CS 301

Lecture 17 – Church–Turing thesis

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March 19, 2018

An abridged modern history of formalizing algorithms

An algorithm is a finite, unambiguous sequence of steps for solving a problem

[1] Polynomial equation with integer coefficients

[2] Statements with $\forall$ and $\exists$

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In 1928, Hilbert posed 3 more problems, the third of which became known as Hilbert's Entscheidungsproblem (decision problem). It asks for an algorithm which takes as input a statement in first-order logic[2]

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Formalizing algorithms

In 1933, Austrian-American mathematician Kurt Gödel and French mathematician Jacques Herbrand[3] defined the class of general recursive functions; these were designed to be intuitively "computable"

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Church and Turing proved that all three of these models of computation are equivalent: A function is λ-computable iff it is computable by a Turing machine iff it is general recursive

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Church–Turing thesis

The Church–Turing thesis is that the intuitive notion of an algorithm (called an "effectively calculable function") is equivalent to a Turing machine

Hilbert's problems

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Unfortunately (depending on your point of view), both of these problems don't have general solutions

Types of problems

There are several types of problems we could consider, consider the problem of paths in a weighted, undirected graph G between two vertices u and v

Decision Is there some path between u and v in G of distance at most k?

Search What is the length of the shortest path between u and v in G?

Counting How many paths between u and v in G have distance at most k?

In this course, we're concerned with the simplest of these: decision problems

Decision problems as languages

We can frame decision problems as membership in a language.

What language corresponds to the decision problem: Is there some path between u and v in graph G of distance at most k?

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$$\{\langle G, u, v, k\rangle \mid G = (V, E) \text{ is an undirected graph, } u, v \in V, \text{ and there is a path from } u \text{ to } v \text{ of distance at most } k\}$$

$\langle G, u, v, k\rangle$ means some fixed representation of the graph G, vertices u and v, and natural number k

Representation

For the rest of the course, we're going to be working with languages whose elements are representations of mathematical objects like

- integers
- graphs
- finite automata
- regular expressions
- CFGs
- PDAs
- Turing machines
- finite sets of mathematical objects
- finite, ordered lists of mathematical objects

We need some way to represent these as strings over some alphabet

Explicit representation

Most of the time, we won't give an explicit representation

We'll mostly assume some representation exists and we can convert from one representation to another easily

Example: Graph representation

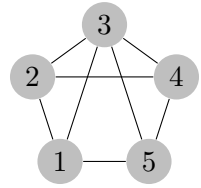
As a mathematical object, an undirected graph[5] is a pair $G = (V, E)$ where

V a finite set of vertices

E a finite set of pairs of vertices

We can specify the graph in a number of ways and easily convert between them

- Vertex and edge lists $G = (V, E)$ where $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1), (1, 3), (2, 4), (3, 5)\}$



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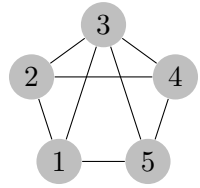

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- Adjacency matrix $a_{ij} = 1$ if $(i, j) \in E$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

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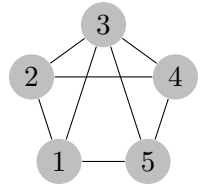

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- Adjacency list: $[\{2, 3, 5\}, \{1, 3, 4\}, \{1, 2, 4, 5\}, \{2, 3, 5\}, \{1, 3, 4\}]$

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Explicit representation for a DFA

Consider $M = (Q, \Sigma, \delta, q_0, F)$ where $Q = \{1, 2, \ldots, m\}$, $\Sigma = \{1, 2, \ldots, n\}$, and $F = \{f_1, f_2, \ldots, f_k\}$

Build it up piece by piece using the alphabet $\Sigma = \{0, 1\}$

$\langle k \rangle = 0^k$ integers

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$$= 101001 \cdots 10^m 1$$

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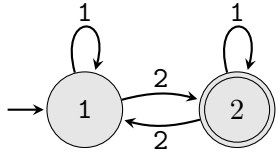
$$\langle M \rangle = 111\langle Q \rangle 111 \langle \Sigma \rangle 111 \langle \delta \rangle 111 \langle q_0 \rangle 111 \langle F \rangle 111$$

Explicit representation

Consider the simple DFA:



$\langle Q \rangle$ = 1 0 1 00 1

$\langle \Sigma \rangle$ = 1 0 1 00 1

$\langle \delta \rangle$ = 11 01010 11 0100100 11 0010100 11 0010010 11

$\langle q_0 \rangle$ = 0

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Representations of other objects

We're not going to bother giving explicit representations of other objects

But we could if we really needed to and that's enough for our purposes

Taking representations as inputs to TMs

We're going to be dealing with languages like

$$A_{\text{DFA}} = \{\langle M, w\rangle \mid M \text{ is a DFA, } w \text{ is a string and } w \in L(M)\}$$

This language is related to the acceptance problem for DFAs which involves testing if the DFA M accepts a string w

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Theorem

The language A_{DFA} is decidable.

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Theorem

The language A_{DFA} is decidable.

Proof.

Let's build a TM to decide A_{DFA}.

M = "On input $\langle B, w\rangle$ where B is a DFA and w is a string,

1. Simulate B on input w.
2. If the simulation ends in an accept state, *accept*. If it ends in a nonaccepting state, *reject*."

□

Some notes about that construction

First, the form of the construction:
M = "On input $\langle \_\_ \rangle$,

1. First step.
2. Second step.
3. Last step."

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Second, note that the input to this TM is supposed to be a representation of a DFA and a string

The representation is not the same thing as the mathematical object

If M is a DFA, then $\langle M \rangle$ is a string

TMs take strings as input, nothing else

How does this construction actually work?

First, the TM M is going to check that the input is a valid representation of a DFA and a string
Which representation is that?

Then, it needs to simulate the DFA by keeping track of the DFA's initial state and how much of the input it has read so far

Initially, the DFA starts in its initial state and it has read none of the input so far

The simulation proceeds step by step where M needs to compare the current state and the next input symbol to the representation of δ to find the next state

Next time

We'll discuss some more decidable languages related to

- the acceptance problem for other models of computation
- The emptiness problem for models of computation (E.g., is the language of a DFA the empty language?)
- The equivalence problem for models of computation (E.g., do two DFAs have the same language?)

Later, we'll discuss some related problems (and languages) that are *not* decidable!