Closed-form Covariance For Generating Mock Redshifted 21cm Signal Fields

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1 Introduction

We want to produce the visibilities that would be the result of observing a redshifted, Gaussian field with known power spectrum $P(k)$. The observed specific intensity I in the direction $\hat{\mathbf{s}}$ at frequency ν is

$$I(\nu, \hat{\mathbf{s}}) = 2k_b \frac{\nu^2}{c^2} T(\nu, \hat{\mathbf{s}}) \text{ [Jy/sr]}. \tag{1}$$

The observed specific intensity $I(\nu)$ is related to the cosmological source specific intensity $I'(\nu')$ by invariance of the photon phase space number density (e.g. Rybicki and Lightman (1991, §4.9)), which produces

$$\frac{I(\nu)}{\nu^3} = \frac{I'(\nu')}{\nu'^3}. \tag{2}$$

In the rest frame of the source, $\nu' = \nu_e = 1420.4057$MHz, and the intensity of the emission is parameterized by a brightness temperature field $T(\vec{\mathbf{r}}, z)$. so that the brightness temperature of the observed redshifted radiation is then

$$\frac{T(\nu, \hat{\mathbf{s}})}{\nu} = \frac{T(\vec{\mathbf{r}}, z)\big|_{r=r_\nu, z=z_\nu}}{\nu_e} \tag{3}$$

where $z_\nu = \frac{\nu_e}{\nu} - 1$ is the redshift, $\vec{\mathbf{r}} = r\hat{\mathbf{s}}$ is the co-moving coordinate vector, and r_ν is the co-moving distance

$$r_\nu = r(\nu) = c\int_0^{z_\nu} \frac{\mathrm{d}z'}{H(z')}. \tag{4}$$

The cosmological brightness temperature field is further decomposed as

$$T(\vec{\mathbf{r}}, z) = \overline{T}(z) + \delta T(\vec{\mathbf{r}}, z) \tag{5}$$

$$= \overline{T}(z) + \int \frac{\mathrm{d}^3\vec{\mathbf{k}}}{(2\pi)^3} \delta\widetilde{T}(\vec{\mathbf{k}}, z) e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}} \tag{6}$$

where $\delta\widetilde{T}$ is a Gaussian random field with

$$\left\langle \delta\widetilde{T}(\vec{\mathbf{k}}, z) \right\rangle = 0 \tag{7}$$

$$\left\langle \delta\widetilde{T}(\vec{\mathbf{k}}, z)\delta\widetilde{T}^*(\vec{\mathbf{k}}', z) \right\rangle = (2\pi)^3 P(k, z)\delta(\vec{\mathbf{k}} - \vec{\mathbf{k}}') \tag{8}$$

Alternatively, the field's statistics are characterized by the spatial covariance function

$$\xi(|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|, z) = \langle \delta T(\vec{\mathbf{r}}, z)\delta T(\vec{\mathbf{r}}', z)\rangle \tag{9}$$

$$= \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k\, k^2 j_0(|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|k) P(k, z) \tag{10}$$

The observed $T(\nu, \hat{\mathbf{s}})$ is then a Gaussian field on the sphere for each ν with angular power spectrum

$$C_\ell(\nu) = \frac{2}{\pi} \int_0^\infty \mathrm{d}k\, k^2 (j_\ell(r_\nu k))^2 P(k, z_\nu) \tag{11}$$

The key feature of the redshifted 21cm emission is its frequency correlation structure, so what we actually need to fully specify the observed field is the cross-frequency angular power spectrum

$$\langle a_{\ell m}(\nu) a^*_{\ell' m'}(\nu') \rangle = C_\ell(\nu, \nu') \delta_{\ell\ell'} \delta_{mm'} \tag{12}$$

where

$$\delta T(\nu, \hat{\mathbf{s}}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\nu) Y_{\ell m}(\hat{\mathbf{s}}). \tag{13}$$

Equivalently, the multi-frequency angular covariance function

$$\xi(\nu, \nu, \hat{\mathbf{s}}\cdot\hat{\mathbf{s}}') = \langle \delta T(\nu, \hat{\mathbf{s}}) \delta T(\nu', \hat{\mathbf{s}}') \rangle \tag{14}$$

$$= \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell + 1) C_\ell(\nu, \nu') \mathcal{P}_\ell(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}'). \tag{15}$$

specifies the angular power spectrum by

$$C_\ell(\nu, \nu') = 4\pi \frac{1}{2} \int_{-1}^{1} \mathrm{d}\mu\, \mathcal{P}_\ell(\mu) \xi(\nu, \nu', \mu), \qquad \mu = \hat{\mathbf{s}}\cdot\hat{\mathbf{s}}'. \tag{16}$$

where $\mathcal{P}_\ell$ is the Legendre polynomial of order ℓ.

Assume for now that the power spectrum is independent of redshift $P(k, z) = P(k)$. Then

$$C_\ell(\nu, \nu') = \frac{2}{\pi} \int_0^\infty \mathrm{d}k\, k^2 j_\ell(r_\nu k) j_\ell(r'_\nu k) P(k) \tag{17}$$

$$\xi(\nu, \nu', \mu) = \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}k\, k^2 j_0(\gamma(\nu, \nu', \mu) k) P(k), \tag{18}$$

$$\gamma(\nu, \nu', \mu) = \sqrt{r_\nu^2 + r'^2_\nu - 2 r_\nu r'_\nu \mu} \tag{19}$$

1.1 Frequency discretization

We observe the sky in discrete frequency channels. The energy flux due to radiation incident from the direction $\hat{\mathbf{s}}$ in the frequency channel centered at ν_c with width $\Delta\nu$ is

$$\delta F(\nu_c, \Delta\nu, \hat{\mathbf{s}}) = \int_{\nu_c - \Delta\nu/2}^{\nu_c + \Delta\nu/2} \delta I(\nu, \hat{\mathbf{s}})\, \mathrm{d}\nu \tag{20}$$

$$= \frac{2 k_b A_{Jy}}{c^2 \nu_e} \int_{\nu_c - \Delta\nu/2}^{\nu_c + \Delta\nu/2} \nu^3 \delta T(\nu, \hat{\mathbf{s}})\, \mathrm{d}\nu\, [\text{Jy Hz / sr}], \qquad A_{Jy} = 10^{26} \text{Jy} / (\text{W/m}^2/\text{Hz}). \tag{21}$$

On the other hand, visibilities tend to be interpreted as measurements in Jy, i.e. have units of flux spectral density, which allows the value to be equivalently described by a brightness temperature. This effectively refers to an average specific intensity on the sky in the frequency channel ν_c

$$\delta \bar{I}(\nu_c, \hat{\mathbf{s}}) = \frac{1}{\Delta\nu} \delta F(\nu_c, \hat{\mathbf{s}}) \tag{22}$$

$$= \frac{2 k_b A_{Jy}}{c^2 \nu_e} \frac{1}{\Delta\nu} \int_{\nu_c - \Delta\nu/2}^{\nu_c + \Delta\nu/2} \nu^3 \delta T(\nu, \hat{\mathbf{s}})\, \mathrm{d}\nu\, [\text{Jy / sr}]. \tag{23}$$

For our purposes the channel width is the same $\Delta\nu$ for all ν_c so we drop the explicit functional dependence, but of course it could vary in general. The angular covariance function of the average specific intensity of the cosmological field is then

$$\overline{\xi}(\nu_c, \nu'_c, \mu) = \left\langle \delta\overline{I}(\nu_c, \hat{\mathbf{s}})\delta\overline{I}(\nu'_c, \hat{\mathbf{s}}')\right\rangle \tag{24}$$

$$= \left(\frac{2k_b A_{Jy}}{c^2\nu_e}\right)^2 \frac{1}{\Delta\nu^2}\int_{\nu_c-\Delta\nu/2}^{\nu_c+\Delta\nu/2}\int_{\nu'_c-\Delta\nu/2}^{\nu'_c+\Delta\nu/2} \mathrm{d}\nu\,\mathrm{d}\nu'\,\nu^3\nu'^3\xi(\nu,\nu',\mu) \tag{25}$$

The cross-frequency angular power spectrum of the discretized observed specific intensity is then

$$\overline{C}_\ell(\nu_c, \nu'_c) = 2\pi\int_{-1}^{1}\mathrm{d}\mu\,\overline{\xi}(\nu_c, \nu'_c, \mu)\mathcal{P}_\ell(\mu)\ [\mathrm{Jy}^2/\mathrm{sr}^2]. \tag{26}$$

2 Useful Closed-form Results

In the

2.1 A "White Noise" Power Spectrum

If the cosmological field has equal power at all scales the power spectrum is

$$P(k) = P_0 = \text{ constant} \tag{27}$$

and cross-frequency angular power spectrum is

$$C_\ell(\nu,\nu') = \frac{2}{\pi}\int_0^\infty \mathrm{d}k\,k^2 j_\ell(r_\nu k)j_\ell(r'_\nu k)P(k) \tag{28}$$

$$= \frac{2P_0}{\pi}\int_0^\infty \mathrm{d}k\,k^2 j_\ell(r_\nu k)j_\ell(r'_\nu k) \tag{29}$$

$$= \frac{2P_0}{\pi}\frac{\pi}{2r_\nu r'_\nu}\delta(r_\nu - r'_\nu) \tag{30}$$

Note the units here are indeed $\mathsf{Temperature}^2$. The delta function simplifies with the multidimensional generalization of the function composition identity

$$\delta(g(\vec{\mathbf{s}})) = \int_{\mathcal{S}}\mathrm{d}\vec{\mathbf{a}}\,\frac{\delta(\vec{\mathbf{s}}-\vec{\mathbf{a}})}{|\boldsymbol{\nabla} g(\vec{\mathbf{a}})|},\ \text{where } \mathcal{S} = \{\vec{\mathbf{a}} : g(\vec{\mathbf{a}}) = 0\}. \tag{31}$$

Let $(\nu,\nu') = \vec{\mathbf{s}}$ and $g(\vec{\mathbf{s}}) = g(\nu,\nu') = r(\nu) - r(\nu')$. Then

$$\delta(r(\nu) - r(\nu')) = \int_0^\infty \mathrm{d}a\,\sqrt{2}\frac{\delta(\nu-a)\delta(\nu'-a)}{\sqrt{\left(\frac{\mathrm{d}r(\nu)}{\mathrm{d}\nu}\right)^2 + \left(\frac{\mathrm{d}r(\nu')}{\mathrm{d}\nu'}\right)^2}} \tag{32}$$

$$= \frac{1}{\left|\frac{\mathrm{d}r(\nu)}{\mathrm{d}\nu}\right|}\delta(\nu-\nu') \tag{33}$$

The factor of $\sqrt{2}$ in the first line is due to the integral being along the ray $\nu = \nu'$ in the (ν,ν') plane. Thus

$$C_\ell(\nu,\nu') = \frac{P_0}{r_\nu r'_\nu\left|\frac{\mathrm{d}r(\nu)}{\mathrm{d}\nu}\right|}\delta(\nu-\nu') \tag{34}$$

We can check that the angular correlation function is then

$$\xi(\nu,\nu',\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}') = \frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)C_\ell(\nu,\nu')\mathcal{P}_\ell(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}') \tag{35}$$

$$= \frac{P_0}{r_\nu r'_\nu\left|\frac{\mathrm{d}r(\nu)}{\mathrm{d}\nu}\right|}\delta(\nu-\nu')\sum_{\ell=0}^{\infty}\frac{(2\ell+1)}{4\pi}\mathcal{P}_\ell(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}') \tag{36}$$

$$= \frac{P_0}{r_\nu r'_\nu\left|\frac{\mathrm{d}r(\nu)}{\mathrm{d}\nu}\right|}\delta(\nu-\nu')\delta(1-\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}') \tag{37}$$

as it should be. Assuming a set of disjoint equal-width frequency channels ν_c, the averaged cross-frequency angular power spectrum is

$$\overline{C}_\ell(\nu_c,\nu'_c) = \left(\frac{2k_b A_{Jy}}{c^2\nu_e}\right)^2\frac{1}{\Delta\nu^2}\int_{\nu_c-\Delta\nu/2}^{\nu_c+\Delta\nu/2}\int_{\nu'_c-\Delta\nu/2}^{\nu'_c+\Delta\nu/2}\mathrm{d}\nu\,\mathrm{d}\nu'\,\nu^3\nu'^3 C_\ell(\nu,\nu') \tag{38}$$

$$= \delta_{\nu_c,\nu'_c}\left(\frac{2k_b A_{Jy}}{c^2\nu_e}\right)^2\frac{P_0}{\Delta\nu^2}\int_{\nu_c-\Delta\nu/2}^{\nu_c+\Delta\nu/2}\mathrm{d}\nu\,\frac{\nu^6}{r(\nu)^2\left|\frac{\mathrm{d}r(\nu)}{\mathrm{d}\nu}\right|} \tag{39}$$

2.2 A Particular Power-law

If the power spectrum takes the form

$$P(k) = A_0 k^{-2} \tag{40}$$

then the angular covariance function is

$$\xi(\nu,\nu',\mu) = \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k\,k^2\frac{\sin(\gamma k)}{\gamma k}A_0 k^{-2}. \tag{41}$$

From the identity $\int_0^\infty \frac{\sin(x)}{x} = \frac{\pi}{2}$, we thus have

$$\xi(\nu,\nu',\mu) = \frac{A_0}{4\pi\gamma} \tag{42}$$

$$= \frac{A_0}{4\pi}\frac{1}{\sqrt{r_\nu^2+r_\nu'^2-2r_\nu r'_\nu\mu}} \tag{43}$$

$$= \frac{A_0}{4\pi}\sum_{\ell=0}^{\infty}\frac{r_\nu'^\ell}{r_\nu^{\ell+1}}\mathcal{P}_\ell(\mu),\ \text{for } r'_\nu < r_\nu \tag{44}$$

Then the multi-frequency angular power spectrum is

$$C_\ell(\nu,\nu') = 2\pi\int_{-1}^{1}\mathrm{d}\mu\,\mathcal{P}_\ell(\mu)\frac{A_0}{4\pi}\sum_{\ell'=0}^{\infty}\frac{r_\nu'^{\ell'}}{r_\nu^{\ell'+1}}\mathcal{P}_{\ell'}(\mu) \tag{45}$$

$$= \frac{A_0}{2\ell+1}\frac{r_\nu'^\ell}{r_\nu^{\ell+1}},\ \text{for } r'_\nu < r_\nu. \tag{46}$$

2.3 A Gaussian

Suppose the power spectrum takes the form

$$P(k) = e^{-\frac{k^2}{2k_0^2}}. \tag{47}$$

and let $g(t) = e^{-tk^2}$ so that

$$P(k) = g\left(\frac{1}{2k_0^2}\right). \tag{48}$$

The angular correlation function is then given by a similar definition $\xi(\gamma) = f(\frac{1}{2k_0^2})$ from

$$\begin{aligned}
f(t) &= \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k\, k^2 \frac{\sin(\gamma k)}{\gamma k} e^{-tk^2} && (49)\\
&= \frac{1}{2\pi^2\gamma}\int_0^\infty \mathrm{d}k\, k \sin(\gamma k) e^{-tk^2} && (50)\\
&= \frac{-1}{4\pi^2\gamma}\frac{\partial}{\partial\gamma}\int_{-\infty}^\infty \mathrm{d}k \cos(\gamma k) e^{-tk^2} && (51)\\
&= \frac{1}{8(\pi t)^{3/2}} e^{-\frac{\gamma^2}{4t}} && (52)\\
&= \frac{1}{8(\pi t)^{3/2}} \exp\left(\frac{-1}{4t}(r^2 + r'^2 + 2rr'\mu)\right) && (53)
\end{aligned}$$

The angular power spectrum is then

$$\begin{aligned}
h_\ell(t) &= 2\pi\int_{-1}^1 \mathrm{d}\mu\, \mathcal{P}_\ell(\mu) f(t,\mu) && (54)\\
&= \frac{\pi}{4}\frac{1}{(\pi t)^{3/2}} \exp\left(\frac{-1}{4t}(r^2 + r'^2)\right)\int_{-1}^1 \mathrm{d}\mu\, \mathcal{P}_\ell(\mu) \exp\left(\frac{-rr'\mu}{2t}\right) && (55)\\
&= \frac{\pi}{2}\frac{1}{(\pi t)^{3/2}} \exp\left(\frac{-1}{4t}(r^2 + r'^2)\right) i_\ell\left(\frac{rr'}{2t}\right) && (56)
\end{aligned}$$

where $i_\ell(z)$ is a modified spherical Bessel function of the first kind with integral definition

$$i_\ell(z) = i^{-\ell} j_\ell(iz) = \frac{1}{2}\int_{-1}^1 \mathrm{d}x\, e^{-xz}\mathcal{P}_\ell(x), \tag{57}$$

$$\tag{58}$$

It is related to the modified Bessel function $I_n(z)$ by

$$i_\ell(z) = \sqrt{\frac{\pi}{2z}} I_{\ell+\frac{1}{2}} \tag{59}$$

and thus we have

$$\begin{aligned}
h_\ell(t) &= \frac{\pi}{2}\frac{1}{(\pi t)^{3/2}} \exp\left(\frac{-1}{4t}(r^2 + r'^2)\right)\sqrt{\frac{\pi}{2}\frac{2t}{rr'}} I_{\ell+\frac{1}{2}}\left(\frac{rr'}{2t}\right) && (60)\\
&= \frac{1}{2t\sqrt{rr'}} \exp\left(\frac{-1}{4t}(r^2 + r'^2)\right) I_{\ell+\frac{1}{2}}\left(\frac{rr'}{2t}\right) && (61)
\end{aligned}$$

or, evaluated at $t = \frac{1}{2k_0^2}$,

$$\begin{aligned}
C_\ell(r, r') &= h_\ell\left(\frac{1}{2k_0^2}\right) && (62)\\
&= \frac{k_0^2}{\sqrt{rr'}} \exp\left(-\frac{k_0^2}{2}(r^2 + r'^2)\right) I_{\ell+\frac{1}{2}}(k_0^2 rr') && (63)
\end{aligned}$$

Denote the exponentially scaled modified Bessel function by

$$\mathcal{I}_n(z) = e^{-z} I_n(z). \tag{64}$$

Inserting a factor of $1 = e^{k_0^2 rr'} e^{-k_0^2 rr'}$ we have a version of the expression which is useful for numerical implementation:

$$C_\ell(r, r') = \frac{k_0^2}{\sqrt{rr'}} \exp\left(-\frac{k_0^2}{2}|r - r'|^2\right) \mathcal{I}_{\ell+\frac{1}{2}}(k_0^2 rr') \tag{65}$$

2.4 A Gaussian Bump

We can produce the cross-frequency angular power spectrum for a power spectrum with a small-k cutoff

$$P(k) = \frac{1}{2}\left(\frac{k}{k_0}\right)^2 e^{-\frac{k^2}{2k_0^2}} \tag{66}$$

by identifying a derivative of the previously defined Gaussian generating function

$$P(k) = -\frac{1}{2k_0^2} \frac{\partial g}{\partial t}\bigg|_{t=\frac{1}{2k_0^2}}. \tag{67}$$

$$-\frac{\partial h_\ell}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{1}{2t} \exp\left(\frac{y}{t}\right) I_{\ell+\frac{1}{2}}\left(\frac{x}{t}\right)\right), \tag{68}$$

$$\text{where } y = \frac{-1}{4}(r^2 + r'^2) \text{ and } x = \frac{rr'}{2} \tag{69}$$

$$= \frac{1}{2t^2} \exp\left(\frac{y}{t}\right) I_{\ell+\frac{1}{2}}\left(\frac{x}{t}\right) - \frac{1}{2t} \exp\left(\frac{y}{t}\right)\left(\frac{-y}{t^2}\right) I_{\ell+\frac{1}{2}}\left(\frac{x}{t}\right) - \frac{1}{2t} \exp\left(\frac{y}{t}\right) I_{\ell+\frac{1}{2}}\left(\frac{x}{t}\right)\left(\frac{-x}{t^2}\right) \tag{70}$$

$$\tag{71}$$

With identity

$$I'_n(z) = I_{n+1}(z) + \frac{n}{z} I_n(z) \tag{72}$$

this simplifies to

$$-\frac{\partial h_\ell}{\partial t} = \frac{1}{2\sqrt{rr'}} \frac{1}{t^2} e^{\frac{y}{t}} \left[I_{\ell+\frac{1}{2}}\left(\frac{x}{t}\right)\left(1 + \frac{y}{t} + \frac{2\ell+1}{2}\right) + \left(\frac{x}{t}\right) I_{\ell+\frac{3}{2}}\left(\frac{x}{t}\right)\right] \tag{73}$$

and thus

$$C_\ell(r, r') = -\frac{1}{2k_0^2} \frac{\partial h_\ell}{\partial t}\bigg|_{t=\frac{1}{2k_0^2}} \tag{74}$$

$$= \frac{k_0^2}{\sqrt{rr'}} \exp\left(\frac{-k_0^2}{2}(r^2 + r'^2)\right)\left[\left(\frac{2\ell+3}{2} - \frac{k_0^2}{2}(r^2 + r'^2)\right) I_{\ell+\frac{1}{2}}(k_0^2 rr') + k_0^2 rr' I_{\ell+\frac{3}{2}}(k_0^2 rr')\right]. \tag{75}$$

Finally, in terms of the exponetially scaled modified Bessel functions,

$$C_\ell(r, r') = \frac{k_0^2}{2\sqrt{rr'}} \exp\left(-\frac{k_0^2}{2}|r - r'|^2\right)\left[(2\ell + 3 - k_0^2(r^2 + r'^2))\mathcal{I}_{\ell+\frac{1}{2}}(k_0^2 rr') + 2k_0^2 rr' \mathcal{I}_{\ell+\frac{3}{2}}(k_0^2 rr')\right] \tag{76}$$

2.5 A "Top-hat"

Suppose the power spectrum is a "top-hat" function centered at $k = k_0$ with width 2Δ

$$P(k) = \begin{cases} 1, & |k - k_0| < \Delta \\ 0, & |k - k_0| > \Delta \end{cases} \tag{77}$$

The angular power spectrum is then

$$C_\ell(r,r') = \frac{2}{\pi}\int_{k_0-\Delta}^{k_0+\Delta} \mathrm{d}k\, k^2 j_\ell(rk) j_\ell(r'k) \tag{78}$$

$$= \int_{k_0-\Delta}^{k_0+\Delta} \mathrm{d}k\, k J_{\ell+\frac{1}{2}}(rk) J_{\ell+\frac{1}{2}}(r'k) \tag{79}$$

where $J_n(x)$ is a Bessel function of the first kind. Define

$$\mathcal{J}_\ell(q) = \int_0^q \mathrm{d}k\, k J_{\ell+\frac{1}{2}}(rk) J_{\ell+\frac{1}{2}}(r'k). \tag{80}$$

The simplification of this integral expression is straightforward and found in (Bowman, 2012, §94) to be

$$\int_0^1 \mathrm{d}x\, x J_n(\alpha x) J_n(\beta x) = \begin{cases} \frac{\alpha\beta}{2n(\beta^2-\alpha^2)}\left(J_{n-1}(\alpha)J_{n+1}(\beta) - J_{n+1}(\alpha)J_{n-1}(\beta)\right), & \text{if } \alpha \neq \beta \\ \frac{1}{2}\left(J_n(\alpha)^2 - J_{n-1}(\alpha)J_{n+1}(\alpha)\right), & \text{if } \alpha = \beta. \end{cases} \tag{81}$$

So with changes of variables $x = k/q$, $\alpha = rq$, $\beta = r'q$,

$$\mathcal{J}_\ell(q) = \begin{cases} q^2 \frac{rr'}{(2\ell+1)(r'^2-r^2)}\left(J_{\ell-\frac{1}{2}}(rq)J_{\ell+\frac{3}{2}}(r'q) - J_{\ell+\frac{3}{2}}(rq)J_{\ell-\frac{1}{2}}(r'q)\right), & \text{if } r \neq r' \\ \frac{q^2}{2}\left(J_{\ell+\frac{1}{2}}(rq)^2 - J_{\ell-\frac{1}{2}}(rq)J_{\ell+\frac{3}{2}}(rq)\right), & \text{if } r = r'. \end{cases} \tag{82}$$

The angular power spectrum is thus

$$C_\ell = \mathcal{J}_\ell(k_0+\Delta) - \mathcal{J}_\ell(k_0-\Delta). \tag{83}$$

3 Redshift Dependent Power Spectrum

We started with a redshift independent power spectrum $P(k)$, but more generally we might consider the unequal-time correlator

$$\left\langle \delta\widetilde{T}(\vec{\mathbf{k}},z)\delta\widetilde{T}^*(\vec{\mathbf{k}}',z')\right\rangle = (2\pi)^3\Gamma(k,z,z')\delta(\vec{\mathbf{k}}-\vec{\mathbf{k}}'). \tag{84}$$

In order to produce

$$\left\langle \delta\widetilde{T}(\vec{\mathbf{k}},z)\delta\widetilde{T}^*(\vec{\mathbf{k}}',z)\right\rangle = (2\pi)^3 P(k,z)\delta(\vec{\mathbf{k}}-\vec{\mathbf{k}}') \tag{85}$$

we can choose to let Γ have the form

$$\Gamma(k,z,z') = \sqrt{P(k,z)}\sqrt{P(k,z')} \tag{86}$$

We can then compute the angular power spectrum for such a field using some of the analytic forms derived in the previous section. For example we can choose to use the Gaussian-bump terms (subsection 2.4) and choose some a_i and k_i such that

$$\sqrt{P(k,z)} = \sum_i a_i(z)\frac{k}{\sqrt{2}k_i}\exp\left(\frac{-k^2}{4k_i^2}\right). \tag{87}$$

Then we have

$$\Gamma(k,z,z') = \sum_{ij} a_i(z)a_j(z')\frac{k^2}{2k_ik_j}\exp\left(\frac{-k^2}{4k_i^2} - \frac{k^2}{4k_j^2}\right) \tag{88}$$

$$= \sum_{ij} a_i(z)a_j(z')\frac{k^2}{2b_{ij}^2}\exp\left(\frac{-k^2}{2k_{ij}^2}\right), \tag{89}$$

$$\text{where } b_{ij} = \sqrt{k_ik_j} \text{ and } k_{ij}^2 = \frac{1}{2k_i^2} + \frac{1}{2k_j^2} \tag{90}$$

$$= \sum_{ij} A_{ij}(z,z')P_{ij}(k), \tag{91}$$

$$\text{with } A_{ij}(z,z') = a_i(z)a_j(z')\frac{k_{ij}^2}{b_{ij}^2} \text{ and } P_{ij}(k) = \frac{k^2}{2k_{ij}^2}\exp\left(\frac{-k^2}{2k_{ij}^2}\right) \tag{92}$$

The cross-frequency angular power spectrum is then

$$C_\ell(\nu,\nu') = \sum_{ij} A_{ij}(z,z')C_\ell^{term}(\nu,\nu',k_{ij}) \tag{93}$$

4 Misc

4.1 Multiple components

$$P(k) = \mathcal{N}\sum_i a_iP_i(k) \tag{94}$$

$$\overline{C}_\ell(\nu,\nu') = \mathcal{N}\sum_i a_i\overline{C}_{\ell,i}(\nu,\nu') \tag{95}$$

4.2 Bessel function asymptotic series

The modified Bessel function has an asymptotic series for large arguments

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}\frac{c_k(\nu)}{z^k} \tag{96}$$

where

$$c_k(\nu) = (-1)^k\prod_{n=0}^{k} a_n \tag{97}$$

$$a_n = \begin{cases} 1 & n = 0, \\ \frac{4\nu^2-(2n-1)^2}{8n} & n > 0. \end{cases} \tag{98}$$

For $\nu = \ell + \frac{1}{2}$ the "large argument" condition is that $z >> \ell(\ell+1)$.

4.3 Gaussian Cut-offs

For the Gaussian-based power spectrum terms in subsection 2.3 and subsection 2.4 we can often take advantage of the psuedo-compact nature of the basis function to avoid evaluating matrix elements $C_\ell(r,r')$ that are effectively zero. Since our ultimate aim is to generate realizations of a Gaussian random field using a matrix decomposition of $\bar{C}_\ell(\nu,\nu')$ for each ℓ we can consider elements that are smaller than the largest matrix elements by a factor of the machine precision to be effectively zero. We will thus consider the expression for the dynamic range resulting from the definition and massage the resulting expression to obtain a useful heuristic for when to avoid expensive evaluations that produce small matrix elements.

Given a power spectrum definition by a set of coefficients a_i, locations k_i and a choice of either type of Gaussian basis function we can write

$$C_\ell(r, r') = \sum_i a_i e^{\frac{-k_i^2}{2}|r-r'|^2} f(k_i, \ell, r, r') \tag{99}$$

where we collect the relevent exponentially-scaled Bessel functions $\mathcal{I}_{n+\frac{1}{2}}$ and algebraic functions of k_i, ℓ, r and r' into the function f which is generally ≤ 1. Denoting the smallest of the k_i's by k_m ($k_m < k_i \forall i \neq m$) we factor out the corresponding term and introduce the change of variables $d = r - r'$, $\bar{r} = \frac{1}{2}(r + r')$ to obtain

$$C_\ell(r, r') = a_m f(k_m, \ell, r, r') e^{\frac{-k_m^2}{2} d^2} \left(1 + \sum_{i \neq m} \frac{a_i}{a_m} \frac{f(k_i, \ell, r, r')}{f(k_m, \ell, r, r')} e^{\frac{-d^2}{2}(k_i^2 - k_m^2)} \right). \tag{100}$$

The ratio of the off-diagonal element $C_\ell(r, r')$ to the diagonal element $C_\ell(\bar{r}, \bar{r})$ is then

$$\frac{C_\ell(r, r')}{C_\ell(\bar{r}, \bar{r})} = e^{\frac{-k_m^2}{2} d^2} \frac{f(k_m, \ell, r, r')}{f(k_m, \ell, \bar{r}, \bar{r})} \frac{\left(1 + \sum_{i \neq m} \frac{a_i}{a_m} \frac{f(k_i, \ell, r, r')}{f(k_m, \ell, r, r')} e^{\frac{-d^2}{2}(k_i^2 - k_m^2)} \right)}{\left(1 + \sum_{i \neq m} \frac{a_i}{a_m} \frac{f(k_i, \ell, \bar{r}, \bar{r})}{f(k_m, \ell, \bar{r}, \bar{r})} \right)}. \tag{101}$$

Heuristically we can expect the two ratio factors to be ~ 1. For typical ranges of r in low frequency radio cosmology we might have $\frac{d}{\bar{r}} \sim \frac{1}{10}$ and since f is slowly varying with r, r', the ratios will be ~ 1, and since $k_m < k_i$ the sum in the numerator will generally be smaller than the denominator. We can thus estimate the ratio of matrix elements as

$$\frac{C_\ell(r, r')}{C_\ell(\bar{r}, \bar{r})} \lessapprox e^{\frac{-k_m^2}{2} d^2} \tag{102}$$

and hence choose an ϵ such that we set $C_\ell(r, r') = 0$ when

$$e^{\frac{-k_m^2}{2}|r-r'|^2} \leq \epsilon \tag{103}$$

The relative errors in the subsequent matrix decomposition will then be roughly proportional to ϵ so if we choose $\epsilon = 10^{-16}$ we will still obtain decompositions that are machine precision limited.

References

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