Math 19 - Exam 3 Review Answers (Sections 5.1-8.1)

Chapter 5, Supplementary Exercises

1. Prove by induction: P(n): $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^n}=1-\frac{1}{3^n}$ whenever n is a positive integer (n ≥ 1).

<u>Proof</u>: Basis step: We check that P(1) is true:

$$\frac{2}{3^1}=1-\frac{1}{3^1}$$
$$\frac{2}{3}=\frac{3}{3}-\frac{1}{3}$$
$$\frac{2}{3}=\frac{2}{3}$$

Inductive step: Assume P(k) is true for some k ≥ 1, which means
$\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^k}=1-\frac{1}{3^k}$. This is our inductive hypothesis.

We then want to show P(k+1) is true, which is the statement:
$\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^k}+\frac{2}{3^{k+1}}=1-\frac{1}{3^{k+1}}$.

We start with the left-hand side of the P(k+1) statement:

$$\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\cdots+\frac{2}{3^k}+\frac{2}{3^{k+1}}$$
$$\overset{IH}{=}1-\frac{1}{3^k}+\frac{2}{3^{k+1}}$$
$$=1-\frac{1}{3^k}\cdot\frac{3}{3}+\frac{2}{3^{k+1}}$$
$$=1-\frac{3}{3^{k+1}}+\frac{2}{3^{k+1}}$$
$$=1+\frac{-3+2}{3^{k+1}}$$
$$=1-\frac{1}{3^{k+1}}$$

which gets us to the right-hand side of P(k+1).

Therefore P(k+1) is true and we can conclude that P(n) is true for all positive integers n.

4. Prove by induction: P(n): $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$
whenever n is a positive integer (n ≥ 1).

<u>Proof</u>: Basis step: We check that P(1) is true:

$$\frac{1}{1\cdot 3}=\frac{1}{2(1)+1}$$
$$\frac{1}{3}=\frac{1}{2+1}$$
$$\frac{1}{3}=\frac{1}{3}$$

Inductive step: Assume P(k) is true for some k ≥ 1, which means
$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}.$$
This is our inductive hypothesis.

We then want to show P(k+1) is true, which is the statement:
$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}=\frac{k+1}{2(k+1)+1}.$$

We start with the left-hand side of the P(k+1) statement:

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$$
$$\overset{IH}{=}\frac{k}{2k+1}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$$
$$=\frac{k}{2k+1}+\frac{1}{(2k+2-1)(2k+2+1)}$$
$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$
$$=\frac{k}{2k+1}\cdot\frac{2k+3}{2k+3}+\frac{1}{(2k+1)(2k+3)}$$
$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$
$$=\frac{k+1}{2k+3}=\frac{k+1}{2(k+1)+1}$$
which gets us to the right-hand side of P(k+1).

Therefore P(k+1) is true and we can conclude that P(n) is true for all positive integers n.

Chapter 5, Supplementary Exercises, continued

45. a) $M(102) = 102 - 10 = 92$ b) $M(101) = 101 - 10 = 91$

c)
$$M(99) = M(M(99+11)) = M(M(110)) = M(110-10) = M(100)$$
$$= M(M(100+11)) = M(M(111)) = M(111-10) = M(101) = 101 - 10 = 91$$

d)
$$M(97) = M(M(97+11)) = M(M(108)) = M(108-10) = M(98)$$
$$= M(M(98+11)) = M(M(109)) = M(109-10) = M(99) = 91$$
where the last equality comes from using part (c).

Chapter 6, Review Questions

6. b) There are k = 10 possible last digits (the digits 0, 1, 2, …, 9) and 11 > 10, so in 11 integers there must be at least two that share a last digit, by the Pigeonhole Principle.

7. b) Here we use the Generalized Pigeonhole Principle, with N = 91, k =10.

This tells us that there are at least $\left\lceil \frac{91}{10} \right\rceil = \lceil 9.1 \rceil = 10$ integers which must end with the same digit.

8. c) C(25,6)=177,100 d) P(25,6)= 127,512,000

12. c) Using the Binomial Theorem, the term in the expansion of $(2x+5y)^{201}$ with $x^{100}y^{101}$ has k = 101 and n = 201, so it is the term $C(201,101)(2x)^{100}(5y)^{101}$, which has the coefficient $C(201,101)\cdot 2^{100}\cdot 5^{101}$.

Chapter 6, Supplementary Exercises

1. a) ordered, no repetition counting can be done with P(10,6) = 151,200 or with the product rule: $10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5 = 151{,}200$.
   b) ordered, repetition allowed counting can be done with the product rule: $10\cdot 10\cdot 10\cdot 10\cdot 10\cdot 10 = 1{,}000{,}000$
   c) unordered, no repetition counting can be done with C(10,6)=210.

3. The student has 3 choices (T, F, blank) for each of the 100 questions, so by the product rule, there are 3^{100} ways to fill out answers to the test.

6. By the product rule, there are $9\cdot 10\cdot 10\cdot 10\cdot 10 = 90{,}000$ phone numbers.

10. By the Generalized Pigeonhole Principle we want to solve for the smallest integer N for which $\left\lceil \frac{N}{12} \right\rceil = 6$. This means we can solve the inequality

$$5 < \frac{N}{12} \le 6$$
$$60 < N \le 72$$

The smallest integer N that satisfies the inequality is N = 61 people.

11. By the Generalized Pigeonhole Principle, we know there will be at least 4 repeated fortunes if $\left\lceil \frac{N}{213} \right\rceil = 4$. This means

$$3 < \frac{N}{213} \le 4$$
$$639 < N \le 852$$

Since we want to know how many times (at most) he can go to the restaurant <u>without</u> getting the same fortune 4 times, we choose N = 639, because we know that once N > 639, he will get the same fortune at least 4 times.

29. Use the Binomial Theorem, with x = 1 and y = 3. We get
$(1+3)^n = 4^n = \sum_{k=0}^{n} C(n,k)\cdot 1^{n-k}\cdot 3^k = \sum_{k=0}^{n} 3^k C(n,k)$, since any power of 1 is just 1.

Chapter 8, Supplementary Exercises

1. Note: This answer uses the initial condition with k = 1, not k = 0. Adjust to your solution accordingly.

a) The next person in the chain sends 4 letters, so $a_n = 4a_{n-1}$, n ≥ 2.

b) The first mailing was 40 letters, so $a_1 = 40$ is the initial condition.

c) By backwards recursion, we have:

$$a_n = 4a_{n-1}$$
$$a_n = 4(4a_{n-2}) = 4^2 a_{n-2}$$
$$a_n = 4(4(4a_{n-3})) = 4^3 a_{n-3}$$
$$a_n = 4(4(4(4a_{n-4}))) = 4^4 a_{n-4}$$
$$\vdots$$
$$a_n = 4^{n-1} a_{n-(n-1)} = 4^{n-1} a_1 = 4^{n-1}(40) = 4^{n-1}(4)(10) = 10(4^n)$$