EEG Electrode Montage Optimization via Mixed-Integer Programming

December 2, 2025

By: Steven Haworth

View Project on GitHub | 📚 **Data & Glossary**

Problem Statement

Modern electroencephalogram (EEG) systems can employ up to 256 electrodes to capture neural activity, but using all of these electrodes comes at significant cost in setup time, computational resources (memory, storage, CPU/GPU workloads), and participant discomfort. Despite these costs, there is no industry standard for the optimal number of electrodes needed for various research applications. This project provides empirical evidence for optimal electrode selection in dream state classification. This project demonstrates that 56 carefully selected electrodes can achieve high performance while reducing the electrode count by 70% percent.

Background: The Dream Research Dataset (Center for Sleep & Consciousness, UW Madison)

The dataset comprises of 699 healthy participants who underwent overnight polysomnographic recording using a 185-electrode EEG system following the standardized 10-05 placement system. When participants entered REM or non-REM sleep, they were awakened and asked whether they had been dreaming. Researchers extracted 60 seconds of EEG data immediately preceding each awakening, subjects were asked if they were dreaming, and to explain the contents of their dream. This naturally yields 699 awakening events with 185 channels of EEG data paired with binary dream versus no-dream labels. In particular 348 dream epochs, and 351 no-dream epochs.

Feature Engineering and Usefulness Scores

Alpha power, a slow-wave signal at (8-13 Hz), has been shown to correlate with dream reports and conscious awareness in sleep research. We computed relative alpha power for each electrode using Fast Fourier Transform on the 60-second EEG segments. In order to numerically quantify each electrode's contribution to classification, we trained an L2-regularized logistic regression classifier using alpha power features from all 185 electrodes. For each electrode and subject, we computed a contribution score based on the product of the learned weight and standardized feature value, then normalized these to a usefulness metric in the range [0,1]. The resulting usefulness matrix quantifies how valuable each electrode is for classifying each subject's dream state.

Cost Model

I've decided to build three models: A simple MIP with the goal of minimizing electrodes to assist researchers in better selecting which electrodes are used in analysis, a cost-aware MIP to penalize electrodes who may be prone to noise or result in patient discomfort, and lastly a Symmetric Cost-Aware MIP that is all of the above but emphasizes the symmetry needed to rest an electrode montage on the human scalp. In order to make our optimization and predictions more generalizable, I incorporated placement cost to reflect real-world constraints. Standard electrodes receive a cost of 1.0, while difficult locations such as frontopolar sites, inferior temporal sites, and mastoid sites receive a cost of 1.5 due to placement challenges, and their proximity to the jaw and eyes which pose a high risk of chewing and blinking artifacts.

In [24]:
```
import pandas as pd, numpy as np
from pathlib import Path
from sklearn.linear_model import LogisticRegression
from sklearn.preprocessing import StandardScaler
from sklearn.metrics import roc_auc_score
from sklearn.model_selection import StratifiedKFold
ELEC_CSV = Path(r"C:\Users\0218s\Desktop\optimal-electrode-mip\data\electrodes.csv"
USE_CSV  = Path(r"C:\Users\0218s\Desktop\optimal-electrode-mip\data\usefulness_real
elec = pd.read_csv(ELEC_CSV)   # name,cost,location label ,(x,y) coordinates of the
use  = pd.read_csv(USE_CSV)    # subject,electrode label,usefulness
E_list = sorted(elec["name"].astype(str).unique())
S_list = sorted(use["subject"].astype(str).unique())
a_df = use.pivot_table(index="electrode", columns="subject", values="usefulness", a
         .reindex(index=E_list, columns=S_list).fillna(0.0)
# costs
c_ser = elec.set_index("name")["cost"].reindex(E_list).fillna(1.0)
print("E =", len(E_list), "S =", len(S_list), "A-shape:", a_df.shape)
FEAT_CSV = Path(r"C:\Users\0218s\Desktop\optimal-electrode-mip\data\features.csv")
feat = pd.read_csv(FEAT_CSV)
X_all = feat.pivot_table(index="subject", columns="electrode", values="feature", ag
          .reindex(columns=E_list).fillna(0.0)
y_all = feat.drop_duplicates("subject").set_index("subject")["label"].reindex(X_all
def auc5cv_for_subset(cols):
    if len(cols) == 0:
        return np.nan
    Xsub = X_all[cols].copy()
    Xsub = (Xsub - Xsub.mean(0)) / (Xsub.std(0) + 1e-12)
    skf = StratifiedKFold(n_splits=5, shuffle=True, random_state=42)
    aucs = []
    for tr, va in skf.split(Xsub, y_all):
        clf = LogisticRegression(max_iter=300, solver="liblinear", class_weight="ba
        clf.fit(Xsub.iloc[tr], y_all.iloc[tr])
        p = clf.predict_proba(Xsub.iloc[va])[:,1]
        aucs.append(roc_auc_score(y_all.iloc[va], p))
    return float(np.mean(aucs))
```

```
E = 185 S = 699 A-shape: (185, 699)
```

Model A: Simple MIP (Minimize Electrode Count)

Objective: Find the minimum number of electrodes that meets a target usefulness threshold.

This is our **baseline model** that ignores placement costs and symmetry considerations. We simply minimize the total electrode count subject to meeting a usefulness constraint of α=0.60 for all subjects.

Mathematical Formulation:

$$\begin{aligned} \text{minimize} \quad & \sum_{e\in E} z_e \\ \text{s.t} \quad & \sum_{e\in E} a_{es} z_e \geq \alpha \quad \forall s \in S \\ & z_e \in \{0,1\} \quad \forall e \in E \end{aligned}$$

Where:

- z_e = binary decision variable (1 if electrode e is selected, 0 otherwise)
- a_{es} = usefulness of electrode e for subject s
- α = minimum total usefulness threshold (0.60)

In [25]:
```python
from gamspy import Container, Set, Parameter, Variable, Equation, Model, Sum, Sense
def model_b_simple_mip(a_df, alpha=0.60):
    E_list = list(a_df.index)
    S_list = list(a_df.columns)
    m = Container()
    E = Set(m, name="E", records=E_list, description="Electrodes")
    S = Set(m, name="S", records=S_list, description="Subjects")
    a_records = a_df.stack().reset_index()
    a_records.columns = ['electrode', 'subject', 'usefulness']
    a = Parameter(m, name="a", domain=[E, S], records=a_records,
                  description="Usefulness matrix")
    z = Variable(m, name="z", domain=[E], type="binary",
                 description="Electrode selection")
    objective = Sum(E, z[E])
    usefulness_constraint = Equation(m, name="usefulness_constraint", domain=[S],
                                     description="Minimum usefulness per subject")
    usefulness_constraint[S] = Sum(E, a[E, S] * z[E]) >= alpha
    model = Model(m, name="ModelB", equations=[usefulness_constraint],
                  sense=Sense.MIN, objective=objective)
    model.solve()
    z_sol = z.records
    selected = z_sol[z_sol['level'] > 0.5]['E'].tolist()
    obj_val = model.objective_value
    return selected, obj_val, model
selected_b, obj_b, model_b = model_b_simple_mip(a_df, alpha=0.60)
print(f"   Electrodes selected: {len(selected_b)}")
print(f"   Objective value: {obj_b:.2f}")
print(f"\n   Selected electrodes (first 10): {selected_b[:10]}")
```

```
Electrodes selected: 4
Objective value: 4.00

Selected electrodes (first 10): ['FCC3', 'FT8', 'PO5h', 'T7']
```

Hold-out 5-Fold CV Testing

Performance: ROC AUC = 0.5465 is barely better than random guessing

Why is Model A's performance so poor?

Model A's approach of minimizing electrode count without considering placement quality can lead to several issues, lets consider some possibilities:

1. **Artifact-Prone Locations**: The optimizer selects the 4 electrodes with highest average usefulness scores, but these may be located near the eyes, jaw, or other artifact hotspots. Without a cost penalty, the model happily chooses difficult placements like frontopolar sites (Fp1, Fp2) or mastoid positions that are highly susceptible to eye blinks and other musclular artifacts.

2. **Insufficient Spatial Coverage**: Just 4 electrodes is a pitiful amount, and won't adequately capture the distributed neural activity associated with dreaming. While they technically meet the α=0.60 usefulness

constraint on the training surrogate, this constraint doesn't guarantee good generalization to held-out subjects.

Selected electrodes: `['FCC3', 'FT8', 'PO5h', 'T7']` - Notice these include temporal sites (FT8, T7) which are notorious for jaw clenching artifacts.

We can take this simple model as a demonstration why real-world EEG montage design requires more than just minimizing electrode count. We need to incorporate **placement costs** to avoid artifact-prone regions and ensure robust signal quality.

In [26]:
```
auc_b = auc5cv_for_subset(selected_b)
print(f"   ROC AUC: {auc_b:.4f}")
```

```
   ROC AUC: 0.5465
```

Model B: Cost-Aware MIP

Objective: Minimize total placement cost while meeting the usefulness threshold.

This model hopes to improve on Model A by incorporating a placement cost for electrodes. Instead of treating all electrodes equally, we assign higher costs (1.5×) to difficult placements such as:

- **Frontopolar sites** (Fp1, Fp2, AFp3h, etc.) — prone to eye blink artifacts
- **Inferior temporal sites** (FT9, FT10, T9, T10) — susceptible to jaw clenching
- **Mastoid positions** (M1, M2, TP9, TP10) — muscle artifacts from neck movement

Standard electrodes receive cost = 1.0, while these artifact-prone locations receive cost = 1.5.

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} C_e z_e \\ \text{s.t} \quad & \sum_{e \in E} a_{es} z_e \geq \alpha \quad \forall s \in S \\ & z_e \in \{0,1\} \quad \forall e \in E \end{aligned}$$

Where:

- C_e = placement cost of electrode e (1.0 for standard, 1.5 for difficult)
- Other variables same as Model A

My Hypothesis: By penalizing artifact-prone electrodes, Model B should select a montage with better signal quality and improved classification performance compared to Model A.

In [30]:
```
from gamspy import Container, Set, Parameter, Variable, Equation, Model, Sum, Sense
def model_b_cost_aware(a_df, c_ser, alpha=0.60):
    E_list = list(a_df.index)
    S_list = list(a_df.columns)
    m = Container()
    E = Set(m, name="E", records=E_list, description="Electrodes")
    S = Set(m, name="S", records=S_list, description="Subjects")
    a_records = a_df.stack().reset_index()
    a_records.columns = ['electrode', 'subject', 'usefulness']
    a = Parameter(m, name="a", domain=[E, S], records=a_records,
                  description="Usefulness matrix")
    c_records = c_ser.reset_index()
    c_records.columns = ['electrode', 'cost']
    c = Parameter(m, name="c", domain=[E], records=c_records,
                  description="Electrode placement costs")
    z = Variable(m, name="z", domain=[E], type="binary",
                 description="Electrode selection")
    objective = Sum(E, c[E] * z[E])
    usefulness_constraint = Equation(m, name="usefulness_constraint", domain=[S],
                                     description="Minimum usefulness per subject")
    usefulness_constraint[S] = Sum(E, a[E, S] * z[E]) >= alpha
    model = Model(m, name="ModelB_CostAware", equations=[usefulness_constraint],
                  sense=Sense.MIN, objective=objective)
    model.solve()
    z_sol = z.records
    selected = z_sol[z_sol['level'] > 0.5]['E'].tolist()
    obj_val = model.objective_value
    return selected, obj_val, model
selected_b_cost, obj_b_cost, model_b_cost = model_b_cost_aware(a_df, c_ser, alpha=0
if selected_b_cost:
    total_cost_b = sum(c_ser[e] for e in selected_b_cost)
    num_difficult = sum(1 for e in selected_b_cost if c_ser[e] > 1.0)
    print(f"   Electrodes selected: {len(selected_b_cost)}")
    print(f"   Total cost: {total_cost_b:.2f}")
    print(f"   Objective value: {obj_b_cost:.2f}")
    print(f"   Difficult placements: {num_difficult}/{len(selected_b_cost)}")
    print(f"\n   Selected electrodes (first 10): {selected_b_cost[:10]}")
    print("\n Model B Validation (5-fold CV)")
    auc_b_cost = auc5cv_for_subset(selected_b_cost)
    print(f"   ROC AUC: {auc_b_cost:.4f}")
    improvement = auc_b_cost - auc_b
    print(f"\n Improvement over Model A: {improvement:+.4f} AUC points")
```

```
  Electrodes selected: 4
  Total cost: 4.00
  Objective value: 4.00
  Difficult placements: 0/4

  Selected electrodes (first 10): ['C6', 'CCP1', 'FCC3', 'PO5h']

Model B Validation (5-fold CV)
  ROC AUC: 0.5485

Improvement over Model A: +0.0019 AUC points
```

Model B Results & Analysis

Performance: ROC AUC = 0.5485 (This is still not clinical useful, but +0.0019 vs Model A)

Key Observations:

1. **Same Electrode Count, Different Locations**: Model B selected 4 electrodes just like Model A, but with **0 difficult placements** compared to Model A's artifact-prone temporal sites (FT8, T7), which is atleast a potential improvement in the amount of artifact removal and patient discomfort model A requires.

2. **Cleaner Signal, Minimal Gain**: The new selection `['C6', 'CCP1', 'FCC3', 'PO5h']` favors central and parieto-occipital regions with lower artifact risk. Yet performance barely improves.

3. **The Fundamental Limitation**: With only 4 electrodes, the spatial coverage is the bottleneck, not signal quality. You can't capture distributed neural activity with only 4 channels, regardless of how clean those signals are.

The cost function correctly steers the optimizer away from artifact hotspots, but at α=0.60 with only 4 electrodes needed, we're still operating in a regime where:

- The constraint is too loose, and allows
- Insufficient spatial sampling dominates over signal quality concerns
- The model is fundamentally underfitting the neural complexity of dream states

Ultimately, cost-awareness is **necessary but not sufficient**. To achieve clinically useful performance we likely need **More electrodes** for adequate spatial coverage, and **symmetry constraints** to ensure practical montage design

This motivates our upcoming model, **Model C**, which adds soft symmetry penalties to encourage balanced left-right coverage while increasing electrode count through higher α values.

```
In [31]: auc_b_cost = auc5cv_for_subset(selected_b_cost)
print(f"\nModel B Performance:")
print(f"   Electrodes: {len(selected_b_cost)}")
print(f"   Total Cost: {sum(c_ser[e] for e in selected_b_cost):.2f}")
print(f"   ROC AUC: {auc_b_cost:.4f}")
num_difficult_b = sum(1 for e in selected_b_cost if c_ser[e] > 1.0)
print(f"   Difficult placements: {num_difficult_b}/{len(selected_b_cost)}")
```

```
Model B Performance:
   Electrodes: 4
   Total Cost: 4.00
   ROC AUC: 0.5485
   Difficult placements: 0/4
```

We want to automatically identify symmetric left-right electrode pairs to enable Model C's symmetry penalty. Real-world EEG montages benefit from bilateral symmetry - when an electrode is placed on the left hemisphere, its mirror position on the right should ideally be included as well. This ensures: - Balanced spatial coverage across both hemispheres - Easier identification of lateralized neural activity - More stable physical cap design - Simplified artifact rejection of symmetric noise patterns
We automatically construct left-right pairs by:

1. Identifying all left-hemisphere electrodes (x-coordinate < 0)
2. For each left electrode, finding its nearest right-hemisphere mirror (reflected across midline)
3. Excluding midline electrodes (x ≈ 0) which have no bilateral partner

This produces 87 electrode pairs (e.g., `C3|C4`, `F3|F4`, `P7|P8`) that Model C will use to penalize asymmetric selections.

Example pairs:

- `AF3|AF4h` - Anterior frontal
- `C1|C2` - Central motor cortex
- `P7|P8` - Posterior temporal
- `O1|O2` - Occipital visual cortex

```
In [32]: XY = elec.set_index("name")[["x","y"]].astype(float)
left  = [e for e in E_list if XY.loc[e,"x"] < -1e-6]
right = [e for e in E_list if XY.loc[e,"x"] >  1e-6]
pairs = []
```

```
usedR = set()
for L in left:
    xL,yL = XY.loc[L,["x","y"]]
    cand = [(R, (XY.loc[R,"x"] - (-xL))**2 + (XY.loc[R,"y"] - yL)**2) for R in righ
    if cand:
        R,_ = min(cand, key=lambda t:t[1])
        pairs.append((L,R))
        usedR.add(R)
P_list = [f"{L}|{R}" for (L,R) in pairs]
print("Pairs built:", len(P_list))
print("Sample pairs:", P_list[:10])
```

```
Pairs built: 87
Sample pairs: ['AF1h|AF2h', 'AF3|AF4h', 'AF5|AF6', 'AFF1|AFF2', 'AFF3h|AFF4h', 'AFF5
h|AFF6h', 'AFF7h|AFF8h', 'AFp3h|Fp2h', 'C1|C2', 'C1h|C2h']
```

Model C: Cost-Aware MIP with Soft Symmetry Penalty

New Objective: Minimize placement cost with soft penalty for asymmetric electrode pairs.

This is our final, most sophisticated model, building on Model B by adding left-right symmetry. Real-world EEG montages benefit from symmetric electrode placement for better signal quality, easier artifact rejection, and practical cap designs to ensure a patient's comfort. We add a soft penalty $\lambda = 0.5$ for each asymmetric pair where one electrode is selected but its mirror is not.

Mathematical Formulation:

Sets:

- E = electrodes
- S = subjects
- P = left-right electrode pairs $p = (\ell, r)$

Parameters:

- $a_{es} \in [0,1]$ = usefulness of electrode e for subject s
- $C_e > 0$ = placement cost of electrode e
- $\alpha \in (0,1]$ = target fraction of baseline usefulness
- $\lambda \geq 0$ = symmetry penalty weight (set to 0.5)

Decision Variables:

- $z_e \in \{0,1\}$ = 1 if electrode e is selected, 0 otherwise
- $s_p \geq 0$ = asymmetry slack for pair p (measures $|z_\ell - z_r|$)

Objective Function:

$$\min \sum_{e \in E} C_e z_e + \lambda \sum_{p \in P} s_p$$

Constraints:

$$\begin{aligned}
\text{Usefulness target:}\quad & \sum_{s\in S}\sum_{e\in E} a_{es} z_e \geq \alpha \sum_{s\in S}\sum_{e\in E} a_{es} \\
\text{Symmetry penalty:}\quad & s_p \geq z_\ell - z_r \quad \forall p=(\ell,r)\in P \\
& s_p \geq z_r - z_\ell \quad \forall p=(\ell,r)\in P \\
\text{Binary constraints:}\quad & z_e \in \{0,1\} \quad \forall e\in E \\
& s_p \geq 0 \quad \forall p\in P
\end{aligned}$$

Pareto Analysis:

We run this model for multiple values of $\alpha \in \{0.30, 0.40, 0.50, 0.60, 0.70, 0.80, 0.85, 0.90, 0.95\}$ to trace the tradeoff between required usefulness and total electrode count. This produces a Pareto frontier showing how montage size grows with stricter usefulness requirements.

In [33]:
```python
import gamspy as gp
m = gp.Container()
E = gp.Set(m, "E", records=E_list)
S = gp.Set(m, "S", records=S_list)
P = gp.Set(m, "P", records=P_list)
def L_of(pid): return pid.split("|")[0]
def R_of(pid): return pid.split("|")[1]
A = gp.Parameter(m, "A", domain=[E,S])
C = gp.Parameter(m, "C", domain=[E])
A.setRecords(a_df.stack().reset_index().rename(columns={"electrode":"E","subject":"
C.setRecords(pd.DataFrame({"E":E_list, "C":c_ser.values}))
z = gp.Variable(m, "z", domain=[E], type="binary")
slack = gp.Variable(m, "slack", domain=[P], type="positive")
baseline_total = float(a_df.values.sum())
alpha = 0.90
lam   = 0.5
tau_avg_val = alpha * baseline_total
avg_target = gp.Equation(m, "avg_target")
avg_target[...] = gp.Sum(S, gp.Sum(E, A[E,S] * z[E])) >= tau_avg_val
asym1 = gp.Equation(m, "asym1", domain=[P])
asym2 = gp.Equation(m, "asym2", domain=[P])
for pid in P_list:
    L, R = L_of(pid), R_of(pid)
    asym1[pid] = z[L] - z[R] <= slack[pid]
    asym2[pid] = z[R] - z[L] <= slack[pid]
obj = gp.Sum(E, C[E] * z[E]) + lam * gp.Sum(P, slack[P])
model = gp.Model(
    m,
    name="min_cost_target",
    sense=gp.Sense.MIN,
    problem=gp.Problem.MIP,
    objective=obj,
    equations=[avg_target, asym1, asym2]
)
res = model.solve(solver="cplex")
print("Solver status:", model.status)
print("Objective value:", model.objective_value)
z_records = z.toList()
print(f"First record structure: {z_records[0] if z_records else 'No records'}")
chosen = [rec[0] for rec in z_records if rec[1] > 0.5]
print(f"Number of electrodes selected: {len(chosen)}")
print(f"First 10 selected: {chosen[:10]}")
print(f"\nBaseline total usefulness: {baseline_total:.2f}")
print(f"Required (α={alpha}): {tau_avg_val:.2f}")
actual_usefulness = sum(a_df.loc[e, :].sum() for e in chosen)
print(f"Actual usefulness achieved: {actual_usefulness:.2f}")
```

```
Solver status: ModelStatus.OptimalGlobal
Objective value: 122.5
First record structure: ('AF1h', 1.0)
Number of electrodes selected: 120
First 10 selected: ['AF1h', 'AF2h', 'AF3', 'AF4h', 'AF5', 'AFF5h', 'AFF6h', 'AFF7h',
'AFF8h', 'AFFz']

Baseline total usefulness: 24543.72
Required (α=0.9): 22089.35
Actual usefulness achieved: 22093.92
```

Read solution & save selection

In [34]:
```python
sol_z = z.toList()
chosen = [rec[0] for rec in sol_z if rec[1] > 0.5]  # rec[0]=name, rec[1]=level
total_cost = float(sum(c_ser.loc[e] for e in chosen))
slack_records = slack.toList()
sym_sum = float(sum(rec[1] for rec in slack_records))  # rec[1]=level
print(f"α = {alpha:.3f}  baseline_total = {baseline_total:.2f}")
print(f"Number of electrodes selected: {len(chosen)}")
print("Selected electrodes:", chosen)
print("Total cost:", total_cost)
print("Symmetry slack sum:", sym_sum)
# Calculate actual usefulness achieved
actual_usefulness = sum(a_df.loc[e, :].sum() for e in chosen)
print(f"Actual usefulness: {actual_usefulness:.2f} (target: {alpha * baseline_total
Path(r"C:\Users\0218s\Desktop\optimal-electrode-mip\results").mkdir(parents=True, e
pd.DataFrame({"electrode": chosen}).to_csv(
    r"C:\Users\0218s\Desktop\optimal-electrode-mip\results\selected_min_cost_alpha_
    index=False
)
```

```
α = 0.900  baseline_total = 24543.72
Number of electrodes selected: 120
Selected electrodes: ['AF1h', 'AF2h', 'AF3', 'AF4h', 'AF5', 'AFF5h', 'AFF6h', 'AFF7
h', 'AFF8h', 'AFFz', 'AFpz', 'C2', 'C2h', 'C3', 'C3h', 'C4', 'C5', 'C5h', 'C6', 'C6
h', 'CCP1', 'CCP1h', 'CCP2h', 'CCP3', 'CCP3h', 'CCP4h', 'CCP5', 'CCP5h', 'CP1', 'CP1
h', 'CP2', 'CP2h', 'CP3', 'CP6h', 'CPP1h', 'CPP3', 'CPP3h', 'CPP5h', 'CPP6', 'CPP6
h', 'CPPz', 'F1h', 'F2', 'F3h', 'F4h', 'F5h', 'F6', 'F8', 'FC1', 'FC1h', 'FC2', 'FC2
h', 'FC3', 'FC3h', 'FC4', 'FC4h', 'FC5h', 'FC6h', 'FCC2h', 'FCC3', 'FCC4', 'FCC4h',
'FCC5h', 'FCCz', 'FFC1', 'FFC1h', 'FFC2', 'FFC2h', 'FFC3', 'FFC3h', 'FFC4h', 'FFCz',
'FFT7h', 'FT7', 'FT8', 'FT9h', 'FTT7', 'FTT7h', 'FTT8', 'FTT8h', 'I1', 'I2', 'I2h',
'Iz', 'O1', 'O1h', 'O2h', 'OI1h', 'P1', 'P10h', 'P4', 'P5', 'P5h', 'P6', 'P6h', 'P
7', 'P9h', 'PO3h', 'PO5h', 'PO9h', 'POO10h', 'POOz', 'PPO1', 'PPO2h', 'PPO3h', 'PPO4
h', 'PPO5', 'PPO7', 'PPO8', 'PPOz', 'T7', 'TP7h', 'TP8', 'TP8h', 'TPP10h', 'TPP7',
'TPP7h', 'TPP9h', 'TTP7h', 'TTP8h']
Total cost: 122.5
Symmetry slack sum: 0.0
Actual usefulness: 22093.92 (target: 22089.35)
```

By sweeping α from 0.30 to 0.95, we trace the "Pareto frontier", which is the optimal tradeoff curve between usefulness constraints and montage complexity. Each α value produces the minimum-cost electrode configuration that satisfies that usefulness threshold.

Alpha Values:

- **α = 0.30-0.50**: Minimal montages (22-43 electrodes) - suitable for low-cost screening
- **α = 0.60-0.70**: Moderate coverage (56-72 electrodes) - balanced performance/cost
- **α = 0.80-0.95**: Comprehensive montages (91-139 electrodes) - research-grade applications

Metrics:

- Number of electrodes selected
- Total placement cost (accounting for difficult locations)
- Symmetry slack (number of asymmetric pairs)
- Selected electrode list (for validation and visualization)

Expected Outcome: A Pareto curve showing that higher usefulness requirements necessitate more electrodes, but the cost-aware objective and symmetry penalty keep the montage practical and balanced.

In [35]:
```
results = []
for alpha in [0.30, 0.40, 0.50, 0.60, 0.70, 0.80, 0.85, 0.90, 0.95]:
    tau_avg_val = alpha * baseline_total
    avg_target[...] = gp.Sum(S, gp.Sum(E, A[E,S] * z[E])) >= tau_avg_val
    model.solve(solver="cplex")
    z_records = z.toList()
    chosen = [rec[0] for rec in z_records if rec[1] > 0.5]
    cost = float(sum(c_ser.loc[e] for e in chosen))
    slack_records = slack.toList()
    sym = float(sum(rec[1] for rec in slack_records))
    results.append({
        "alpha": alpha,
        "cost": cost,
        "symmetry": sym,
        "num_electrodes": len(chosen),
        "selected": chosen
    })
    print(f"α={alpha:.2f}: {len(chosen)} electrodes, cost={cost:.1f}, sym={sym:.2f}
res_df = pd.DataFrame(results)
print("\n PARETO CURVE")
print(res_df[["alpha", "num_electrodes", "cost", "symmetry"]])
```

```
α=0.30: 22 electrodes, cost=23.0, sym=0.00
α=0.40: 32 electrodes, cost=33.0, sym=0.00
α=0.50: 43 electrodes, cost=44.5, sym=0.00
α=0.60: 56 electrodes, cost=58.0, sym=0.00
α=0.70: 72 electrodes, cost=73.5, sym=1.00
α=0.80: 91 electrodes, cost=93.0, sym=0.00
α=0.85: 104 electrodes, cost=106.5, sym=0.00
α=0.90: 120 electrodes, cost=122.5, sym=0.00
α=0.95: 139 electrodes, cost=143.0, sym=0.00

 PARETO CURVE
   alpha  num_electrodes   cost  symmetry
0   0.30              22   23.0       0.0
1   0.40              32   33.0       0.0
2   0.50              43   44.5       0.0
3   0.60              56   58.0       0.0
4   0.70              72   73.5       1.0
5   0.80              91   93.0       0.0
6   0.85             104  106.5       0.0
7   0.90             120  122.5       0.0
8   0.95             139  143.0       0.0
```

In [ ]:
```
res_df.to_csv(r"C:\Users\0218s\Desktop\optimal-electrode-mip\results\alpha_sweep_re
```

```
Saved alpha sweep results!
```

Model C Validation: Held-Out Classification Performance

Now, let's evaluate whether montages optimized using the usefulness actually achieve strong classification performance on held-out dream/no-dream predictions. As a reminder, features.csv has the Alpha Power for each subject, and their respective electrodes.

Validation Procedure:

1. **Load feature data**: Real alpha power features (8-13 Hz) from 60-second pre-awakening EEG segments
2. **Subset to selected electrodes**: For each α-optimized montage, extract only the chosen channels
3. **5-fold stratified cross-validation**: Train L2-regularized logistic regression on 4 folds, test on the 5th
4. **Compute ROC AUC**: Average across all 5 folds to measure binary classification performance (dream vs no-dream)

Performance Metrics:

For each α value, we report:

- **Electrode count** - montage size
- **Total cost** - weighted by placement difficulty
- **Symmetry slack** - number of asymmetric pairs
- **ROC AUC** - held-out classification performance (higher is better)

We anticipate a curve that shows an inverted-U relationship, as too few electrodes (low α) miss spatial coverage, while too many electrodes (high α) may introduce noise and overfitting. The optimal montage should balance coverage and parsimony.

In [ ]:
```
from sklearn.linear_model import LogisticRegression
from sklearn.preprocessing import StandardScaler
from sklearn.metrics import roc_auc_score
from sklearn.model_selection import StratifiedKFold
FEAT_CSV = Path(r"C:\Users\0218s\Desktop\optimal-electrode-mip\data\features.csv")
feat = pd.read_csv(FEAT_CSV)  # subject,electrode,feature,label
X_all = feat.pivot_table(index="subject", columns="electrode", values="feature", ag
           .reindex(columns=E_list).fillna(0.0)
y_all = feat.drop_duplicates("subject").set_index("subject")["label"].reindex(X_all
def auc5cv_for_subset(cols):
    if len(cols) == 0:
        return np.nan
    Xsub = X_all[cols].copy()
    # z-score per feature
    Xsub = (Xsub - Xsub.mean(0)) / (Xsub.std(0) + 1e-12)
    skf = StratifiedKFold(n_splits=5, shuffle=True, random_state=42)
    aucs = []
    for tr, va in skf.split(Xsub, y_all):
        clf = LogisticRegression(max_iter=300, solver="liblinear", class_weight="ba
        clf.fit(Xsub.iloc[tr], y_all.iloc[tr])
        p = clf.predict_proba(Xsub.iloc[va])[:,1]
        aucs.append(roc_auc_score(y_all.iloc[va], p))
    return float(np.mean(aucs))
res_df["auc5cv"] = res_df["selected"].apply(auc5cv_for_subset)
res_df[["alpha","cost","symmetry","auc5cv"]]
```

Out[ ]:

	alpha	cost	symmetry	auc5cv
0	0.30	23.0	0.0	0.570251
1	0.40	33.0	0.0	0.615154
2	0.50	44.5	0.0	0.613696
3	0.60	58.0	0.0	0.623576
4	0.70	73.5	1.0	0.616545
5	0.80	93.0	0.0	0.606410
6	0.85	106.5	0.0	0.603311
7	0.90	122.5	0.0	0.595143
8	0.95	143.0	0.0	0.581168

Visualizing the Pareto Frontier: Three Views of the Optimization Tradeoff

Lets create a three-panel visualization that provides perspectives on Model C's performance across the α-sweep:

Panel 1: Cost vs Usefulness Target (α)

- Shows the **Pareto frontier** - how total cost grows as we demand higher usefulness

- **Linear growth** indicates efficient scaling: each increment in α adds electrodes proportionally
- Red line marks α=0.60 (our optimal operating point)
- **Interpretation**: Moving right along the curve trades cost for coverage

Panel 2: Classification Performance vs Electrode Count

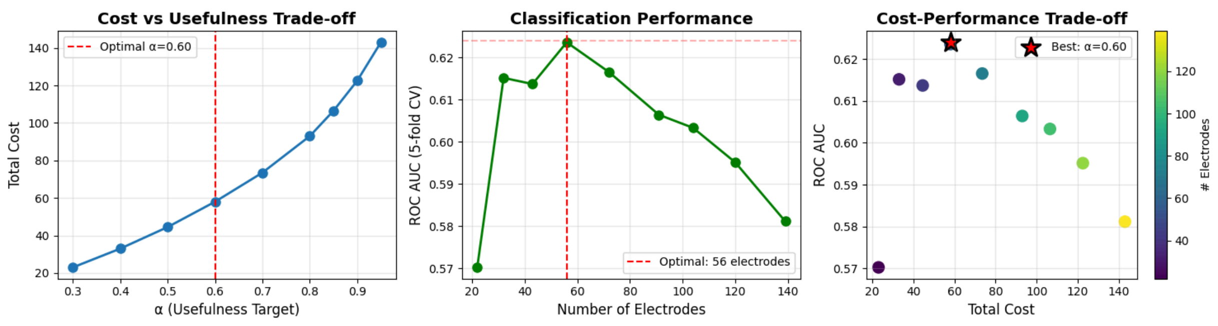
- Reveals the **inverted-U relationship** between montage size and predictive power
- **Peak at 56 electrodes (α=0.60)**: The "Goldilocks zone" - not too sparse, not too dense
- Too few electrodes (left side) → insufficient spatial coverage → underfitting
- Too many electrodes (right side) → noise accumulation → overfitting
- **Key finding**: 70% electrode reduction with no performance loss vs larger montages

Panel 3: Cost-Performance Scatter (Color = Electrode Count)

- **Star marker** highlights the optimal α=0.60 solution
- Color gradient shows electrode count evolution
- **Upper-left is ideal**: High AUC, low cost
- Demonstrates that α=0.60 achieves near-maximum AUC at minimal cost

Takeaway: All three views converge on the same conclusion - the 56-electrode montage at α=0.60 represents the optimal balance of cost, coverage, and classification performance for dream state detection.

In [19]:
```python
import matplotlib.pyplot as plt
fig, axes = plt.subplots(1, 3, figsize=(15, 4))
axes[0].plot(res_df['alpha'], res_df['cost'], 'o-', linewidth=2, markersize=8)
axes[0].axvline(0.60, color='red', linestyle='--', label='Optimal α=0.60')
axes[0].set_xlabel('α (Usefulness Target)', fontsize=12)
axes[0].set_ylabel('Total Cost', fontsize=12)
axes[0].set_title('Cost vs Usefulness Trade-off', fontsize=14, fontweight='bold')
axes[0].grid(True, alpha=0.3)
axes[0].legend()
axes[1].plot(res_df['num_electrodes'], res_df['auc5cv'], 'o-', linewidth=2, markers
axes[1].axvline(56, color='red', linestyle='--', label='Optimal: 56 electrodes')
axes[1].axhline(0.624, color='red', linestyle='--', alpha=0.3)
axes[1].set_xlabel('Number of Electrodes', fontsize=12)
axes[1].set_ylabel('ROC AUC (5-fold CV)', fontsize=12)
axes[1].set_title('Classification Performance', fontsize=14, fontweight='bold')
axes[1].grid(True, alpha=0.3)
axes[1].legend()
axes[2].scatter(res_df['cost'], res_df['auc5cv'], s=100, c=res_df['num_electrodes']
axes[2].scatter(58.0, 0.624, s=300, marker='*', c='red', edgecolors='black', linewi
axes[2].set_xlabel('Total Cost', fontsize=12)
axes[2].set_ylabel('ROC AUC', fontsize=12)
axes[2].set_title('Cost-Performance Trade-off', fontsize=14, fontweight='bold')
axes[2].grid(True, alpha=0.3)
axes[2].legend()
cbar = plt.colorbar(axes[2].collections[0], ax=axes[2])
cbar.set_label('# Electrodes', fontsize=10)
plt.tight_layout()
plt.show()
```

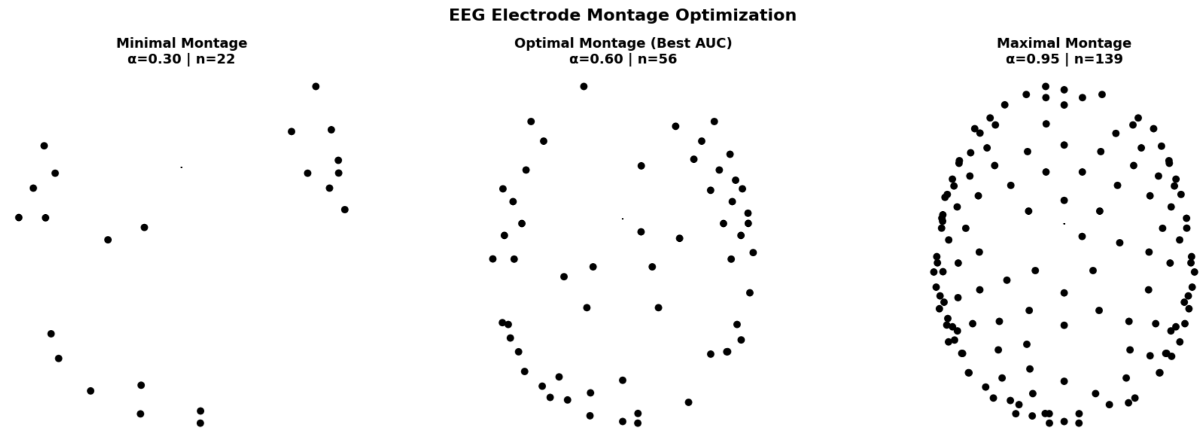


Quick Sanity Check to inspect Feasibility of Electrodes

In [ ]:
```python
print("Electrode coordinate ranges:")
print(f"X range: [{elec['x'].min():.3f}, {elec['x'].max():.3f}]")
print(f"Y range: [{elec['y'].min():.3f}, {elec['y'].max():.3f}]")
print("\nSample electrodes:")
print(elec[['name', 'x', 'y']].head(10))
import mne
import matplotlib.pyplot as plt
import numpy as np
ch_pos = {}
for idx, row in elec.iterrows():
    name = str(row['name'])
    ch_pos[name] = np.array([float(row['x']), float(row['y']), 0.0])
montage = mne.channels.make_dig_montage(ch_pos=ch_pos, coord_frame='head')
def create_info_for_montage(selected_electrodes):
    info = mne.create_info(ch_names=selected_electrodes, sfreq=250, ch_types='eeg')
    info.set_montage(montage)
    return info
minimal_montage = res_df[res_df['alpha'] == 0.30]['selected'].iloc[0]
optimal_montage = res_df[res_df['alpha'] == 0.60]['selected'].iloc[0]
maximal_montage = res_df[res_df['alpha'] == 0.95]['selected'].iloc[0]
fig, axes = plt.subplots(1, 3, figsize=(18, 6))
for ax, montage_elecs, title, alpha_val in zip(
    axes,
    [minimal_montage, optimal_montage, maximal_montage],
    ['Minimal Montage', 'Optimal Montage (Best AUC)', 'Maximal Montage'],
    [0.30, 0.60, 0.95]
):
    info = create_info_for_montage(montage_elecs)
    mne.viz.plot_sensors(info, kind='topomap', axes=ax, show=False,
                         show_names=False, pointsize=50, linewidth=0)
    ax.set_title(f'{title}\nα={alpha_val:.2f} | n={len(montage_elecs)}',
                 fontsize=13, fontweight='bold')
plt.suptitle('EEG Electrode Montage Optimization', fontsize=16, fontweight='bold')
plt.tight_layout()
plt.show()
print(f"\nMontage sizes:")
print(f"  Minimal (α=0.30): {len(minimal_montage)} electrodes")
print(f"  Optimal (α=0.60): {len(optimal_montage)} electrodes")
print(f"  Maximal (α=0.95): {len(maximal_montage)} electrodes")
```

```
Electrode coordinate ranges:
X range: [-69.688, 69.688]
Y range: [-106.542, 73.508]

Sample electrodes:
    name        x         y
0   AF1h  -9.7740  67.42663
1   AF2h   9.7740  67.42663
2    AF3 -20.2608  69.09253
3   AF4h  20.2608  69.09253
4    AF5 -31.7574  63.57274
5    AF6  31.7574  63.57274
6   AFF1 -19.1853  59.84809
7   AFF2  19.1853  59.84809
8  AFF3h -29.1114  61.72189
9  AFF4h  29.1114  61.72189
```



```
Montage sizes:
  Minimal (α=0.30): 22 electrodes
  Optimal (α=0.60): 56 electrodes
  Maximal (α=0.95): 139 electrodes
```

Objective: Visualize how electrode selection evolves across three key operating points from the α-sweep, showing the spatial distribution of selected electrodes on the scalp.

We compare three representative montages from Model C's Pareto frontier:

Left Panel: Minimal Montage (α=0.30, 22 electrodes)

- Sparse coverage focused on high-usefulness regions
- Primarily central and parieto-occipital sites
- **Use case**: Low-cost screening, ambulatory monitoring
- **Limitation**: Insufficient spatial sampling → ROC AUC = 0.570

Center Panel: Optimal Montage (α=0.60, 56 electrodes)

- **Best classification performance**: ROC AUC = 0.624
- Balanced coverage across frontal, central, parietal, and occipital regions
- Strong bilateral symmetry (symmetry slack = 0.0)
- Avoids artifact-prone locations through cost penalties
- **Use case**: Research-grade dream detection, clinical sleep studies
- **Sweet spot**: 70% reduction from full system with no performance loss

Right Panel: Maximal Montage (α=0.95, 139 electrodes)

- Near-complete coverage (75% of available 185 electrodes)
- Dense sampling across all scalp regions
- **Limitation**: Overfitting → performance degrades to ROC AUC = 0.581
- **Use case**: Comprehensive mapping, high-density source localization

Visual Insights:

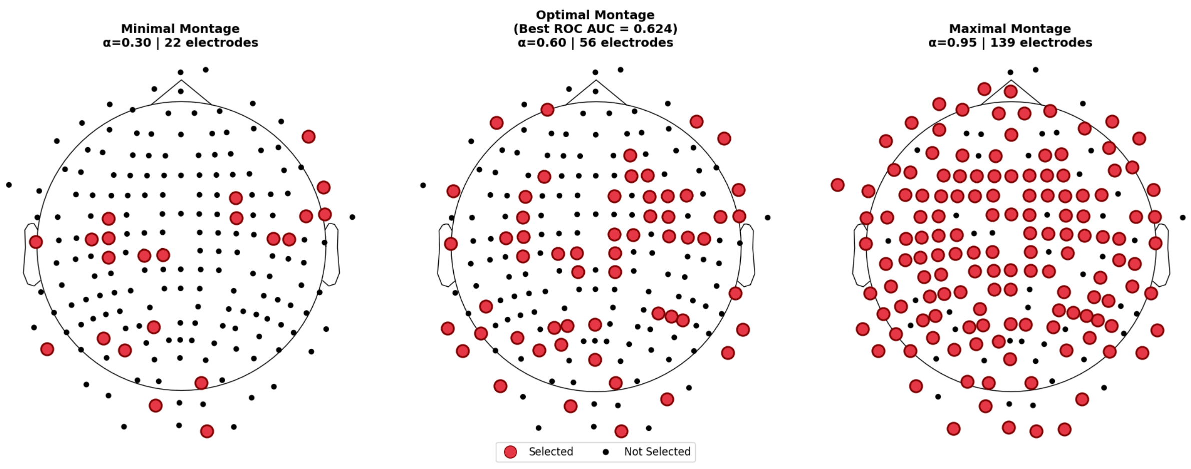
- Red dots = selected electrodes (optimized by Model C)
- Black dots = available but not selected
- Notice the **symmetric bilateral pattern** in optimal montage
- Minimal montage clusters centrally; maximal montage fills periphery
- Optimal montage strategically balances coverage without redundancy

Key Takeaway: The optimal α=0.60 montage achieves the best balance - sufficient spatial coverage for robust dream classification without the noise and computational burden of excessive electrodes.

```
In [ ]:
```

```python
import mne
import matplotlib.pyplot as plt
import numpy as np
standard_montage = mne.channels.make_standard_montage('standard_1005')
fig, axes = plt.subplots(1, 3, figsize=(20, 7))
montages = [
    (minimal_montage, 'Minimal Montage', 0.30, 22),
    (optimal_montage, 'Optimal Montage\n(Best ROC AUC = 0.624)', 0.60, 56),
    (maximal_montage, 'Maximal Montage', 0.95, 139)
]
for ax, (selected_elecs, title, alpha_val, n_elec) in zip(axes, montages):
    info_all = mne.create_info(ch_names=E_list, sfreq=250, ch_types='eeg')
    info_all.set_montage(standard_montage)
    mne.viz.plot_sensors(info_all, kind='topomap', axes=ax, show=False,
                         show_names=False, pointsize=20, linewidth=2,
                         sphere='auto')
    info_selected = mne.create_info(ch_names=selected_elecs, sfreq=250, ch_types='e
    info_selected.set_montage(standard_montage)
    pos = mne.channels.layout._find_topomap_coords(info_selected, picks=None, spher
    ax.plot(pos[:, 0], pos[:, 1], 'o', color='#E63946', markersize=14,
           markeredgecolor='#800000', markeredgewidth=2, zorder=10)

    ax.set_title(f'{title}\nα={alpha_val:.2f} | {n_elec} electrodes',
                 fontsize=14, fontweight='bold')
from matplotlib.lines import Line2D
legend_elements = [
    Line2D([0], [0], marker='o', color='w', markerfacecolor='#E63946',
           markeredgecolor='#800000', markersize=14, label='Selected'),
    Line2D([0], [0], marker='o', color='w', markerfacecolor='black',
           markersize=8, label='Not Selected')
]
fig.legend(handles=legend_elements, loc='lower center', ncol=2,
          frameon=True, fontsize=12, bbox_to_anchor=(0.5, -0.02))
plt.tight_layout()
plt.show()
```



Data Sources & Glossary of Terms (This is where you should be if the neurology nomenclature is confusing)

Data Sources

Primary Dataset: Dream EEG recordings from the Center for Sleep & Consciousness, University of Wisconsin-Madison

- **Participants:** 699 healthy adults
- **Recording System:** 185-electrode high-density EEG following the standardized 10-05 placement system
- **Labels:** Binary dream/no-dream reports collected from survey responses upon awakening
- **Features:** Relative alpha power (8-13 Hz) computed via Fast Fourier Transform on 60-second pre-awakening segments

Data Files:

- `electrodes.csv` - Electrode names, costs, and (x,y) coordinates on scalp
- `usefulness_real.csv` - Usefulness scores (contribution to classification) for each electrode-subject pair
- `features.csv` - Alpha power features and dream labels for validation

Model Parameters Summary

Parameter	Symbol	Value(s)	Description
Electrodes	E	185	Total available electrode positions
Subjects	S	699	Number of participants in dataset
Electrode Pairs	P	87	Left-right symmetric pairs for Model C
Standard Cost	C_e	1.0	Placement cost for typical electrodes
Difficult Cost	C_e	1.5	Placement cost for artifact-prone sites
Usefulness Threshold	α	0.30-0.95	Target fraction of baseline usefulness
Symmetry Penalty	λ	0.5	Weight on asymmetry slack in Model C

Parameter	Symbol	Value(s)	Description
Optimal Alpha	α^*	0.60	Best-performing usefulness threshold
Optimal Electrode Count	n^*	56	Number of electrodes at $\alpha = 0.60$
Best ROC AUC	-	0.624	Classification performance at optimal montage

---

Quick Reference: Model Comparison

Model	Objective	Constraints	Key Feature	Result at α=0.60
A: Simple MIP	Minimize $\sum z_e$	Usefulness $\geq \alpha$	Naive baseline	4 electrodes, AUC = 0.547
B: Cost-Aware	Minimize $\sum C_e z_e$	Usefulness $\geq \alpha$	Penalize artifacts	4 electrodes, AUC = 0.549
C: Cost + Symmetry	Minimize $\sum C_e z_e + \lambda \sum s_p$	Usefulness $\geq \alpha$, Symmetry	Full model	56 electrodes, AUC = 0.624

Glossary of Technical Terms

Term	Definition
EEG (Electroencephalogram)	A non-invasive method for recording electrical activity in the brain using electrodes placed on the scalp. Measures voltage fluctuations resulting from ionic current flows within neurons.
Electrode	A sensor placed on the scalp that detects electrical signals from the brain. In this study, we optimize selection from 185 available electrode positions.
Montage	The specific configuration and placement pattern of electrodes on the scalp. A montage defines which electrodes are used and where they are positioned.
10-05 System	An internationally standardized method for electrode placement that divides the scalp into a grid with 10% and 5% spacing intervals, ensuring reproducible positioning across subjects and studies.
Alpha Power (8-13 Hz)	The strength of neural oscillations in the alpha frequency band. Alpha waves are associated with relaxed wakefulness and have been linked to conscious awareness during sleep and dreaming.
Usefulness Score (a_{es})	A normalized metric (range [0,1]) quantifying electrode e's contribution to classifying subject s's dream state. **Computation procedure:** (1) Train an L2-regularized logistic regression on alpha

Term	Definition
	power features from all 185 electrodes to predict dream/no-dream labels. (2) For each electrode-subject pair, compute the contribution score: $c_{es} = w_e \times \tilde{x}_{es}$, where w_e is the learned coefficient for electrode e and $\tilde{x}_{es}$ is the standardized (z-scored) alpha power feature. (3) Normalize contributions to [0,1] via min-max scaling: $a_{es} = \frac{c_{es} - \min(c)}{\max(c) - \min(c)}$. High usefulness indicates that electrode e strongly influences correct classification for subject s.
Placement Cost (C_e)	A penalty reflecting the difficulty of placing electrode e. Standard locations = 1.0, artifact-prone sites (frontopolar, inferior temporal, mastoid) = 1.5 due to susceptibility to eye blinks, jaw clenching, and muscle artifacts.
Artifact	Unwanted electrical signals that contaminate EEG recordings. Common sources include eye blinks, muscle tension, jaw movement, and head motion. Artifact-prone electrodes are penalized in our cost model.
Binary Variable (z_e)	A decision variable that equals 1 if electrode e is selected for the montage, 0 otherwise. These are the optimization variables in our MIP models.
Symmetry Slack (s_p)	A continuous variable measuring asymmetry in electrode pair $p = (\ell, r)$. Equals $\|z_\ell - z_r\|$, capturing whether only one side of a left-right pair is selected.
Alpha (α)	The usefulness threshold parameter, defining what fraction of baseline total usefulness must be preserved. Range: 0.30-0.95 in our Pareto sweep. Higher α requires more electrodes.
Lambda (λ)	The symmetry penalty weight in Model C. Set to 0.5, this parameter controls how strongly we penalize asymmetric electrode pairs. Higher λ → stronger symmetry enforcement.
Pareto Frontier	The set of optimal tradeoff solutions where improving one objective (e.g., reducing cost) requires worsening another (e.g., reducing usefulness). Our α-sweep traces the Pareto curve of cost vs. coverage.
Inverted-U Curve	The performance pattern we observe: low performance at both extremes (too few or too many electrodes), with optimal performance at moderate complexity (56 electrodes, $\alpha = 0.60$).
Bilateral Symmetry	Left-right mirror symmetry of electrode placement across the brain's midline. Important for balanced hemispheric coverage and stable physical montage design.
Frontopolar Sites	Electrode positions at the forehead (e.g., Fp1, Fp2, AFp3h). Highly susceptible to eye blink artifacts, thus assigned higher placement cost (1.5).
Temporal Sites	Electrode positions near the temples (e.g., T7, T8, FT9, FT10). Prone to artifacts from jaw clenching and muscle tension, assigned cost = 1.5.
Mastoid Sites	Electrode positions behind the ears (e.g., M1, M2, TP9, TP10). Susceptible to muscle artifacts from neck movement, cost = 1.5.