} + x_{45} &\leq 2 \\ -x_{12} - x_{13} + x_{24} + x_{34} + x_{45} &\geq 0 \end{aligned}$$

This example shows that even for small problems, it is hard to transform the inequalities to a form that corresponds to one of the classes introduced in section 3.1. Thus, for the cases $n = 6$ and $n = 7$, we only identify some examples of inequalities instead of attempting to categorize all inequalities.

For $n = 6$, polymake outputs 100 facet-defining inequalities. We make the following observations:

1. The TSP polytope has dimension 9 while the points $x \in S$ have dimension 15. Hence, an additional 6 implicit equalities are required to fully define the polyhedron.

2. We identified nine nonnegativity inequalities.

3. We identified a total of 15 subtour elimination inequalities. Out of these, nine had $|S| = 2$, five had $|S| = 3$ and one inequality had $|S| = 4$. Below are examples for (4) $S = \{4, 5, 6\}$ and (5) $S = \{2, 4, 5, 6\}$:

$$x_{45} + x_{46} + x_{56} \leq 2 \quad (4)$$
$$x_{24} + x_{25} + x_{26} + x_{45} + x_{46} + x_{56} \leq 3 \quad (5)$$

4. We found four comb inequalities. All inequalities had a handle containing three nodes and three teeth containing two nodes each. For example, $H = \{2, 4, 5\}$, $T_1 = \{2, 3\}$, $T_2 = \{1, 4\}$ and $T_3 = \{5, 6\}$:

$$x_{14} + x_{23} + x_{24} + x_{25} + x_{45} + x_{56} \leq 4$$

Overall, we identified 28 out of the 100 facet-defining inequalities.

For $n = 7$, polymake gave 3437 facet-defining inequalities, which agrees with the theoretical results by Boyd and Cunningham (1991). We make the following observations:

1. The TSP polytope has dimension 14 while the points $x \in S$ have dimension 21. Hence, an additional 7 implicit equalities are required to fully define the polyhedron.

2. We found 14 nonnegativity inequalities.

3. We found a total of 24 subtour elimination inequalities. Out of these, 14 had $|S| = 2$, nine had $|S| = 3$ and one inequality had $|S| = 4$.

4. By searching for specific types of comb inequalities, we were able to identify a total of 105 comb inequalities. As our search didn't include all possible types of comb inequalities over 7 nodes, we expect that more comb inequalities can be identified. The most frequent type of comb inequality we found contained 42 inequalities. This type had a handle containing three nodes and three teeth with two nodes each. The least common type of comb inequality we found had four nodes in the handle, two teeth with two nodes each and one tooth with three nodes. We only found two such inequalities.

5. No inequalities could immediately be identified as envelope inequalities.

We were able to classify a total of 143 out of the 3437 facet-defining inequalities.

3.3. Observations

We were able to obtain the correct number of facets in the examples studied. This gives us confidence that our method works correctly. However, we found that the facets we found are hard to categorize, especially as the size of examples increases. We believe that since there are many equivalent ways to express facet-defining inequalities, it is hard to recognize specific formulations of inequalities.

We observed several patterns specific to the TSP polytope. First, while number of facet-defining inequalities increased very rapidly, the number of implicit equalities only increased linearly between $n = 4$ and $n = 7$. Second, we found a number of nonnegativity inequalities equal to the dimension of the TSP polytope. We also found an equal number of subtour elimination inequalities with $|S| = 2$.

4. Set Cover Problem

4.1. Problem Definition and Simpler Results

Let the universe be $U = \{u_1, u_2, \ldots, u_m\}$ and let $S_1, S_2, \ldots, S_n$ be subsets of U. The unweighted set cover problem (SCP) aims to minimize the cardinality of $S \subseteq \{S_1, S_2, \ldots, S_n\}$ such that $\bigcup_{S_i \in S} S_i = U$.

Consider the linear programming problem

$$\min\{c^T x | Ax \geq 1, 0 \leq x \leq 1\}$$

where c is an objective function assigning weights to each set S_j and $A = [x(S_1), \ldots, x(S_n)]$. Here, the columns $x(S_j)$ are defined as

$$x_i(S_j) = \begin{cases} 1 & u_i \in S_j \\ 0 & u_i \notin S_j \end{cases}$$

for $0 \leq i \leq m$ and $0 \leq j \leq n$.
The SCP can be phrased as the $x_i \in \{0, 1\}$ solutions to this linear programming problem. Analogously, the focus is on inequalities of the integer convex hull, the polytope

$$P_I(A) = \{x | Ax \geq 1, x_i \in \{0, 1\}\}.$$

Definition 4.1. ***Odd holes*** *are matrices with exactly 2 ones in each row and 2 ones in each column.*

The SCP has multiple characterized classes of inequalities, but the only fully characterized polytopes are those with no odd holes as a submatrix of A (BALAS & NG, 1989; Kuo & Leung, 2016).

For simplicity, we begin by reducing the problem to full dimensional cases, this can be accomplished by successively:

1. Removing $u_i \in U$ contained by exactly a single set S_j from consideration.

2. Removing S_j and all elements of the universe $u_k \in S_j$ from consideration.

Balas and Ng provide our first inequalities, intuitive results derived from the definition $P_I(A)$ (1989). We provide proofs for results taken as trivial.

Definition 4.2 (Row and Column indexing)**.** *Let M, N denote the row and column index sets of A respectively.*
For $i \in M$, denote

$$N^i := \{j \in N | a_{ij} = 1\}.$$

Theorem 4.1. *$P_I(A)$ is full dimensional if and only if $|N^i| \geq 2$ for all $i \in M$.*

Proof. $|N^i| = 1$ if and only if u_i is an element of one set S_j. This provides an implicit inequality $Ax_j = 1$, hence $P_I(A)$ is not full dimensional. □

Theorem 4.2. *Assume $P_I(A)$ is full dimensional. These theorems are numbered as they appear in*

2. *The inequality $x_j \geq 0$ defines a facet of $P_I(A)$ if and only if $|N^i \backslash \{j\}| \geq 2$ for all $i \in M$.*
3. *All inequalities $x_j \geq 1$ defines facets of $P_I(A)$.*
4. *All facet defining inequalities $\alpha x \geq \alpha_0$ for $P_I(A)$ have $\alpha \geq 0$ if $\alpha_0 > 0$.*
5. *Inequality $\sum_{j \in N^i} x_j \geq 1$ defines a facet of $P_I(A)$ if and only if:*
   i.) *there exists no $k \in M$ with $N^k \subsetneq N^i$, and*
   ii.) *for each $k \in N \backslash N^i$, there exists $j(k) \in N^i$ such that $a_{hj(k)} = 1$ for all $h \in M^0(k) := \{h \in M | a_{hk} = 1$ and $a_{hj} = 0, \forall j \in N \backslash N^i \cup \{k\}\}$.*
6. *The only minimal valid inequalities for $P_I(A)$ with integer coefficients and right-hand side equal to 1 are those of the system $Ax \geq 1$.*

(BALAS & NG, 1989)

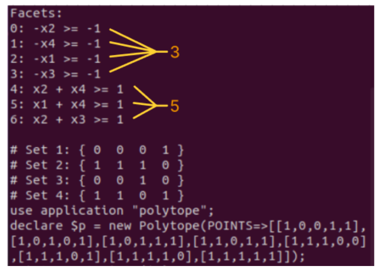
Proof. (2): When $|N^i \backslash \{j\}| = 1$, face at $x_j = 0$ not only $n-1$ dimensional, but $n-2$ dimensional as we have an implicit equality by same arguement as Theorem 4.1.
(3): Restricting $x_j = 1$ does not restrict other $x_i, i \neq j$.
(4): Otherwise $\alpha < 0, x_i \in [0,1] \implies \alpha x_i < 0$.
(5) and (6) are proven formally by Balas and Ng. □

These results along with odd holes allow us to fully characterize some SCP polytopes.

Definition 4.3 (Odd Holes)**.** *An **odd hole** is an odd order 0,1-matrix with exactly two 1s for each row and two 1s for each column.*

Theorem 4.3 (Fulkerson, Hoffman, and Oppenheim (1994))**.** *If A does not contain an odd hole as a submatrix, then no additional facets are required in the description of the polytope.*

We retroactively find an example of a fully characterized SCP by identifying polytopes with only these simple facets, see `Polymake` generated example p447 of Figure 3. Matrix A of SCP p447 has no possible odd holes as a submatrix, and correspondingly has relatively simple facets defined by Theorem 4.2. However, Theorem 4.3 applies only to a subset of SCPs and we find many examples of inequalities which do not fit the description of Theorem 4.2, such as the SCP of Figure 4.



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Figure 3. SCP p447 with 4 elements and 4 sets.
Each facet is labeled by a corresponding numbered inequality result.

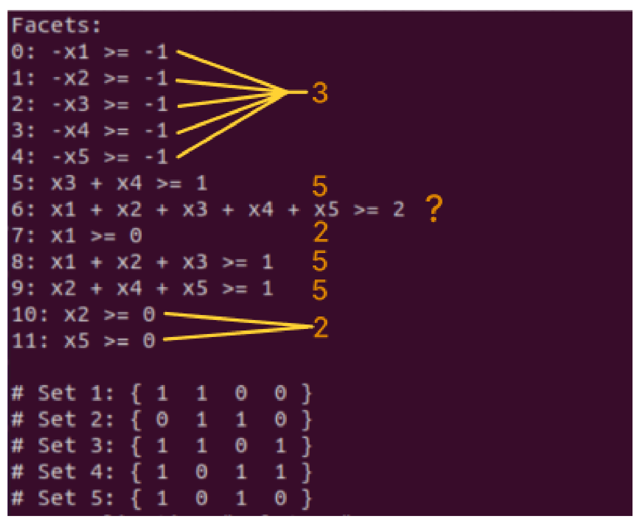


Figure 4. SCP p453 with 4 elements and 5 sets.
Each facet is labeled by a corresponding numbered inequality result.
Facet 6 is not classified by Theorem 4.2.

While SCPs where Theorem 4.3 does not apply are not fully characterized, there exist other results characterizing classes of inequalities that can occur.

4.2. Inequalities With Integers {0,1,2}

Balas and Ng provide a result which classifies all inequalities of the form

$$\alpha x \geq 2,\ a_j \in \{0,1,2\}\ (*)$$

Definition 4.4. *For $\neq S \subseteq M$, define inequality $\alpha^S x \geq 2\ (**)$ as*

$$\alpha_j^S = \begin{cases} 0 & if\ a_{ij} = 0\ \forall i \in S, \\ 2 & if\ a_{ij} = 1\ \forall i \in S \\ 1 & otherwise \end{cases}$$

Inequality $(**)$ is proven to dominate other inequalities of form $(*)$.(BALAS & NG, 1989) Additionally, Balas and Ng provide conditions to when inequality $(**)$ is facet defining.

Definition 4.5. *For any $Q \subseteq N$ define*

$$M(Q) := \{i \in M | a_{ij} = 0,\ \forall j \in Q\}.$$

<u>Idea</u>: $M(Q)$ the elements of the universe not covered by Q. For the inequality $\alpha x \geq 2,\ a_j \in \{0,1,2\}$, denote

$$J_t = J_t(\alpha) = \{j \in N | \alpha_j = t\},\ t = 0,1,2$$

<u>Idea</u>: J_t are the sets with coefficient t in α.
*For any valid $\alpha^S x \geq 2$ a pair $j,h \in J_1$ the **2-cover** of $A^{J_1}_{M(J_0)}$, the submatrix of A with row set $M(J_0)$ and column set J_1, if $a_{ij} + a_{ih} \geq 1$ for all $i \in M(J_0)$.*
*<u>Idea</u>: $S_j \cup S_h$ a feasible solution for the SCP induced by submatrix $A^{J_1}_{M(J_0)}$. The **2-cover graph** of $A^{J_1}_{M(J_0)}$ is the graph that has a vertex for every $j \in J_1$ and an edge for every 2-cover of $A^{J_1}_{M(J_0)}$.*
We define $T(k)$ to be the elements of U such that $S_k, k \in J_0$ is the only set to cover $T(k)$;

$$T(k) = \{i \in M | a_{ik} = 1, a_{ij} = 0\, for\ all\ j \in J_0 \backslash \{k\}\}.$$

We define permutations $j, h : J_0 \longrightarrow J_1$.

Theorem 4.4. *Assume $P_I(A)$ is full dimensional and let $\alpha^S x \geq 2$ be a valid inequality for $P_I(A)$, with $S = M(J_0)$. Then $\alpha^S x \geq 2$ defines a facet if and on if:*

i.) every component of the 2-cover graph $A^{J_1}_{M(J_0)}$ has an odd cycle; and

ii.) for every $k \in J_0$ such that $T(k) \neq$ there exists either:

a) some $j(k) \in J_2$ such that $a_{ij(k)} = 1$ for all $i \in T(k)$; or

b) some pair $j(k), h(k) \in J_1$ such that $a_{ij(k)} + a_{ih(k)} \geq 1$ for all $i \in T(k) \cup M(J_0)$

(BALAS & NG, 1989)
This result proves to be difficult to follow, but is very strong in combination with other results from Balas and Ng to classify all facets of form $\alpha^S \geq 2$. However, many smaller sized SCPs have all facets of form $\alpha x \geq 2$ and inequalities of Theorem 4.2.
Table 4.2 shows the proportion of SCPs where all facets are of forms Theorem 4.4 and Theorem 4.2:

	p44	p45	p66	p68	p99	p9[15]
Proportion	9/9	8/9	7/9	5/9	1/5	0/1

Table 1. SCPs where all Facets from Theorem 4.2 & 4.4

4.3. Further Inequalities

Another approach to classifying the inequalities from Cornuéjols and Sassano explores the graphs of SCPs.

Definition 4.6. *For the {0,1}-matrix A, the **bipartite incidence graph** $B = (V, U, E)$ is the graph with:*

- *a node $i \in U$ for each $u_i \in U$,*
- *a node $j \in V$ for each set S_j, and*
- *an edge between nodes $i \in U$ and $j \in V$ if $u_i \in S_j$.*

$\beta(T)$ to be the minimum cardinality of a cover of $T \subseteq U$, and $\beta(U)$ the covering number of A.

*$G^* = (V, E^*)$ to be the **critical graph** of B with nodes V and edges*

$$E^* = \{\{v_i, v_j\} | \beta(U \backslash U_{ij}) < \beta(U)\}$$

with $U_{ij} \subseteq U$ the set of shared neighbors of v_i, v_j.

Theorem 4.5 (Sassano (1985))**.** *If G^* connected, then $\sum_{j=1}^n x_j \geq \beta(U)$ defines a facet of $P_I(A)$.*

(Sassano, 1989) Figure 6 of p686 provides an example of a bipartite incidence graph, critical graph, and application of Theorem 4.5 as the critical graph of p686 is connected.

Additionally, Cornuéjols and Sassano generalize this theorem to consider the covering number of submatrices of A with subsets of the sets $S \subseteq S_1, S_2, ..., S_n$,and provide both sufficient and necessary conditions for when this inequality is facet defining.(Cornuéjols & Sassano, 1989) This allows the classification of some facets of the form

$$\alpha x \geq d,\ alpha_i \in \{0,1\},\ d \geq 3$$

.

Finally, Odd holes can be applied to derive another class of facets.

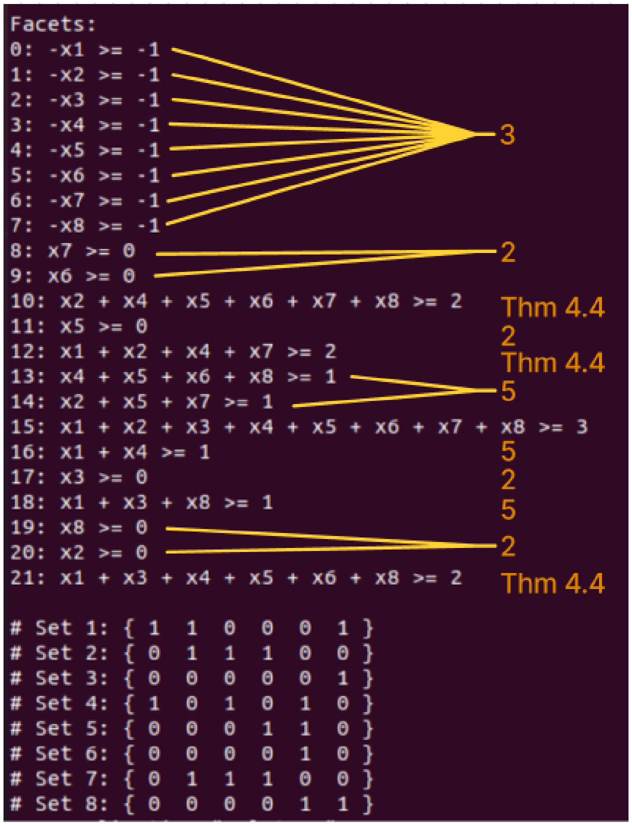


Figure 5. SCP p668.
Each facet labeled with corresponding inequality result.
Facet 15 classified by Theorem 4.5.

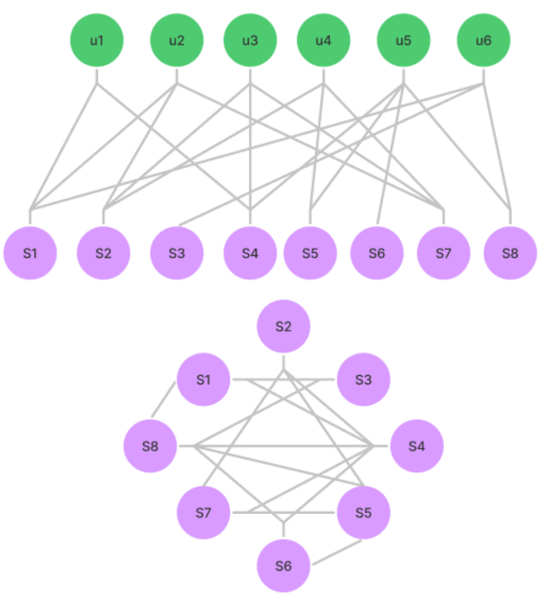


Figure 6. Bipartite Incidence Graph and Critical Graphs of p686.

Theorem 4.6. *If $H \leq A \leq H^*$ where H is an odd hole and H^* is $(n+1)/2$-maximal, then*

$$\sum_{j=1}^{n} x_j \geq \frac{n+1}{2}$$

defines a facet on $P_I(A)$.

(Fulkerson et al., 2009)(Cornuéjols & Sassano, 1989)

4.4. Observations and Conclusions

The preceeding results classify multiple types of inequalities, and lead to necessary and sufficient results on facets of the SCP polytope. However, infinitely more inequalities still are not classified, and still need more research. Facet 13 of p669 in Figure 7 presents such a facet that has still not been classified. The general form of these unexplored inequalities

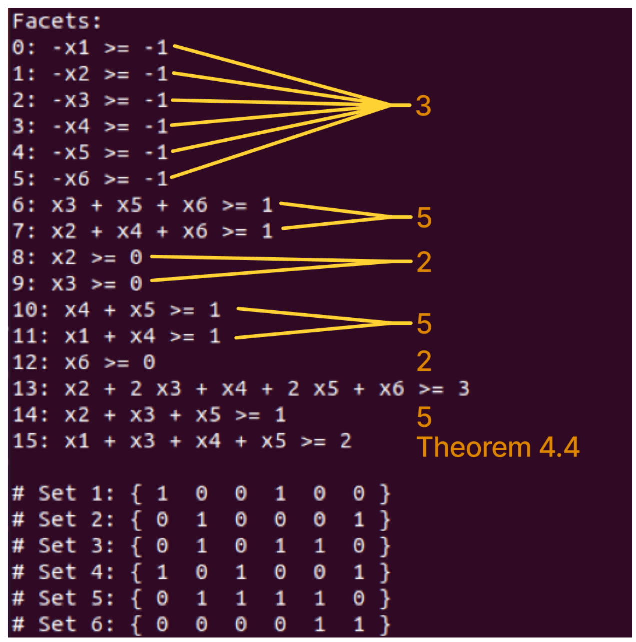


Figure 7. SCP p669.
Each facet labeled with corresponding inequality result.

is

$$\beta x \geq d, \ \exists \beta_i \notin \{0,1\}, \ d > 2.$$

Many of the classifications of inequalities only dealt with coefficients of x in $\{0,1\}$. Only Balas and Ng featured inequalities with another coefficient, 2. A reasonable direction would be to try to reproduce the unclassified inequalities from $P_I(A)$ using a procedure similar to that for Theorem 4.4 inequalities.

Larger problems had larger coefficients, and more occurrences of unclassified inequalities.

	p44	p45	p66	p68	p99	p9[15]
Proportion	0/9	1/9	2/9	1/9	4/5	1/1

Table 2. SCPs with unclassified inequalities as facets.

4.5. Application and Observations

We used `Polymake` to generate multiple small problems, varying both the size of the universe, $|U|$, and the number of sets, n, see Table **??**
Naming of each example is standardized as

$$p[|U|][n][\text{iteration}]$$

where the iteration is used to index SCPs generated with the same characteristics.

	p44	p45	p66	p68	p99	p9[15]
Amount Generated	9	9	9	9	5	1

Table 3. Count of SCPs generated of varying sizes

Each SCP generated is full dimensional with where sets are both nonempty and proper subsets.

5. Conclusion

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