CS 383

Lecture 03 – Nondeterministic Finite Automata (NFAs)

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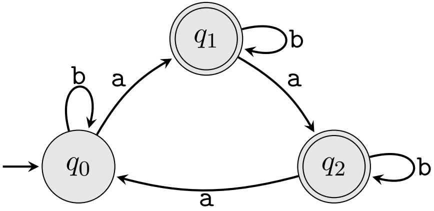
Fall 2023

Review from last time

DFAs are 5-tuples $M = (Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of states
- Σ is an alphabet (finite, nonempty set of symbols)
- $\delta : Q \times \Sigma \to Q$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accepting states



A language A is regular if it is recognized by some DFA M, i.e., $A = L(M) = \{w \in \Sigma^* \mid M \text{ accepts } w\}$

Operations on languages

We can define operations on languages which are functions that map from one or more languages to a new language

Unary operations are functions that map one language to another

- Complement: $\overline{A} = \{w \in \Sigma^* \mid w \notin A\}$
- Reverse: $A^{\mathcal{R}} = \{w^{\mathcal{R}} \mid w \in A\}$
- Kleene star: $A^* = \{w_1 w_2 \cdots w_k \mid k \geq 0 \text{ and } w_i \in A \text{ for all } i\}$
- $\textsc{EndsWith}(A) = \{xw \mid x \in \Sigma^* \text{ and } w \in A\}$
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Binary operations are functions that map a pair of languages to a new language

- Union: $A \cup B$
- Intersection: $A \cap B$
- Concatenation: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$
- ...

Complement

Theorem

If A is a regular language, then $\overline{A}$ is a regular language.

General proof technique

1. Start by assuming that A is a regular language
2. Since (by assumption) A is regular, there is a DFA $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes A (i.e., $L(M) = A$)
3. Construct a new DFA $M' = (Q', \Sigma, \delta', q_0', F')$ that recognizes the language we want to show is regular
4. Since the language is recognized by a DFA, it is regular

Complement

Theorem

If A is a regular language, then $\overline{A}$ is a regular language.

Proof.

1. Assume A is a regular language recognized by DFA $M = (Q, \Sigma, \delta, q_0, F)$

Complement

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Proof.

1. Assume A is a regular language recognized by DFA $M = (Q, \Sigma, \delta, q_0, F)$
2. Construct a new DFA $M' = (Q, \Sigma, \delta, q_0, F')$ that is identical to M except that the accepting and nonaccepting states have been swapped.
   That is, $F' = Q \setminus F$.

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3. If M accepts w, then when M is run on w, it ends in a state $q \in F$. Thus, when M' is run on w, it ends in state $q \notin Q \setminus F = F'$ so M' rejects w.

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4. If M rejects w, then when M is run on w, it ends in state $q \notin F$. Thus, when M' is run on w, it ends in state $q \in Q \setminus F = F'$ so M' accepts w.
5. Therefore, $L(M') = \overline{A}$. Since DFA M' recognizes $\overline{A}$, $\overline{A}$ is regular. $\square$

Complement example

Let $A = \{w \mid \texttt{aba} \text{ is a substring of } w\}$

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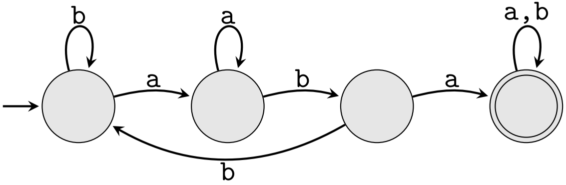

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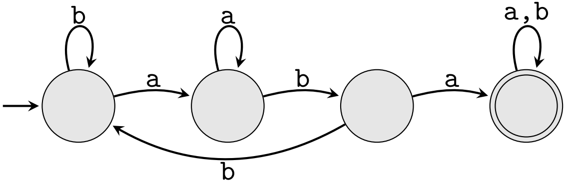

$\overline{A} = \{w \mid \texttt{aba} \text{ is } \textit{not} \text{ a substring of } w\}$

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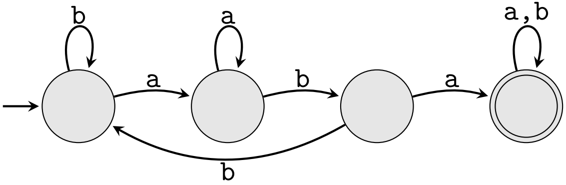

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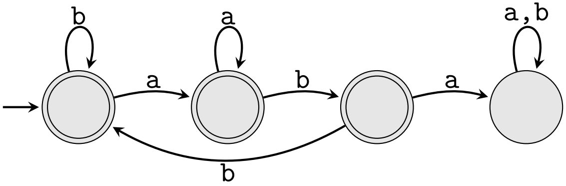

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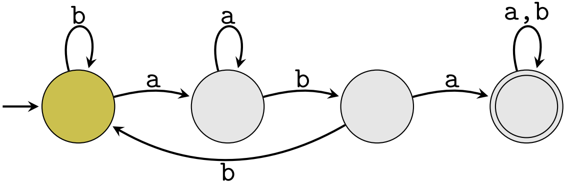

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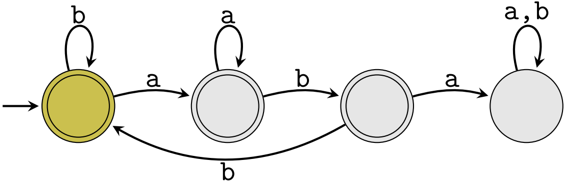

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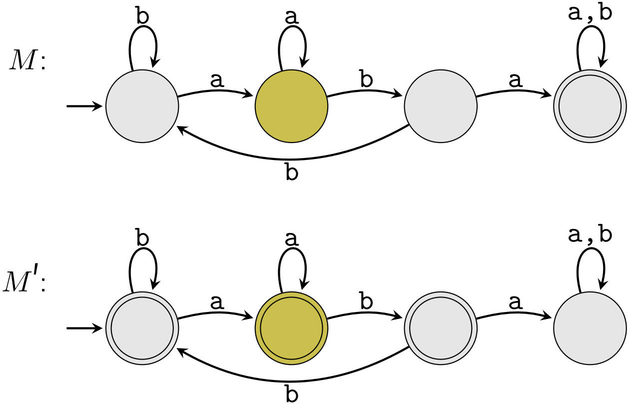

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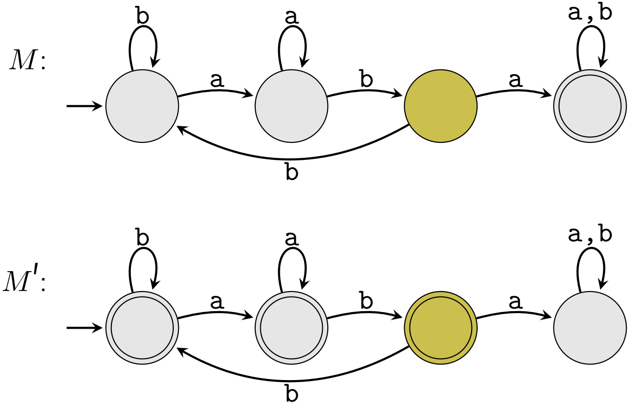

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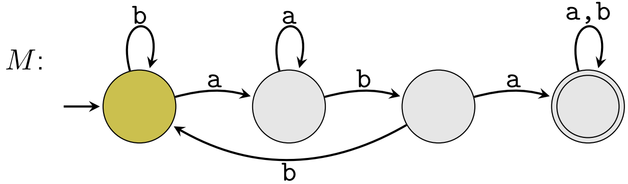

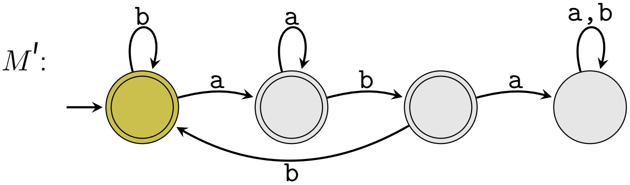

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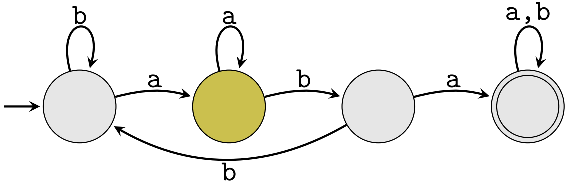

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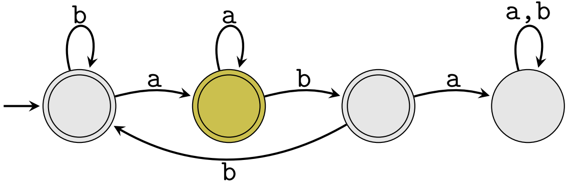

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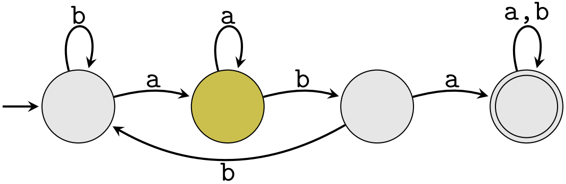

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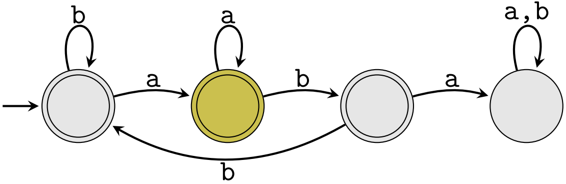

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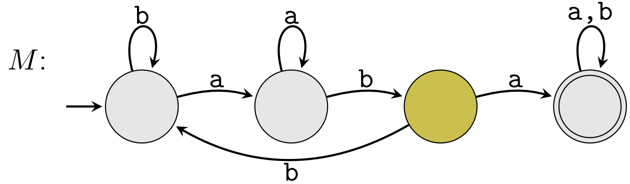

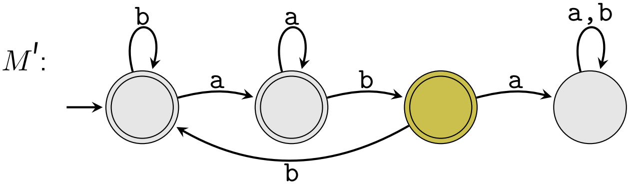

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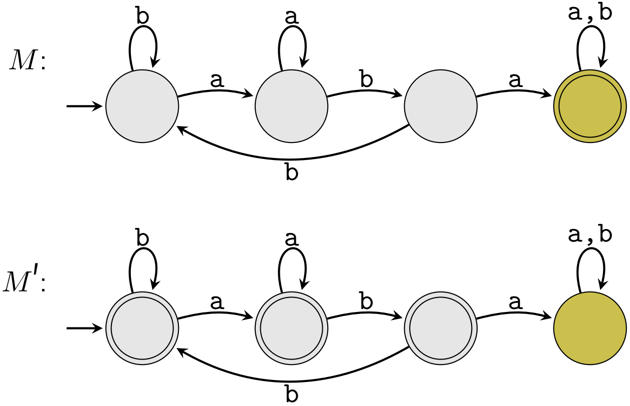

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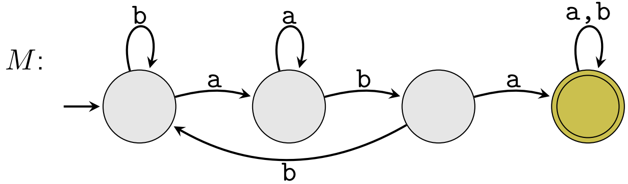

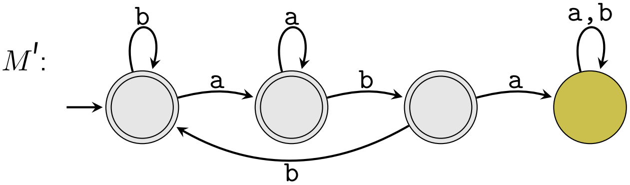

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Union

Theorem

If A and B are regular languages, then $A \cup B$ is regular.

Proof.

1. Assume DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognizes A and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognizes B.

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2. Build a new DFA $M = (Q, \Sigma, \delta, q_0, F)$ with states consisting of pairs of states from M_1 and M_2. Formally,

$$\begin{aligned} Q &= Q_1 \times Q_2 \\ q_0 &= (q_1, q_2) \\ \delta\big((q, r), t\big) &= \big(\delta_1(q, t),\ \delta_2(r, t)\big) \\ F &= \{(q, r) \mid q \in F_1 \text{ or } r \in F_2\}. \end{aligned}$$

As M transitions from state (q, r) to state (q', r'), the first element changes according to δ_1 and the second according to δ_2.

Union

3. Consider running M_1, M_2, and M on string w. The three DFAs end in states q, r, and (q, r), respectively. If $w \in A$, then M_1 accepts w so $q \in F_1$ and thus $(q, r) \in F$ so M accepts w. Similarly, if $w \in B$, then M_2 accepts w so $r \in F_2$ and thus $(q, r) \in F$. If w is in neither A nor B, then $q \notin F_1$ and $r \notin F_2$ so $(q, r) \notin F$.
4. Therefore, $L(M) = A \cup B$ so $A \cup B$ is regular. $\square$

Union example

Let $A = \{w \mid \texttt{aba} \text{ is not a substring of } w\}$ and M_1 recognize A
Let $B = \{w \mid |w| \text{ is even}\}$ and M_2 recognize B

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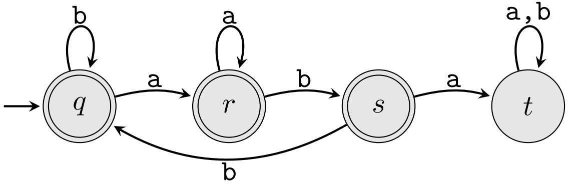
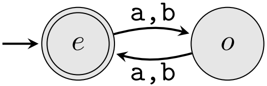


M_2:



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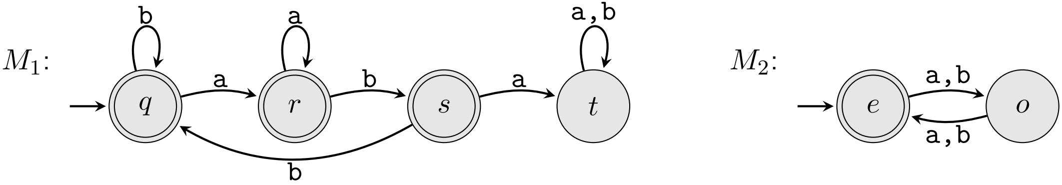

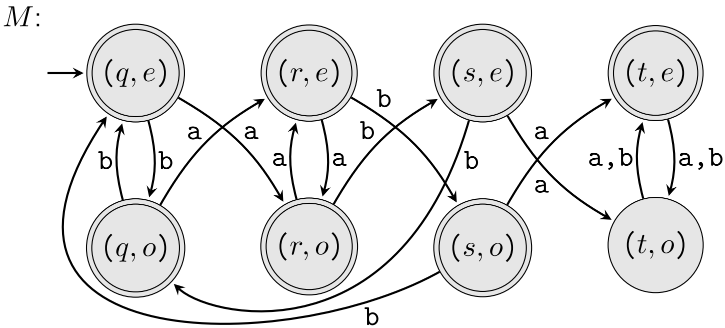

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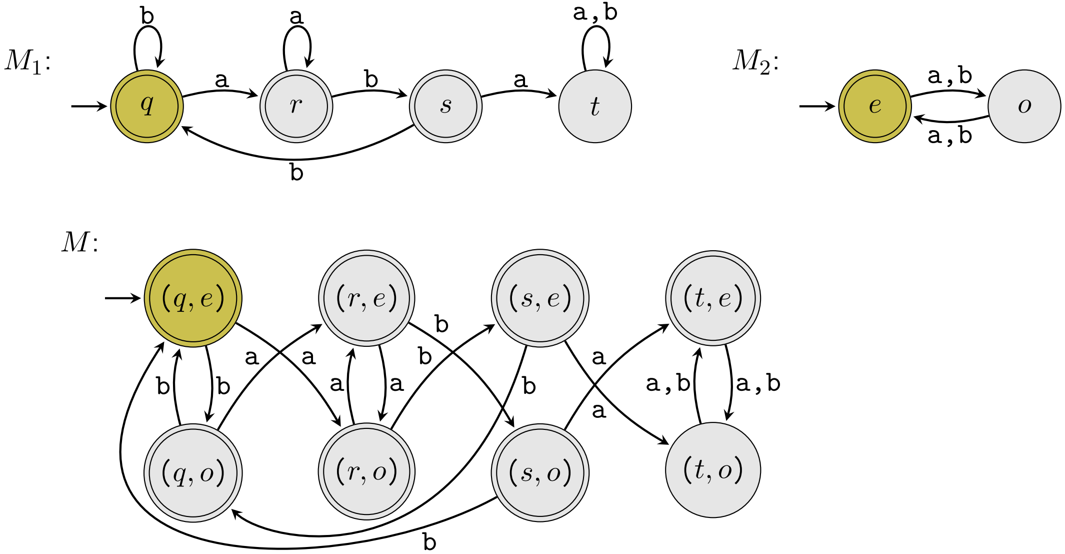

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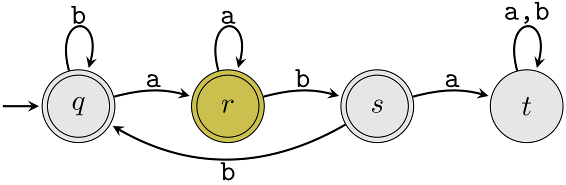

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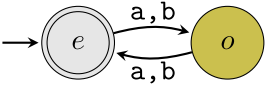

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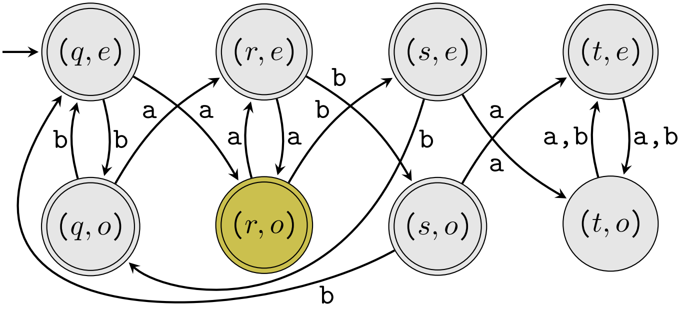

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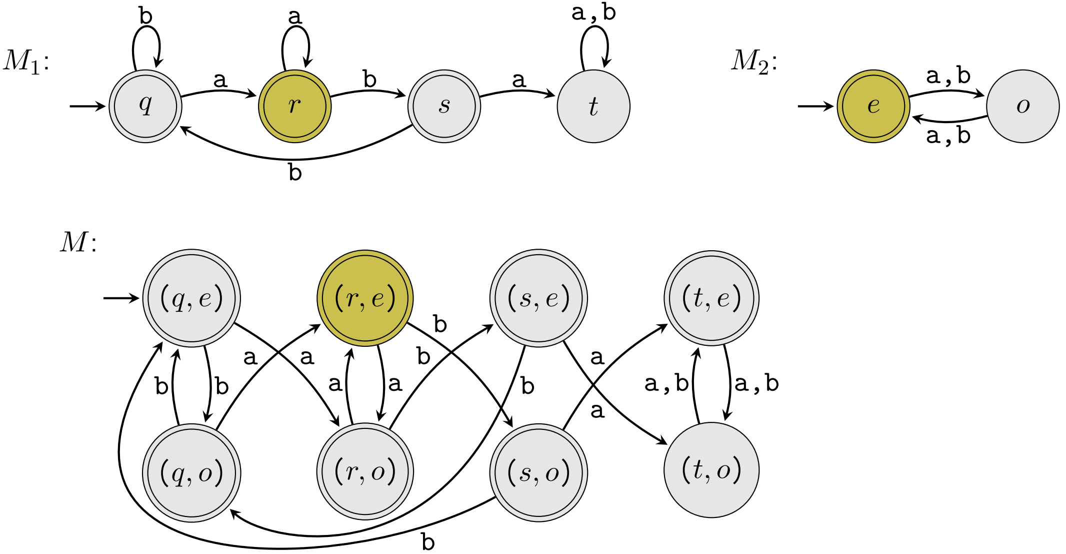

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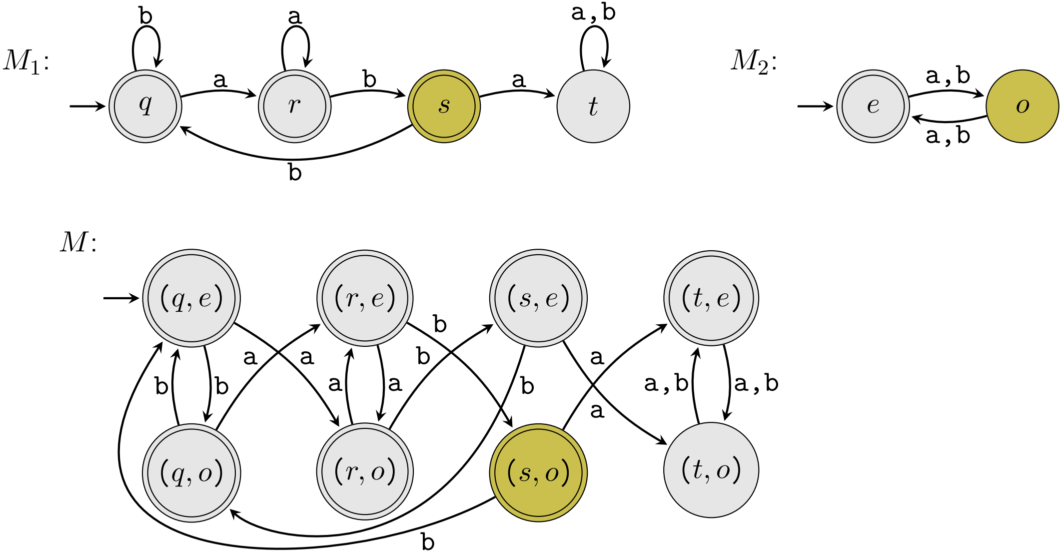

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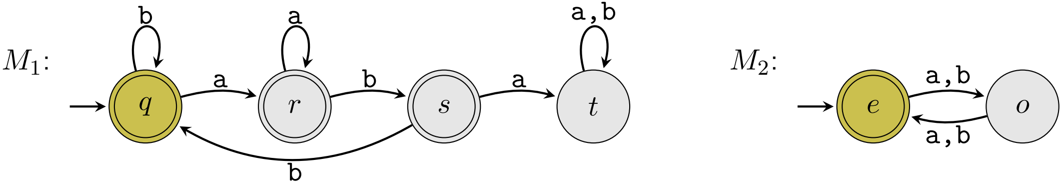

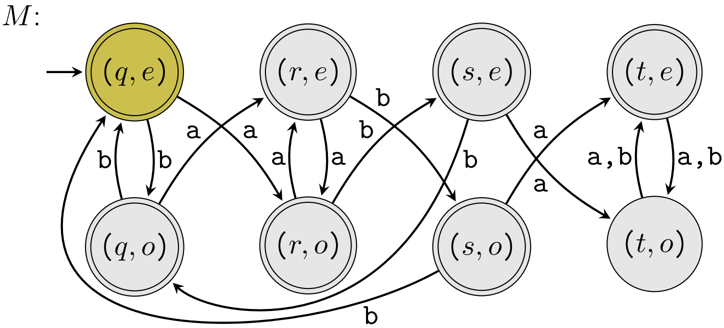

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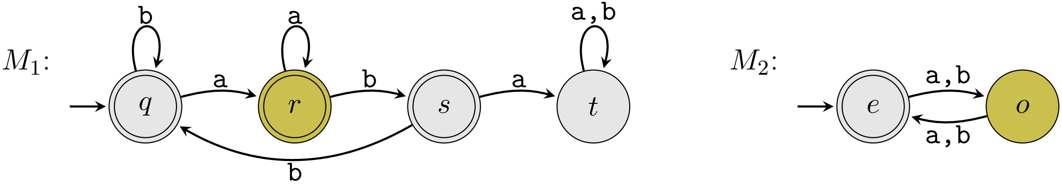

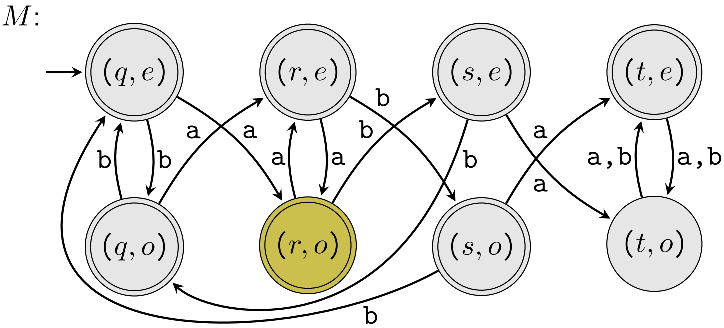

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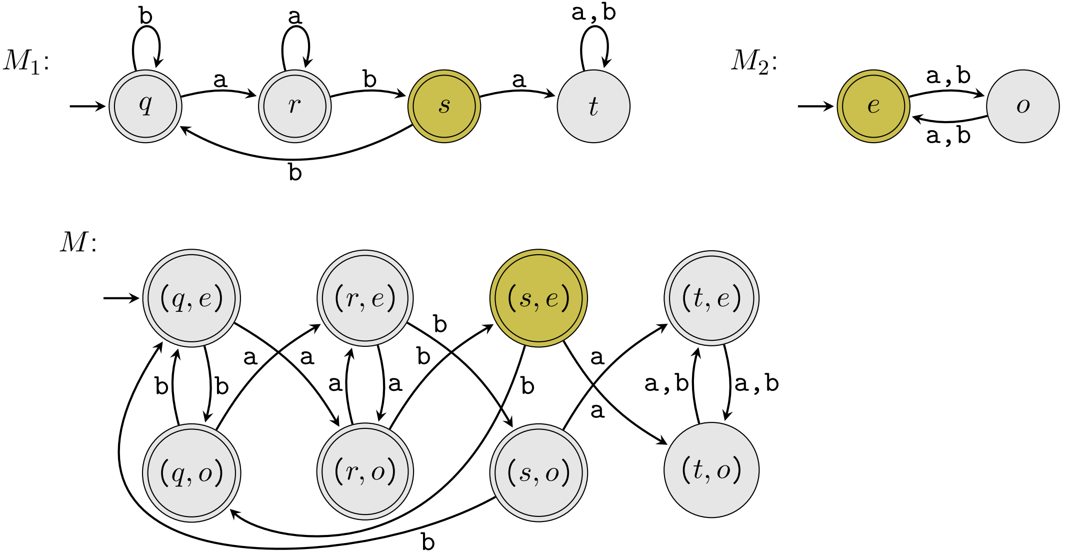

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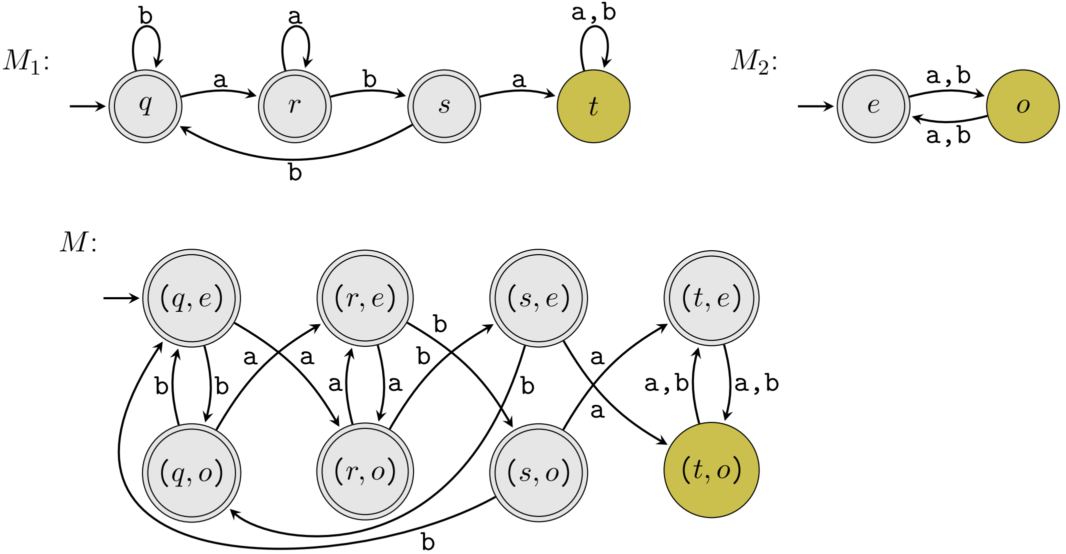

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EndsWith

$$\textsc{EndsWith}(A) = \{xw \mid x \in \Sigma^* \text{ and } w \in A\}$$

- $A = \{\texttt{a}, \texttt{aab}, \texttt{bab}\}$; $\textsc{EndsWith}(A) = \{w \mid w \text{ ends with } \texttt{a}, \texttt{aab}, \text{or } \texttt{bab}\}$
- $B = \{\texttt{b}^k \mid k > 0\}$; $\textsc{EndsWith}(B) = \{w \mid w \text{ ends with 1 or more } \texttt{b}\}$
- $C = \{\texttt{a}^k\texttt{b}^k \mid k \geq 0\}$;
  $\textsc{EndsWith}(\text{C}) = \{w \mid w \text{ ends with } \texttt{a}^k\texttt{b}^k \text{ for some } k \geq 0\} = \Sigma^*$ [Why?]

A simple theorem

Theorem

If A is regular, then $\textsc{EndsWith}(A) = \{xw \mid x \in \Sigma^* \text{ and } w \in A\}$ *is regular.*

Proof technique

Start by assuming that A is regular and thus there exists a DFA M such that $L(M) = A$

Now construct a new DFA M' such that $L(M') = \textsc{EndsWith}(A)$.

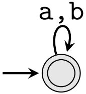
Ideally, this new DFA would have two parts:

1. some states that read symbols from Σ^* (i.e., matching the symbols of x)
2. a copy of M to accept the last part of the string which should be in A

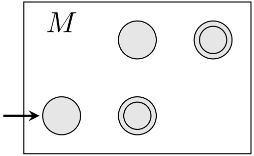
A simple theorem proof difficulty

The two parts are individually easy

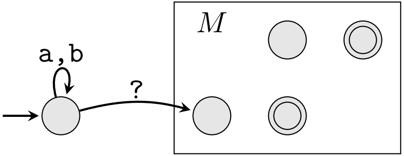
1. Match symbols from Σ^* (assume $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, easy to generalize)



2. A copy of M



But how can we combine them?



Determinism

DFAs are deterministic because at every step, the DFA has exactly one thing it can do

When M is in some state $q \in Q$ and the next input symbol is $t \in \Sigma$, the only thing it can do is move to state $\delta(q, t)$

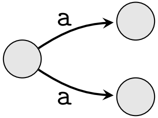
Graphically, we don't allow any state to have multiple edges (transitions) labeled with the same symbol going to different states

Similarly, we don't allow a state to not have a transition labeled with a symbol of Σ

Nondeterminism

Let's build a new type of machine, a nondeterministic finite automaton (NFA), where at each step, it has zero or more things it can do

Three new options

1. Multiple transitions from a state on the same symbol 

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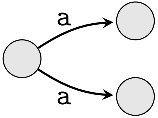
1. Multiple transitions from a state on the same symbol 

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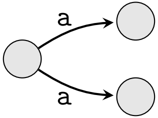
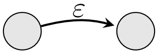
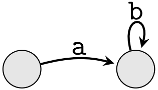
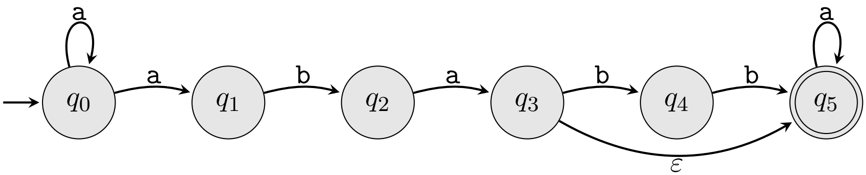
1. Multiple transitions from a state on the same symbol 

2. Transitions on no input 

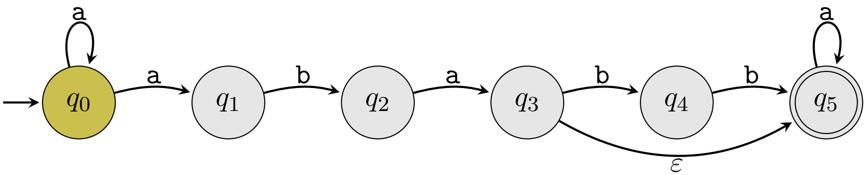
3. States without transitions on some (or all) symbols 

Example



Let's run this on input `ababb`

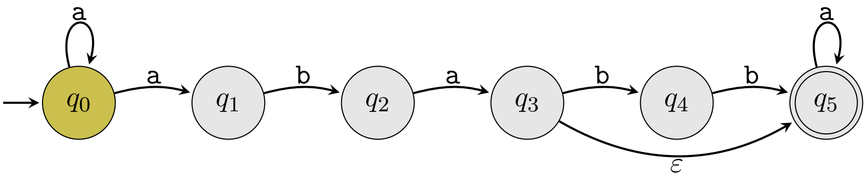
Example



Let's run this on input `ababb`

1. Start in q_0, first symbol is a, two choices, let's stay in q_0

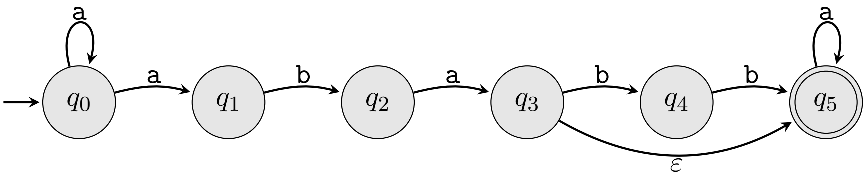
Example



Let's run this on input `ababb`

1. Start in q_0, first symbol is `a`, two choices, let's stay in q_0
2. Next symbol is `b`, but there are no transitions labeled `b`

Example

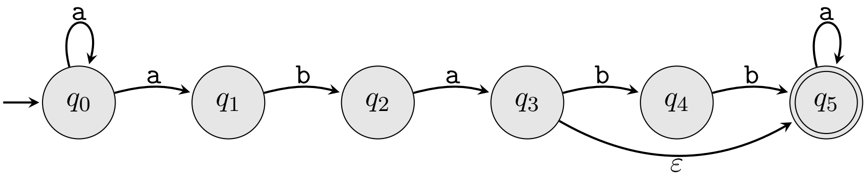


Let's run this on input ababb

1. Start in q_0, first symbol is a, two choices, let's stay in q_0
2. Next symbol is b, but there are no transitions labeled b
3. Now the machine is dead because there's no active state

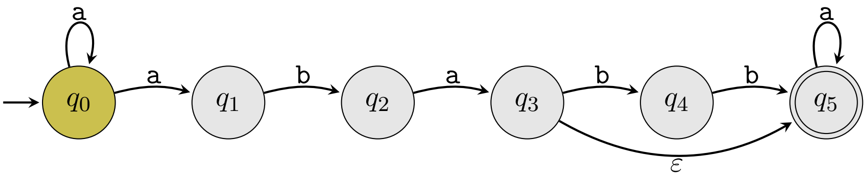
Since the machine didn't end in an accepting state. Is ababb ✖Rejected?

Example again



Let's run this on input `ababb` again

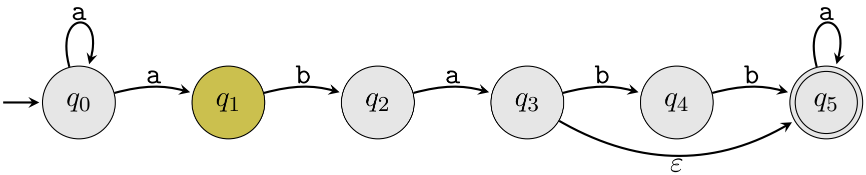
Example again



Let's run this on input `ababb` again

1. Start in q_0, first symbol is a, two choices, let's go to q_1

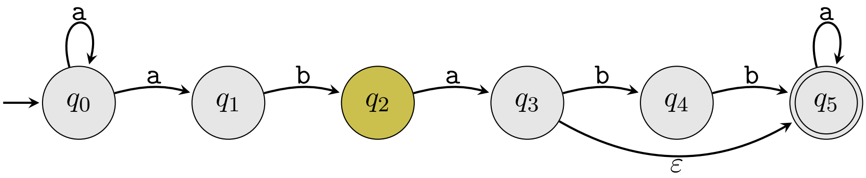
Example again



Let's run this on input ababb again

1. Start in q_0, first symbol is a, two choices, let's go to q_1
2. Next symbol is b, go to q_2

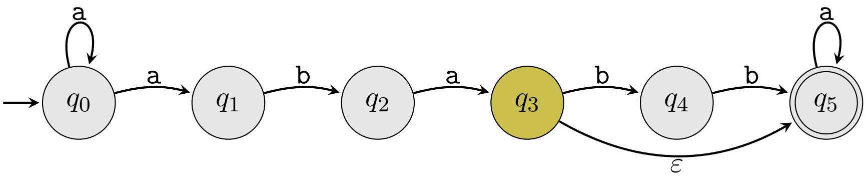
Example again



Let's run this on input `ababb` again

1. Start in q_0, first symbol is `a`, two choices, let's go to q_1
2. Next symbol is `b`, go to q_2
3. Next symbol is `a`, go to q_3

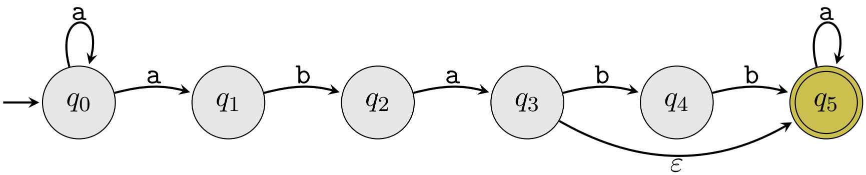
Example again



Let's run this on input `ababb` again

1. Start in q_0, first symbol is `a`, two choices, let's go to q_1
2. Next symbol is `b`, go to q_2
3. Next symbol is `a`, go to q_3
4. We have two choices: follow the ε transition or not, let's follow it

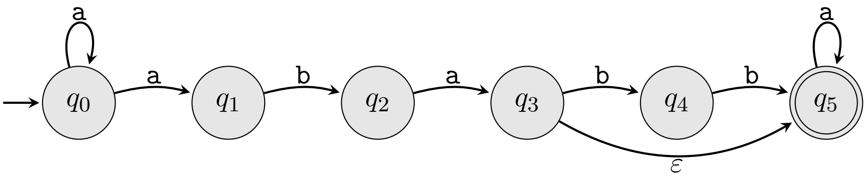
Example again



Let's run this on input `ababb` again

1. Start in q_0, first symbol is `a`, two choices, let's go to q_1
2. Next symbol is `b`, go to q_2
3. Next symbol is `a`, go to q_3
4. We have two choices: follow the ε transition or not, let's follow it
5. Next symbol is `b`, but there are no transitions labeled `b`

Example again

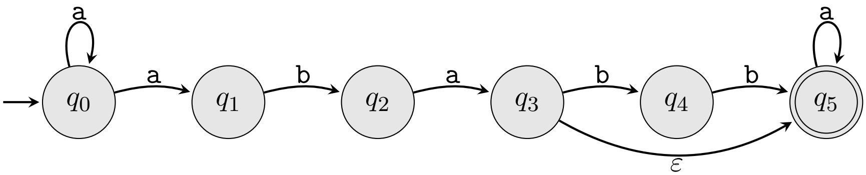


Let's run this on input ababb again

1. Start in q_0, first symbol is a, two choices, let's go to q_1
2. Next symbol is b, go to q_2
3. Next symbol is a, go to q_3
4. We have two choices: follow the ε transition or not, let's follow it
5. Next symbol is b, but there are no transitions labeled b
6. Now the machine is dead because there's no active state

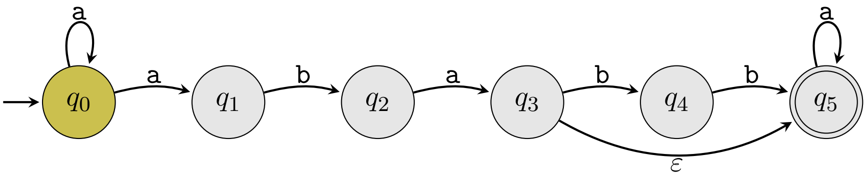
Once again, it didn't end in an accepting state.

Example yet again



Let's run this on input `ababb` a third time

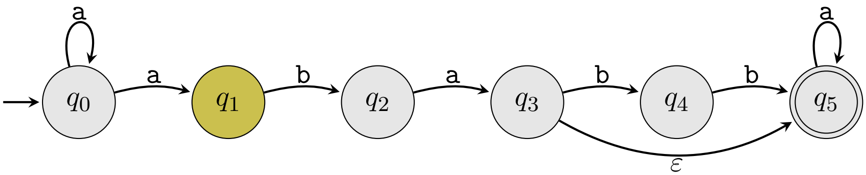
Example yet again



Let's run this on input `ababb` a third time

1. Start in q_0, first symbol is `a`, two choices, let's go to q_1

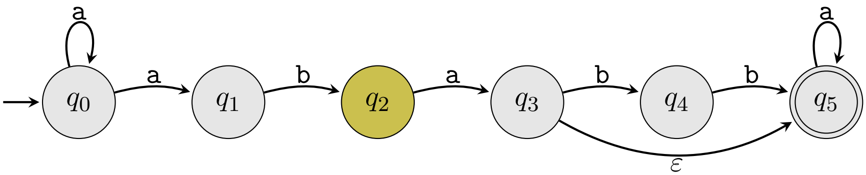
Example yet again



Let's run this on input `ababb` a third time

1. Start in q_0, first symbol is `a`, two choices, let's go to q_1
2. Next symbol is `b`, go to q_2

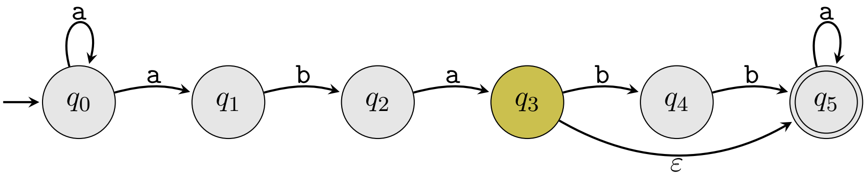
Example yet again



Let's run this on input `ababb` a third time

1. Start in q_0, first symbol is `a`, two choices, let's go to q_1
2. Next symbol is `b`, go to q_2
3. Next symbol is `a`, go to q_3

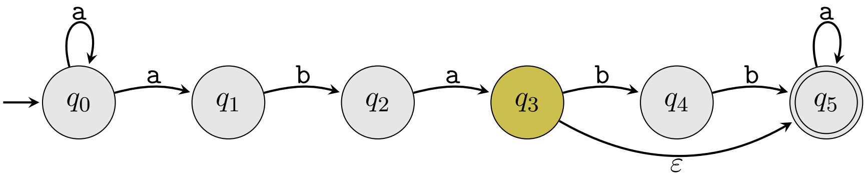
Example yet again



Let's run this on input `ababb` a third time

1. Start in q_0, first symbol is `a`, two choices, let's go to q_1
2. Next symbol is `b`, go to q_2
3. Next symbol is `a`, go to q_3
4. We have two choices: follow the ε transition or not, let's *not* follow it

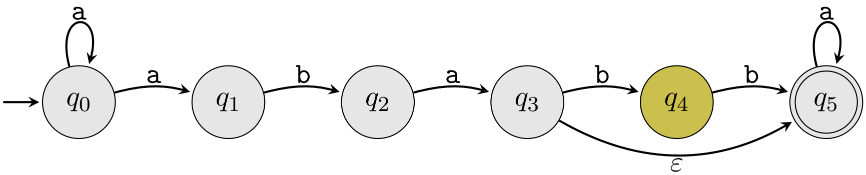
Example yet again



Let's run this on input `ababb` a third time

1. Start in q_0, first symbol is `a`, two choices, let's go to q_1
2. Next symbol is `b`, go to q_2
3. Next symbol is `a`, go to q_3
4. We have two choices: follow the ε transition or not, let's *not* follow it
5. Next symbol is `b`, go to q_4

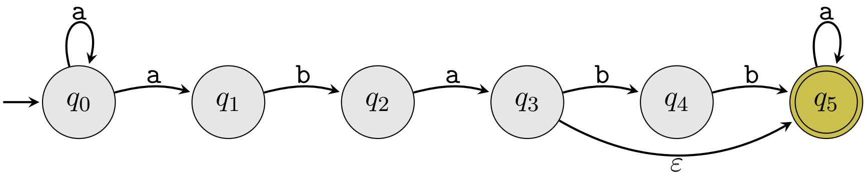
Example yet again



Let's run this on input `ababb` a third time

1. Start in q_0, first symbol is `a`, two choices, let's go to q_1
2. Next symbol is `b`, go to q_2
3. Next symbol is `a`, go to q_3
4. We have two choices: follow the ε transition or not, let's *not* follow it
5. Next symbol is `b`, go to q_4
6. Next symbol is `b`, go to q_5

Example yet again



Let's run this on input `ababb` a third time

1. Start in q_0, first symbol is `a`, two choices, let's go to q_1
2. Next symbol is `b`, go to q_2
3. Next symbol is `a`, go to q_3
4. We have two choices: follow the ε transition or not, let's *not* follow it
5. Next symbol is `b`, go to q_4
6. Next symbol is `b`, go to q_5
7. There's no more input and the machine ended in an accepting state so `ababb` is ✔Accepted

Was `ababb` accepted or rejected?

Two choices we made led to the machine dying because it couldn't follow a transition

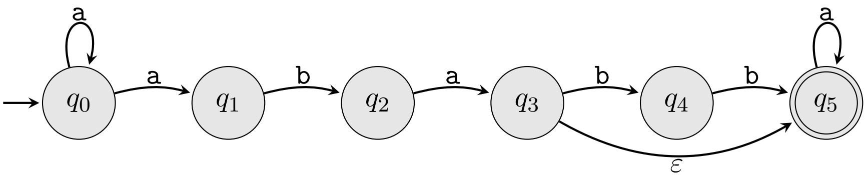
The third choice we made ended in an accepting state

Let's say an NFA accepts a string if *any* path through the NFA ends in an accepting state

So `ababb` was ✔Accepted

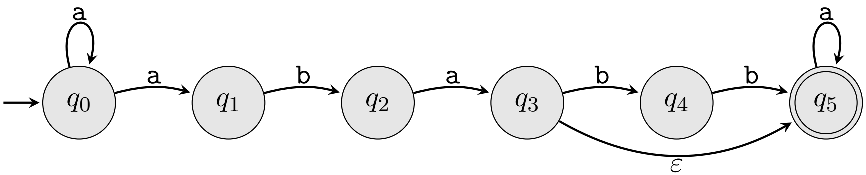
Language of the NFA

What strings are accepted by this NFA?



Language of the NFA

What strings are accepted by this NFA?



Strings starting with at least 1 a, followed by ba, optionally followed by bb, followed by any number of as: $\{\mathtt{a}^m \mathtt{ba} w a^n \mid m \geq 1 \text{ and } n \geq 0 \text{ and } w \in \{\varepsilon, \mathtt{bb}\}\}$

Running NFAs

It was a pain to run the NFA multiple times on the same input, making difference choices

Let's instead keep track of all possible states the NFA N can be in at each point in its computation

Rather than having a single current state, let's have a set of current states, call it C

At each step, we're going to update C

Procedure for running NFAs

Procedure

1. Set C = $\{q_0\}$, the set containing only the start state
2. Set C = $\{q \mid q$ is reachable from C by following 0 or more ε-transitions$\}$
3. For each successive symbol t in the input w,
4.   Set C = $\{q \mid$ there is a transition to q on symbol t from some state in $C\}$
5.   Set C = $\{q \mid q$ is reachable from C by following 0 or more ε-transitions$\}$
6. If C contains any accepting states, N accepts w, otherwise N rejects w

Running our NFAs

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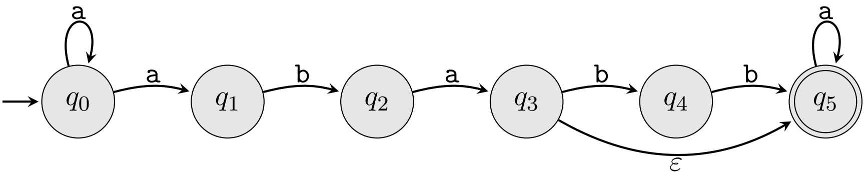

`ababb`

Running our NFAs

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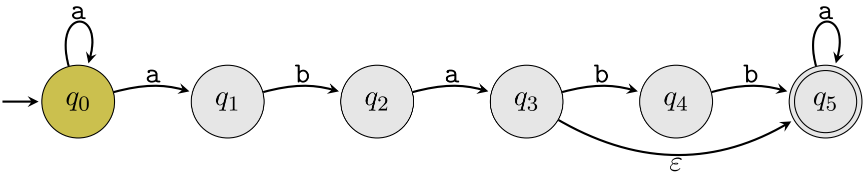

ababb

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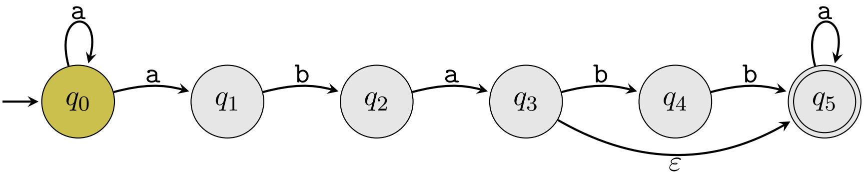

ababb

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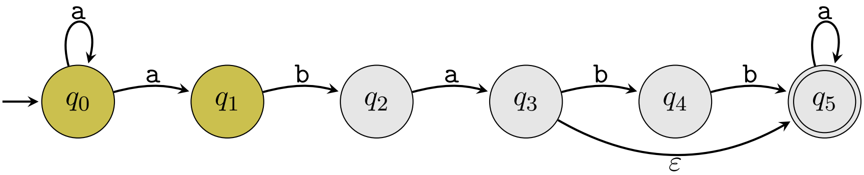

ababb

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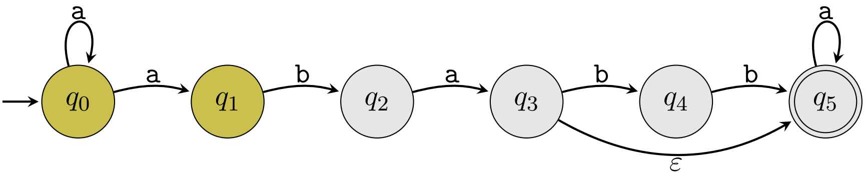

ababb

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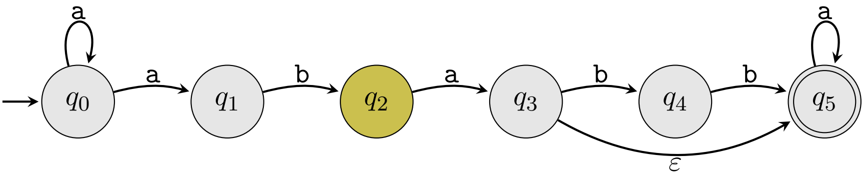

ababb

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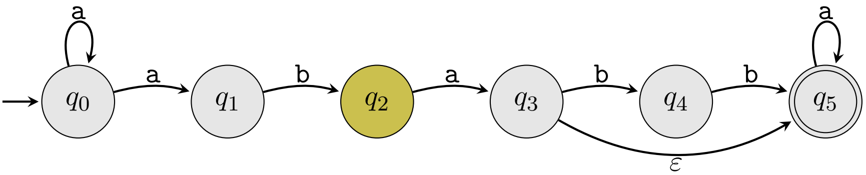

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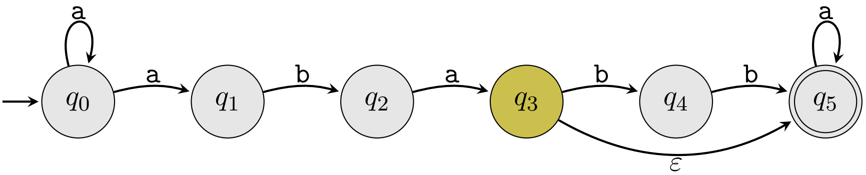

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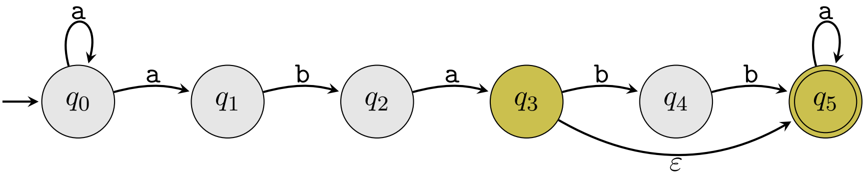

ababb

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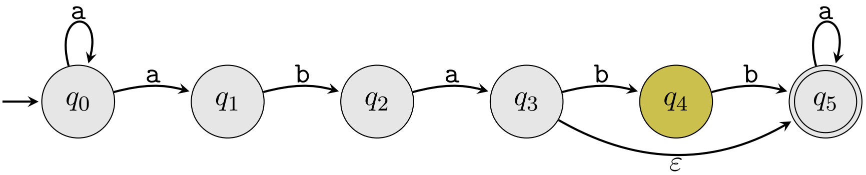

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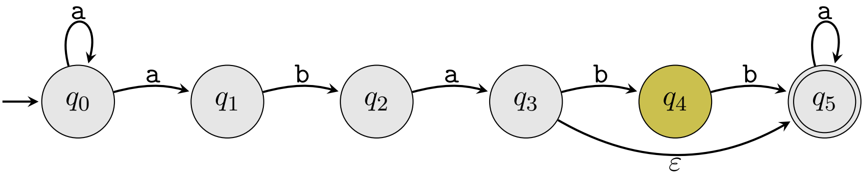

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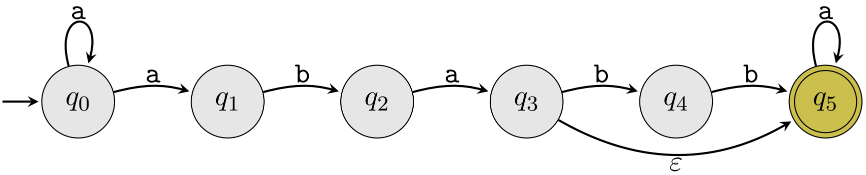

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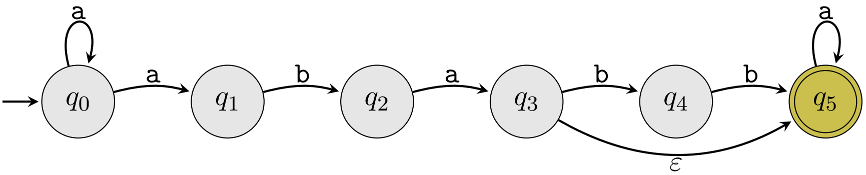

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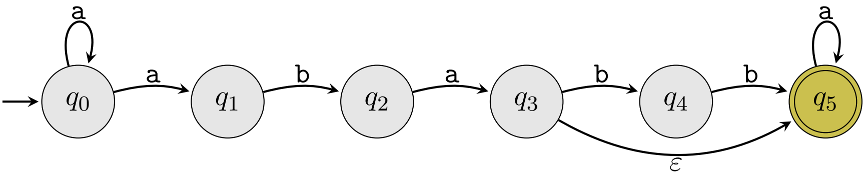

ababb ✔ Accepted

Nondeterministic finite automaton (NFA)

A nondeterministic finite automaton (NFA) is a 5-tuple $N = (Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of states
- Σ is an alphabet
- $\delta : Q \times \Sigma_\varepsilon \to P(Q)$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accepting (or final) states

$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$ is the alphabet Σ augmented with an additional symbol ε which we use to denote transitions on no input

$P(Q)$ is the power set of Q so δ returns a set of next states

Transition functions

DFAs have transitions of the form $\delta : Q \times \Sigma \to Q$

For each (state, symbol) pair, δ returns a single state

NFAs have transitions of the form $\delta : Q \times \Sigma_\varepsilon \to P(Q)$

For each (state, symbol) pair, δ returns 0 or more states

For each (state, ε), δ returns 0 or more states

Formalizing NFA computation

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA and let $w = w_1 w_2 \cdots w_n$ be a string where $w_i \in \Sigma_\varepsilon$

N accepts w if there exist states $r_0, r_1, \ldots, r_n \in Q$ such that

1. $r_0 = q_0$
   [The NFA starts in the start state]
2. $r_i \in \delta(r_{i-1}, w_i)$ for $i \in \{1, 2, \ldots, n\}$
   [The NFA moves from state r_{i-1} to one of the possible next states according to δ]
3. $r_n \in F$
   [The NFA ends in an accepting state]

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Two key differences from DFAs

1. w_i is either an alphabet symbol or ε
   E.g., if $w = \texttt{abaa}$, then we can write $w = \varepsilon\texttt{ab}\varepsilon\varepsilon\varepsilon\texttt{a}\varepsilon\texttt{a}$
2. $r_i \in \delta(r_{i-1}, w_i)$ since δ returns a set of next possible states

The sequence of $n + 1$ states $r_0, r_1, \ldots, r_n$ is one of the possible sequences of states that the NFA moves through on input w

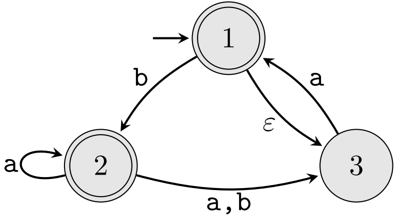
Language of an NFA

The language of an NFA N is $L(N) = \{w \mid N \text{ accepts } w\}$

We say N recognizes a language A to mean $L(N) = A$

[This is analogous to DFAs]

Example



$N = (Q, \Sigma, \delta, q_0, F)$ where

$$Q = \{1, 2, 3\}$$

$$\Sigma = \{\mathtt{a}, \mathtt{b}\}$$

$$q_0 = 1$$

$$F = \{1, 2\}$$

δ :

	a	b	ε
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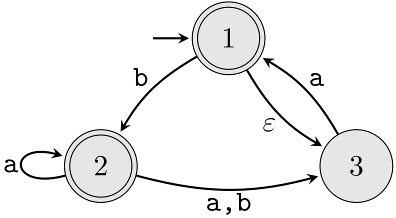

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Consider string w = abaa

Write w as εabaa then one of the possible sequences of states N moves through is

$r_0 \quad r_1 \quad r_2 \quad r_3 \quad r_4 \quad r_5$

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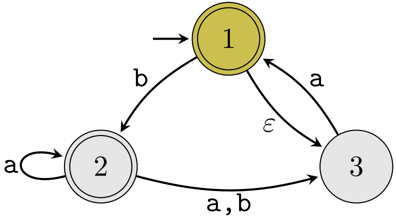

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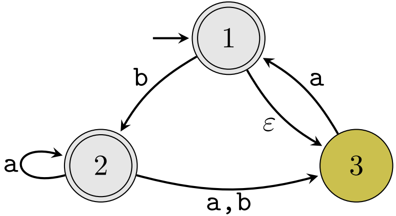

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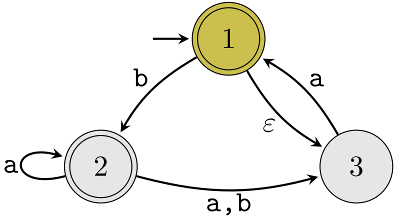

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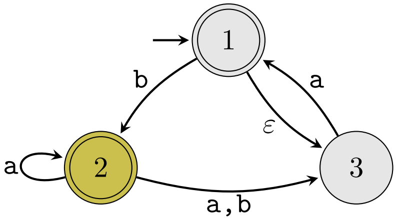

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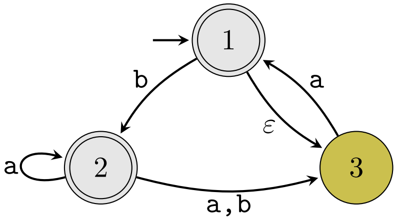

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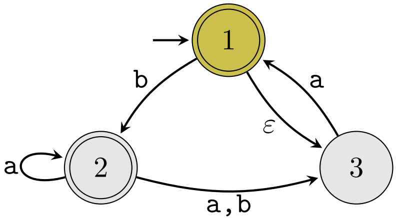

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All three conditions for acceptance hold

1. $r_0 = q_0$
2. $r_i \in \delta(r_{i-1}, w_i)$ for $i \in \{1, 2, \ldots, n\}$
3. $r_n \in F$

Converting NFAs to DFAs

Theorem

For every NFA N, there exists a DFA M such that $L(M) = L(N)$.

We can prove this by following our procedure for running NFAs

Procedure

1. Set $C = \{q_0\}$, the set containing only the start state
2. Set $C = \{q \mid q$ is reachable from C by following 0 or more ε-transitions$\}$
3. For each successive symbol t in the input w,
4. Set $C = \{q \mid$ there is a transition to q on symbol t from some state in $C\}$
5. Set $C = \{q \mid q$ is reachable from C by following 0 or more ε-transitions$\}$
6. If C contains any accepting states, N accepts w, otherwise N rejects w

Some helpful notation

Given an NFA $N = (Q, \Sigma, \delta, q_0, F)$, define a new function E that takes a set of states $S \subseteq Q$ as input and returns the set of states reachable by following 0 or more ε-transitions from states in S

Formally, $E : P(Q) \to P(Q)$ given by
$E(S) = \{q \mid q \text{ is reachable from some } r \in S \text{ by following 0 or more } \varepsilon\text{-transitions}\}$

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Procedure (ver. 2)

1. Set $C = E(\{q_0\})$
2. For each successive symbol t in the input w,
3. Set $C = \left\{q \;\middle|\; q \in E(\delta(r,t)) \text{ for some } r \in C\right\}$
4. If $C \cap F \neq \varnothing$, N accepts w, otherwise N rejects w

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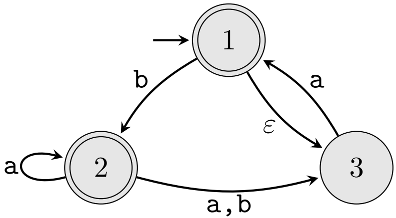

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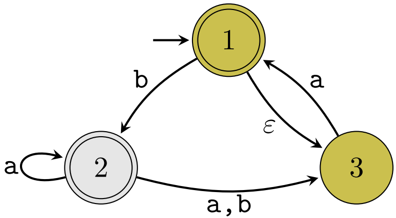

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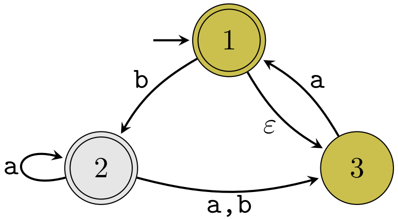

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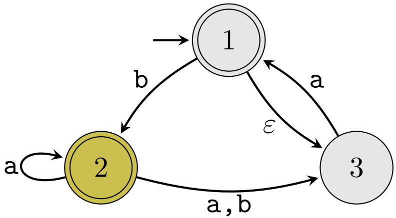

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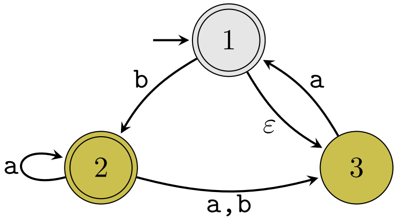

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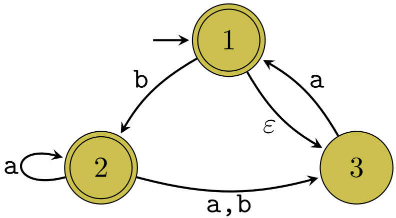

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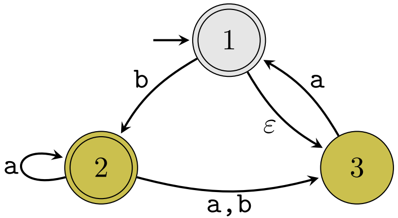

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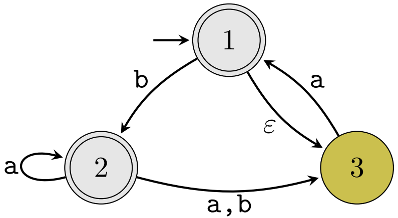

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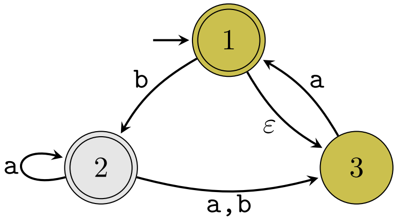

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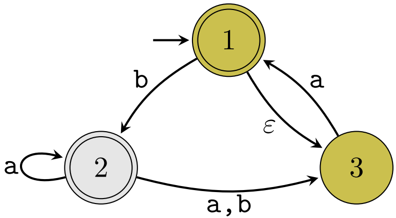

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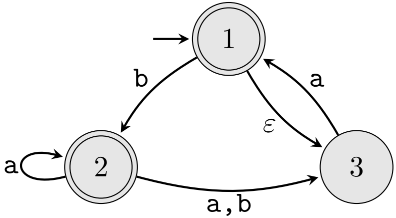

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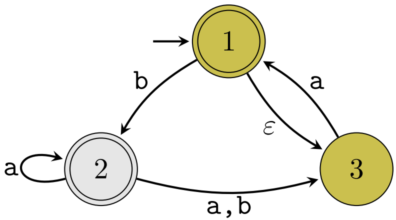

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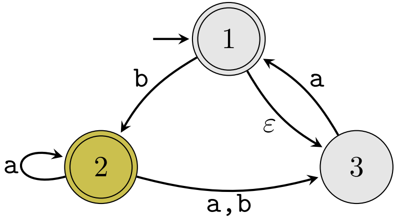

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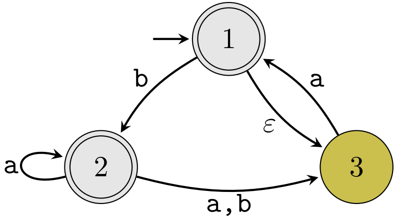

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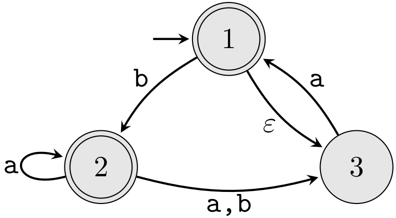

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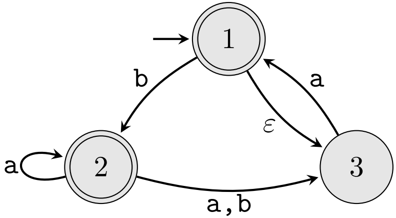

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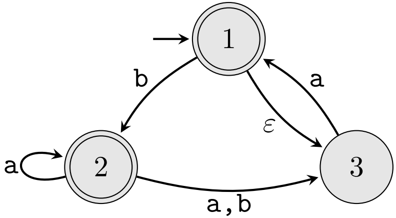

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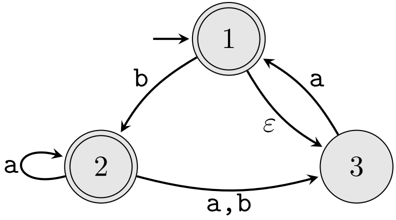

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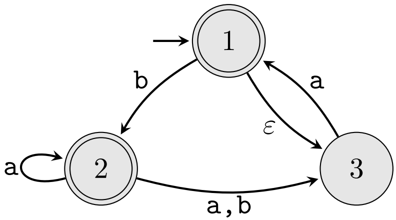

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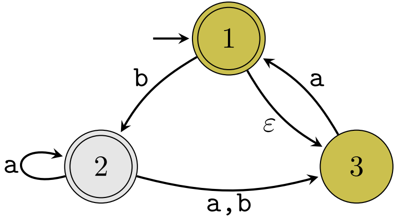
- `bbbab` ✖Rejected
- `bb`

Running the procedure again

Procedure (ver. 2)

1. Set $C = E(\{q_0\})$
2. For each successive symbol t in the input w,
3. Set $C = \{q \mid q \in E(\delta(r, t)) \text{ for some } r \in C\}$
4. If $C \cap F \neq \varnothing$, N accepts w, otherwise N rejects w



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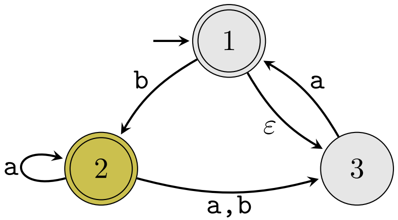

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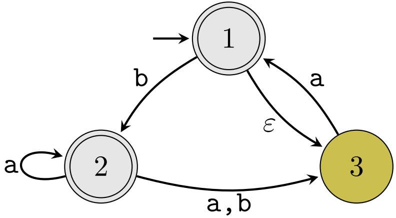

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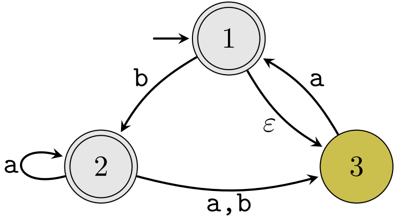

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Converting an NFA to a DFA

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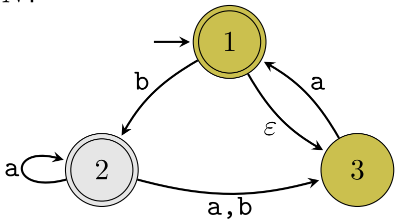
- M's accepting states are every subset of Q that contains at least one of N's accepting states: $F' = \{S \mid S \subseteq Q \text{ and } S \cap F \neq \varnothing\}$

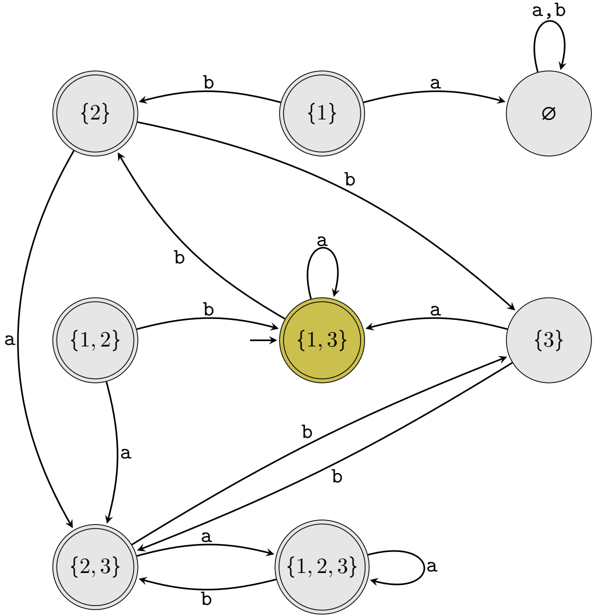
Converting our example to a DFA

N:



abaababb

M:



Converting our example to a DFA

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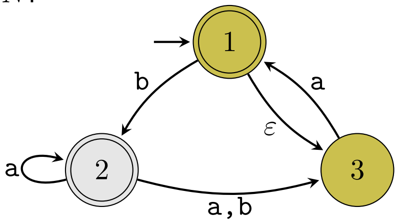

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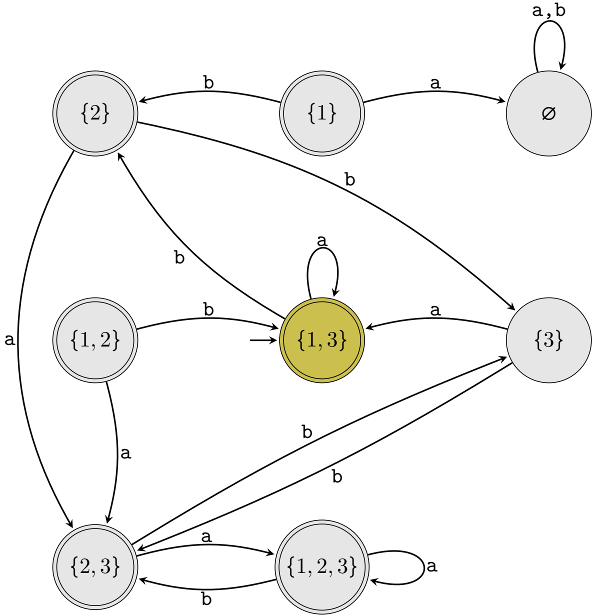

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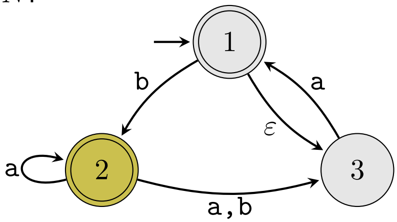

abaababb

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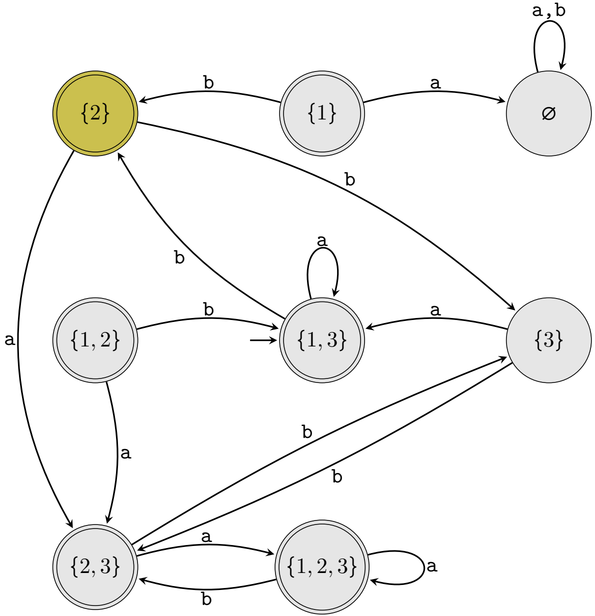

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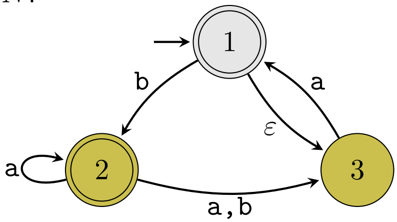

abaababb

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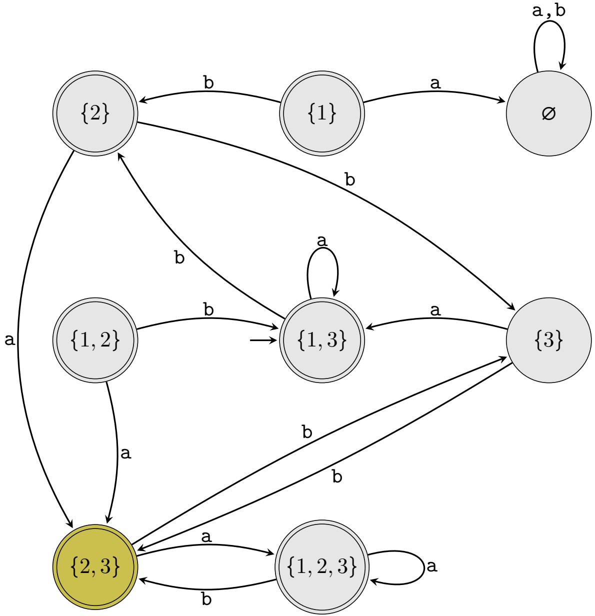

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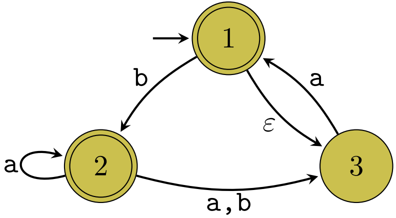

abaababb

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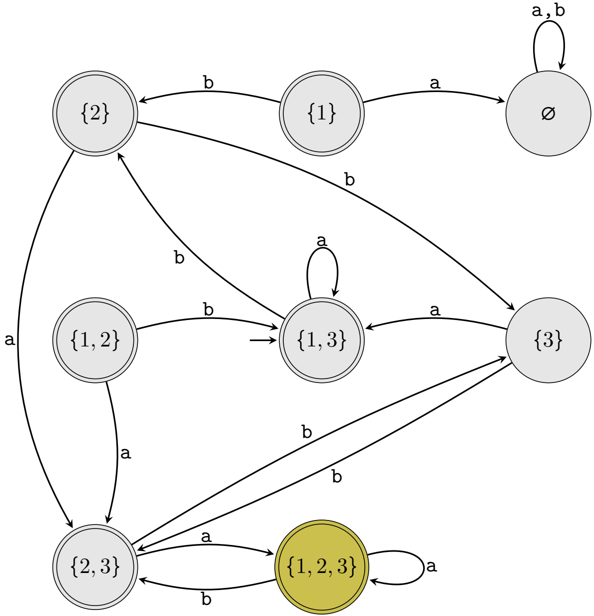

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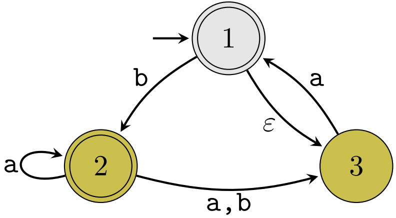

abaababb

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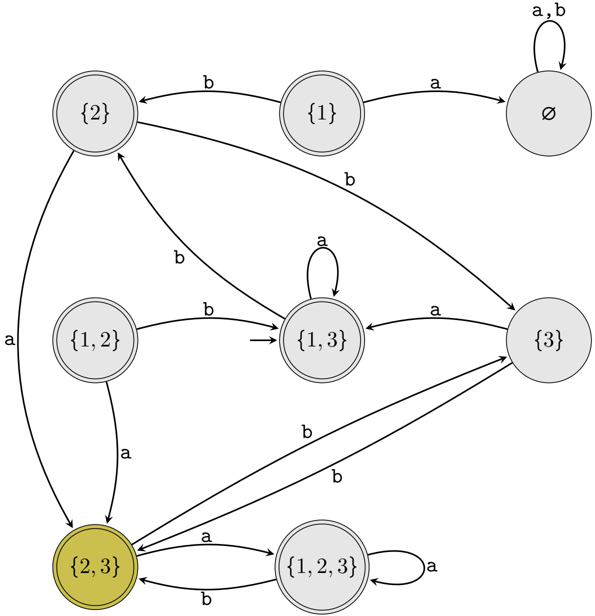

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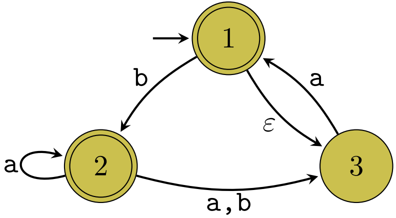

abaababbb

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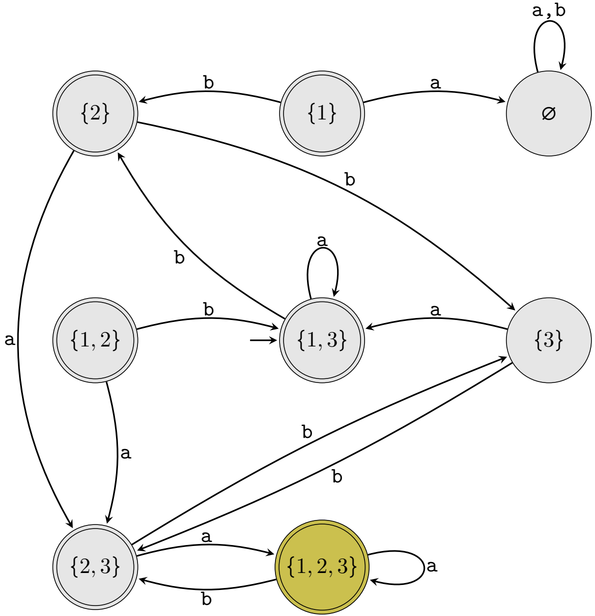

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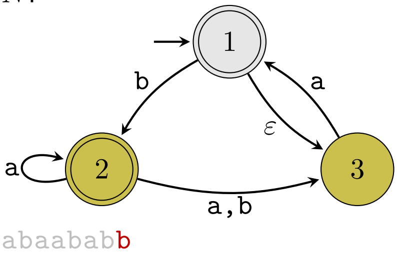

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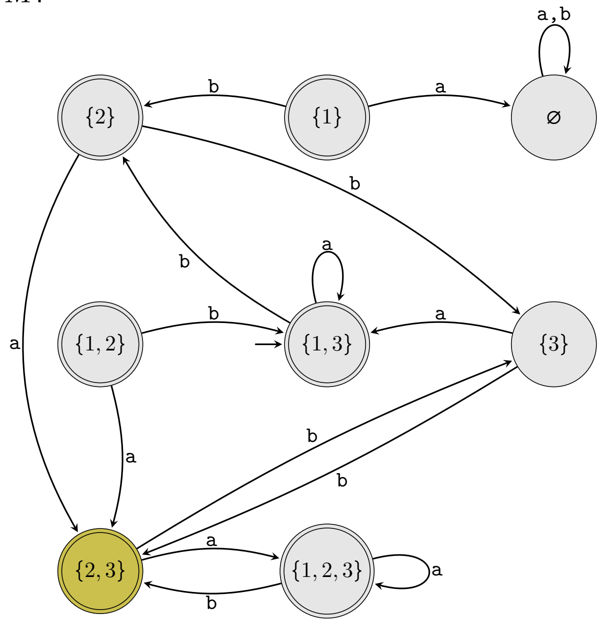

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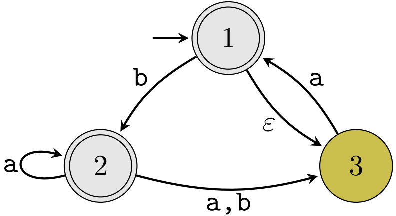

abaababb

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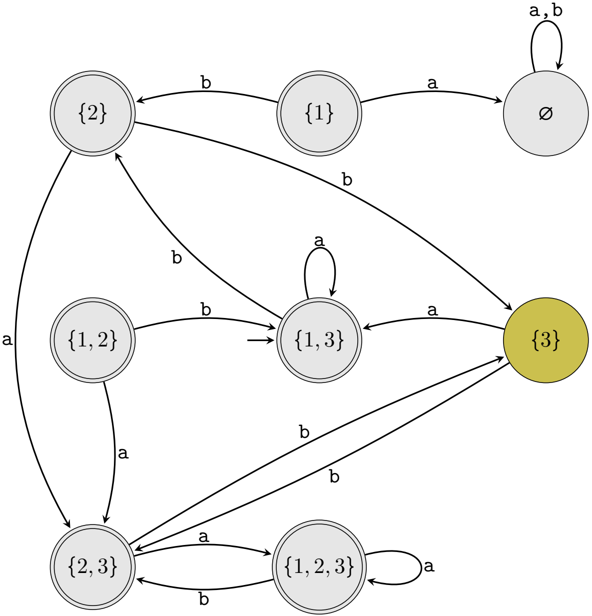

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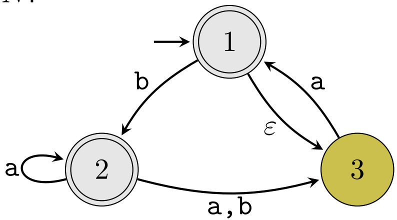

abaababb ✖Rejected

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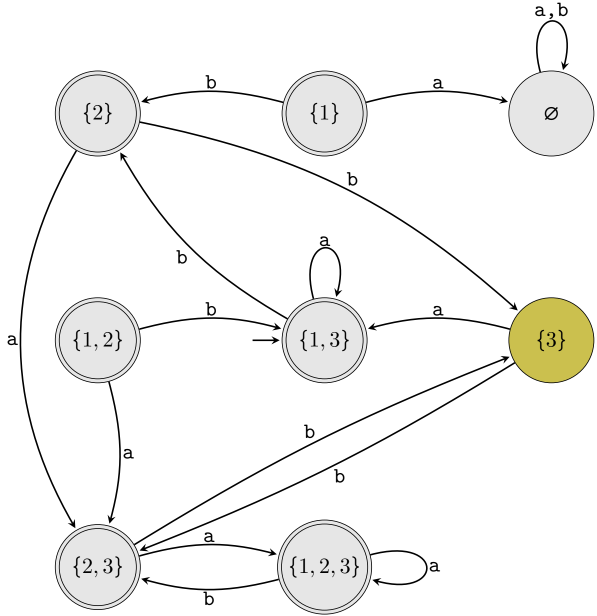


Regular languages

Theorem

A language A is regular if and only if it is recognized by some NFA N.

Proof.

$\implies$

If A is regular, then it is recognized by a DFA M. DFAs are NFAs where each state has exactly one next state for each alphabet symbol so M is an NFA.

$\impliedby$

If NFA N recognizes A, then using the NFA to DFA construction, we can build an DFA M such that $L(M) = A$. Therefore, A is regular. $\square$

Regular languages closed under operations

Let f be an operation on languages
[Recall that means f takes some languages as input and produces a new language as output]

We say regular languages are closed under f to mean

- **Unary** If A is regular, then $f(A)$ is regular
- **Binary** If A and B are regular, then $f(A, B)$ is regular
- **n-ary** If $A_1, A_2, \ldots, A_n$ are regular, then $f(A_1, A_2, \ldots, A_n)$ is regular

Regular languages are closed under regular operations

Regular operations

Union	$A \cup B = \{w \mid w \in A \text{ or } w \in B\}$
Concatenation	$A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$
Kleene star	$A^* = \{w_1 w_2 \cdots w_k \mid k \geq 0 \text{ and } w_i \in A \text{ for all } i\}$

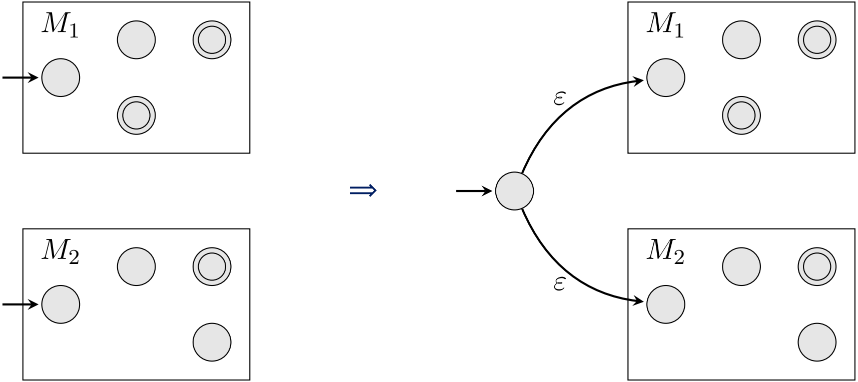
Theorem

Regular languages are closed under union, concatenation, and Kleene star.

In other words, if A and B are regular languages, then $A \cup B$, $A \circ B$, and A^* are regular.

Union

Let A and B be regular languages recognized by DFAs M_1 and M_2



Regular languages are closed under union

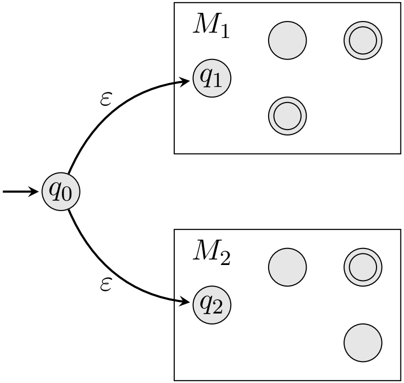
Proof.

Let A and B be regular languages recognized by DFAs

$$M_1 = (Q_1, \Sigma, \delta, q_1, F_1)$$
$$M_2 = (Q_2, \Sigma, \delta, q_2, F_2).$$

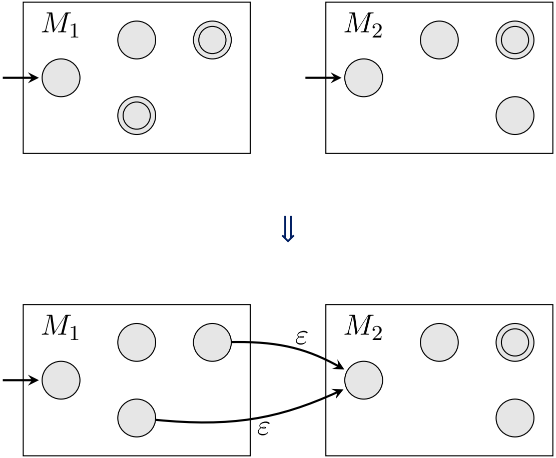
Build NFA $N = (Q, \Sigma, \delta, q_0, F)$ where

$$Q = Q_1 \cup Q_2 \cup \{q_0\}$$
$$F = F_1 \cup F_2$$
$$\delta(q, \varepsilon) = \begin{cases} \{q_1, q_2\} & \text{if } q = q_0 \\ \varnothing & \text{otherwise} \end{cases}$$
$$\delta(q, t) = \begin{cases} \varnothing & \text{if } q = q_0 \\ \{\delta_1(q, t)\} & \text{for } q \in Q_1 \\ \{\delta_2(q, t)\} & \text{for } q \in Q_2 \end{cases} \quad \square$$



Concatenation

Let A and B be regular languages recognized by DFAs M_1 and M_2



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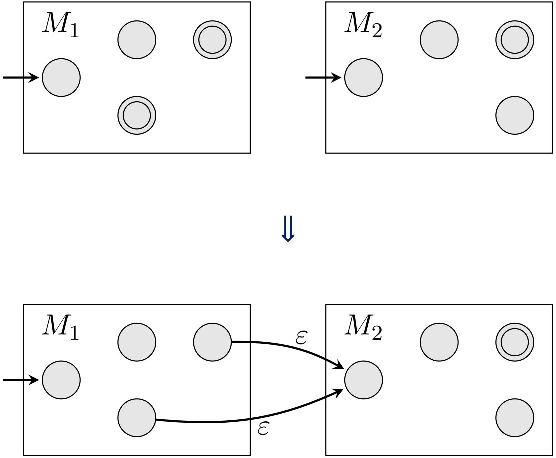


Let

$$M_1 = (Q_1, \Sigma, \delta, q_1, F_1)$$
$$M_2 = (Q_2, \Sigma, \delta, q_2, F_2).$$

Build NFA $N = (Q, \Sigma, \delta, q_1, F_2)$ where

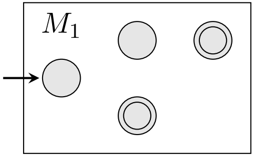
$$Q = Q_1 \cup Q_2$$

$$\delta(q, \varepsilon) = \begin{cases} \{q_2\} & \text{if } q \in F_1 \\ \varnothing & \text{otherwise} \end{cases}$$

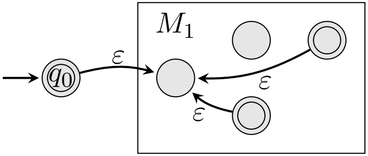
$$\delta(q, t) = \begin{cases} \{\delta_1(q, t)\} & \text{for } q \in Q_1 \\ \{\delta_2(q, t)\} & \text{for } q \in Q_2. \end{cases}$$

Kleene Star

Let A be a regular language recognized by DFA M_1



$\Downarrow$



Kleene Star

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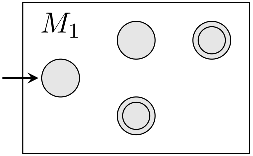

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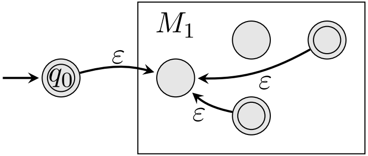

Let $M_1 = (Q_1, \Sigma, \delta, q_1, F_1)$.
Build NFA $N = (Q, \Sigma, \delta, q_0, F)$ where

$$Q = Q_1 \cup \{q_0\}$$

$$F = F_1 \cup \{q_0\}$$

$$\delta(q, \varepsilon) = \begin{cases} \{q_1\} & \text{if } q \in F \\ \varnothing & \text{otherwise} \end{cases}$$

$$\delta(q, t) = \begin{cases} \varnothing & \text{if } q = q_0 \\ \{\delta_1(q, t)\} & \text{for } q \in Q_1 \end{cases}$$

Let's build some NFAs!

- $A = \{w \mid w$ starts with $\mathtt{a}$ and ends with $\mathtt{b}\}$
- $B = \varnothing$
- $C = \{\varepsilon\}$
- $D = \{w \mid w$ has an even number of $\mathtt{a}$s or exactly 2 $\mathtt{b}$s$\}$
- $E = \{\mathtt{aa}, \mathtt{aba}, \mathtt{bab}, \mathtt{bbb}\}$