HW 10 **Prof. Caldwell**
Due: 27 September 2011 **COSC 3015**

1 Datatypes and Structural Induction

As discussed in class today, Haskell datatypes have corresponding structural induction princples.

1.1 Nat

The datatype definition for natural numbers was given as:

datatype Nat = Zero | Succ Nat

The induction principle for the type *Nat* is

$$(P(\mathit{Zero}) \wedge \forall k :: Nat.\ P(k) \Rightarrow P(\mathit{Succ}\, k)) \Rightarrow \forall n :: Nat.\ P(n)$$

Thus to show a property P holds for every finite natural number, show two things:

Case *Zero:* Show $P(\mathit{Zero})$.

Case *Succ:* For arbitrary $k :: Nat$ assume $P(k)$ and show $P(\mathit{Succ}\, k)$.

Note, if we want to reason about possibly infinte elements of the type we add the case to show $P(\bot)$. This is true for each of the induction principles below.

1.2 Lists

Recall the defintion of lists containing elements of type a.

datatype List a = Nil | Cons a (List a)

The induction principle for finite lists is

$$(P(\mathit{Nil}) \quad \wedge \quad \forall x :: a.\ \forall xs :: \mathit{List}\ a.\ P(xs) \Rightarrow P(\mathit{Cons}\, x\, xs)) \quad \Rightarrow \forall ys :: \mathit{List}\ a.\ P(ys)$$

Thus to show a property P holds for every list, show two things:

Case *Nil***:** Show $P(\mathit{Nil})$.

Case *Cons***:** For arbitrary $x :: a$ and $xs :: \mathit{List}\ a$ assume $P(xs)$ and show $P(\mathit{Cons}\ x\ xs)$.

1.3 Binary Trees

Recall the definition of the datatype for binary trees containing elements of type a.

datatype BTree a = Leaf | Node a (BTree a) (BTree a)

The induction principle for finite binay trees is

$$\begin{array}{l}(P(\mathit{Leaf}) \\ \ \wedge\ \forall x :: a.\ \forall t_l, t_r :: BTree\ a.\ (P(t_l) \wedge P(t_r)) \Rightarrow P(\mathit{Node}(v, t_l, t_r))) \\ \ \ \Rightarrow \forall t :: BTree\ a.\ P(t)\end{array}$$

Thus to show a property P holds for every $BTree$, show two things:

Case *Leaf***:** Show $P(\mathit{Leaf})$.

Case *Node***:** For arbitrary $x :: a$ and $t_l, t_r :: BTree\ a$ assume $P(t_l)$ and assume $P(t_r)$ and show $P(\mathit{Node}(v, t_l, t_r))$.

2 Assignment

Write the structural induction principles for the finite instances of the following types.

1. Trees having nodes with both one and with two children.

   $$\mathit{OneTwoTree}\ a = \mathit{Empty} \mid \mathit{Single}\ a\ (\mathit{OneTwoTree}\ a) \mid \mathit{Branch}\ a\ (\mathit{OneTwoTree}\ a)\ (\mathit{OneTwoTree}\ a)$$

2. Trees whose nodes have exactly three children.

   $$\mathit{ThreeTree}\ a = \mathit{Leaf} \mid \mathit{Node}\ a\ (\mathit{ThreeTree}\ a)\ (\mathit{ThreeTree}\ a)\ (\mathit{ThreeTree}\ a)$$

3. Formulas of the following form.

   $$\mathit{Formula} = \mathit{Bottom} \mid \mathit{Not}\ \mathit{Formula} \mid \mathit{And}\ \mathit{Formula}\ \mathit{Formula}$$