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# $\mathcal{L}_1$ Adaptive Controller for MIMO Systems with Unmatched Uncertainties using Modified Piecewise Constant Adaptation Law

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***Abstract*— This paper presents the $\mathcal{L}_1$ adaptive controller for a class of multi-input multi-output (MIMO) systems with both matched and unmatched nonlinear uncertainties. A modification of the piecewise constant adaptive law is introduced based on the history of the past samples. Compared with the standard $\mathcal{L}_1$ adaptive law with piecewise constant adaptive law, here the performance bounds decay much faster as the sample period decreases, yielding better performance at a given sample period, or less stringent requirements on the computation and measurement frequency for a given performance level. Simulation results verify the theoretical findings.**

## I. Introduction

The philosophy of the $\mathcal{L}_1$ adaptive controller [1], [2] is to decouple the estimation and the control lops by using a predictor-based fast adaptation scheme, while compensating for the uncertainty only within the bandwidth of the control channel [3]–[5].

There are two types of fast estimation schemes for the $\mathcal{L}_1$ adaptive control: the gradient descent adaptive law, which achieves fast adaptation by increasing the adaptive gain, and the piecewise constant adaptive law [6]–[8], which achieves the same goal by reducing the sample period (i.e., increasing the updating frequency). In both approaches, by increasing the adaptation rate, the input and output of the uncertain system can be rendered arbitrarily close to the corresponding signals of a closed-loop reference system that defines the desired achievable performance, in both transient and steady-state. Refer to [9] for a comprehensive introduction of the theory, algorithms for different classes of uncertainty structure and various applications of the $\mathcal{L}_1$ adaptive controller.

While the general framework of the $\mathcal{L}_1$ adaptive control shows that the performance bounds (in particular, the prediction error) tend to zero as the adaptation rate is increased to infinity, in practice, the adaptation rate cannot be increased arbitrarily large, as the CPU cannot run arbitrarily fast. This issue is particularly challenging in the application of networked control systems, in which both the computation and the sensing resources are limited.

This paper proposes an $\mathcal{L}_1$ adaptive controller with a modified piecewise constant adaptive law that imposes significantly less stringent requirements on the CPU. The main idea is to more efficiently exploit the information of the uncertainties from the past sample time, and use this estimation to compensate for the uncertainty in the future. As a result, for a given sample period, the new adaptive law leads to improved performance bounds with significantly less requirements for computation and measurement frequency, as compared with the standard piecewise constant adaptive law.

This paper is organized as follows. Section II gives the problem formulation and specifies the control objective. Section III presents the $\mathcal{L}_1$ adaptive controller structure and defines the bounds for performance specification. Section IV analyzes the stability and performance of the $\mathcal{L}_1$ adaptive controller. Section V shows the simulation results, which demonstrate the theoretical findings. Section VI concludes the paper and proposes future research directions.

## II. Problem Formulation

Consider the following MIMO system with nonlinear unmatched uncertainties:

$$\begin{aligned}&\dot{x}(t) = A_m x(t) + B_m \omega u(t) + f(x(t),t)\,,\ x(0) = x_0,\\ &y(t) = Cx(t)\,,\end{aligned} \tag{1}$$

where the system state vector $x(t) \in \mathbb{R}^n$ is measured for feedback; $u(t) \in \mathbb{R}^m$ is the control signal ($n \geq m$); $y(t) \in \mathbb{R}^m$ is the regulated output; $A_m$ is a known Hurwitz $n \times n$ matrix that defines the desired dynamics for the closed-loop system; $B_m \in \mathbb{R}^{n\times m}$ is a known constant matrix with linearly independent columns, and $(A_m\,, B_m)$ is controllable; $C \in \mathbb{R}^{m\times n}$ is a known full-rank constant matrix with $(A_m, C)$ observable; $\omega \in \mathbb{R}^{m\times m}$ is the unknown high-frequency gain matrix; $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is an unknown nonlinear function. The initial condition $x_0$ is bounded according to $\|x_0\|_\infty \leq \rho_0$ for some known $\rho_0 > 0$.

The system in (1) can be rewritten in the form

$$\begin{aligned}\dot{x}(t) =& A_m x(t) + B_m \left(\omega u(t) + f_1(x(t),t)\right) + B_{um} f_2(x(t),t)\\ y(t) =& Cx(t)\,,\quad x(0) = x_0,\end{aligned} \tag{2}$$

where $B_{um} \in \mathbb{R}^{n\times(n-m)}$ is a constant matrix such that $B_m^\top B_{um} = 0$ and also $\mathrm{rank}([B_m\ B_{um}]) = n$; and $f_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, $f_2 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n-m}$, are unknown nonlinear functions that verify

$$\begin{bmatrix} f_1(x(t),t) \\ f_2(x(t),t) \end{bmatrix} = B^{-1} f(x(t),t)\,, \tag{3}$$

where $B = [B_m\ B_{um}]$. In this problem formulation, $f_1(\cdot)$ represents the matched component of the uncertainties, whereas $B_{um} f_2(\cdot)$ represents the cross-coupling dynamics.

The system uncertainties $f(x(t),t)$ and $\omega$ are assumed to satisfy the following conditions.

*Assumption 1:* There exists $B_i > 0$, $i = 1, 2$, such that $\|f_i(0,t)\|_\infty \leq B_{i0}$ for all $t \geq 0$.

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*Assumption 2:* For any $\delta > 0$, there exist positive constants $K_{i\delta}$, $i = 1, 2$, such that

$$\|f_i(x_1,t) - f_i(x_2,t)\|_\infty \leq K_{i\delta}\|x_1 - x_2\|_\infty$$

for all $\|x_1\|_\infty, \|x_2\|_\infty \leq \delta$, uniformly in $t$.

*Assumption 3:* For any $\delta > 0$, there exists a positive constant $L_\delta$, such that

$$\left\|\frac{\partial f(x,t)}{\partial t}\right\|_\infty \leq L_\delta$$

holds for all $\|x\|_\infty \leq \delta$, uniformly in $t$.

*Assumption 4:* The system input gain matrix $\omega$ is assumed to be an unknown (non-singular) strictly row-diagonally dominant matrix with $\mathrm{sgn}(\omega_{ii})$ known. Also, we assume that there exists a known compact convex set $\Omega$, such that $\omega \in \Omega \subset \mathbb{R}^{m\times m}$, and that a nominal system input gain $\omega_0 \in \Omega$ is known.

The control objective is to design an adaptive state feedback controller to ensure that $y(t)$ tracks the output response of a *desired system* $M(s) \triangleq C\left(s\mathbb{I} - A_m\right)^{-1} B_m K_g(s)$, where $K_g(s)$ is a feedforward prefilter, to a given bounded reference signal $r(t)$ *both in transient* and *steady-state*, while all other signals remain bounded.

## III. $\mathcal{L}_1$ Adaptive Controller

### A. Definitions and Sufficient Condition for Stability

Define the following transfer matrices

$$H_{xm}(s) \triangleq (s\mathbb{I}_n - A_m)^{-1} B_m\,, \qquad H_m(s) \triangleq C H_{xm}(s)\,,$$
$$H_{xum}(s) \triangleq (s\mathbb{I}_n - A_m)^{-1} B_{um}\,, \quad H_{um}(s) \triangleq C H_{xum}(s)\,,$$

and let $x_{in}(t)$ be the signal with Laplace transform $x_{in}(s) \triangleq (s\mathbb{I} - A_m)^{-1}x_0$, and $\rho_{in} \triangleq \|s(s\mathbb{I} - A_m)^{-1}\|_{\mathcal{L}_1}\rho_0$, where $\rho_0$ is defined in Section II. Since $A_m$ is Hurwitz and $x_0$ is finite, $\|x_{in}\|_{\mathcal{L}_\infty} \leq \rho_{in}$.

The design of the $\mathcal{L}_1$ adaptive controller involves a gain matrix $K \in \mathbb{R}^{m\times m}$ and an $m \times m$ strictly-proper transfer matrix $D(s)$, which lead to a strictly-proper stable

$$C(s) \triangleq \omega K D(s)(\mathbb{I}_m + \omega K D(s))^{-1} \tag{4}$$

with DC gain $C(0) = \mathbb{I}_m$ for all $\omega \in \Omega$. The choice of $D(s)$ needs to ensure also that $C(s)H_m^{-1}$ is proper and stable.

For the proofs of stability and performance bounds, the choice of $K$ and $D(s)$ also needs to ensure that, for a given $\rho_0$, there exists $\rho_r > \rho_{in}$ such that the following $\mathcal{L}_1$-norm condition holds:

$$\|G_m(s)\|_{\mathcal{L}_1} + \|G_{um}(s)\|_{\mathcal{L}_1}\,\ell_0 \leq \frac{\rho_r - \|H_{xm}(s)C(s)K_g(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} - \rho_{in}}{K_{1\rho_r}\rho_r + \bar{B}}\,, \tag{5}$$

where $\ell_0 \triangleq K_{2\rho_r}/K_{1\rho_r}$, $\bar{B} \triangleq \max\{B_{10}, B_{20}/\ell_0\}$, and

$$G_m(s) = H_{xm}(s)(\mathbb{I}_m - C(s))\,,$$
$$G_{um}(s) = \left(\mathbb{I}_n - H_{xm}C(s)H_m^{-1}(s)C\right) H_{xum}(s)\,,$$

and $K_g(s)$ is the feedforward prefilter. Further, for an arbitrary constant $\bar{\gamma}_x > 0$, let

$$\rho_x = \rho_r + \bar{\gamma}_x,$$

and let $\gamma_x$ be given by

$$\gamma_x \triangleq \frac{\left\|H_{xm}(s)C(s)H_m^{-1}(s)C\right\|_{\mathcal{L}_1}}{1 - \|G_m(s)\|_{\mathcal{L}_1} K_{1\rho_r} - \|G_{um}(s)\|_{\mathcal{L}_1} K_{2\rho_r}}\bar{\gamma}_0 \;+\; \beta,$$

where $\bar{\gamma}_0$ and $\beta$ are arbitrarily small positive constants, such that $\gamma_x \leq \bar{\gamma}_x$. Let

$$\rho_u = \rho_{u_r} + \gamma_u,$$

where $\rho_{u_r}$ and $\gamma_u$ are defined as

$$\begin{aligned}
\rho_{u_r} \triangleq\;& \left\|\omega^{-1}C(s)\right\|_{\mathcal{L}_1}(K_{1\rho_r}\rho_r + B_{10}) \\
&+ \left\|\omega^{-1}C(s)H_m^{-1}(s)H_{um}(s)\right\|_{\mathcal{L}_1}(K_{2\rho_r}\rho_r + B_{20}) \\
&+ \left\|\omega^{-1}C(s)K_g(s)\right\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} \\
\gamma_u \triangleq\;& \Big(\left\|\omega^{-1}C(s)\right\|_{\mathcal{L}_1} K_{1\rho_r} \\
&+ \left\|\omega^{-1}C(s)H_m^{-1}(s)H_{um}(s)\right\|_{\mathcal{L}_1} K_{2\rho_r}\Big)\gamma_x \\
&+ \left\|\omega^{-1}C(s)H_m^{-1}(s)C\right\|_{\mathcal{L}_1}\bar{\gamma}_0\,.
\end{aligned}$$

### B. Definitions for the Performance Bounds

Let $T_s > 0$ be the adaptation sample period, which can be directly related to the computation capabilities of the CPU and the sensor sample period.

For $i = 1, 2$, let

$$B_0 = \max_i B_{i0}\,,\;\; K_\delta = \max_i K_{i\delta}\,,$$

for any $\delta > 0$.

Let $\alpha_1(t)$, $\alpha_2(t)$ and $\alpha_3(t)$ be defined as

$$\begin{aligned}
\alpha_1(t) &= \|\mathrm{e}^{A_m t}\|_\infty, \\
\alpha_2(t) &= \int_0^t \|\mathrm{e}^{A_m(t-\xi)}B\Phi^{-1}(T_s)\mathrm{e}^{A_m T_s}\|_\infty d\xi, \\
\alpha_3(t) &= \int_0^t \|\mathrm{e}^{A_m(t-\xi)}B\|_\infty d\xi,
\end{aligned}$$

where $\Phi(t)$ is an $n \times n$ matrix

$$\Phi(t) = \int_0^t \mathrm{e}^{A_m(t-\xi)}B d\xi\,,$$

for $t \in [0\,,\; T_s]$.

Define the following bounds

$$\begin{aligned}
&\bar{\alpha}_1(T_s) = \max_{t\in[0,T_s]} \alpha_1(t), \qquad \bar{\alpha}_2(T_s) = \max_{t\in[0,T_s]} \alpha_2(t), \\
&\bar{\alpha}_3(T_s) = \max_{t\in[0,T_s]} \alpha_3(t)\,.
\end{aligned} \tag{7}$$

Define

$$\begin{aligned}
d_x \triangleq\;& \|A_m\|_\infty\rho_x + \max_{\omega\in\Omega}\|B_m(\omega - \omega_0)\|_\infty\rho_u \\
&+ \|B\|_\infty(K_{\rho_x}\rho_x + B_0)\,, \\
\rho_\sigma \triangleq\;& \max_{\omega\in\Omega}\|\omega - \omega_0\|_\infty\rho_u + (K_{\rho_x}\rho_x + B_0)\,, \\
d_u \triangleq\;& \|s(\mathbb{I}_m + KD(s)\omega_0)^{-1}KD(s)\|_{\mathcal{L}_1}(\|K_g(s)\|_{\mathcal{L}_1}\|r\|_{\mathcal{L}_\infty} \\
&+ (1 + \|H_m^{-1}(s)H_{um}(s)\|_{\mathcal{L}_1})(2\bar{\alpha}_1(T_s) + 1)\rho_\sigma)\,, \\
d_\sigma \triangleq\;& \max_{\omega\in\Omega}\|\omega - \omega_0\|_\infty d_u + (K_{\rho_x}d_x + L_{\rho_x})\,.
\end{aligned}$$

Finally, let

$$\varsigma(T_s) = \bar{\alpha}_3(T_s) d_\sigma T_s,$$

and

$$\gamma_0(T_s) \triangleq (\bar{\alpha}_1(T_s) + \bar{\alpha}_2(T_s))\bar{\alpha}_3(T_s) d_\sigma T_s + 2\bar{\alpha}_3(T_s) d_\sigma T_s .$$

*Lemma 1:* The following limit holds:

$$\lim_{T_s \to 0} \bar{\gamma}_0(T_s) = 0 .$$

### C. $\mathcal{L}_1$ Adaptive Control Architecture

The key idea of the $\mathcal{L}_1$ adaptive controller is to estimate the signals due to the mismatch between the plant and the desired model, and to utilize this estimation in the controller, which compensates for the effects of the uncertainties only within the bandwidth of a low-pass filter. The main elements of it are defined next.

**State-predictor:** Consider the following state predictor

$$\dot{\hat{x}}(t) = A_m \hat{x}(t) + B_m \omega_0 u(t) + B\hat{\sigma}(t) , \ \hat{x}(0) = x_0 , \quad (8)$$

where $\hat{\sigma} = [\hat{\sigma}_1^\top, \hat{\sigma}_2^\top]^\top$, $\hat{\sigma}_1 \in \mathbb{R}^m$, and $\hat{\sigma}_2 \in \mathbb{R}^{n-m}$.

**Adaptive Laws:** $\hat{\sigma}(t)$ is updated by the following modified piecewise constant adaptive laws:

$$h(t) = h(kT_s) \quad (9a)$$
$$\hat{\sigma}(t) = \hat{\sigma}(kT_s), \quad t \in [kT_s, (k+1)T_s) \quad (9b)$$
$$h(kT_s) = -\tilde{x}(kT_s) + h((k-1)T_s) , \quad h(0) = 0 \quad (9c)$$
$$\hat{\sigma}(kT_s) = -\Phi^{-1}(T_s) \mathrm{e}^{A_m T_s} \tilde{x}(kT_s) + \Phi^{-1}(T_s) h(kT_s) \quad (9d)$$

for $k = 0, 1, 2, \ldots$, where $\tilde{x}(t) \triangleq \hat{x}(t) - x(t)$ is the prediction error.

**Control Law:** The control signal $u(t)$ is generated in frequency domain by:

$$u(s) = -KD(s) \left(\omega_0 u(s) + \hat{\sigma}_1(s) + \hat{\sigma}_{2m}(s) - K_g(s) r(s)\right) , \quad (10)$$

where $\hat{\sigma}_{2m}(s) = H_m^{-1}(s) H_{um}(s) \hat{\sigma}_2(s)$.

*Remark 1:* The main difference from the controller in [10] is the form of the adaptive law. Here, the function of $h(t)$ is to record the influence of the uncertainty from the past step and to use it to improve the prediction error in the next step. Lemma 3 clarifies its derivation.

## IV. Analysis of $\mathcal{L}_1$ Adaptive Controller

### A. Closed-Loop Reference System

To analyze the stability and performance of the $\mathcal{L}_1$ adaptive controller, we first assume that the input gain matrix $\omega$ and the uncertain nonlinear function $f(x(t), t)$ are known, and design a non-adaptive controller, which yields the closed-loop reference system:

$$\begin{aligned} \dot{x}_{\mathrm{ref}}(t) = \ & A_m x_{\mathrm{ref}}(t) + B_m(\omega u_{\mathrm{ref}}(t) + f_1(x_{\mathrm{ref}}(t), t)) \\ & + B_{um} f_2(x_{\mathrm{ref}}(t), t) , \ x_{\mathrm{ref}}(0) = x_0 \quad (11a) \\ u_{\mathrm{ref}}(s) = \ & \omega^{-1} C(s) (K_g(s) r(s) - \eta_{1\mathrm{ref}}(s) \\ & H_m^{-1}(s) H_{um}(s) \eta_{2\mathrm{ref}}(s)) \quad (11b) \\ y_{\mathrm{ref}}(t) = \ & C x_{\mathrm{ref}}(t) , \quad (11c) \end{aligned}$$

where $\eta_{i\mathrm{ref}}(t) \triangleq f_i(x_{\mathrm{ref}}(t), t)$ for $i = 1, 2$.

*Lemma 2:* For the closed-loop reference system in (11), subject to the $\mathcal{L}_1$-norm condition (5), if $\|x_0\|_\infty \leq \rho_0$, then

$$\|x_{\mathrm{ref}}\|_{\mathcal{L}_\infty} < \rho_r , \ \text{and} \ \|u_{\mathrm{ref}}\|_{\mathcal{L}_\infty} < \rho_{u_r} . \quad (12)$$

**Proof.** The proof is similar to the one for Lemma 2 of [10], and is thus omitted here.

### B. Prediction Error Signal

Subtracting (2) from (8), we obtain the prediction error dynamics

$$\dot{\tilde{x}}(t) = A_m \tilde{x}(t) + B_m \tilde{\eta}_1(t) + B_{um} \tilde{\eta}_2(t) , \ \tilde{x}(0) = 0, \quad (13)$$

where

$$\begin{aligned} \tilde{\eta}_1(t) &= \hat{\sigma}_1(t) - ((\omega - \omega_0) u(t) + \eta_1(t)) \\ \tilde{\eta}_2(t) &= \hat{\sigma}_2(t) - \eta_2(t) \end{aligned} \quad (14)$$

with $\eta_i(t) = f_i(x(t), t)$ for $i = 1, 2$.

Next we show that if $T_s$ is chosen to satisfy

$$\gamma_0(T_s) < \bar{\gamma}_0 , \quad (15)$$

then the prediction error $\tilde{x}(t)$ can be systematically reduced both in transient and steady-state by reducing $T_s$. More importantly, an estimate of the information of the uncertainty can be extracted from the prediction error signal, and be used in the controller to compensate for the effect of the uncertainty.

*Lemma 3:* Let the adaptation sample period $T_s$ be selected to satisfy (15). For the prediction error dynamics (13), if

$$\|x_\tau\|_{\mathcal{L}_\infty} \leq \rho_x , \quad \|u_\tau\|_{\mathcal{L}_\infty} \leq \rho_u , \quad (16)$$

for some $\tau > 0$, then

$$\|\tilde{x}_\tau\|_{\mathcal{L}_\infty} < \bar{\gamma}_0 .$$

**Proof.**

If the bounds in (16) hold, Assumptions 2 and 3 lead to $\|\eta_{i\tau}\|_{\mathcal{L}_\infty} \leq K_{i\rho_x} \rho_x + B_{i0}$, for $i = 1, 2$, which implies that

$$\|\eta_\tau\|_{\mathcal{L}_\infty} \leq K_{\rho_x} \rho_x + B_0 , \quad (17)$$

where $\eta = [\eta_1^\top, \eta_2^\top]^\top$.

From the system dynamics in (2) and the bounds of $\|x_\tau\|_{\mathcal{L}_\infty}$ and $\|u_\tau\|_{\mathcal{L}_\infty}$, $\dot{x}(t)$ is bounded by

$$\begin{aligned} \|\dot{x}_\tau\|_{\mathcal{L}_\infty} \leq & \|A_m\|_\infty \rho_x + \max_{\omega \in \Omega} \|B_m(\omega - \omega_0)\|_\infty \rho_u \\ & + \|B\|_\infty (K_{\rho_x} \rho_x + B_0) = d_x . \end{aligned}$$

For any $t_1, t_2 \in [0, \tau]$ with $|t_1 - t_2| \leq T_s$,

$$\begin{aligned} & \|\eta(t_1) - \eta(t_2)\|_\infty \\ = & \|f(x(t_1), t_1) - f(x(t_2), t_2)\|_\infty \\ = & \|f(x(t_1), t_1) - f(x(t_2), t_1) + f(x(t_2), t_1) - f(x(t_2), t_2)\|_\infty \\ \leq & K_{\rho_x} \|x(t_1) - x(t_2)\|_\infty + L_{\rho_x} T_s \\ \leq & (K_{\rho_x} d_x + L_{\rho_x}) T_s . \end{aligned} \quad (18)$$

The error dynamics (13) can be rewritten as

$$\dot{\tilde{x}}(t) = A_m \tilde{x}(t) + B(\hat{\sigma}(t) - \sigma(t)) , \quad (19)$$

where

$$\sigma(t) = \begin{bmatrix} (\omega-\omega_0)u(t)+\eta_1(t) \\ \eta_2(t) \end{bmatrix} . \tag{20}$$

For any $t \in [0\,,\,T_s]$, integrating the error dynamics (19) from time $kT_s$ to $kT_s+t$ yields

$$\begin{aligned}\tilde{x}(kT_s+t) =& \mathrm{e}^{A_m t}\tilde{x}(kT_s) + \int_0^t \mathrm{e}^{A_m(t-\xi)}Bd\xi\hat{\sigma}(kT_s) \\ &- \int_0^t \mathrm{e}^{A_m(t-\xi)}B\sigma(kT_s+\xi)d\xi\,. \end{aligned} \tag{21}$$

Thus, at the sample time we have

$$\begin{aligned}\tilde{x}((k+1)T_s) =& \mathrm{e}^{A_m T_s}\tilde{x}(kT_s) + \Phi(T_s)\hat{\sigma}(T_s) \\ &- \int_0^{T_s} \mathrm{e}^{A_m(T_s-\xi)}B\sigma(kT_s+\xi)d\xi\,. \end{aligned} \tag{22}$$

From the recursion of the adaptive laws (9) one can verify

$$\tilde{x}((k+1)T_s) = h(kT_s) - h((k+1)T_s)\,, \tag{23}$$

and

$$h(kT_s) = \int_0^{T_s} \mathrm{e}^{A_m(T_s-\xi)}B\sigma((k-1)T_s+\xi)d\xi\,.$$

From (20), (17) and the bound on $u$, $\sigma(t)$ is bounded by

$$\|\sigma_\tau\|_{\mathcal{L}_\infty} \leq \max_{\omega\in\Omega}\|\omega-\omega_0\|_\infty\rho_u + (K_{\rho_x}\rho_x+B_0) = \rho_\sigma\,,$$

and $h(kT_s)$ satisfies

$$\|h(kT_s)\|_\infty \leq \bar{\alpha}_3(T_s)\rho_\sigma\,, \quad \forall\; kT_s \leq \tau\,.$$

Now we have a bound on $\tilde{x}$ at the sample time

$$\|\tilde{x}((k+1)T_s)\|_\infty \leq 2\bar{\alpha}_3(T_s)\rho_\sigma\,,\; \forall (k+1)T_s < \tau\,,$$

which will be replaced by a more precise bound later with better convergence properties. From the adaptive laws (9) we have

$$\|\hat{\sigma}_\tau\|_{\mathcal{L}_\infty} \leq (2\bar{\alpha}_1(T_s)+1)\rho_\sigma\,.$$

The control law can be written as

$$u(s) = (\mathbb{I}+KD(s)\omega_0)^{-1}KD(s)(K_g r(s) - \hat{\sigma}_1(s) - \hat{\sigma}_{2m}(s))\,.$$

Further, $\dot{u}(t)$ can be bounded by

$$\|\dot{u}_\tau\|_{\mathcal{L}_\infty} \leq d_u\,, \tag{24}$$

and thus for any $t_1, t_2 \in [0,\tau]$ with $|t_1-t_2| \leq T_s$, we have

$$\|\sigma(t_1)-\sigma(t_2)\|_\infty \leq d_\sigma T_s\,. \tag{25}$$

Now we can derive a new bound for $\tilde{x}$ at the sample time,

$$\|\tilde{x}((k+1)T_s)\| \leq \bar{\alpha}_3(T_s)d_\sigma T_s\,.$$

Thus, the bound in (16) holds for arbitrary $kT_s \leq \tau$. Yet we still need to show the bounds for $kT_s+t \leq \tau$, with $t\in[0,T_s]$. Plugging (9) into (21), we have

$$\begin{aligned}\tilde{x}(kT_s+t) =& \mathrm{e}^{A_m t}\tilde{x}(kT_s) \\ &- \int_0^t \mathrm{e}^{A_m(t-\xi)}B\Phi^{-1}(T_s)\mathrm{e}^{A_m T_s}d\xi\tilde{x}(kT_s) + \zeta(t)\,,\end{aligned}$$

where

$$\begin{aligned}\zeta(t) =& \Phi(t)\Phi^{-1}(T_s)\int_0^{T_s}\mathrm{e}^{A_m(T_s-\xi)}B\sigma((k-1)T_s+\xi)d\xi \\ &- \int_0^t \mathrm{e}^{A_m(t-\xi)}B\sigma(kT_s+\xi)d\xi\,.\end{aligned}$$

Define $\delta_1(\xi) = \sigma((k-1)T_s+\xi) - \sigma(kT_s)$ and $\delta_2(\xi) = \sigma(kT_s+\xi)-\sigma(kT_s)$, which by (25) satisfy

$$\|\delta_i(\xi)\|_\infty \leq d_\sigma T_s\,,\; i=1,2\,,\forall\;\xi\in[0\,,T_s]\,.$$

Then we have

$$\begin{aligned}\zeta(t) =& \Phi(t)\Phi^{-1}(T_s)\int_0^{T_s}\mathrm{e}^{A_m(T_s-\xi)}B\left(\sigma(kT_s)+\delta_1(\xi)\right)d\xi \\ &- \int_0^t \mathrm{e}^{A_m(t-\xi)}B\left(\sigma(kT_s)+\delta_2(\xi)\right)d\xi \\ =& \Phi(t)\Phi^{-1}(T_s)\int_0^{T_s}\mathrm{e}^{A_m(T_s-\xi)}B\delta_1(\xi)d\xi \\ &- \int_0^t \mathrm{e}^{A_m(t-\xi)}B\delta_2(\xi)d\xi\,.\end{aligned}$$

From the definition of $\bar{\alpha}_i(T_s)$, we have

$$\begin{aligned}\|\tilde{x}(kT_s+t)\|_\infty \leq& (\bar{\alpha}_1(T_s)+\bar{\alpha}_2(T_s))\bar{\alpha}_3(T_s)d_\sigma T_s \\ &+ 2\bar{\alpha}_3(T_s)d_\sigma T_s = \gamma_0(T_s)\,,\end{aligned}$$

which implies that

$$\|\tilde{x}(t)\|_\infty \leq \bar{\gamma}_0\,, \quad \forall t\in[0,\tau]\,.$$

Thus, we have

$$\|\tilde{x}_\tau\|_{\mathcal{L}_\infty} \leq \bar{\gamma}_0\,.$$

□

*Remark 2:* The bound $\bar{\gamma}_0$ for the prediction error dynamics decreases as the sample period $T_s$ decreases, similar to the case in [10]. However, the decay rate here is faster due to the multiplication by $T_s$. The benefit is obvious: given the same adaptation sample period, the performance bound is tighter in this paper; or with the same performance bound requirement, the controller requires less computational power (CPU).

### C. Transient and Steady-State Performance

Next, we show the stability of the closed-loop system with the $\mathcal{L}_1$ adaptive controller and derive the performance bounds.

*Theorem 1:* Let $T_s$ be chosen to satisfy (15). Given the closed-loop system with the $\mathcal{L}_1$ adaptive controller, subject to the $\mathcal{L}_1$-norm condition in (5), and the closed-loop reference system in (11), if $\|x_0\|_\infty \leq \rho_0$, then

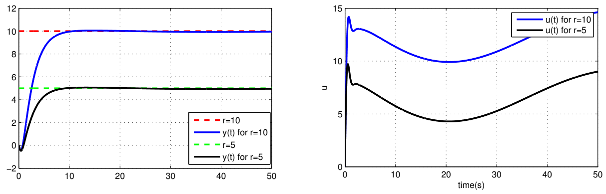
$$\|x\|_{\mathcal{L}_\infty} \leq \rho_x\,, \qquad \|u\|_{\mathcal{L}_\infty} \leq \rho_x \tag{26}$$

$$\|\tilde{x}\|_{\mathcal{L}_\infty} \leq \bar{\gamma}_0 \tag{27}$$

$$\|x_{\mathrm{ref}}-x\|_{\mathcal{L}_\infty} \leq \gamma_x\,, \qquad \|u_{\mathrm{ref}}-u\|_{\mathcal{L}_\infty} \leq \gamma_u \tag{28}$$

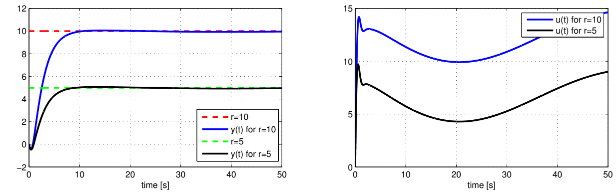
$$\|y_{\mathrm{ref}}-y\|_{\mathcal{L}_\infty} \leq \|C\|_\infty\gamma_x \tag{29}$$

**Proof.** The proof is similar to the proof in [9] and is thus omitted.




(a) $y(t)$ (b) $u(t)$

Fig. 1: Standard $\mathcal{L}_1$ Controller, $T_s = 0.001s$




(a) $y(t)$ (b) $u(t)$

Fig. 2: Modified $\mathcal{L}_1$ Controller, $T_s = 0.001s$

## V. Simulation Examples

This section shows the simulation results to demonstrate the performance of the proposed $\mathcal{L}_1$ adaptive controller. We start with a simple SISO system with unmatched nonlinear uncertainties and highlight the improvements of performance using the controller in this paper compared with the standard $\mathcal{L}_1$ adaptive controller with piecewise constant adaptive law from [10]. Then we show simulation results for a more complicated MIMO system.

The systems considered in both cases have the following form

$$\dot{x}(t) = (A_m + A_\Delta)x(t) + B_m \omega u(t) + f_\Delta(x(t), t)\,,$$
$$y(t) = Cx(t)\,,\; x(0) = x_0\,,$$

where $A_m$, $B_m$, $C$ are known, $A_\Delta$, $\omega$ are unknown constant matrices, and $f_\Delta$ is an unknown nonlinear function. The control objective is to design a control law $u(t)$ so that the output of the system $y(t)$ tracks the output of the desired model $M(s)$, defined by $(A_m, B_m, C)$, to a bounded reference signal $r(t)$.

### A. *Performance Improvements of the $\mathcal{L}_1$ Adaptive Controller*

Consider a SISO system with

$$A_m = \begin{bmatrix} -0.5 & 1 \\ 0 & -1.5 \end{bmatrix}, B_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and uncertainties given by

$$A_\Delta = \begin{bmatrix} -0.2 & -0.2 \\ 0.1 & 0.1 \end{bmatrix}, f_\Delta(x(t), t) = \begin{bmatrix} -1 + 2\cos(0.05t) \\ 2\sin(0.1t) \end{bmatrix}$$

and $\omega = 1$.

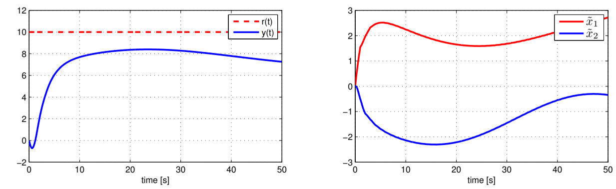
The parameters used in the design of the $\mathcal{L}_1$ adaptive controller are listed as follows

$$\omega_0 = 0.9\,, K = 4\,, D(s) = \frac{1}{0.2s^2 + s}\,,$$

and $K_g(s) = K_g = -(CA_m^{-1}B_m)^{-1} = 0.75$.

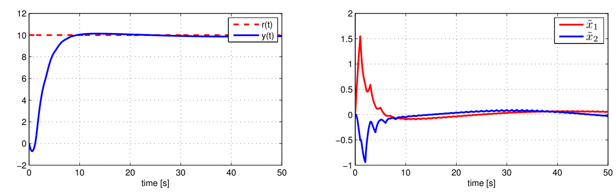
First, we let $T_s = 0.001s$ and plot the simulation results for the standard $\mathcal{L}_1$ adaptive controller and the controller in this paper in Figure 1 and Figure 2, respectively. We can see that both controllers have satisfactory tracking performance and have *scaled response for scaled reference signals* $r(t)$.

However, when the sample period is set to $T_s = 1s$, the performances of the two controllers are quite different. One can observe that the system output in Figure 3-(a) is oscillating and has a steady-state error, while in Figure 4-(a)



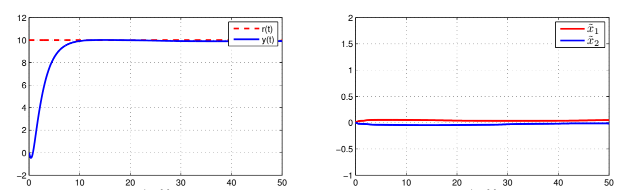

(a) $y(t)$ (b) $\tilde{x}(t)$

Fig. 3: Standard $\mathcal{L}_1$ Controller, $T_s = 1s$



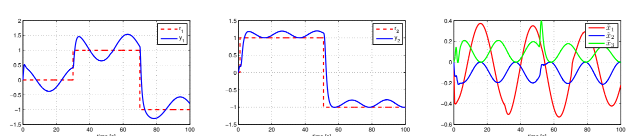

(a) $y(t)$ (b) $\tilde{x}(t)$

Fig. 4: Modified $\mathcal{L}_1$ Controller, $T_s = 1s$



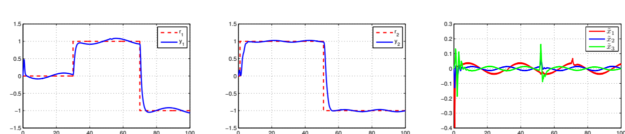

(a) $y(t)$ (b) $\tilde{x}(t)$

Fig. 5: Standard $\mathcal{L}_1$ Controller, $T_s = 0.01s$



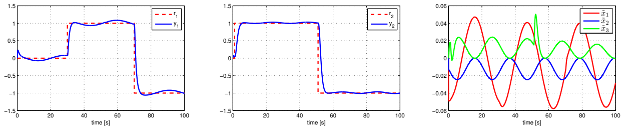
(a) $y_1(t)$ (b) $y_2(t)$ (c) $\tilde{x}(t)$

Fig. 6: Standard $\mathcal{L}_1$ Controller, $T_s = 0.5s$



(a) $y_1(t)$ (b) $y_2(t)$ (c) $\tilde{x}(t)$

Fig. 7: Modified $\mathcal{L}_1$ Controller, $T_s = 0.5s$



(a) $y_1(t)$ (b) $y_2(t)$ (c) $\tilde{x}(t)$

Fig. 8: Standard $\mathcal{L}_1$ Controller, $T_s = 0.05s$

the tracking performance is much better. The difference can also be observed from Figure 3-(b) and Figure 4-(b), which show the prediction error signals for the two controllers, respectively. One can see that in Figure 4-(b), the prediction error has much smaller bound after the first several samples, when no information of the uncertainty is available to the estimation components (state predictor and adaptive law) yet.

The question is: to achieve the same level of tracking error, what are the sample periods required by each controller, respectively? Using the performance in Figure 4 as a benchmark, we decrease the sample period $T_s$ for the standard $\mathcal{L}_1$ adaptive controller, until similar performance is achieved, which is shown in Figure 5, with corresponding sample period $T_s = 0.01s$. In other words, to achieve the same level of performance for this example, the sample frequency required by the controller in this paper is roughly $\frac{1}{100}$ of the frequency required by the standard $\mathcal{L}_1$ adaptive controller.

### B. Simulations for a MIMO System

Consider a MIMO system with

$$A_m = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1.5 \end{bmatrix}, B_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

where the uncertainties are given by $A_\Delta = 0_{3\times 3}$,

$$f_\Delta(x,t) = \begin{bmatrix} 0.033x^\top x + 0.1\tanh(\frac{1}{2}x_1)x_1 \\ -0.015x_3^2 - 0.01(1-\mathrm{e}^{-0.3t}) \\ -x_3\cos(0.1t) \end{bmatrix}, \omega = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1.1 \end{bmatrix}$$

and the unknown matrix $\omega$ is assumed to be within the convex set $\Omega = \{\omega | \omega = \omega^\top, \omega_{11} \in [1, 1.3], \omega_{12} \in [-0.2, 0.1], \omega_{22} \in [1, 1.3]\}$. The reference signals for outputs $y_1$ and $y_2$ are series of steps.

The parameters used in the design of the $\mathcal{L}_1$ adaptive controller are: $\omega_0 = \mathbb{I}_2$, $K = 4\mathbb{I}_2$,

$$D(s) = \frac{1}{s(s/25+1)(s/70+1)(s^2/40^2+1.8s/40+1)},$$

$$K_g(s) = K_g = -(CA_m^{-1}B_m)^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

First, we let $T_s = 0.5s$ and plot the outputs and the prediction errors using the standard $\mathcal{L}_1$ adaptive controller and the modified controller from this paper, in Figure 6 and Figure 7, respectively. One can observe that the tracking performance using the modified adaptive law is much better than the standard version. Next, by decreasing the sample period for the standard $\mathcal{L}_1$ adaptive controller to $0.05s$, we obtain tracking performance similar to Figure 7, as shown in Figure 8.

## VI. Conclusion and Future Work

This paper designs an $\mathcal{L}_1$ adaptive controller for a class of MIMO systems with unmatched uncertainties using a new piecewise constant adaptive law. Compared with the standard $\mathcal{L}_1$ adaptive controller, the new controller achieves faster decay rate of the performance bounds as the sample period decreases, which results in better performance given the hardware (CPU and sensor) condition, or less stringent requirement for computation for a given performance level. The theoretical findings are demonstrated by extensive simulations.

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