# CS 383

## Lecture 18 – Decidable languages

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# Decidable language

Recall, a language $A$ is decidable if there is some TM $M$ that

1. recognizes $A$ (i.e., $L(M) = A$), and
2. halts on every input (i.e., $\forall w \in \Sigma^*$, $M$ either accepts or rejects $w$)

Such a TM is called a decider

# Acceptance

The acceptance problem for a type of machine[1] asks whether a given machine of the specified type accepts a string

Examples:

$$A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts the string } w\}$$
$$A_{\text{NFA}} = \{\langle B, w\rangle \mid B \text{ is an NFA that accepts the string } w\}$$
$$A_{\text{REX}} = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates the string } w\}$$
$$A_{\text{PDA}} = \{\langle B, w\rangle \mid B \text{ is a PDA that accepts the string } w\}$$
$$A_{\text{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates the string } w\}$$
$$A_{\text{TM}} = \{\langle B, w\rangle \mid B \text{ is a TM that accepts the string } w\}$$

[1] For models of computation like regular expressions and grammars, the acceptance problems asks if it *generates* the string, rather than accepts it

# Emptiness and equivalence

The emptiness problem asks whether the language of the machine is empty

Examples:

$E_{\text{DFA}} = \{\langle B \rangle \mid B \text{ is a DFA and } L(B) = \varnothing\}$
$E_{\text{CFG}} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \varnothing\}$
$E_{\text{TM}} = \{\langle B \rangle \mid B \text{ is a TM and } L(B) = \varnothing\}$

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$$E_{\text{CFG}} = \{\langle G \rangle \mid G \text{ is a CFG and } L(G) = \varnothing\}$$
$$E_{\text{TM}} = \{\langle B \rangle \mid B \text{ is a TM and } L(B) = \varnothing\}$$

The equivalence problem asks whether two machines have the same language

Examples:

$$EQ_{\text{DFA}} = \{\langle A, B \rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$$
$$EQ_{\text{CFG}} = \{\langle G, H \rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$$
$$EQ_{\text{TM}} = \{\langle A, B \rangle \mid A \text{ and } B \text{ are TMs and } L(A) = L(B)\}$$

# $A_{\text{DFA}}$ is decidable

**Theorem**

*The language*

$$A_{DFA} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts the string } w\}$$

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**Proof.**

We want to build a TM $M$ that decides $A_{\text{DFA}}$:
$M$ = "On input $\langle B, w\rangle$,

1. Run (or simulate) $B$ on $w$
2. If $B$ ends in an accept state, *accept*; otherwise *reject*"

Since the simulation always ends after $|w|$ steps, $M$ is a decider.
If $w \in L(B)$, then $M$ will accept. If $w \notin L(B)$, then $M$ will reject. □

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Note that the form of the input to $M$ matches *exactly* the form of the strings in the language we want to decide

# Using a TM as a subroutine in another TM

Given a TM $R$, we can construct another TM $M$ that uses $R$

The language $A = \{\langle B, w\rangle \mid B$ is a DFA, $w \in L(B)$, and $w^{\mathcal{R}} \in L(B)\}$ is decidable

Let $R$ be the decider for $A_{\text{DFA}}$.
$M$ = "On input $\langle B, w\rangle$,

1. Run $R$ on $w$ and if $R$ rejects, *reject*
2. Run $R$ on $w^{\mathcal{R}}$ and if $R$ accepts, *accept*; otherwise *reject*"

How does this work?

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The details aren't important, but here's one way we could incorporate $R$'s implementation into $M$'s:

1. Construct $M'$ as a 2-TM that uses its second tape for the implementation of $R$

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1. Construct $M'$ as a 2-TM that uses its second tape for the implementation of $R$
2. So $M'$ first copies $w$ to the second tape and then simulates $R$ on it. If $R$ doesn't reject, it copies $w^{\mathcal{R}}$ to the second tape and simulates $R$ a second time

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3. Construct the basic TM $M$ by the procedure that lets us simulate a $k$-TM on a basic TM

# TM subroutines

We're going to be doing this a lot so there are some important things to watch out for

- If $R$ isn't a decider, then the simulation of $R$ on some input may not halt; if it doesn't the machine we're building won't either!
- We can't say, "Run $R$ on some input and if it loops, do something" because we don't know how to tell if a TM is looping

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We want to build a TM $M$ that decides $A_{\text{NFA}}$. Let $R$ be the TM that decides $A_{\text{DFA}}$ and build
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$M$ = "On input $\langle B, w\rangle$,

1. Convert the NFA $B$ to an equivalent DFA $C$ using the power set construction
2. Run $R$ on $\langle C, w\rangle$. If $R$ accepts, *accept*; otherwise *reject*"

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Since $R$ is a decider, the simulation in step 2 will always halt so $M$ is a decider.

If $w \in L(B)$, then $w \in L(C)$ so $R$ will accept $\langle C, w\rangle$ and thus $M$ accepts. If $w \notin L(B)$, then $w \notin L(C)$ so both $R$ and $M$ will reject. □

# $A_{\mathsf{REX}}$ is decidable

**Theorem**

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$M$ = "On input $\langle R, w\rangle$,

1. Convert the regular expression $R$ to an equivalent NFA $N$ using the standard construction
2. Run $R$ on $\langle N, w\rangle$. If $R$ accepts, *accept*; otherwise *reject*"

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If $w \in L(R)$, then $w \in L(N)$ so $R$ will accept $\langle N, w\rangle$ and thus $M$ accepts. If $w \notin L(R)$, then $w \notin L(N)$ so both $R$ and $M$ will reject. □

# DFAs, NFAs, and regular expressions are equivalent

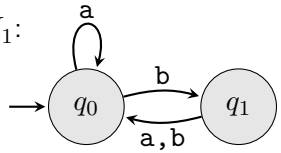
For the purposes of constructing TMs, it doesn't matter if the input is a DFA, NFA, or regular expression

The TM can always convert from one to another using the procedures we've been learning about all semester

# Emptiness problem

Consider the DFAs

$M_1$:




Is $L(M_1) = \varnothing$?

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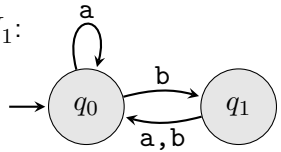


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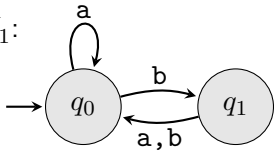


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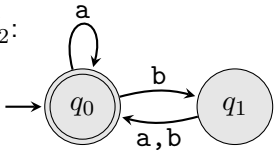


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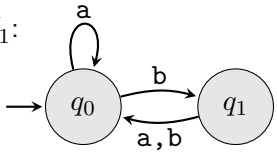


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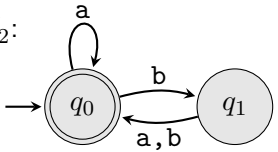


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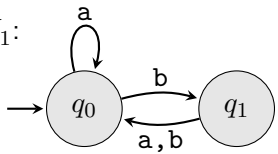


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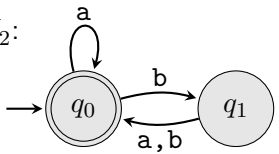


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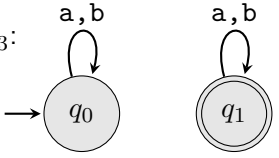


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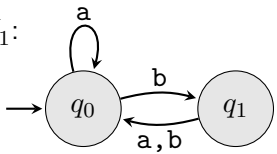


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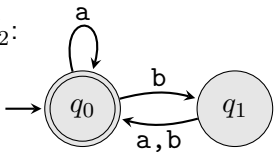


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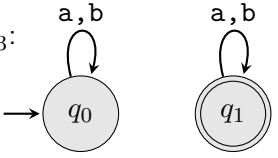


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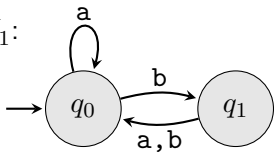


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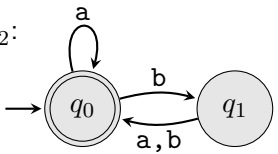


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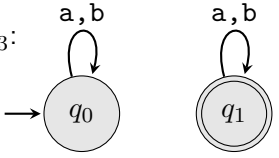


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What are the necessary and sufficient conditions for a DFA to recognize the empty language?

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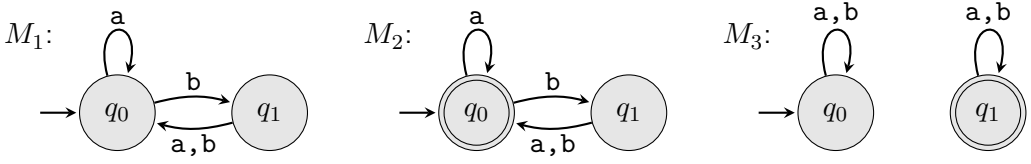


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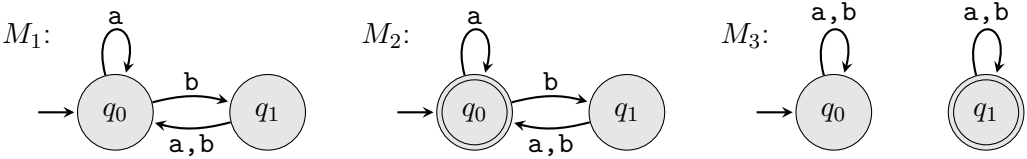
What are the necessary and sufficient conditions for a DFA to recognize the empty language? There must be no path from the initial state to an accepting state

# Algorithm to check if the language of a DFA is empty

We can use a graph algorithm to determine which states are reachable from the initial state

Start by marking the initial state in some way, then iteratively mark all of the neighbors of all marked states until no new states are marked

If an accepting state is marked, the language is not empty, otherwise it is empty



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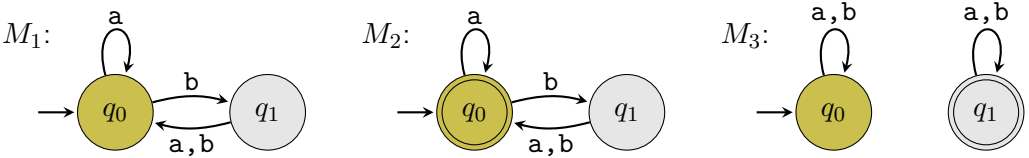

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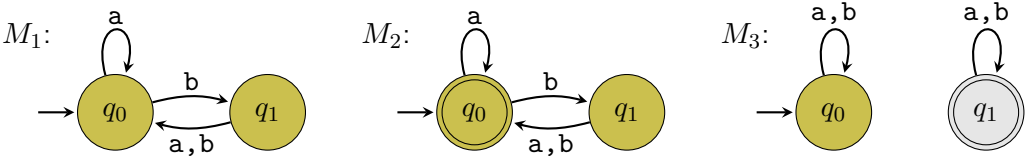

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Let's build a TM $M$ to decide $E_{\text{DFA}}$:

$M$ = "On input $\langle B \rangle$,

1. Mark the initial state of $B$
2. Repeat until no new states are marked
3. Mark any state that has an incoming transition from a marked state
4. If any accept state is marked, *reject*; otherwise *accept*"

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If $L(B) \neq \varnothing$, then there must be a path from the initial state to an accept state so $M$ will reject. If $L(B) = \varnothing$, then no such path exists and $M$ will accept

$M$ is a decider because step 2 can happen at most $|Q| - 1$ times before no new states are marked $\square$

# Emptiness problems for NFAs and regular expression

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How would we show that the analogous emptiness problems for NFAs and regular expressions are decidable?

Let $R$ be a decider for $E_{\text{DFA}}$ and then build a TM that first converts the NFA or regular expression to a DFA and then runs $R$ on it

# Equivalence of DFAs

The equivalence problem for DFAs asks if two DFAs recognize the same language

Using a decider for $E_{\text{DFA}}$, we could check if both languages are empty, but that doesn't really help

We also have a decider to check if a string is in a language

# An idea that doesn't work

Let's try to build a decider for
$EQ_{\text{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$

Let $R$ be a decider for $A_{\text{DFA}}$ and build
$M$ = "On input $\langle A, B\rangle$,

1. Repeat for each string $w \in \Sigma^*$
2. Run $R$ on $\langle A, w\rangle$
3. Run $R$ on $\langle B, w\rangle$
4. If one of $A$ or $B$ accepts $w$ and the other rejects, *reject*
5. Otherwise, $A$ and $B$ agree on all strings so *accept*"

If there is a string $w$ that is accepted by $A$ but rejected by $B$ or vice versa, $M$ will eventually reject in step 4. Otherwise $L(A) = L(B)$ so $M$ will accept in step 5

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## Why doesn't this idea work?

There are infinitely many strings in $\Sigma^*$ so the loop on line 1 will never end if $L(A) = L(B)$

# A better idea

Given two languages $X$ and $Y$, consider the language $D = (X \setminus Y) \cup (Y \setminus X)$

Three cases:

1. If there is a string $x$ which is in $X$ but not in $Y$, then $X \setminus Y \neq \varnothing$ so $D \neq \varnothing$

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If $X$ and $Y$ are regular languages, then $D$ is regular because $X \setminus Y = X \cap \overline{Y}$ and regular languages are closed under union, intersection, and complement

# $EQ_{\text{DFA}}$ is decidable

**Theorem**

*The language $EQ_{DFA} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$ is decidable*

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**Proof.**

Let $R$ be a decider for $E_{\text{DFA}}$ and build
$M$ = "On input $\langle A, B\rangle$,

1. Construct DFA $C$ where $L(C) = (L(A) \setminus L(B)) \cup (L(B) \setminus L(A))$
2. Run $R$ on $\langle C\rangle$
3. If $R$ accepts, *accept*; otherwise *reject*"

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3. If $R$ accepts, *accept*; otherwise *reject*"

The construction in step 1 can be carried out using the constructions we've seen earlier in the semester
$M$ is a decider because $R$ is a decider
$L(C) = \varnothing$ iff $L(A) = L(B)$ so $M$ accepts iff $L(A) = L(B)$ □

# Recapitulation

The acceptance, emptiness, and equivalence problems for DFAs/NFAs/regular expressions are all decidable

For some of these, we used deciders for other languages as subroutines

# Analogous problems for context-free languages

We can consider the acceptance, emptiness, and equivalence problems for context-free languages

Like the situation with DFAs, NFAs, and regular expressions, we can easily convert between CFGs and PDAs, so it suffices to consider CFGs

# $A_{\mathsf{CFG}}$ is decidable

**Theorem**

*The language $A_{CFG} = \{\langle G, w\rangle \mid G$ is a CFG and $G$ generates $w\}$ is decidable.*

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This is trickier than DFAs, but we can use Chomsky normal form (CNF)

If $w \in L(G)$ and $G$ is in CNF, how many steps does it take for $G$ to derive $w$?

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This is trickier than DFAs, but we can use Chomsky normal form (CNF)

If $w \in L(G)$ and $G$ is in CNF, how many steps does it take for $G$ to derive $w$?
If $w = \varepsilon$, it takes $1$ step
If $w \neq \varepsilon$, it takes $2|w| - 1$ steps

Let's use this fact along with our procedure to convert a CFG to CNF

# Proof

Proof.

Build $M$ = "On input $\langle G, w \rangle$,

1. Convert $G$ to CNF
2. If $w = \varepsilon$, then if $S \to \varepsilon$ is a rule in $G$, *accept*; otherwise *reject*
3. Otherwise, list all derivations of length $2|w| - 1$
4. If any of these derivations is $w$, *accept*; otherwise *reject*"

This is a decider because there are finitely many derivations of length $2|w| - 1$.
If $w \in L(G)$, then $M$ will accept, otherwise $M$ will reject. □

# Emptiness problem for CFGs

Given a CFG, we want to figure out if the CFG can derive *any* string of terminals

Let's figure out which variables in $G$ can derive strings of terminals and then we can look the start variable to see if it's one that can

# $E_{\mathsf{CFG}}$ is decidable

**Theorem**

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**Proof.**

$M$ = "On input $\langle G \rangle$,

1. Mark all terminal symbols in the rules of $G$
2. Repeat until no new variables get marked
3. Mark any variable $A$ in $G$ where $A \to u_1 u_2 \cdots u_k$ is a rule and each $u_i$ is marked
4. If the start variable is marked, *reject*; otherwise *accept*"

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4. If the start variable is marked, *reject*; otherwise *accept*"

$M$ is a decider because the loop can happen only finitely many times
If the start variable is marked, then it can derive a string of terminals so $L(G) \neq \varnothing$ and $M$ rejects
If the start variable is not marked, then it cannot derive any string of terminals so $L(G) = \varnothing$ and $M$ accepts □

# What about equivalence of CFGs?

**Theorem**

*The language $A = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) \neq L(H)\}$ is Turing-recognizable (RE)*

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**Theorem**

*The language $A = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) \neq L(H)\}$ is Turing-recognizable (RE)*

**Proof.**

Let $R$ be a decider for $A_{\text{CFG}}$ and build $M$ = "On input $\langle G, H\rangle$,

1. For each $w \in \Sigma^*$,
2. Run $R$ on $\langle G, w\rangle$
3. Run $R$ on $\langle H, w\rangle$
4. If $w$ was generated by only one of $G$ or $H$, *accept*

If $L(G) \neq L(H)$, then $M$ will accept at some iteration of the loop
If $L(G) = L(H)$, then $M$ will run forever □

# Co-Turing-recognizable (coRE)

The language $A$ is clearly related to the complement of $EQ_{\mathsf{CFG}}$

$\overline{EQ_{\mathsf{CFG}}} = A \cup \{w \mid w \text{ is not a valid representation of } \langle G, H \rangle \text{ for CFGs } G \text{ and } H\}$

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We say that a language is co-Turing-recognizable (coRE) if its complement is Turing-recognizable (RE)

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The language $EQ_{\mathsf{CFG}}$ is coRE

We'll see later that $EQ_{\mathsf{CFG}}$ is not RE and thus not decidable (think about what that means for grading homework problems about CFGs)