Part III Gauge/Gravity Duality

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Release Notes

Last updated: 27th April 2022.

- Main examinable texts finished.
- Non-examinable sections to be added.
- Missing figures to be added.
- Detailed references to be added.

Any suggestions and corrections, please email zy286@cam.ac.uk.

Course Information

Gauge/Gravity duality (also known as AdS/CFT) is a surprising duality that relates theories of quantum gravity (with a negative cosmological constant) to certain quantum field theories living in a spacetime with fewer dimensions. This is the most precise known realisation of the holographic principle, the idea that all information in the universe is encoded somehow at the boundary of the universe. These lectures will describe in detail the "dictionary" used to relate observables on the bulk side to observables on the boundary side.

The course will cover a selection of topics including:

- Anti-de Sitter spacetime;
- Conformal field theory;
- Wave equations in AdS, and their relationship to CFT operators and sources;
- The duality between black holes and thermal states;
- Holographic entanglement entropy;
- Bulk reconstruction from boundary data.

If time permits, we will also discuss Maldacena's original derivation [1] of the duality, but this topic is non-examinable. **Non-examinable sections* are starred!**

> Additional Information
>
> Additional and supplementary information is provided in such boxes.

Prerequisites

Required: General Relativity, Black Holes, Advanced Quantum Field Theory

Helpful: Some basic aspects of quantum information theory and conformal symmetry will play an important role in this course, but the relevant aspects will be reviewed in a self-contained manner.

Not Required: String Theory, Supersymmetry. Although most of the specific known examples of AdS/CFT come from superstring theories, these aspects will not be emphasised in these lectures.

Literature

Our approach of the AdS/CFT dictionary part is inspired by Raman Sundrum's review [2]. Other very useful reviews on AdS/CFT include [3, 4, 5]. For reviews on gravity and entanglement, see [6]. A more detailed discussion on bulk emergence is offered in [7]. Some possible gaps in background knowledge can be filled by Hartman's notes [8]. For those who want to know more about CFT, [9, 10, 11] may be good choices to read.

Original papers on AdS/CFT correspondence include [1, 12, 13].

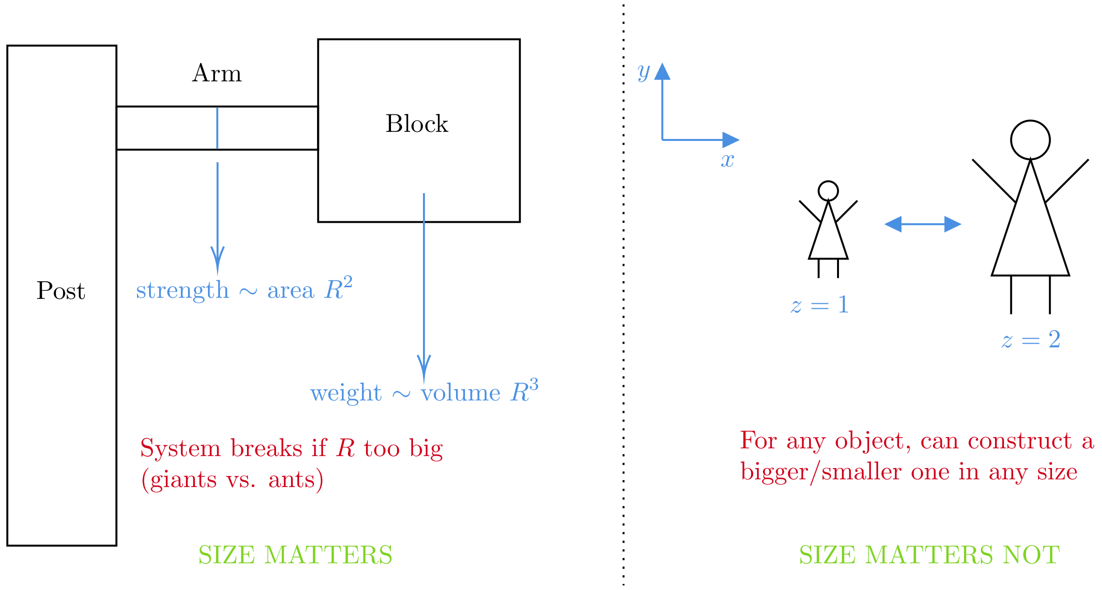
Ian Lim's notes taken in 2019 is also useful to look at. We have used some texts collected there.

Contents

0 Inspirational Remarks

In this course we will study an amazing duality between gravitational and non-gravitational field theories, first discovered by Juan Maldacena [1], known as the *Gauge/Gravity Duality.* To motivate[1] this duality, let's go back in time and discuss a conflict between two seminal figures in science — Galileo and Yoda.

Galileo wrote a book called *Two New Sciences*, one such new science is a proto-atomic theory, suggesting that the laws of physics depend on *scale*, e.g. area-volume laws.



Yoda, in his book *Empire Strikes Back*, asserted that "size matters not" — we can construct laws of physics with scaling symmetry. This implies that for any object, you can construct a bigger/smaller one in any size z. The crackpot idea here is:

SIZE IS AN EXTRA DIMENSION!

We add an extra space coordinate z.

However, this is an idea too wild to be accepted and easy to refute.

Objection 1 We need a conjugate momentum to z.

Resolution: By Noether's theorem, scaling symmetry $\ln z \to \ln z + C$ admits a corresponding conserved momentum $P(\ln z)$.

Objection 2 How can we rotate in z–x plane?

Resolution: In d dimensions, We would require the transformations which consist of *Poincaré* + *dilations* to be enhanced to a bigger group with d more generators. There exists such group, known as the *conformal group*.

Objection 3 (Einsteinian) Speed of light should be constant while the light crossing time is dependent on size z.

[1] The approach of the first few sections of the lectures is inspired by Raman Sundrum's review [2].

Resolution: We take a cue from Einstein and realise that the clock goes slower for the bigger girl. This suggests there is a gravitational field along z-direction with metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}z^2 + \eta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j}{z^2}, \quad \eta_{ij} = \text{Minkowski metric.} \tag{1}$$

This is known as the Anti-de Sitter (AdS) metric (to be precise, this only covers the *Poincaré patch* of AdS). It is a unique metric (up to an overall factor) that has

- Poincaré symmetry in x^i directions;
- translation symmetry of $\ln z$;
- redshift factor $\mathrm{d}s \sim \mathrm{d}t\,/z$.

Objection 4 We cannot put objects on top of each other without interaction.

Resolution: We can put objects on top of each other and they won't interact much *if* there are a large number N of species of particles/fields in field theory (e.g. $\mathrm{SU}(N_c)$ Yang-Mills theory has $N \sim N_c^2$), especially if objects have to be in singlet states, e.g. trivial $\mathrm{SU}(N_c)$ irreps, enforced by *gauge* symmetry.

Objection 5 OK, but even if N is large but finite, there will be small interactions over large $\Delta \ln z$ value.

Resolution: We see there is a *long range force* between *any* 2 objects — congratulations, we've just discovered gravity!

Objection 6 OK, but if a gauge theory gets really hot, it deconfines (e.g. quark-gluon plasma) and you will get a hyperentropic object with huge entropy $\sim N$.

Resolution: It's a *black hole*!

It seems all our objections have failed to refute the identification of an extra dimension. Indeed, recent studies have shown abundant evidence of the central topic of this course — AdS/CFT duality. We present the idea in the following diagram

$$\begin{array}{rcl}
\underset{\text{quantum field theory}}{\mathbf{CFT}_d} & \longleftrightarrow & \underset{\text{quantum gravity}}{\text{asymptotically } \mathbf{AdS}_{d+1} \times F} \\[2ex]
\text{large } N & \longleftrightarrow & \text{classical limit } l_{\mathrm{P}} \ll R_{\mathrm{AdS}} \\[2ex]
\text{strongly coupled} & \longleftrightarrow & \text{local (point-like) fields below} \\
& & \text{curvature scale (e.g. } l_{\mathrm{s}} \ll R_{\mathrm{AdS}})
\end{array}$$

where the asymptote is with respect to the spacelike infinity and F is some compact fibre.

The quantum field theory side probably obeys some known axioms, but they are hard to define and study analytically at strong coupling regime. Also, we have poor knowledge to treat quantum gravity non-perturbatively in G_{N}. Now, the duality lets us study poorly understood regimes on one side using calculable physics on the other side. (Which may triple your citations!).

1 Conformal Field Theories and Conformal Group

1.1 Conformal Field Theories

We suppose *size matters not.* A concrete example is 4d Maxwell theory defined by action

$$I = \frac{1}{4}\int \mathrm{d}^4x\,\sqrt{-g}g^{ac}g^{bd}F_{ab}F_{cd} \tag{2}$$

which is invariant under Weyl transformation

$$g_{ab} \mapsto \Omega^2(x)g_{ab} \tag{3}$$

as $\sqrt{-g} \mapsto \Omega^d\sqrt{-g}$ and $g^{ab} \mapsto \Omega^{-2}g^{ab}$.

If this symmetry holds at quantum level up to trace anomaly

$$\frac{\delta \ln \mathcal{Z}}{\delta\Omega(x)} \propto \langle T\rangle \sim \text{curvatures} \tag{4}$$

then we call it a *conformal field theory* (CFT). This requires the beta functions of the theory to vanish $\beta_i = 0$, i.e. the theory is a fixed point of the renormalisation group (RG) flow. This generically happens at *isolated points* of parameter space. The famous exception is supersymmetry, where you can have a fixed subspace rather than just isolated points.

Trace Anomaly

Consider under infinitesimal Weyl transformation $\delta g_{ab} = \omega(x)g_{ab}$, the change in $\ln\mathcal{Z}$ is

$$\delta\ln\mathcal{Z} = \frac{1}{\mathcal{Z}}\delta\mathcal{Z} = -\frac{1}{\mathcal{Z}}\int \mathcal{D}\phi\,\mathcal{D}g\,\delta S\,e^{-S[g,\phi]} = \int \mathrm{d}^dx\,\sqrt{-g}\,\omega(x)\,\langle T^a{}_a(x)\rangle$$

by $\delta S/\delta g^{ab} = \sqrt{-g}\,T_{ab}$ up to different conventions for numerical constants.

1.2 Conformal Symmetry Group

The *conformal symmetry group* consists of diffeomorphisms that preserve the metric up to conformal factors $\Omega(x)$. The generators are known as the *conformal Killing vectors* (CKV) ξ^a that obey the *conformal Killing equation*

$$\underbrace{\nabla_a\xi_b + \nabla_b\xi_a}_{\delta_\xi g_{ab}} - \frac{2}{d}g_{ab}\nabla\cdot\xi = 0 \tag{5}$$

where d is the dimension of the spacetime.

If we restrict our attention to Minkowski spacetime $g_{ab} = \eta_{ab} = \mathrm{diag}(-1,+1,\cdots,+1)$ then when $d = 1,2$ there are infinitely many generators for the conformal group. When $d > 2$ there are $\frac{1}{2}(d+1)(d+2)$ generators. These consist of

Transformation		Generator
d translations	$x^a \mapsto x^a + c^a$	$P_a = -\mathrm{i}\partial_a$
$\frac{d(d-1)}{2}$ Lorentz	$x^a \mapsto \Lambda^a{}_b x^b$ ($\Lambda^\dagger\Lambda = \mathbb{1}$)	$M_{ab} = \mathrm{i}(x_a\partial_b - x_b\partial_a)$
1 scaling	$x^a \mapsto \Omega x^a$	$D = -\mathrm{i}(x\cdot\partial)$
d special conformal	$x^a \mapsto \dfrac{x^a - x^2b^a}{(x-x^2b)^2}$, $\Omega = \dfrac{x^2}{(x-x^2b)^2}$	$K_a = \mathrm{i}(x^2\partial_a - 2x_a(x\cdot\partial))$

Notice that we have an inversion transformation defined by

$$x^a \mapsto \frac{x^a}{x^2}, \quad \Omega = \frac{1}{x^2} \tag{6}$$

so the special conformal transformation is equivalent to an inversion followed by a translation followed by another inversion. We also observe potential problems here: for Euclidean space, the inversion diverges when $x = 0$; for Lorentzian spacetime, the inversion diverges on lightcone as $x^2 = 0$. These are discussed later.

We calculate the commutators of the generator as below

$$\begin{aligned}
[M_{ab}, M_{cd}] &= \mathrm{i}\left(\eta_{ac}M_{bd} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac} - \eta_{bc}M_{ad}\right) \\
[M_{ab}, P_c] &= \mathrm{i}\left(\eta_{ac}P_b - \eta_{bc}P_a\right) \\
[D, P_a] &= \mathrm{i}P_a \\
[D, K_a] &= -\mathrm{i}K_a \\
[K_a, P_b] &= 2\mathrm{i}\left(\eta_{ab}D - M_{ab}\right) \\
[K_a, M_{bc}] &= \mathrm{i}\left(\eta_{ab}K_c - \eta_{ac}K_b\right) \\
\text{all others} &= 0.
\end{aligned} \tag{7}$$

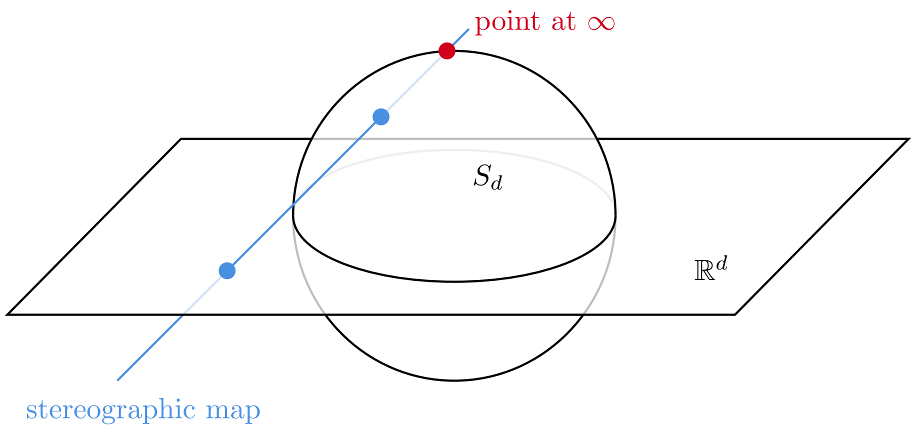
From these we can see, the conformal group for spacetime $\mathbb{R}^{d-1,1}$ is equivalent to $\mathrm{SO}(d,2)$. Later we will see why. (Really, we want the *universal cover*.)

1.3 A Brief Revision of Noether's Theorem

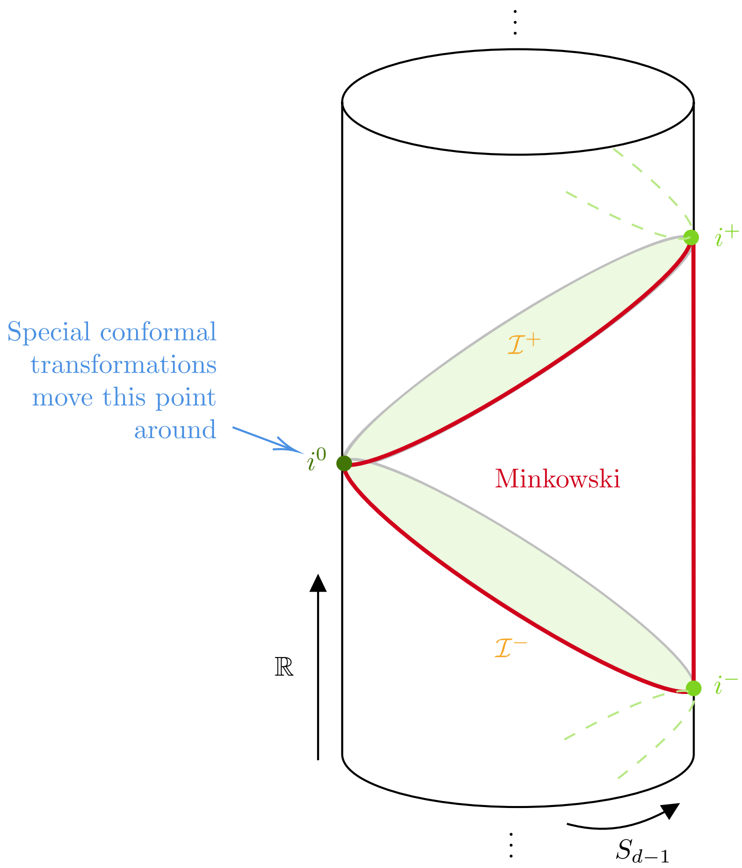
Under Construction.

1.4 Maximal Conformal Extension

As we saw above, inversion sends $|x| = 0$ to $|x| = \infty$ — we have to *extend* spacetime to act with (non-infinitesimal) special conformal transformations. In Euclidean case, we add one point at infinity, this is known as *1-point compactification*.



For Lorentzian case, $|x| = 0$ is an entire lightcone of points. Also, an inversion naively reverses time order of timelike points with $x^2 < 0$. We resolve this by considering the maximal conformal extension $S_{d-1} \times \mathbb{R}$ with Minkowski a partial patch.



Here,

$$
\begin{aligned}
i^{\pm} &: \text{ future/past timelike infinity,} \\
i^{0} &: \text{ spacelike infinity,} \\
\mathcal{I}^{\pm} &: \text{ future/past null infinity.}
\end{aligned}
$$

Finite conformal transformations may map points between different patches on the cylinder, particularly, special conformal transformation maps between isomorphic patches. Working on the maximal conformal extension, the causality is preserved.

1.5 Conformal Representations

Having discussed conformal symmetry, we would like to see how it act on fields — we need to study its representation. Unitary, positive-energy irreps come from *fields* (i.e. local operators). In QFT, states are constructed by acting on vacuum with operators $\mathcal{O}(x)$ smeared out by test functions $f(x)$:

$$
|\Psi\rangle = \int \mathrm{d}^d x\, f(x)\mathcal{O}(x)\,|0\rangle\,. \tag{8}
$$

These are classified by $\mathrm{SO}(d)$ spin and *weight* of "primary" field $\mathcal{O}$. By *primary*, we mean the field transforms as $\varphi \mapsto \Omega^{\Delta}\varphi$ with weight Δ. This definition differs from the usual one in $d = 2$ for Virasoro algebra. The derivatives $\partial^n\varphi$ are known as the *descendants* whose transformations depends on the derivatives of $\Omega(x)$.

Gauge invariant operators $\mathcal{O}$ must satisfy certain "unitary bounds" the details depend on spacetime dimension d and here are some common examples:

$$
\begin{aligned}
\Delta &\geq \frac{d-2}{2} && \text{for scalars} \\
\Delta &\geq \frac{d-1}{2} && \text{for spinors} \\
\Delta &\geq d-1 && \text{for vectors} \\
\Delta &\geq d && \text{for symmetric traceless tensors.}
\end{aligned}
$$

The bound are saturated by free fields $\Box\phi = 0, \not{\partial}\psi = 0$ and conserved currents $\nabla_a J^a, \nabla_a T^{ab} = 0$, respectively.

More on Conformal Representations

In QM/QFT with symmetry group G, a symmetry transformation $g \in G$ of the system can be represented as an operator $U[g]$ such that the norm of the state $|\psi\rangle$ is preserved $\langle\psi|U[g]^\dagger U[g]|\psi\rangle = \langle\psi|\psi\rangle$, i.e. $U[g]$ is a *unitary representation* of g. In this set-up, a state $|\psi\rangle$ and an operator $\mathcal{O}(x)$ transform according to

$$
|\psi\rangle \mapsto U[g]\,|\psi\rangle\,, \quad \mathcal{O}(x) \mapsto U[g]\mathcal{O}(x)U[g]^\dagger,
$$

respectively. We can think as if the states live in "fundamental representation" of U and the operators live in "adjoint representation". This is clearer to see if we go down to the Lie algebra level. Write U in terms of the generators (Lie algebra bases) T, we have, schematically, $U = \exp(-\mathrm{i}\lambda \cdot T)$ so

$$
\delta\,|\psi\rangle = -\mathrm{i}T\,|\psi\rangle\,, \quad \delta\mathcal{O}(x) = -\mathrm{i}[T, \mathcal{O}(x)].
$$

Now when considering the "adjoint representations" of conformal algebra, we can start with operator at origin $\mathcal{O}(0)$ as we can translate it to any point using P_a or using relevant Jacobi identities. First we define the *conformal weight* Δ of a field as the "eigenvalue" under ad_D action:

$$
[D, \mathcal{O}(0)] = -\mathrm{i}\Delta\mathcal{O}(0).
$$

Then using the commutation relations (7) we have

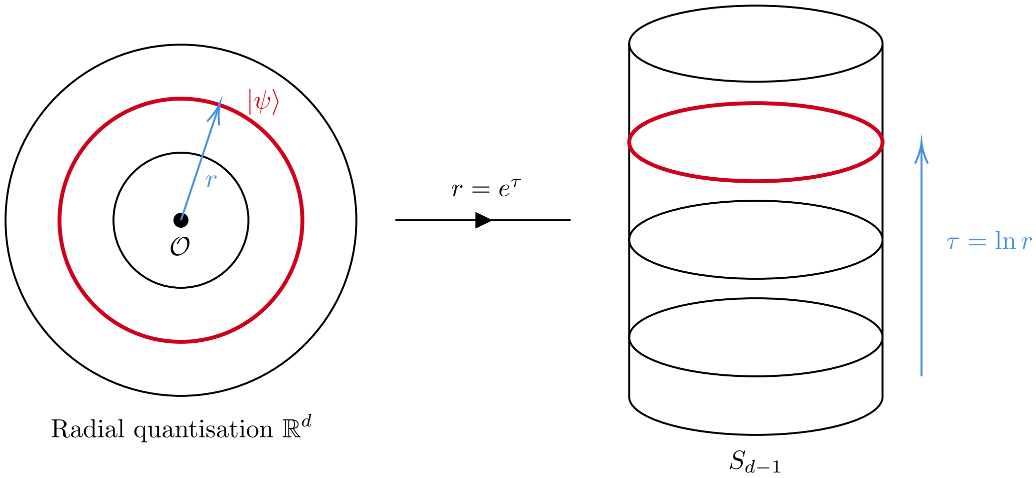
$$
\begin{aligned}
[D, [K_a, \mathcal{O}(0)]] &= -\mathrm{i}(\Delta - 1)\mathcal{O}(0), \\
[D, [P_a, \mathcal{O}(0)]] &= -\mathrm{i}(\Delta + 1)\mathcal{O}(0),
\end{aligned}
$$

meaning that K_a lowers the conformal weight while P_a raises it. However, for $\mathcal{O}(x)$ to be valid unitary representation, there is a lowest value for Δ known as the *unitary bound* (for a sketch see page 8 of [2]). Hence, there are $\mathcal{O}(x)$'s such that their weight cannot be further lowered, these are known as the *primary* operators, obeying

$$
[K_a, \mathcal{O}(0)] = 0.
$$

The *descendants* are derivatives of primaries, they have higher weights by identifying derivatives ∂_a as P_a operators.

1.6 Operator-State Correspondence



$$
\begin{array}{rcl}
g_{ab} = \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_{d-1}^2 & \xrightarrow{\Omega=1/r,\quad r=e^\tau} & \mathrm{d}\tau^2 + \mathrm{d}\Omega_{d-1}^2 \\
\text{Spectrum of Operators} & \longrightarrow & \text{Spectrum of States on Cylinder} \\
\text{Dilation } r \mapsto e^a r & \longrightarrow & \text{Time Translation } \tau \mapsto \tau + a
\end{array}
$$

$$\Delta = \text{radius} \times (E - E_0) \tag{9}$$

where E_0 is Casimir energy, E is the energy of vacuum $|0\rangle \leftrightarrow \mathbb{1}$. $E_0 \neq 0$ in even dimensions from trace anomalies. Note that

$$|0\rangle_{\text{cyl.}} = |0\rangle_{\text{Mink.}}\,. \tag{10}$$

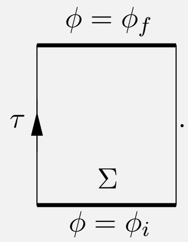
Euclidean Path Integral

For convenience in our discussion, we Wick rotate $\tau = it$ to Euclidean signature.
Path integrals are derived from transition amplitudes in Euclidean time 0 to τ with respect to some Hamiltonian H by inserting many "identity operators" in the form of $\int \mathcal{D}\phi_\tau\, |\phi_\tau\rangle \langle\phi_\tau|$ on consecutive time-slices Σ_τ:

$$
\begin{aligned}
\langle \phi_f | e^{-\tau H} | \phi_i \rangle &= \int \mathcal{D}\phi_{\tau_n} \cdots \int \mathcal{D}\phi_{\tau_1}\, \langle \phi_f | e^{-(\tau-\tau_n)H} | \phi_{\tau_n} \rangle \, \langle \phi_{\tau_n} | \cdots | \phi_{\tau_1} \rangle \, \langle \phi_{\tau_1} | e^{-\tau_1 H} | \phi_i \rangle \\
&= \int_{\phi(0)=\phi_i}^{\phi(\tau)=\phi_f} \mathcal{D}\phi\, e^{-S[\phi]}
\end{aligned}
$$

where $S[\phi]$ is the Euclidean action.
Using diagrams we can represent the path integral as

$$\int_{\phi(0)=\phi_i}^{\phi(\tau)=\phi_f} \mathcal{D}\phi\, e^{-S[\phi]} =$$



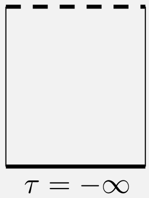
If we set one boundary condition but leave the other one open, we get a state

$$|\Psi\rangle = e^{-\tau H}|\phi\rangle = \int_{\phi(0)=\phi}^{\phi(\tau)=?} \mathcal{D}\phi\, e^{-S[\phi]}.$$

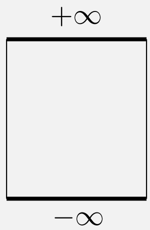
If we take the initial state $|\phi\rangle$ at Euclidean time $\tau \to -\infty$, then the state we get at $\tau = 0$ is the ground state $|0\rangle$. This is because

$$\lim_{\tau\to\infty} e^{-\tau H}|\phi\rangle = \lim_{\tau\to\infty} \sum_n e^{-\tau E_n} c_n |n\rangle \propto |0\rangle$$

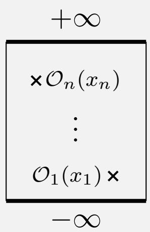
by writing $|\phi\rangle$ in terms of Hamiltonian eigenstates $|n\rangle$ with component c_n. So up to normalisation, we have

$$|0\rangle = \int_{\tau=-\infty} \mathcal{D}\phi\, e^{-S[\phi]} =$$


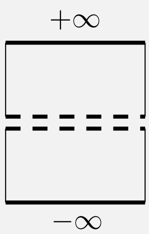
Using this, we can consider the vacuum correlation functions. First settle the normalisation

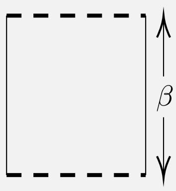
$$\langle 0|0\rangle = \int \mathcal{D}\phi\, e^{-S[\phi]} =$$


and the n-point function is then

$$\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle \equiv \langle 0|\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)|0\rangle$$
$$= \int \mathcal{D}\phi\, e^{-S[\phi]}\,\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n) =$$


Also, leaving two ends open, we can obtain density operators, for example:

$$\rho_{\text{vac}} = |0\rangle\langle 0| =$$


$$\rho_{\text{thermal}} = e^{-\beta H} =$$


1.7 Correlation Functions in Minkowski Spacetime

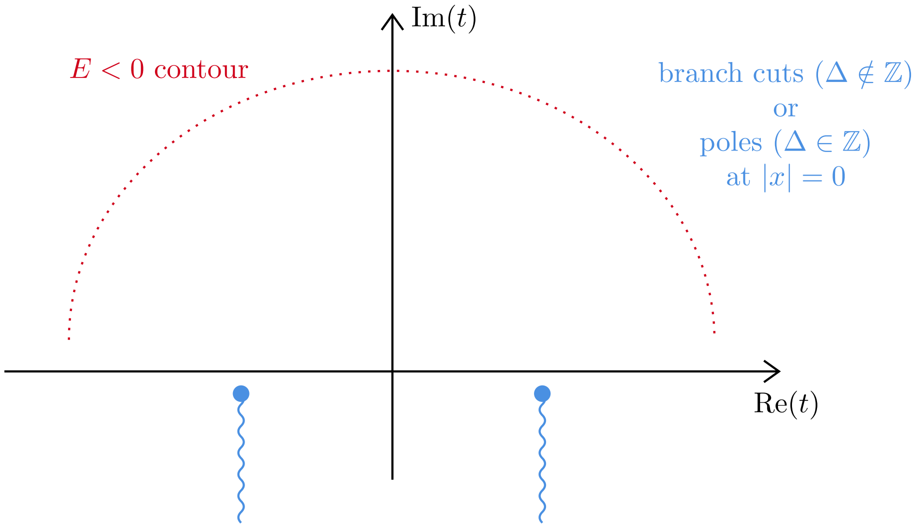
Using conformal symmetry, we can fix the form of the 2-point functions from some field $\varphi(x)$ as

$$\langle 0|\varphi(x)\varphi(y)|0\rangle = \frac{1}{\left|x-y+\mathrm{i}\epsilon\hat{t}\right|^{2\Delta}} \tag{11}$$

up to normalisation. Here, the iϵ-prescription in time direction $\hat{t}$ is to prevent the singular behaviour when the separation is null, i.e. $(x-y)^2 = 0$. We will justify this specific contour as below. Consider only the Fourier transform in time of the 2-point function. Without loss of generality, we can choose $y = 0$. Then

$$\int_{-\infty}^{+\infty} |x|^{-2\Delta} e^{-\mathrm{i}Et}\,\mathrm{d}t = \int_{-\infty}^{+\infty} (-t^2 + |\mathbf{x}|^2)^{-\Delta} e^{-\mathrm{i}Et}\,\mathrm{d}t \tag{12}$$

is suppressed for $E < 0$ as $t \to +\mathrm{i}\infty$ (i.e. in the upper half-plane) so the contour integral vanishes — hence the spectrum of $|\Psi\rangle$ is *positive*!



The form of 3-point functions is also fixed:

$$\langle 0|\varphi_3(z)\varphi_2(y)\varphi_1(x)|0\rangle \propto \frac{1}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}|x-z|^{\Delta_1+\Delta_3-\Delta_2}|y-z|^{\Delta_2\Delta_3-\Delta_1}}. \tag{13}$$

Spectral Decomposition Now we consider the spectral decomposition of the 2-point function. We want to know the possible momenta of states like

$$|\Psi\rangle = \int \mathrm{d}^d x\, f(x)\varphi(x)\,|0\rangle \tag{14}$$

so we Fourier transform x, y of the 2-point function

$$\varphi(p) = \int \mathrm{d}^d x\, \varphi(x) e^{\mathrm{i}p\cdot x} \tag{15}$$

to obtain

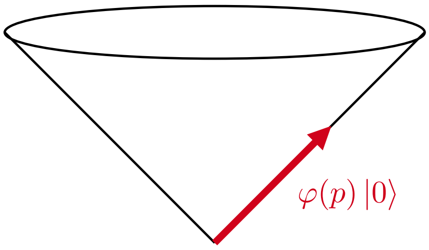
$$\langle 0|\varphi(-q)\varphi(p)|0\rangle \propto \delta^d(p-q)\Theta(E-|\mathbf{p}|)\begin{cases} \left|-p^2\right|^{\Delta-d/2} & \text{for } \Delta > \frac{d-2}{2}, \\ \delta(p^2) & \text{for } \Delta = \frac{d-2}{2}. \end{cases} \tag{16}$$

Here, the $\delta^d(p-q)$ term is from translational invariance, $\left|-p^2\right|^{\Delta-d/2}$ is from dimensional analysis, and the $\delta(p^2)$ because when unitarity bound is saturated we have $p^2=0$. We note that

$$\frac{1}{p^2}=\frac{1}{p_y^2-2p_up_v} \tag{17}$$

would give log divergence if integrated to $p^2=0$.

Hence, as unitarity bound, states are on E-p lightcone:

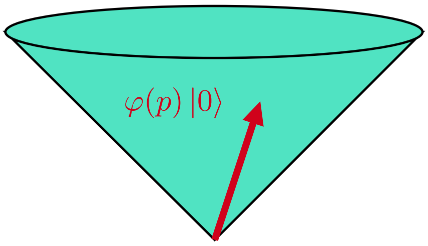


as befits free field for which

$$\Box\varphi=0 \quad\Rightarrow\quad E^2-|\mathbf{p}|^2=0 \tag{18}$$

which is a codimension-1 locus of solutions.

But *above* the unitarity bound, the states *fill* the E-p lightcone (in fact, no normalisable states have $E^2=|\mathbf{p}|^2$)



which is a codimension-0 locus and therefore these states do not obey a wave equation,

UNLESS IT IS IN ONE HIGHER DIMENSION!

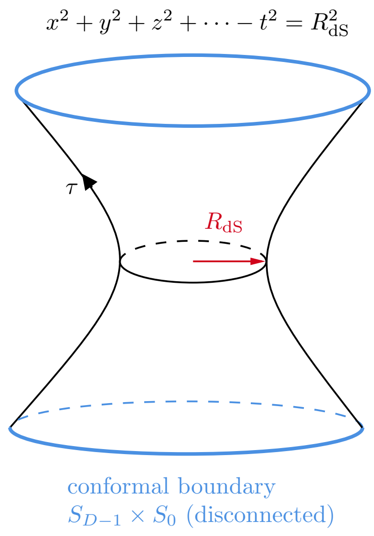
2 Maximally Symmetric Spacetimes in D Dimensions

We denote $D=d+1$ as the bulk dimension throughout the lectures. In D-dimension, there can be at most $D(D+1)/2$ Killing vectors. Spacetimes with the maximum number of Killing vectors are *maximally symmetric* spacetimes. The ansatz of Ricci tensor here is

$$R_{ab}=\frac{16\pi G}{D-2}g_{ab}\Lambda. \tag{19}$$

2.1 De Sitter

For $\Lambda>0$ we have *de Sitter* spacetime, which can be considered as a unit hyperboloid embedded in $\mathbb{R}^{D,1}$:



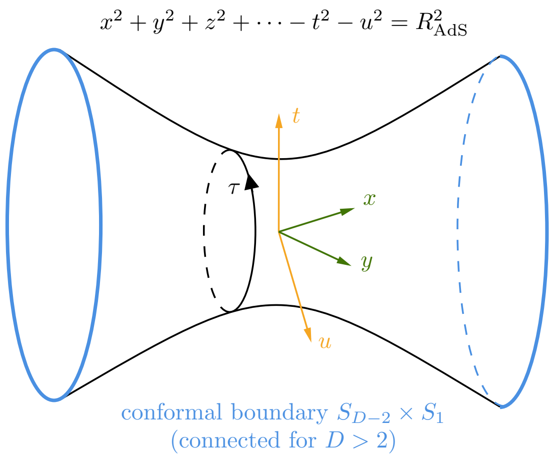
with symmetry group $\mathrm{SO}(D,1)$. The metric is

$$\mathrm{d}s^2 = R_{\mathrm{dS}}^2 \left(-\,\mathrm{d}\tau^2 + \cosh^2\tau\,\mathrm{d}\Omega_{D-1}^2 \right) \tag{20}$$

corresponding to contracting and expanding cosmology.

2.2 Anti-de Sitter

For $\Lambda < 0$ we have *anti-de Sitter* spacetime, which can be considered as a unit hyperboloid embedded in $\mathbb{R}^{D-1,2}$:

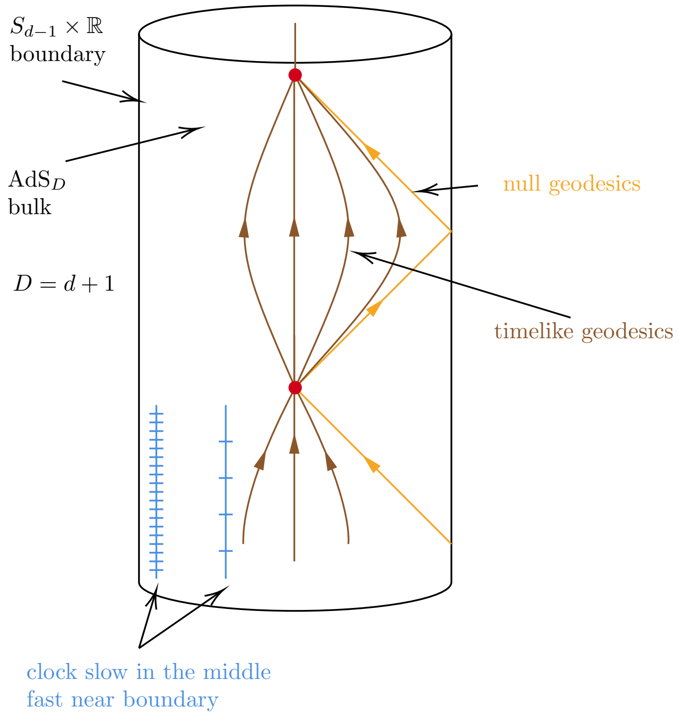


with symmetry group $\mathrm{SO}(D-1,2)$. We must take the *universal cover* to avoid closed timelike curves (CTC). The metric is

$$\mathrm{d}s^2 = R_{\mathrm{AdS}}^2 \left(\mathrm{d}\rho^2 - \cosh^2\rho\,\mathrm{d}\tau^2 + \sinh^2\rho\,\mathrm{d}\Omega_{D-2}^2 \right) \tag{21}$$

corresponding to hyperbolic space with redshift factor.

We can unroll AdS and conformally compactify it by sending $\rho = \infty$ to $\theta = \frac{\pi}{2}$ to get "tin can" (label on boundary corresponds to soup in bulk)



which has the following properties:

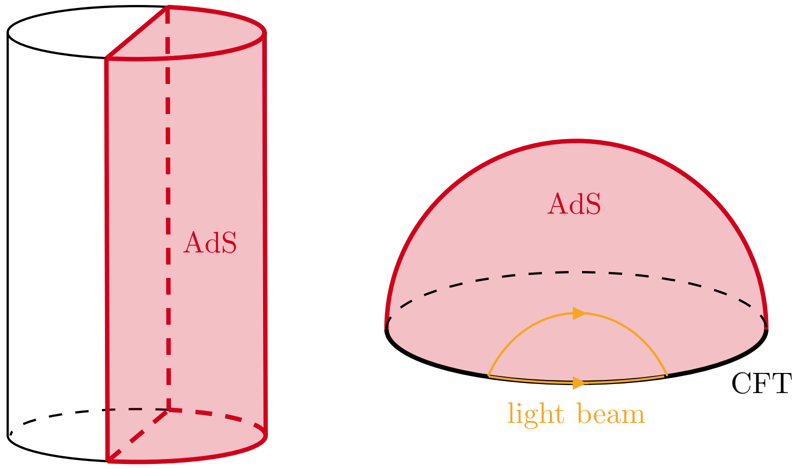
- Centre of AdS$_D$ group is $\mathbb{Z}$;
- Each point has infinite number of antipodal points;
- Timelike geodesics are attracted to spatial origin ($\rho = 0$);
- Lightlike rays make it to $\rho = \infty$ and with reflecting boundary condition they return in finite time as measured at $\rho = 0$.

Note that the location of origin is an arbitrary coordinate choice.

Comments on the Centre of AdS$_D$

The centre of the group being $\mathbb{Z}$ is slightly more subtle due to the option to include in the universal cover of SO$(2, d)$ either time reflections (T) or spatial reflections (P) or both. The group element which translates you to the next antipodal point consists of a π rotation in the t-u plane times a reflection of the $x, y, z, \ldots$ coordinates. This commutes with all elements of the AdS group only if we exclude discrete time reversal symmetry (T). Furthermore, if AdS has an odd number of spatial dimensions, then the map to the next antipodal point is only an element of the group if we include discrete spatial reflection symmetry (P). However, if we don't include P or T in our group, then the centre is still $\mathbb{Z}$, but now the smallest element of the centre is a 2π rotation in the t-u plane (to the next cover of the AdS hyperboloid).

The boundary is $S_{d-1} \times \mathbb{R}$ and AdS is conformal to $\frac{1}{2} S_d \times \mathbb{R}$



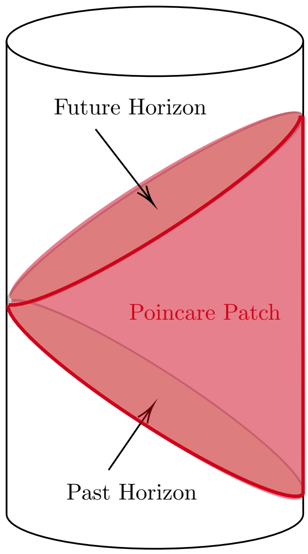
and we can easily take a consistency check: light cannot go faster through bulk than boundary. Hence

$$\mathrm{d}s^2 = \frac{R_{\mathrm{AdS}}^2}{\cos^2\theta}\left(-\,\mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\Omega_{d-1}^2\right) \tag{22}$$

where $1/\cos\theta$ is the Ω factor.

2.3 Poincaré Patch

Sometimes it is still convenient to restrict to Mink_d on CFT side, which corresponds to boundary of a patch with horizons in bulk AdS.



In Poincaré coordinates valid on the patch, the metric reads

$$\mathrm{d}s^2 = R_{\mathrm{AdS}}^2\left(\frac{\mathrm{d}z^2 + (-\,\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2)}{z^2}\right), \tag{23}$$

which is invariant under rescaling

$$\left.\begin{aligned} t &\mapsto \Omega t \\ \mathbf{x} &\mapsto \Omega \mathbf{x} \end{aligned}\right\} D \text{ operator on boundary} \qquad z \mapsto \Omega z \tag{24}$$

thus z represents “scale”.

The horizons corresponds to $z \to \infty$ limit so matter there is *arbitrarily redshifted*, which corresponds that CFT has *arbitrarily low energy excitations* on Minkowski spacetime (whereas CFT_d on $S_{d-1} \times \mathbb{R}$ is typically *gapped* for $d > 2$ due to curvature effects — see next topic).

3 Scalar Waves in AdS, Operators and Sources

3.1 Klein-Gordon Field

Use $\Omega = z$ rescaling, we can transform AdS-Poincaré to half Minkowski space:

$$\frac{\mathrm{d}z^2 + \eta_{ij}^{(d)}\,\mathrm{d}x^i\,\mathrm{d}x^j}{z^2} \mapsto \eta_{ij}^{(D)}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{25}$$

where we chose $R_{\mathrm{AdS}} = 1$ for convenience.

Start with Klein-Gordon equation $\Box\varphi = m^2\varphi$. We notice

- The dimension of φ is $\Delta_\varphi = \frac{D-2}{2}$ so the rescaled field is $\tilde{\varphi} = z^{-(D-2)/2}\varphi$;
- $\Box\varphi = 0$ is *not* conformal (it is covariant under uniform scaling but not $\Omega(x)$), but

  $$\left(\Box - \frac{D-2}{4(D-1)}R\right)\varphi \tag{26}$$

  is conformal, since $R = -D(D-1)$ in AdS, it shifts mass by

  $$\tilde{m}^2 = m^2 + \frac{D(D-2)}{4} \tag{27}$$

- $m^2\varphi$ is dimensionful, but we can promote it to position dependent mass: $m^2(z) = \frac{1}{z^2}$, then the differential equation becomes

  $$\left(\partial_z^2 - \frac{\tilde{m}^2}{z^2} + \Box^{(d)}\right)\tilde{\varphi} = 0. \tag{28}$$

  We use ansatz $\tilde{\varphi} \sim z^\nu + \mathcal{O}(z^{\nu+2})$ as $z \to 0$ and find that

  $$\nu(\nu-1)z^{\nu-2} - \tilde{m}^2 z^{\nu-2} + \mathcal{O}(z^\nu) = 0 \tag{29}$$

  to get

  $$\boxed{\nu(\nu-1) = \tilde{m}^2} \tag{30}$$

  i.e. there are two solutions at given $\tilde{m}^2$, depending on choices of boundary condition. Notice that $\Box^{(d)}\tilde{\varphi}$ term is subleading as $z \to 0$, but we need ${}^{(d)}p^2 < 0$ for normalisability as $z \to \infty$.

  For example, we can consider solutions like $\tilde{\varphi} = z^{-1}$ for $\varphi = \text{const.}$ in AdS_4 ($\nu = -1 \leftrightarrow \Delta = 0$).

3.2 Dictionary

We have a dictionary that maps bulk to boundary concepts.

Entry #1:

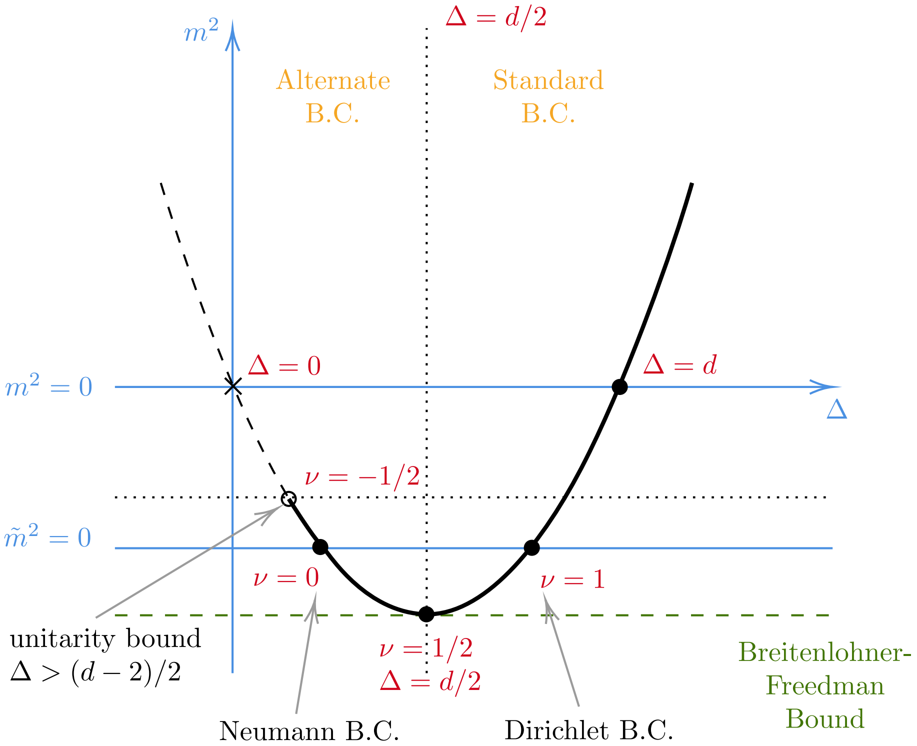
$$\boxed{\mathcal{O}(x^i) = \lim_{z\to 0} z^{-\nu}\tilde{\varphi}(z,x^i) = \lim_{z\to 0} z^{-\Delta_{\mathcal{O}}}\varphi(z,x^i)} \tag{31}$$

and by dimensional analysis

$$\Delta_{\mathcal{O}} = \Delta_\varphi + \nu = \frac{D-2}{2} + \nu = \frac{d-2}{2} + \nu + \frac{1}{2}. \tag{32}$$

We can find the correspondence

$$\nu(\nu-1) = \tilde{m}^2 \qquad \leftrightarrow \qquad \Delta_{\mathcal{O}}(\Delta_{\mathcal{O}} - d) = m^2. \tag{33}$$



$\nu > -1/2$ corresponds to unitary bound for holographic fields. The inequality is *strict* since the Klein-Gordon norm of φ must converge to make sense of quantum theory.

Also, when ν is complex, it corresponds to unstable exponentially growing modes.

Other solutions below the unitarity bound correspond to sources added to CFT

$$\begin{aligned}\mathcal{Z}_{\text{QFT}} &= \int \mathcal{D}\chi \, \exp\left(-I[\chi] - \int \mathrm{d}^d x \, \mathcal{J}\cdot\mathcal{O}\right) \\ &= \mathcal{Z}_{\text{CFT}}^{(\mathcal{J}=0)} \left\langle \mathcal{T} \exp\left(-\int \mathrm{d}^d x\, \mathcal{J}\cdot\mathcal{O}\right)\right\rangle = \mathcal{Z}_{\text{bulk}}[\mathcal{J}_{\text{bdy}}]\end{aligned} \tag{34}$$

the second line makes sense even if CFT doesn't come from action. By dimensional analysis, the sources have dimension

$$\Delta_{\mathcal{J}} = d - \Delta_{\mathcal{O}}. \tag{35}$$

3.3 Holographic Dictionary with Sources

$$\mathcal{J}(x^i) = \lim_{z\to 0} z^{-\Delta_\mathcal{J}}\varphi(x^i, z) \tag{36}$$

naively, however, the different z-powers in $\mathcal{J}, \mathcal{O}$ would contaminate each other (may blow up). Hence, we have

$$\begin{aligned} \mathcal{J}(x^i) &= \lim_{z\to 0}\left(z^{-\Delta_\mathcal{J}}\varphi(x^i,z) - \mathcal{O} \text{ profile}\right) \qquad \text{if } \Delta_\mathcal{O} < \Delta_\mathcal{J}, \\ \mathcal{O}(x^i) &= \lim_{z\to 0}\left(z^{-\Delta_\mathcal{O}}\varphi(x^i,z) - \mathcal{J} \text{ profile}\right) \qquad \text{if } \Delta_\mathcal{J} < \Delta_\mathcal{O}, \end{aligned} \tag{37}$$

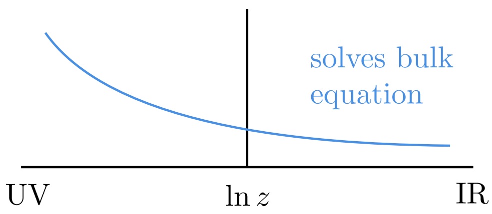
by writing, say

$$\varphi(x^i, z) = \mathcal{J}(x^i) z^{\Delta_\mathcal{J}} + \mathcal{O}(x^i) z^{\Delta_\mathcal{O}} + \text{subleading}. \tag{38}$$

If $\Delta_\mathcal{J} - \Delta_\mathcal{O} \in 2\mathbb{Z}$, the mixing leads to log terms.

- $\Delta_\mathcal{J} > 0$ relevant: important in IR, unimportant in UV, QFT still makes sense. $\varphi \to 0$, $z \to 0$;
- $\Delta_\mathcal{J} = 0$ marginal: still have a CFT at least to leading order. $\varphi \to$ const., $z \to 0$;
- $\Delta_\mathcal{J} < 0$ irrelevant: UV potentially ill-defined, corresponding to IR problem in the bulk. $\varphi \to \infty$, $z \to 0$.

If the bulk φ's have interaction, there will be additional non-linear terms in z-ODE, giving a *holographic RG flow.*



One bulk theory describes multiple boundary field theories, it is *scheme dependent* — z doesn't exactly match any boundary RG scheme, so the agreement is qualitative in universal features.

4 Vectors and Tensors, Conserved Currents

4.1 Bulk Vector Field

Consider Proca field

$$I = \int \mathrm{d}^D x \sqrt{-g}\left(\frac{1}{4} g^{ac} g^{bd} F_{ab} F_{cd} + \frac{1}{2} m^2 g^{ab} A_a A_b\right) \tag{39}$$

where the massive term adds extra degree of freedom: the number of degree of freedom is $D-2$ when $m^2 = 0$, it is $D-1$ for $m^2 > 0$. When $m^2 < 0$ there will be longitudinal mode ghost.
In AdS-Poincaré patch, we use $g_{ab} = \Omega^2 \eta_{ab}$ with $\sqrt{-\eta} = 1$ and $\Omega = 1/z$ to get

$$I = \int \mathrm{d}^D x \left(\frac{1}{4}\Omega^{D-4} F_{ab} F^{ab} + \frac{1}{2}\Omega^{D-2} m^2 A_a A^a\right) \tag{40}$$

whose equation of motion is

$$\partial_a \left(z^{4-D} F^{ab} \right) = z^{2-D} m^2 A^b. \tag{41}$$

We find A^a component by component. First consider the i-components (d-index). We use ansatz

$$A^i(x,z) = z^\nu J^i(x) + O(z^{\nu+2}) \tag{42}$$

to get

$$\underbrace{\partial_i F^{ib}}_{\text{subleading}} + \underbrace{\partial_z F^{zb}}_{\partial_z^2 A^i + \text{sub.}} + \underbrace{(4-D) z^{-1} F^{zb}}_{(3-d)z^{-1}\partial_z A^i} = z^{-2} m^2 A^b. \tag{43}$$

so

$$\nu(\nu-1) z^{\nu-2} J^i + (3-d)\nu z^{\nu-2} J^i = m^2 z^{\nu-2} J^i \tag{44}$$

we get

$$\nu(\nu-1) + (3-d)\nu = m^2. \tag{45}$$

Now we undo the conformal transformation by

$$A^i = \eta^{ij} A_j = (\pm) A_i = z^2 g^{ij} A_j \tag{46}$$

so we conclude that A_i has weight 1 under dilation $x^i \mapsto \Omega x^i$, $z \mapsto \Omega z$, just like $\partial_i \phi$.
Hence the weight for J is

$$\Delta_{J=z^{-\nu}A} = 1 + \nu \tag{47}$$

and

$$\begin{aligned} (\Delta-1)(\Delta-2) + (3-d)(\Delta-1) &= m^2 \\ (\Delta-1)(\Delta+1-d) &= m^2 \end{aligned} \tag{48}$$

so

$$m^2 = 0 \qquad \Leftrightarrow \qquad \Delta = d-1 \quad \text{or} \quad \Delta = 1 \tag{49}$$

where $\Delta = d-1$ corresponds to boundary currents J^i, $\Delta = 1$ corresponds to boundary potential (background source) A_i.
If $A_i = 0$, then it corresponds to a *global* conserved current.
What about A^z?

- For $m^2 = 0$, we have gauge symmetry $\delta A_a = \nabla_a \alpha$ so can impose gauge condition, e.g. $A_z = 0$, giving $D-2 = d-1$ degrees of freedom. Use z-equation of motion

$$\begin{aligned} \partial_i F^{iz} &= 0 \qquad (F_{zz} = 0) \\ \partial_z(\partial_i A^i) &= 0 \qquad \Rightarrow \qquad \partial_i J^i = \nabla_i J^i = 0 \end{aligned} \tag{50}$$

  where we used the fall-off condition at $z = \infty$.

- For $m^2 > 0$, vary with respect to α:

$$\delta_\alpha I = m^2 \int \mathrm{d}^D x \, \nabla\alpha \cdot A = -m^2 \int \mathrm{d}^D x \, \alpha (\nabla \cdot A) \overset{\text{EOM}}{=} 0 \tag{51}$$

  notice here $\nabla \cdot A = 0$ is not a gauge choice but the genuine equation of motion! Hence

$$\partial_a \left(\sqrt{g} g^{ab} A_b \right) = \partial_a \left(\sqrt{-\eta} \eta^{ab} z^{D-2} A_b \right) = 0. \tag{52}$$

  Solve for divergence, after a little algebra, we get

$$A_z = \frac{z^{\Delta_J} (\partial \cdot J)}{\Delta_J - (d-1)} \tag{53}$$

  where we have $D-1 = d$ degrees of freedom.

4.2 New Entries in Dictionary

Then we have new entries in our dictionary:

$$\begin{array}{rcll} \text{boundary global current} & \longleftrightarrow & \text{bulk gauge field} & \\ J_I^i(x) & \longleftrightarrow & A_I^a(x,z) & \text{(possible Lie index } I) \\ T^{ij}(x) & \longleftrightarrow & g^{ab}(x,z) & \text{(dynamical graviton)} \end{array}$$

Here we put the entry $T \leftrightarrow g$: every decent CFT should have a stress-tensor — this suggests bulk AdS description should involve *gravity*. Hence, g_{ab} is only asymptotically AdS for excited states on boundary; vacuum state corresponds to exact AdS (up to quantum fluctuation).

Also we can consider entries such as

$$\text{SUSY current} \quad \longleftrightarrow \quad \text{gravitino}$$

if we consider dualities involving superconformal/supergravity theories.

4.3 Tensors

Conformal irrep of $T_{ij}(x)\,|0\rangle$ corresponds to linear spin-2 equation:

$$\nabla^2 h_{ab} - \nabla_{(a}\nabla_{c)}h^c_b - \nabla_{(b}\nabla_{c)}h^c_a + \nabla_a\nabla_b h^c_c + g^{\text{AdS}}_{ab}(\nabla^c\nabla^d h_{cd} - \nabla^2 h^c_c) = m^2(h_{ab} - g^{\text{AdS}}_{ab}h^c_c) \tag{54}$$

where the right hand side is known as the *Fierz-Pauli mass term*. Here, ∇ and lowering/raising of indices are defined using g^{AdS}_{ab}.

The field equation restricts to traceless-transverse (TT) modes and implies

$$\nabla^2 h_{ab} = m^2 h_{ab}. \tag{55}$$

The numbers of degrees of freedom are

$$\begin{cases} \dfrac{(D-2)(D-1)}{2} - 1 & m^2 = 0, \\ \\ \dfrac{(D-1)D}{2} - 1 & m^2 > 0. \end{cases}$$

The equation is obtainable from quadratic part of GR/FP action:

$$I_{\text{GR/FP}} = \int \mathrm{d}^D x\,\sqrt{-g}\left(-\frac{1}{2}R[g] + \Lambda - \frac{1}{2}m^2(h_{ab}h^{ab} - h^a_a h^b_b)\right) \tag{56}$$

where $g_{ab} = g^{\text{AdS}}_{ab} + h_{ab}$. Note that the indices in the Fierz-Pauli term is raised/lowered using g^{AdS}_{ab} and the action is *not* background independent unless $m^2 = 0$ — this would give problematic "ghost modes" beyond quadratic order.

Similar calculations as before give

$$h_{ij} = z^{\Delta-2}T_{ij} + O(z^\Delta) \qquad \text{so} \qquad T_{ij} = \lim_{z\to 0} z^{2-\Delta}h_{ij} \tag{57}$$

and

$$\Delta(\Delta - d) = m^2, \quad T^i_i = 0 \tag{58}$$

where

$$m^2 = 0 \qquad \Leftrightarrow \qquad \Delta = d \quad \text{or} \quad \Delta = 0 \tag{59}$$

corresponding to conserved stress tensor or boundary metric source, respectively.

Note that massive graviton is still dual to traceless operator as

$$\frac{D(D-1)}{2} - 1 = \frac{d(d+1)}{2} \tag{60}$$

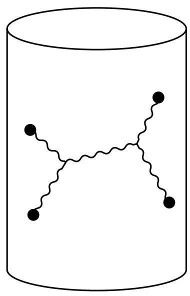
i.e. the configuration d.o.f. of graviton matches with the traceless tensor d.o.f. which is *not* conserved.

Aside: Why Linear Spin-2 Equation?

- Quantum Mechanics is linear, so irreps of conformal group $\mathcal{H}_{\text{irrep}} \subset \mathcal{H}_{\text{CFT}}$ obey a *linear* AdS wave equation $\mathrm{D}\varphi = 0$ for some differential operator D.
- In particular, $T_{ij}\left|0\right\rangle =$ 1-graviton bulk state, but non-linear aspects of GR are still encoded in the theory, e.g. in n-point functions

$$\langle TTTT \cdots \rangle \tag{61}$$

  which we can calculate with *Witten diagrams.*



  These are like Feynman diagrams, but some edges go to boundary of AdS. We also have the following correspondence:

$$G_{\text{bulk}\to\text{bdy}}(z,x;x') = \lim_{z'\to 0} (z')^{-\Delta} G_{\text{bulk}\to\text{bulk}}(z,x;z',x') \tag{62}$$

  where $G_{\text{bulk}\to\text{bdy}}(z,x;x')$ is solution to $\mathrm{D}\varphi = 0$ with boundary source $J = \delta^d(x)$ and $G_{\text{bulk}\to\text{bulk}}(z,x;z',x')$ is solution to $\mathrm{D}\varphi = \delta^D(z,x)/\sqrt{-h}$.

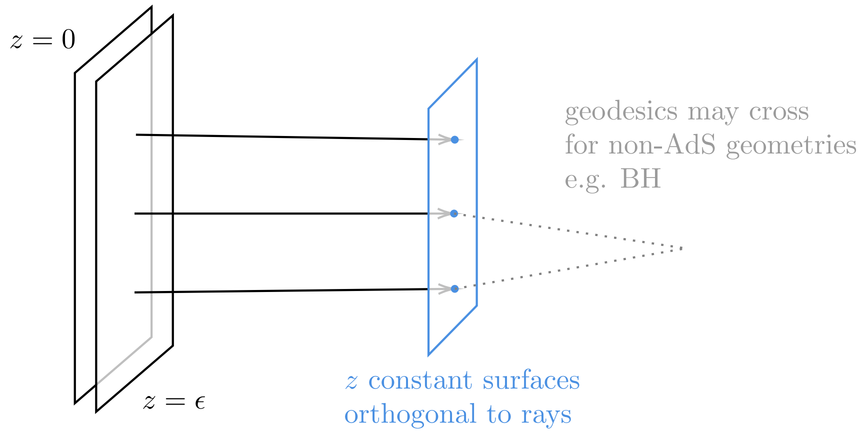
4.4 Fefferman-Graham Expansion

For massless case (GR), it is usually convenient to use *Fefferman-Graham gauge fixing* with

$$g_{zz} = \frac{1}{z^2}, \quad g_{zi} = 0. \tag{63}$$

We can check that these impose D conditions, so the residuals are responsible for diffeomorphism and conformal symmetry of the boundary.

For non-AdS geometry (e.g. black holes) normal geodesics may cross resulting in coordinate breakdown, however, there is no problem with asymptotic expansion in z.



The gauge choice corresponds to shooting out geodesics normal to $z=\epsilon$ surface and we let

- $\ln z = \ln \epsilon +$ proper distance along ray;
- $x^i =$ const. along each ray.

Solving non-linear bulk equations for vacuum GR, we obtain the *Fefferman-Graham expansion*

$$ {}^{(D)}g_{ij}(x,z) = \frac{1}{z^2}\left(g_{ij}^{(0)} + z^2 g_{ij}^{(2)} + z^4 g_{ij}^{(4)} + \cdots + \begin{cases} z^d T_{ij} + O(z^{d+2}) & \text{odd } d, \\ z^d T_{ij} + (z^d \ln z) h_{ij}^{(d)} + O(z^{d+2}\ln z) & \text{even } d. \end{cases} \right) \tag{64} $$

where

$$ \begin{aligned} g_{ij}^{(0)} &= {}^{(d)}g_{ij}, \\ g_{ij}^{(2)} &= \frac{1}{D-2}\left({}^{(d)}R_{ij} - \frac{1}{2(D-1)} {}^{(d)}g_{ij}\ {}^{(d)}R \right), \\ &\cdots \\ h_{ij}^{(d)} &= -\frac{2}{d} T_{ij}^{\text{trace anomaly}} \propto {}^{(d)}g_{ij}. \end{aligned} \tag{65} $$

For more, c.f. de Haro, Skenderis & Solodukhin [14].

5 Large N Gauge Theories

Suppose the bulk is weakly coupled (i.e. approximately free[2] when $N_{\text{quanta}} \sim 1$, and thus approximately *classical* in non-linear regime), then in addition to φ we can consider products of fields $\prod_i \varphi_i$ with

$$ \varphi_i \sim z^{\Delta_i}, \qquad \prod_i \varphi_i \sim z^{\sum_i \Delta_i} \tag{66} $$

giving an operator $\mathcal{O}$ of weight $\sum_i \Delta_i$ in free theory limit.

Thus:

1. There exists a *special* collection of operators $\{\mathcal{O}_s\}$ whose spectrum is Fock($\{\mathcal{O}_s\}$);
2. $\langle \mathcal{O}_s \mathcal{O}_s \mathcal{O}_s \mathcal{O}_s \cdots \rangle$ is approximately *Gaussian* — "generalised free field" — we have Wick contractions for even number of operators;

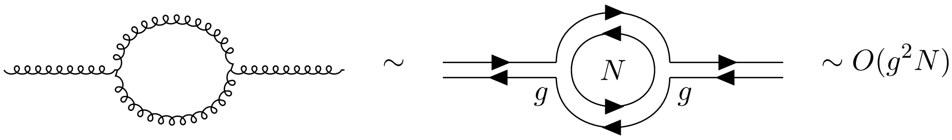
[2] In large N limit, two-point functions dominate.

3. Yet the CFT is not truly free since there is no $\Delta = (d-2)/2$ field!

It turns out this is typical of *single trace* operators $\mathrm{tr}(\cdots)$ in large N gauge theories.
Consider the Yang-Mills theory, the Lagrangian is

$$\mathcal{L}_{\mathrm{YM}} = \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \tag{67}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu]$. This could be $\mathrm{SU}(N)$ for example, we take the $N \to \infty$ limit and we hold the *'t Hooft coupling* $\lambda = g^2 N$ fixed. Now consider gluon interaction, each gluon carries 1 colour and 1 anti-colour, so there are $\sim N^2$ of species.



As $N \to \infty$, we are in the *planar limit*: diagrams with smaller genus dominate. Consider a vacuum bubble, we count

- A factor of N for each face;
- A factor of $g \sim N^{-1/2}$ for each 3-vertex;
- A factor of $g^2 \sim N^{-1}$ for each 4-vertex.

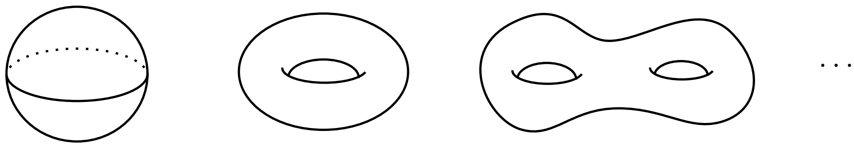
Then the amplitude is proportional to

$$f(\lambda)N^{F-E+V} = f(\lambda)N^{\chi} \tag{68}$$

where F, E, V are the numbers of faces, edges, vertices of the diagram, respectively, and $\chi = F - E + V$ is the *Euler number*.

$$\chi = 2 - 2g \tag{69}$$

for a diagram with genus g.



$g = 0$ $\quad$ $g = 1$ $\quad$ $g = 2$ $\quad$ $\cdots$

In the limit of $N \to \infty$, $g = 0$ diagrams dominate (hence planar limit) and they behave like weakly coupled strings.
Operator like $\mathcal{O} = [\mathrm{tr}(\cdots)]^n$ creates/annihilates n strings.

Large N Derivation

Under Construction.

6 Black Holes and Thermal States

6.1 AdS/CFT Correspondence

We've seen evidence that spectra of AdS and CFT agree, at least at low energy

$$\mathcal{H}_{\text{AdS}} = \mathcal{H}_{\text{CFT}}. \tag{70}$$

We can also state duality using partition functions

$$\mathcal{Z}_{\text{AdS}}[J_{\text{bdy}}] = \mathcal{Z}_{\text{CFT}}[J_{\text{CFT}}] \tag{71}$$

where this *implies* agreement of n-point functions:

$$\frac{\delta}{\delta J_1(x)} \frac{\delta}{\delta J_2(y)} \cdots \ln \mathcal{Z} = \langle \mathcal{O}_1(x) \mathcal{O}_2(y) \cdots \rangle_{\text{conn}}. \tag{72}$$

Conversely, if n-point functions agree, $\mathcal{Z}$ is fixed up to topological invariant of sources, but such constants also seem to agree...

But in large N (and large λ) limits, bulk becomes *classical* (i.e. we can do saddle point approximation around solution to equation of motion $g_{ab} = g_{ab}^{\text{classical}} + h_{ab}$)

$$\mathcal{Z}_{\text{AdS}} = \underbrace{[\det(\cdots)]^{-1/2}}_{\text{quantum fluct.}} \exp(\mathrm{i} I_{\text{grav}}[g_{ab}]) \quad \Rightarrow \quad \ln \mathcal{Z}_{\text{AdS}} = \mathrm{i} I_{\text{grav}} + \underbrace{\text{loop corrections}}_{\text{subleading in } 1/N}, \tag{73}$$

or, in Euclidean signature, $\ln \mathcal{Z} = -I_{\text{grav}}$.

Hence we have a new entry in the dictionary:

Euclidean QFT log partition function over boundary

$\updownarrow$

gravitational action for least action "instanton" solution to Einstein EOM with specified boundary metric γ_{ab}.[3]

6.2 Gravitational Path Integral

The Euclidean gravitational action is

$$I_{\text{grav}}^{(\text{Euc.})} = -\frac{1}{16\pi G} \int_M \mathrm{d}^D x \sqrt{g}(R - 2\Lambda). \tag{74}$$

Evaluating on-shell

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab} \Lambda = 0 \quad \Rightarrow \quad \frac{D-2}{2} R = D\Lambda \tag{75}$$

so

$$I_{\text{grav}} \sim \int_M \mathrm{d}^D x \sqrt{g} = \text{Vol}(M) = +\infty \text{ for asymptotically AdS!} \tag{76}$$

What went wrong? We didn't treat boundary carefully enough. We place a cut-off at $z = \epsilon$ to regulate, let $g_{ij} = \gamma_{ij}$ along ∂M.

[3] Gravitational action is not bounded below for off-shell metrics. Legit calculation would check that you can deform contour from Lorentzian signature properly, but few people do this!

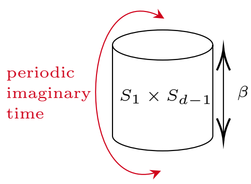
1. We should have included *Gibbons-Hawking term*
$$I_{\text{GH}} = \frac{1}{8\pi G}\int_{\partial M} \mathrm{d}^d x\, \sqrt{\gamma} K_{ij}\gamma^{ij} \tag{77}$$
where $K_{ij} = \frac{1}{2}\mathcal{L}_n g_{ij}$ is the extrinsic curvature, to cancel $\delta K_{ij}\gamma^{ij}$ variation from Einstein-Hilbert action (we do not want $\gamma^{ij} = 0$ EOM);

2. Subtract off local counter-terms:
$$I_{\text{ct}} = \int_{\partial M} \mathrm{d}^d x\, \sqrt{\gamma} h[\gamma] \tag{78}$$
where h depends on up to $d/2$ derivatives of γ.

This is what usually happens in QFT:
$$\ln \mathcal{Z}_{\text{phys}} = \ln \mathcal{Z}_{\text{reg}}(\epsilon) - \text{local divergences} \begin{cases} \epsilon^n \\ \ln \epsilon \end{cases} . \tag{79}$$
We notice in a CFT the coefficient of log divergence is *universal*.

6.3 Thermal States

Using Euclidean partition functions, we can define the thermal states. Start with $\mathbb{R}^d$ in coordinates $(\tau, \mathbf{x})$, consider cylinder metric from compactifying τ-direction $\tau \sim \tau + \beta$, the path integral becomes
$$\mathcal{Z}_{\text{cyl.}}[\beta] = \int_{\phi(\tau=\beta)=\phi(\tau=0)} \mathcal{D}\phi\, e^{-I[\phi]}. \tag{80}$$
[Need figure 18: Cylinder here.]

Evaluating using Hamiltonian methods, first we do the path integral on the strip [Need figure 19: Strip here.]
$$\mathcal{Z}_{\text{strip}} = \int \mathcal{D}\phi\, e^{-I[\phi]}\Psi[\phi(\tau=0)]\Psi^*[\phi(\tau=\beta)] = \langle\Psi|e^{-\beta H}|\Psi\rangle \tag{81}$$
then, by identifying $\tau = 0$ and $\tau = \beta$ of the strip, we get
$$\mathcal{Z}_{\text{cyl.}}[\beta] = \sum_n \langle n|e^{-\beta H}|n\rangle = \operatorname{tr}\left(e^{-\beta H}\right) \tag{82}$$
which is the thermal state (up to normalisation) with temperature $T = 1/\beta$. For example, the two-point function on the cylinder is the thermal correlation function
$$\langle \mathcal{O}(\tau=0,\mathbf{x}_1)\mathcal{O}(\tau=0,\mathbf{x}_2)\rangle = \frac{\mathcal{Z}[\cdot\bullet\mathcal{O}\cdots\bullet\mathcal{O}]}{\mathcal{Z}[\cdots\cdots]} = \frac{\operatorname{tr}\left(e^{-\beta H}\mathcal{O}(\mathbf{x}_1)\mathcal{O}(\mathbf{x}_2)\right)}{\operatorname{tr}(e^{-\beta H})}. \tag{83}$$
The limit $\beta \to \infty$ turns to the usual vacuum path integral in $\mathbb{R}^d$. The vacuum density operator can be seen as
$$\rho_{\text{vac}} = |0\rangle\langle 0| \propto \text{“}e^{-\infty H}\text{”}. \tag{84}$$
We can also calculate the thermal partition function with of compact space S_{d-1}:
$$\mathcal{Z}[\beta] = \operatorname{tr}\left(e^{-\beta H}\right) = \tag{85}$$


From thermal partition function, we can find all of the thermodynamic quantities: e.g. the free energy F from

$$\ln \mathcal{Z} = -\beta F = S - \beta \langle H \rangle \,, \tag{86}$$

the energy expectation value

$$\langle H \rangle = -\partial_\beta \ln \mathcal{Z} \tag{87}$$

and hence the von Neumann entropy

$$S = -\operatorname{tr}(\rho \ln \rho) = (1 - \beta \partial_\beta) \ln \mathcal{Z}. \tag{88}$$

Half of a thermal circle is special thermofield double state

$$|\mathrm{TFD}\rangle = \sum_i e^{-\beta E_i/2} \, |E_i\rangle_{\mathrm{L}} \, |\bar{E}_i\rangle_{\mathrm{R}} \tag{89}$$

which is pure and invariant under $t_{\mathrm{L}} \mapsto t_{\mathrm{L}} + c, t_{\mathrm{R}} \mapsto t_{\mathrm{R}} - c$. If we restrict it to one system, it is a thermal state.

There are 2 types of classical GR solutions dual to the thermal field theory:

(A) Pinch off S_{d-1} to point: [Need figure 20: Pinch S_{d-1} here.] This is the thermal AdS, with the same g_{ab} as vacuum but with period it. The $t = 0$ slice is disconnected. The entropy is

$$S_{\mathrm{CFT}} = 0 + \text{subleading in } \frac{1}{N}. \tag{90}$$

(B) Pinch off S_1 to point: [Need figure 21: Pinch S_1 here.]

The $t = 0$ slice is a wormhole! The bulk QFT is in *Hartle-Hawking state*. If we continue back to Lorentz signature, we obtain an eternal black hole! [Need figure 22: Penrose diagram of eternal black hole here.] This means

ENTANGLEMENT IN CFT = WORMHOLE IN ADS

The entropy is

$$S_{\mathrm{CFT}} = \underbrace{\frac{\mathrm{Area}[\mathcal{H}^+]}{4G\hbar}}_{S_{\mathrm{BH}}} + \underbrace{\text{subleading in } \frac{1}{N}}_{\text{matter outside}} \tag{91}$$

Note: by "pinching off", we mean topologically replace S_1 with a disc B_2 or replace S_{d-1} with a ball B_d to classify the two types of bulk geometries.

6.4 AdS-Schwarzschild as an Example

For concreteness, the formulae below are AdS$_5$ case with $\Lambda = -6/L^2$. The AdS-Schwarzschild metric is

$$\mathrm{d}s^2 = -f \, \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f} + r^2 \, \mathrm{d}\Omega_3{}^2 \,, \qquad f = 1 + \frac{r^2}{L^2} - \frac{\mu}{r^2} \tag{92}$$

with horizon radius

$$r_{\mathrm{H}} = \frac{L^2}{2} \left(\sqrt{1 + \frac{4\mu}{L^2}} - 1 \right) \tag{93}$$

and inverse temperature

$$\beta = \frac{2\pi L^2 r_{\rm H}}{2r_{\rm H}^2 + L^2} \quad \text{with min. temperature } \beta \leq \frac{L\pi}{\sqrt{2}}. \tag{94}$$

[Need figure 23: plot of T-$r_{\rm H}$ here.]

It turns out that for

$$\begin{aligned} \beta > \beta_{\rm HP} = \frac{2\pi L}{3}, &\quad I^{\rm AdS} < I^{\text{large BH}} \\ \beta < \beta_{\rm HP} \qquad\quad , &\quad I^{\text{large BH}} < I^{\rm AdS} \end{aligned} \tag{95}$$

this is the *Hawking-Page phase transition* (1st order) which is a large N effect.

$$\begin{aligned} S \sim O(1) &\quad \longleftrightarrow \quad \textit{confining} \text{ at low } T \text{ (on } S_3) \sim \text{color singlets} \\ S \sim O(N^2) &\quad \longleftrightarrow \quad \textit{deconfinement at high } T \sim \text{(super)gluon plasma}^4 \end{aligned}$$

At high T,

$$S_{\rm BH} = \frac{3}{4} S_{\rm free}^{\mathcal{N}=4\ \rm SYM} \sim VT^3 \quad \text{(Boltzmann)}. \tag{96}$$

Analogous 1+1 CFT calculation gives exact match!

7 Entanglement Entropy

7.1 Mixed States

We consider Quantum Mechanics of a bipartite system. Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ be a Hilbert space factorisation, and let Ψ be a pure but entangled state of $\mathcal{H}$. If we restrict to $\mathcal{H}_1$ only, we obtain a mixed state with density operator

$$\rho_1 = \operatorname{tr}_2(|\Psi\rangle\langle\Psi|) = \begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix} \tag{97}$$

which can be diagonalised via some unitary transformation U. Note that the density operator here is normalised $\operatorname{tr}\rho = 1$.

7.2 Von Neumann Entropy

One particularly nice attribute is von Neumann entropy

$$S(\rho) = -\operatorname{tr}(\rho \ln \rho) \tag{98}$$

and specifically,

$$S(\rho) = \begin{cases} 0 & \text{for pure state,} \\ \ln D & \text{for max mixed state } \rho = 1/D \text{ in } D\text{-dim } \mathcal{H}. \end{cases} \tag{99}$$

Properties of von Neumann entropy:

[4] Technically it is still in singlet state, but so complicated S sees N^2 degrees of freedom.

1. Positivity: $S(\rho) \geq 0$;
2. Invariant under unitary transformations:
   (a) $S(U\rho U^\dagger) = S(\rho)$;
   (b) adding extra $p = 0$ states;
3. Additivity: $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$; (For later convenience, define $S(A) \equiv S(\rho_A)$.)
4. Triangle inequalities:
   (a) Subadditivity: $S(A) + S(B) \geq S(AB)$;
   (b) Araki-Lieb: $S(AB) \geq |S(A) - S(B)|$;
5. Continuous for finite dimension, and lower semicontinuous for infinite dimension: if $\rho = \lim \rho_n$ then $S(\rho) \leq \liminf S(\rho_n)$;

Up to now properties 2, 3, 4(a) and 5 can uniquely pick out S up to scaling.

6. Strong subadditivity: $S(AB)+S(BC) \geq S(ABC)+S(B)$ and $S(AB)+S(BC) \geq S(A)+S(C)$;
7. Concavity: Let $\sum_i \lambda_i = 1$, then $S(\sum_i \lambda_i \rho_i) \geq \sum_i \lambda_i S(\rho_i)$;
8. Chain rule: Let $\rho = \bigoplus_{\substack{i \text{ block} \\ \text{diagonal}}} \lambda_i \rho_i$, then $S(\rho) = \langle S(\rho_i) \rangle_\lambda - \sum_i \lambda_i \ln \lambda_i$.

7.3 Entanglement Entropy

Here we focus on the scope of QFT, where the space is continuous and does not factorise into clean Hilbert spaces as in QM. Consider a Cauchy slice which is divided into two regions $\Sigma = \mathcal{R} \cup \bar{\mathcal{R}}$ by some entangling surface $\mathcal{E}$, then the entanglement entropy between $\mathcal{R}$ and $\bar{\mathcal{R}}$ is defined as

$$S(\rho_\mathcal{R}) = -\operatorname{tr}(\rho_\mathcal{R} \ln \rho_\mathcal{R}) \tag{100}$$

[Need figure 24: bipartite here.] which depends on the choice of

(i) QFT;
(ii) region $\mathcal{R}$;
(iii) state Ψ of the system via restriction to $\rho_\mathcal{R}$;
(iv) a short distance cut-off ϵ to regulate UV divergence.

The entanglement entropy defined in QFT should have properties similar to QM modulo the cut-off issues:

- $S(\mathcal{R}) = S(\bar{\mathcal{R}})$ with “same” cut-off on both sides;
- $S(\mathcal{R}_1) = S(\mathcal{R}_2)$ if $D[\mathcal{R}_\infty] = D[\mathcal{R}_2]$ (D means domain of dependence) although cut-off can mess this up in theories with gravitational anomaly. [Need figure 25: D[R] here.]

7.4 Expected Divergence Structure

$D = 2$ CFT

For length r and central charge c,

$$S = \frac{c}{3} \ln\left(\frac{r}{\epsilon}\right) + \text{finite} \tag{101}$$

where the $c/3$ factor is universal and the finite term is scheme and/or state dependent.

$D > 2$: Area Law

$$S = (\#) \frac{\text{Area}[\mathcal{E}]}{\epsilon^{D-2}} + \text{subleading} + \text{finite} \tag{102}$$

where the subleading divergence can be deduced by dimensional analysis, looking for products of $R, K^2, m^2, \ldots$ with weight up to $D - 2$. The divergence structure can also be expressed as

$$\int_\epsilon x^{2-D} \frac{\mathrm{d}x}{x} \left(a_0 + a_2 x^2 + a_4 x^4 + \cdots\right) = \text{power} + \log \text{ (even dim.)}. \tag{103}$$

[Need figure 26: Zoom in on E close to Rindler here.]

Divergence Structure in Details

Under Construction.

7.5 Geometric Entropy

Conical Method

This method is valid when the state is produced by a *path integral* with U(1) (rotational) invariance around E. Consider the partition function (unnormalised version of $\rho = e^{-\beta K}/\mathcal{Z}$)

$$\mathcal{Z} = \mathrm{tr}\left(e^{-\beta K}\right) \tag{104}$$

with $\beta = 2\pi$ (usually so but not required). Varying β away from 2π corresponds to introducing conical singularity. Then

$$\frac{\partial}{\partial\beta} \ln \mathcal{Z} = \frac{1}{\mathcal{Z}} \mathrm{tr}\left(e^{-\beta K} K\right) = -\beta^{-1} \left(\mathrm{tr}(\rho \ln \rho) + \mathrm{tr}(\rho) \ln \mathcal{Z}\right) \tag{105}$$

where we used $e^{-\beta K} = \rho \mathcal{Z}$ and $K = -\beta^{-1} \ln(\rho\mathcal{Z})$. Thus

$$S(\beta') = (1 - \beta\partial_\beta) \ln \mathcal{Z}\Big|_{\beta=\beta'}. \tag{106}$$

If we linearly interpolate $\ln \mathcal{Z}$ back to $\beta = 0$, anything linear in β drop out, suggesting that zero-point energy carries no entropy.

Replica Trick

What if we have no U(1) symmetry (e.g. a curved $\mathcal{E}$, unsymmetric sources/operator insertions)? In this case, we must do *replica trick* by copying the space for n times [Need figure 27: Replica Trick here.]

This defines

$$\mathcal{Z}_n = \mathrm{tr}(\rho^n) \tag{107}$$

and the same formula works as before:

$$S = (1 - n\partial_n) \ln \mathcal{Z}_n \Big|_{n=1} = \lim_{n\to 1} \frac{1}{1-n} \ln \mathrm{tr}(\rho^n) \tag{108}$$

where $\ln \mathrm{tr}(\rho^n)/(1-n)$ is known as the n-th Rényi entropy, which obeys some but not all properties of S. However, we see an immediate problem: we have to analytically continue to non-integer n (the conical method is a special case where this is easy to do). So how to continue from integer n? It seems ambiguous due to factors such as $\sin(\pi n m)$ but this blows up exponentially as $n \to \mathrm{i}\infty$. Carlson's theorem guarantees unique continuation if the function does not blow up at infinity too fast, as expected for $\mathrm{tr}(\rho^n)$ when $\mathrm{Re}(n) \geq 1$.

An alternative view point is that the manifold has $\mathbb{Z}_n$ replica symmetry. We can add *twist operators* at $\mathcal{E}$ which implement twist. This is very useful in 1+1 CFT[5] where they behave just like any local operator for $n \in \mathbb{Z}$ which is primary with conformal weight

$$\Delta = \frac{c}{12}\left(n - \frac{1}{n}\right). \tag{109}$$

7.6 Holographic Entanglement Entropy

[Need figure 28: AdS/CFT here.]

The AdS/CFT set-up can be summarised as

$$\begin{tikzcd}
\text{String/M-Theory on } \mathrm{AdS}_{d+1} \times F \arrow[r, leftrightarrow] \arrow[d, "L_{\mathrm{AdS}} \gg l_{\mathrm{P}}, l_s", "G\to 0"'] & \mathrm{CFT}_d \arrow[d, "\text{large } N", "\text{(strongly coupled)}"'] \\
\text{Supergravity} \arrow[r, leftrightarrow] & \text{Holographic } \mathrm{CFT}_d
\end{tikzcd}$$

Ryu-Takayanagi (RT) Formula

The *Ryu-Takayanagi formula* is a formula for leading order piece of the entanglement entropy. It applies to static or $t \mapsto -t$ symmetric slice Σ. We find the minimal area surface γ anchored to $\mathcal{E}$ that divides $\mathcal{R}$ from $\bar{\mathcal{R}}$ and the entanglement entropy is then

$$S_{\text{ent}} = \min_{\gamma} \frac{\mathrm{Area}[\gamma]}{4G\hbar}. \tag{110}$$

Note that γ should satisfy the *homology constraint* such that γ can be smoothly deformed through the bulk to get $\mathcal{R}$.

[Need figure 29: RT here.]

Also, by doing the bulk calculation naively, we get infinity as there is infinite redshift when moving towards the boundary $z \to 0$. Hence, we still need a "UV" cut-off as we did before in

[5] 1+1 replica manifold is conformally flat.

the CFT side. However, this actually plays the role of an *IR regulator* in the bulk by stopping at $z = \epsilon$. In fact, if we are careful, we can use theories with large amounts of supersymmetry to check e.g. in $d = 4$ and $d = 2$ that the log divergences from the CFT calculation agree with the log divergences in the bulk calculation.

A simplest example is to take CFT × CFT wormhole, which is the timeslice of an eternal AdS black hole with bifurcate horizon. Then, if we take $\mathcal{R}$ to be the CFT on one side, the minimal area surface will be the throat of the wormhole (which satisfies the homology condition), with the same area as the bifurcate surface — the entanglement entropy of the two copies of CFTs is the black hole entropy! [Need figure 30: BH example here.]

More Examples of AdS/CFT

Some more examples are summarised in the following table.

d	CFT	Bulk	S strong	S weak	Comments
2	D1-D5	$\mathrm{AdS}_3 \times S_3 \times T_4$ (IIB)	c	c	c protected
3	ABJM	$\mathrm{AdS}_4 \times S_7$ (M)	$N^{3/2}$ (IR)	N^2 (UV)	-
4	$\mathcal{N} = 4$ SYM	$\mathrm{AdS}_5 \times S_5$ (IIB)	N^2	N^2	c, a protected
6	$(2,0)$ model	$\mathrm{AdS}_7 \times S_4$ (M)	N^3 (UV)[6]	N^2 (IR)	-

The evidences are from universal pieces, consistency conditions (e.g. strong subadditivity) and Lewkowycz-Maldacena path integral [15]. For more information, see section 10.

7.7 Phase Transitions

There can exist *multiple* local minima, e.g. consider region $\mathcal{R}$ consisting of 2 disjoint intervals for $d = 2$. Let each be angle θ wide. [Need figure 31: Multiple local minima here.]

We can see the entropy has discontinuous first derivative $\partial S/\partial\theta$. This suggests a phase transition, which is sharp at large N by $O(N^2)$ but smoothed out at finite N.

Another picture of this related to an open string that one end lives in A but the other lives in B. The local Rindler temperature leads to deconfinement of the two ends of a string: for long intervals (large θ) $I_{A,B} \sim O(N^2)$ and for short intervals (small θ) $I_{A,B} \sim O(1)$. [Need figure 32: String Deconfinement here.]

Similar story happens for a black hole with a single interval. The homology constraint is crucial here. [Need figure 33: BH Phase Transition here.] This is also related to deconfinement phase transition: [Need figure 34: BH Deconfinement here.]

Phase transition makes it difficult to resolve sub-AdS structure with RT surfaces.

7.8 Proof of Strong Subadditivity

The strong subadditivity of entanglement entropy is very difficult to prove in Quantum Information Theory. However, it is easy using holography. We draw [Need figure 35: SSA here.] and

$$S(AB) + S(BC) \geq S(ABC) + S(B) \tag{111}$$

[6] Proportional to $1/G + O(1)$ corrections.

is proved by using global minimisation. Also it is important that

$$S(AB) + S(BC) \geq S(A) + S(C) \tag{112}$$

is checked for consistency.

Similarly, using holography, *monogamy of mutual information* (Hayden-Headrick-Maloney) is shown [16]:

$$S(AB) + S(BC) + S(CA) \geq S(A) + S(B) + S(C) + S(ABC). \tag{113}$$

This is true only holographically and can be violated for general QM systems.

7.9 Covariant Version

The covariant generalisation of RT formula is the *Hubeny-Rangamani-Takayanagi proposal* (HRT) [] which is needed when spacetime is dynamical or $\mathcal{E}$ is time dependent. The difficulty of time dependence is that minimum surface makes no sense in (Lorentzian) spacetime and we cannot simply Wick rotate to Euclidean signature to simplify the problem. Hence we generalise by looking for *extremal surface* so the entanglement entropy is

$$S_{\text{ent}} = \min_{\gamma} \operatorname{ext} \frac{\text{Area}[\gamma]}{4G\hbar} \tag{114}$$

and we still require the homology condition as in RT formula. [Need figure 36: HRT here.]

Equivalently, we have the *maximin formulation* []:

#1 On each Cauchy slice Σ that passes throught $\mathcal{E}$, we choose a minimum area surface.

#2 We choose Σ to *maximise* the value of this minimum area.

[Need figure 37: Maximin here.] The logic of equivalence is

$$\begin{gathered}\text{null curvature condition } (=\text{NEC}^7)\ R_{ab}k^a k^b \geq 0 + \text{AdS hyperbolicity} \\ \Downarrow \\ \text{maximin}(\mathcal{R}) = \min \operatorname{ext}(\mathcal{R}).\end{gathered}$$

The maximin formulation is easier to prove the existence and global results such as strong subadditivity and monogamy of mutual information, etc. These are violated if NEC is violated.

We can also prove that if $B \supset A$, the extremal surface lies deeper in the bulk. [Need figure 38: $B \supset A$ here.]

8 HKLL Bulk Reconstruction

Step 1

We want to reconstruct the bulk using a formula like

$$\varphi_{\text{bulk}}(z, x) = \int \mathrm{d}^d x' \, K(x'|z, x)\mathcal{O}(x') \tag{115}$$

[7] Null energy condition.

where $K(x'|z,x)$ is the kernel. We can obtain K by solving the non-standard sideways Cauchy problem (in AdS-Poincaré) of equation

$$\left(\partial_z^2 - \frac{\tilde{m}^2}{z^2} + \Box^{(d)}\right)\tilde{\varphi} = 0 \tag{116}$$

with $\tilde{\varphi} = z^{(d-1)/2}\varphi$. We want $\varphi(z>0)$ to be *uniquely* determined by $z=0$. Does this work? The differential equation looks *hyperbolic* in t-z plane but *hyperbolic* in t-$\mathbf{x}$ plane. The trick to first use $\mathbf{x}$ translation symmetry to expand $\varphi, \mathcal{O}$ in plane waves

$$\varphi_{\mathbf{p}}(z,t) = \int \mathrm{d}^{d-1}\mathbf{x}\, e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\varphi(z,t,\mathbf{x}) \tag{117}$$

which obeys the 1+1 Lorentzian equation. It also propagates causally if roles of z,t are reversed (with exponentially growing modes like tachyons, but that does not stop it from being well-defined). We then obtain

$$\varphi_{\mathbf{p}}(z,t) = \int \mathrm{d}t'\, K_{\mathbf{p}}(t'|z,t)\mathcal{O}_{\mathbf{p}}(t'). \tag{118}$$

[Need figure 39: Sideways evolution here.]

Note: $K_{\mathbf{p}}$ blows up at large spacelike $\mathbf{p}$, but this is okay since $\mathcal{O}_{\mathbf{p}_i}$ vanishes for spacelike $\mathbf{p}_i$ and $K(x'|z,x)$ is distributional.

Step 2

Evolve $\mathcal{O}(t,\mathbf{x})$ back to one Cauchy slice by

$$\mathcal{O}(t,\mathbf{x}) = e^{\mathrm{i}Ht}\mathcal{O}(0,\mathbf{x})e^{-\mathrm{i}Ht} \tag{119}$$

[Need figure 40: Back Evolution here.]

That is, we can evolve the operator back to some preferred moment in time corresponding to a single Cauchy slice on the boundary. Having behavior in the bulk determined by a codimension-1 surface is standard field theory. Having behavior in the bulk determined by a codimension-2 surface is surprising. That's the *holographic principle*. Initial data on the boundary not only predicts time evolution on the boundary but the entire interior of the bulk.

In the complete AdS (tin can) picture, in order to reconstruct the bulk field at some point, we need data from a cylindrical chunk of AdS corresponding to how long it takes for null rays to reach the boundary. We can do this, provided that we sum over spherical harmonics (it suffices to add over s-waves). [Need figure 41: Spherical AdS here.] We could also look at the Rindler patch of AdS, in which case we could determine a bulk field just from the intersection of its "light cone" with the boundary in one direction. [Need figure 42: Rindler here.]

We can also include interactions perturbatively in $1/N$ expansion by using spacelike Green's functions (not the same as Witten diagrams). [Need figure 43: Green's function here.]

There is some subtlety with gauge freedom, for example, diffeomorphisms in the bulk. They are allowed as long as they vanish on the boundary to be gauge freedom.

Now the questions is: $\varphi(z,x)$ is coordinate dependent, how can it be dual to a well-defined $\mathcal{O}$ integral? The answer is: we must use gauge-fixing (e.g. Fefferman-Graham gauge) to specify how φ is related to boundary. The boundary data creates φ together with a gravitational field line (like a Wilson line for charge), equivalent to non-local diffeomorphism invariant operator. [Need figure 44: φ with grav line here.]

What can we reconstruct from a general CFT region or bulk region? We first define the *causal wedge* C_W: take a slice $\mathcal{R}$ on the boundary and find its domain of dependence $D[\mathcal{R}]$. Then determine the intersection of its causal future and past within the bulk, i.e.

$$C_W = I^-(D[\mathcal{R}]) \cap I^+(D[\mathcal{R}]). \tag{120}$$

Causality says you can get *at most* C_W from local fields on $D[\mathcal{R}]$. [Need figure 45: Causal Wedge here.]

Holmgrem's uniqueness theorem implies C_W is uniquely specified by boundary data if the bulk background metric or sources are analytic. In practice it is often assumed that the conclusion holds more generally[8]. There are a few known counterexamples for *unphysical* EOM e.g. $\Box\phi = f(z,x)\phi$ with f complex and decays sufficiently quickly near the boundary, in which case a geometric optics approach shows that the causal wedge cannot be completely reconstructed.

Moreover, it turns out non-local operators let us reconstruct a bigger region known as the *entanglement wedge* that goes up to the HRT surface

$$E_W = D[\Sigma_{\mathcal{R}\to\gamma}]. \tag{121}$$

[Need figure 46: EW vs CW here.] Generically, HRT lies deeper in the bulk but the E_W and C_W can coincide in very symmetric situations.

9 Proof of RT and Entanglement Wedge Duality

9.1 Path Integral Derivation of RT

The original derivation is by Lewkowycz and Maldacena [15].

Step 1

Start with replica trick on boundary [Need figure 47: Replica here.]

$$S = \lim_{n\to 1} S_n, \quad S_n = \frac{1}{1-n}\ln \mathrm{tr}(\rho^n) \tag{122}$$

where $\mathrm{tr}(\rho^n) = \mathcal{Z}_n/\mathcal{Z}_1^n$ so

$$S = \partial_n(\ln \mathcal{Z}_n - n\ln \mathcal{Z}_1)\Big|_{n=1}. \tag{123}$$

It is generally hard to analytically continue to non-integer n.

Step 2

Find *smooth* instanton interior bulk and *assume* the $\mathbb{Z}_n$ symmetry is not spontaneously broken. So there exists codimension-2 locus L of fixed points in bulk (by continuity from boundary). The end points of interval are fixed points of the $\mathbb{Z}_n$ *replica symmetry* that cycles through the n copies. [Need figure 48: Replica Manifold here.] Using holographic principle and take large N limit, we can use classical saddle point approximation in the bulk to take

$$\ln \mathcal{Z}_n = -I_{\mathrm{grav}}[\mathcal{M}_n]. \tag{124}$$

[8] This may be an interesting research project.

Step 3

Construct orbifold solution by identifying points related by $\mathbb{Z}_n$ symmetry:

$$O_n = \mathcal{M}_n/\mathbb{Z}_n. \tag{125}$$

The fixed points now have $\beta_{(k)} = 2\pi/n$ conical singularity, i.e. one can go around the fixed points in the bulk by travelling an angle less than 2π. [Need figure 49: Orbifold here.]

Define

$$\tilde{I}[O_n] = \frac{I_{\text{grav}}[\mathcal{M}_n]}{n} \tag{126}$$

and we note that there is *no* contribution from the tip of the cone — O_n is fictitious, $\mathcal{M}_n$ is the real solution. Yet, O_n solves an enhanced action

$$I = \underbrace{\tilde{I}[O_n] + I_{\text{tip}}}_{I_{\text{grav}}[O_n]} + I_{\text{brane}} \tag{127}$$

where

$$I_{\text{brane}} = (1-n)\frac{\text{Area}[L]}{4G} = -I_{\text{tip}}. \tag{128}$$

Step 4

Finally, we take the limit $n \to 1$ as O_n can be smoothly continued to all n. Since $\mathcal{M}_1 = O_1$ solves EOM for I_{grav} variation, $O_{1+\epsilon}$ has the same action to first order

$$I_{\text{grav}}[O_{1+\epsilon}] = I_{\text{grav}}[O_1] + \mathcal{O}(\epsilon^2). \tag{129}$$

Then at $n = 1$

$$0 = \partial_n I_{\text{grav}} = \partial_n \tilde{I} + \partial_n I_{\text{tip}} = \partial_n \tilde{I} - \partial_n I_{\text{brane}} \tag{130}$$

hence we can replace $\tilde{I}$ variation with I_{brane} variation. After a little algebra

$$S = \frac{\text{Area}[X]}{4G} \tag{131}$$

for X extremal (solves I_{brane}). There is no backreaction as $n \to 1$.

9.2 Quantum Corrections to Holographic Entanglement Entropy

The original derivation is by Faulkner, Lewkowycz and Maldacena [17].

RT only gives the leading order term in S_{CFT}, e.g. $\mathcal{O}(N^2)$ in 4d SYM. $\mathcal{O}(1) \sim \hbar$ term corresponds to loop corrections in the bulk. The problem here is $\ln \mathcal{Z}^{\text{loop}}_{\text{bulk}}$ is not local so there is no simple relation such as

$$\ln \mathcal{Z}[O_n] = \frac{\ln \mathcal{Z}[\mathcal{M}_n]}{n}. \tag{132}$$

Solution: we just evaluate QFT on $\mathcal{M}_n$ by doing QFT in curved spacetime. Now tautologically

$$\mathcal{Z}^{\text{loop}}[\mathcal{M}_n] = \mathcal{Z}^{\text{loop}}_n[O_n] \tag{133}$$

where $\mathcal{Z}_n$ is the partition function on n-replicated manifold (but this is related to Rényi entropy $S_n[O_n]$).

$$\begin{aligned} S &= \partial_n (\ln \mathcal{Z}_n[O_n] - n \ln \mathcal{Z}_1[O_1])\Big|_{n=1} \\ &= \underbrace{\partial_n \left(\ln \mathcal{Z}_n[\mathcal{M}_1] - n \ln \mathcal{Z}_1[\mathcal{M}_1]\right)}_{S_{\text{bulk}}} - \partial_n \ln \mathcal{Z}_1[O_n]\Big|_{n=1} \end{aligned} \tag{134}$$

where the last term is present when background metric is no longer a stationary point of the action when we turn on quantum fields. To deal with the last term, we also need to evaluate the gravitational backreaction of QFT on the geometry which can affect Area$[X]$ directly. The end result of FLM is

$$S_{\text{CFT}} = \frac{\langle \text{Area[X]} \rangle}{4G\hbar} + S_{\text{bulk}}[X] + \int_X \text{local counter terms} =: S_{\text{gen}}[X] \tag{135}$$

where the divergences of entanglement entropy are absorbed into $1/G$ and subleading counter terms. $S_{\text{gen}}[X]$ increases on BH horizons. [Need figure 50: S gen here.] For higher order corrections, X should extremise S_{gen}, not its area.

9.3 Modular Hamiltonian

Let us now define the *modular Hamiltonian*

$$K^{(\sigma)} = -\ln \sigma \tag{136}$$

for some state σ. For any variation $\rho = \sigma + \Delta\rho$,

$$S(\rho) = \left\langle K^{(\sigma)} \right\rangle_\rho \tag{137}$$

to first order in $\Delta\rho$. This is sometimes known as the first law of entanglement entropy, it is like the Clausius relation in thermodynamics.

By varying FLM we get the JLMS relation []:

$$K^{(\sigma)}_{\text{CFT}} = \frac{\text{Area}[X]}{4G\hbar} + K^{(\sigma)}_{\text{bulk}} + \text{local counter terms.} \tag{138}$$

For any ρ with the same classical geometry as σ, since K and Area are linear operators, this does *not* require $\Delta\rho$ to be small! It holds as an operator equation.

This further implies the *relative entropy* agrees:

$$S(\rho|\sigma)^{\mathcal{R}}_{\text{CFT}} = S(\rho|\sigma)^{E_W[\mathcal{R}]}_{\text{bulk}} \tag{139}$$

where

$$S(\rho|\sigma) := \text{tr}(\rho \ln \rho) - \text{tr}(\rho \ln \sigma) = \left\langle K^{(\sigma)} \right\rangle_\rho - S(\rho) \geq 0 \tag{140}$$

and the equality holds iff $\rho = \sigma$, measuring the distinguishability between states.

9.4 Bulk Reconstruction Theorem

Theorem by Dong, Harlow and Wall [].

Let $\mathcal{H}_{\text{CFT}} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, $\mathcal{H}_{\text{code}} = \mathcal{H}_a \otimes \mathcal{H}_{\bar{a}}$, $\mathcal{H}_{\text{code}} \subset \mathcal{H}_{\text{CFT}}$ and $\rho_{\bar{a}} = \sigma_{\bar{a}}$

9.5 BH from Collapse and Information Paradox

10 Examples of AdS/CFT

From now on, D is the *total* bulk dimension.

10.1 p-form Potentials

For a p-form potential $A^{(p)}_{a_1 a_2 \cdots a_p}$ we can derive the $(p+1)$-curvature by

$$F^{(p+1)} = \mathrm{d}A^{(p)} \,. \tag{141}$$

Taking the Hodge dual of $F^{(p+1)}$, a $(D-p-1)$-curvature is constructed

$$\star\, F^{(p+1)} = \tilde{F}^{(D-p-1)} = \mathrm{d}\tilde{A}^{(D-p-2)} \tag{142}$$

so p-form potential is dual to $(D-p-2)$-form potential. These couples to branes with $(p-1)$ or $(D-p-3)$ spatial dimensions (more on branes later).

Self-dual potentials exist only when $D = 4k+2, k \in \mathbb{N}$ in Lorentzian signature as for odd D there is no $p = D-p-2$ and for $D = 4k$, $\star\star F = -F$.

10.2 Crash Course on Branes*

Under Construction.

10.3 Stack of N Coincident D3 Branes*

Under Construction.

10.4 Unit Matching*

Under Construction.

10.5 Some Evidence for Duality*

Under Construction.

10.6 Other Versions of AdS/CFT*

Under Construction.

— *The End* —

References

[1] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] R. Sundrum, *From Fixed Points to the Fifth Dimension*, *Phys. Rev. D* **86** (2012) 085025 [1106.4501].

[3] G.T. Horowitz and J. Polchinski, *Gauge/gravity duality*, gr-qc/0602037.

[4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[5] V.E. Hubeny, *The AdS/CFT Correspondence*, *Class. Quant. Grav.* **32** (2015) 124010 [1501.00007].

[6] M. Van Raamsdonk, *Lectures on Gravity and Entanglement*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings*, pp. 297–351, 2017, DOI [1609.00026].

[7] D. Harlow, *TASI Lectures on the Emergence of Bulk Physics in AdS/CFT*, *PoS* **TASI2017** (2018) 002 [1802.01040].

[8] T. Hartman, *Lectures on Quantum Gravity and Black Holes*, 2018, http://www.hartmanhep.net/topics2015/gravity-lectures.pdf.

[9] P.H. Ginsparg, *APPLIED CONFORMAL FIELD THEORY*, in *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena*, 9, 1988 [hep-th/9108028].

[10] S. Rychkov, *EPFL Lectures on Conformal Field Theory in $D \geq 3$ Dimensions*, Springer Briefs in Physics (1, 2016), 10.1007/978-3-319-43626-5, [1601.05000].

[11] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics, Springer-Verlag, New York (1997), 10.1007/978-1-4612-2256-9.

[12] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[13] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [hep-th/9802109].

[14] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [hep-th/0002230].

[15] A. Lewkowycz and J. Maldacena, *Generalized gravitational entropy*, *JHEP* **08** (2013) 090 [1304.4926].

[16] P. Hayden, M. Headrick and A. Maloney, *Holographic Mutual Information is Monogamous*, *Phys. Rev. D* **87** (2013) 046003 [1107.2940].

[17] T. Faulkner, A. Lewkowycz and J. Maldacena, *Quantum corrections to holographic entanglement entropy*, *JHEP* **11** (2013) 074 [1307.2892].