HW 7 **Prof. Caldwell**
Due: 25 September 2011 **COSC 3015**

Recall the induction principle for finite lists.

$$\begin{array}{l}[P([\,])\wedge \\ \ \ \forall x :: a.\, \forall xs :: [a].\, P(xs) \Rightarrow P(x : xs)] \\ \ \ \ \Rightarrow \forall ys :: [a].P(ys)\end{array}$$

Thus, for a property P of lists, to show that $\forall xs :: [a].\ P(xs)$ it is enough to show two things:

$$\begin{array}{rl}\textbf{i}.) & P([\,]) \\ \textbf{ii}.) & \forall x :: a.\, \forall xs :: [a].\ P(xs) \Rightarrow P(x : xs)\end{array}$$

Here are the definitions of some list functions.

$$\begin{array}{l}length\,[\,] = 0 \\ length\,(x : xs) = 1 + length\,xs \\ \\ append\,[\,]ys = ys \\ append\,(x : xs)ys = x : (append\,xsys) \\ \\ map\ f\ [\,] = [\,] \\ map\ f\ (x : xs) = (f\ x)\ :\ map\ f\ xs \\ \\ (f\ .\ g)\ x = f\ (g\ x)\end{array}$$

Definition of the append function shows directly that $[\,]$ is a left identity for *append* , is it a right identity as well? The following theorem establishes this fact.

Theorem 0.1 (Nil right identity for *append* **)**

$$\forall ys :: [a].\ append\ ys\ [\,] = ys$$

Proof: By list induction on ys. The property P of ys is given as:

$$P(ys) \stackrel{\text{def}}{=} append\ ys\ [\,] = ys$$

Base case: Show $P([\,])$, *i.e.* that $append\ [\,]\ [\,] = [\,]$. This follows immediately from the definition of append.
Induction Step: Assume $P(xs)$ (the induction hypothesis) and show $P(x : xs)$ for arbitrary x of type a and arbitrary xs of type $[A]$. The induction hypothesis is:

$$append\ xs\ [\,] = xs$$

We must show $append\ (x : xs)\ [\,] = (x : xs)$. Starting with the left side of the equality we get the following:

$$append\ (x : xs)\ [\,] \stackrel{\langle\langle def.of\,append\ \rangle\rangle}{=} x : (append\ xs\ [\,]) \stackrel{\langle\langle ind.hyp.\rangle\rangle}{=} x : xs$$

So the induction step holds and the proof is complete.
□

Exercise 0.1. Prove the following by list induction.

$$\begin{array}{rl}1.) & \forall ys :: [a].\ length\,(map\,f\,ys) = length\,ys \\ 2.) & \forall ys :: [a].\ \forall xs :: [a].\ length\,(append\ xs\,ys) = (length\ \ xs) + (length\ \ ys)\end{array}$$

For the second proof - choose an arbitrary lets ys of type $[a]$ and then do list induction on xs.