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HW 1.8

Problems = { 1, 6 (do proof by cases), 9, 11, 14, 29, 31, 40 }

1) Prove that $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$ We will do a proof by exhaustion:

a) $(1^2 + 1) = 3 \geq 2^1 = 2$. Since 1 squared plus 1 is three and 2 to the first power is 2 and 3 is greater than or equal to 2, the equation has been proven for this case.

b) $(2^2 + 1) = 5 \geq 2^2 = 4$. Since 2 squared plus 1 equals 5 and 2 squared is 4 and 5 is greater than or equal than 4, the equation has been proven for this case.

c) $(3^2 + 1) = 10 \geq 2^3 = 8$. Since 3 to the 2nd power + 1 is 10 and 2 cubed is 8 and 10 is greater than or equal to 8, the equation has been proven for this case.

d) $(4^2 + 1) = 3 \geq 2^4 = 16$. Since 4 squared + 1 equals 17 and 2 to the 4th power is 16 and 17 is greater than or equal to 16, the equation has been proven for this case.

6) Prove using the notion of without loss generality that 5x + 5y is an odd integer when x and y are integers of opposite parity.

We know that for any integer k, 2k+1 yields an odd integer and 2k yields an even integer.
If we substitute x and y for 2k+1 and 2j and multiply both by 5 we get: $10k + 10j + 5$
Since the integers x and y, or in the case of the substituted equation, k and j, are of opposite parity (meaning one integer is even and the other is odd) we have one case to test.

`Case i`: if k is 0 (even) and j is odd (1) then $10(0) + 10(1) + 5 = 15$. 15 is an odd integer, thus proving the sum of 5x + 5y is odd given two integers of opposite parity.

Since the two integers must be of opposite parity and we are only working with a domain of integers, we have also proven it without loss of generality. If we created a case involving a negative symbol, the output would still always remain odd even if the output may potentially be negative.

9) Prove that there are 100 consecutive integers that are not perfect squares. Is your proof constructive or non-constructive?

Let k be an integer. This represents two numbers that are perfect squares $k^2 and (k + 1)^2$.
Since we are searching for 100 consecutive integers between two perfect squares we can use the above to create: $(k + 1)^2 - k^2 - 1$. We subtract one because subtracting k squared from k squared + 1 would yield 1 which is a perfect square.
Then $(k + 1)^2 - k^2 - 1 > 100$. We then square and solve. $k^2 + 2k + 1 - k^2 - 1$.
Everything cancels except for $2k > 100$. We then divide by 2 finding $k = 50$.
Thus the integers we are looking for must be between 50^2 and 51^2.
$50^2 = 2500$ and $51^2 = 2601$. We have proven by construction (by showing 100 consecutive integers) that there are 100 consecutive integers that are not perfect squares.

11) Prove that there exists a pair of consecutive integers such that one is a perfect square and one is a perfect cube.

We will use a constructive proof by showing an example of two consecutive integers where one is a perfect square and one is a perfect cube.
$2^3 = 8$ and $3^2 = 9$. 8 and 9 are our consecutive integers.

14) Prove that if a and b are rational numbers then a^b is also rational.

We will use a constructive proof by showing a specific example where a^b yields an irrational number.
If $a = 2$, which is rational and $b = \frac{1}{2}$ which is rational, then raise a to the bth power, we are left with an irrational number which is equal to $\sqrt{2}$.

29) Prove that there is no positive integer n such that $n^2 + n^3 = 100$.

We will use a constructive proof by showing specific examples.
We will test values between 1 and 4 since $5^3 = 125$.

$1^2 + 1^3 = 5$
$2^2 + 2^3 = 12$
$3^2 + 3^3 = 36$
$4^2 + 4^3 = 80$
Since all integers above 4 will exceed 100, we have shown that there are no positive integers such that $n^2 + n^3$ will equal 100.

31) Prove that there are no positive integers x and y to the equation $x^4 + y^4 = 625$.
Since $5^4 = 625$ we know that x and y must be less than 5.
If both x and y were 4 then $4^4 + 4^4 = 512$ which is less than 625.
Thus the sum of the integers x and y raised to the 4th power don't equal 625.

40) Verify the $3x + 1$ conjecture for the following integers.
The conjecture states that while the starting number isn't 1, if it's even then divide it by 2 and if the number is odd multiply it by 3 and add 1 to it.
a) 16
16 / 2 = 8 / 2 = 4 / 2 = 2 / 2 = 1
b) 11
3(11)+1 = 34 / 2 = 17 = 3(17)+1 = 52 / 2 = 26 / 2 = 13 = 3(13) + 1 = 40 / 2 = 20 / 2 = 10 / 2 = 5 = 5(3)+1 = 16 /2 = 8 / 2 = 4 / 2 = 2 / 2 = 1.
b) 35
35 = 3(35) + 1 = 106 / 2 = 53 = 3(53)+1 = 160 / 2 = 80 / 2 = 40 / 2 = 20 / 2 = 10 / 2 = 5 = 3(5) + 1 = 16 / 2 = 8 / 2 = 4 / 2 = 2 / 2 = 1.
d) 113
113 = 3(113)+1 = 340 / 2 = 170 / 2 = 85 = 3(85) + 1 = 256 / 2 = 128 / 2 = 64 / 2 = 32 / 2 = 16 / 2 = 8 / 2 = 4 / 2 = 2 / 2 = 1.