Problem Set #2

Problem 1 Give a CFG that generates each of the following languages.

    a. $\{w \mid w$ has twice as many $\mathtt{a}$'s as $\mathtt{b}$'s$\}$.

    b. The complement of $\{\mathtt{a}^n \mathtt{b}^n \mid n \geqslant 0\}$.

Problem 2 We proved that the language $\{\mathtt{a}^i \mathtt{b}^j \mathtt{c}^k \mid$ if $i = 1$, then $j = k\}$ is not regular. Show that it is context-free, either by giving a CFG that generates it or by using closure properties of CFLs.

Problem 3 Convert the following CFG into CNF.

$$S \to TST \mid T \mid \varepsilon$$
$$T \to \mathtt{a}T\mathtt{b} \mid \varepsilon$$

Problem 4 Prove that the class of context-free languages is closed under reversal.

Problem 5 Prove that the class of context-free languages is closed under homormophism. [*Hint: To simplify the notation, consider a CFG that's in CNF and construct one that isn't necessarily in CNF.*]

Problem 6 Use the result from Problem 5 to show that $L = \{\mathtt{a}^n \mathtt{b}^n \mathtt{c}^n \mathtt{d}^n \mid n \geqslant 0\}$ is not context-free.

Problem 7 Prove that every regular language is context-free by using the fact that CFLs are closed under union, concatenation, and Kleene star and that every regular language is generated by a regular expression. [*Hint: There are 6 cases to consider.*]

Problem 8 Prove that the following CFG generates the language

$$\{xy \mid x, y \in \Sigma^*, \ |x| = |y|, \ \text{and } x \neq y\}.$$

$$S \to AB \mid BA$$
$$A \to XAX \mid \mathtt{a}$$
$$B \to XBX \mid \mathtt{b}$$
$$X \to \mathtt{a} \mid \mathtt{b}$$

[Hint: Consider m applications of the rule $A \to XAX$ and n applications of the rule $B \to XBX$ in the derivation of a string and note that each instance of an X gives you a single terminal. So

$$S \Rightarrow AB \overset{*}{\Rightarrow} X^m \mathtt{a} X^m B \overset{*}{\Rightarrow} X^m \mathtt{a} X^m X^n \mathtt{b} X^n$$

where

$$X^i = \underbrace{XX \cdots X}_{i} \, .$$

All strings derived from the rightmost expression have length $2(m+n+1)$. Now divide such a string into two $m+n+1$ parts and show that the two parts differ in at least one position. There's a similar argument when the first step in the derivation is $S \Rightarrow BA$.]

Problem 9 We have used the fact that a CFG in CNF derives a string w of length $|w| = n > 0$ in exactly $2n - 1$ steps. Prove this fact.

Problem 10 Show that the language $\{ww \mid w \in \{\mathtt{a}, \mathtt{b}\}^*\}$ is not context free. *[Hint: Consider the string $w = \mathtt{a}^p \mathtt{b}^p \mathtt{a}^p \mathtt{b}^p$. Remember, in the definition of CF-pumpable, $|vxy| \leqslant p$. Use that fact to limit the number of cases you consider.]*