Part III – **Statistical Field Theory**

Based on Lectures by *C. Thomas*
Notes taken by Zihan Yan

These notes may not reflect the full format and content that are actually lectured. I usually modify the notes after the lectures and sometimes my own thinking or interpretation might be blended in. Any mistake or typo should surely be mine. Be cautious if you are using this for self-study or revision.

Course Information

This course introduces the renormalization group, focusing on statistical systems such as spin models with further connections to quantum field theory.

After introducing the Ising Model, Landau's mean field theory is introduced and used to describe phase transitions. The extension to the Landau-Ginzburg theory reveals broader aspects of fluctuations whilst consolidating connections to quantum field theory. At second order phase transitions, also known as 'critical points', renormalisation group methods play a starring role. Ideas such as scaling, critical exponents and anomalous dimensions are developed and applied to a number of different systems.

Pre-requisites

Background knowledge of Statistical Mechanics at an undergraduate level is essential. This course complements the Quantum Field Theory and Advanced Quantum Field Theory courses.

Overview

1. Phase Transitions (discontinuous, sudden change of properties)

2. Critical Points, e.g.

$$C \sim \frac{1}{|T - T_C|^{0.11008...}}$$

3. Universality:

- Symmetry
- Physics of different length scales, Renormalisation Group

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1   FROM SPIN TO FIELDS

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1.1   The Ising Model

Think about a simple model for a magnet. Consider a lattice in d spatial dimensions with N sites. On each site $i = 1, \ldots, N$ there is a discrete variable, called 'spin', which takes values:

$$S_i = +1 = \uparrow$$
$$S_i = -1 = \downarrow$$

[Need figure Ising Lattice here.]

The system of spins $\{S_i\}$ has energy

$$E = -B \sum_i S_i - J \sum_{\langle ij \rangle} S_i S_j$$

where $\langle ij \rangle$ = nearest neighbour pairs, B is an external magnetic field:

$$B > 0 \qquad \Rightarrow \qquad \text{spins prefer } \uparrow$$
$$B < 0 \qquad \Rightarrow \qquad \text{spins prefer } \downarrow$$

$$J > 0 \qquad \Rightarrow \qquad \text{spins want to align}$$
$$J < 0 \qquad \Rightarrow \qquad \text{spins want to anti-align}$$

and we call $\uparrow\uparrow$ ferromagnet and $\downarrow\uparrow$ anti-ferromagnet.

In the scope of this course, we take $J > 0$.

For a finite temperature T, spins want to randomise to increase entropy.

For canonical ensemble, the probability of configuration $\{S_i\}$ is

$$P[S_i] = \frac{e^{-\beta E[S_i]}}{Z}$$

where $Z(T, J, B) = \sum_{\{S_i\}} e^{-\beta E[S_i]}$ is the *partition function*, $\beta = 1/T$ and we choose units such that $k_B = 1$.

All physics can be extracted from Z.

> EXAMPLE 1.1.
>
> - Thermodynamic free energy $F_{\text{thermo}}(T, J, B) = \langle E \rangle - TS = -T \ln Z$. For this we have
>
> $$dF = -S \, dT - p \, dV - M \, dB$$
>
> $$-S = \left. \frac{\partial F}{\partial T} \right|_{V,B} \qquad -p = \left. \frac{\partial F}{\partial V} \right|_{T,B} \qquad -M = \left. \frac{\partial F}{\partial B} \right|_{T,V}$$

- Equilibrium magnetisation

$$[-1, +1] \ni m = \frac{1}{N} \sum_i \langle S_i \rangle = \frac{1}{N} \sum_{\{S_i\}} \frac{e^{-\beta E[S_i]}}{Z} \sum_i S_i = \frac{1}{N\beta} \frac{\partial \ln Z}{\partial B}$$

We want to understand how this changes as we vary T and B (equation of state).

All we need to do is compute Z. But this is very difficult and sometimes impossible. We need a different approach...

Effective Free Energy

We rewrite Z as

$$Z = \sum_m \sum_{\{S_i\}|m} e^{-\beta E[S_i]} := \sum_m e^{-\beta F(m)}$$

where the expression $\{S_i\}|m$ means we choose $\{S_i\}$ such that $m = \frac{1}{N} \sum_i S_i$.

For large $N (N \approx 10^{23})$, m is essentially continuous, we can write

$$Z = \frac{N}{2} \int_{-1}^{+1} dm \, e^{-\beta F(m)}$$

where $N/2$ is there to make the normalisation correct.

It is easy to see that effective free energy $F(m, T, J, B)$ contains more information than F_{thermo}.

We can determine the equilibrium value of m from it. Define

$$f(m) := \frac{F(m)}{N} \qquad \text{such that} \qquad Z = \frac{N}{2} \int_{-1}^{+1} dm \, e^{\beta N f(m)}$$

where N is large and $\beta f(m)$ is of order 1.

The above integral is dominated by the minimum value of $f(m)$ where $m = m_{\text{min}}$ at $\left. \frac{\partial f}{\partial m} \right|_{m=m_{\text{min}}} = 0$. This is known as the saddle point/steepest descent technique. With this approximation, we have

$$Z \approx e^{-\beta N f(m_{\text{min}})} \qquad \text{so} \qquad F_{\text{thermo}} \approx F(m_{\text{min}})$$

We want to compute $f(m)$ but this is also hard.

Mean Field 'Approximation'

For each $\{S_i\}$ s.t. $\sum_i S_i = Nm$, we guess that the energy is approximately

$$E \approx -B \sum_i m - J \sum_{\langle ij \rangle} m^2$$

$$= -BNm - \frac{1}{2} N J q m^2$$

where the factor of half comes from the fact that we are summing over pairs but not sites, and q is the number of nearest neighbours. (E.g. $q = 2$ for $d = 1$, $q = 4$ for $d = 2$ square lattice, $q = 6$ for $d = 3$ simple cubic lattice.)

Now we just need to count the number of configurations. If there are $N_\uparrow$ up spins and $N_\downarrow = N - N_\uparrow$ down spins, we have

$$m = \frac{N_\uparrow - N_\downarrow}{N} = \frac{2N_\uparrow - N}{N}$$

and the number of configurations is

$$\Omega = \frac{N!}{N_\uparrow!(N - N_\downarrow)!}$$

Using Stirling's formula

$$\ln \Omega \approx N \ln N - N_\uparrow \ln N_\uparrow - (N - N_\uparrow)\ln(N - N_\uparrow) - N + N_\uparrow + N - N_\uparrow + \cdots$$

so that

$$\frac{1}{N}\ln \Omega = \ln 2 - \frac{1}{2}(1 + m)\ln(1 + m) - \frac{1}{2}(1 - m)\ln(1 - m)$$

Therefore, in the mean field approximation, we have

$$e_{-\beta N f(m)} = \sum_{\{S_i\}|m} e^{-\beta E[S_i]}$$

$$\approx \Omega(m)e^{-\beta E(m)}$$

$$-\beta N f(m) \approx \ln \Omega(m) - \beta E(m)$$

Then we have

$$f(m) \approx -Bm - \frac{1}{2}Jqm^2 - \frac{1}{\beta}\left[\ln 2 - \frac{1}{2}(1 + m)\ln(1 + m) - \frac{1}{2}(1 - m)\ln(1 - m)\right]$$

Then to get the minimum value of m,

$$\frac{\partial f}{\partial m} = 0 \qquad \Rightarrow \qquad \beta(B + Jqm) = \frac{1}{2}\ln\left(\frac{1 + m}{1 - m}\right)$$

and rearrange, we get

$$m = \tanh[\beta(B + Jqm)]$$

We can think about this as if

$$B \to B_{\text{eff}} = B + Jqm$$

as the background magnetic field seen from some site can be seen as modified by the magnetisation of neighbouring sites.

1.2   Landau Theory of Phase Transitions

Rec 1-2-1
No-Revise
A phase transition occurs when some quantity changes discontinuously. For us this *order parameter* is m.

Landau theory is a simple yet effective way to understand phase transitions qualitatively. Based on free energy and symmetry.

Recall the free energy for the Ising model in the mean field approximation

$$f(m) \approx -Bm - \frac{1}{2}Jqm^2 - \frac{1}{\beta}\left[\ln 2 - \frac{1}{2}(1+m)\ln(1+m) - \frac{1}{2}(1-m)\ln(1-m)\right]$$

Taylor expand for small m and discarding the constant term (as it is not important), we get

$$f(m) \approx -Bm + \frac{1}{2}\left(\frac{1}{\beta} - Jq\right)m^2 + \frac{1}{12}\frac{1}{\beta}m^4 + \cdots$$

The equilibrium value of m is $m_{\min}$. Landau theory investigates how $m_{\min}$ changes as we vary the parameters.

1.2.1 $B = 0$

$$f(m) \approx \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4 + \cdots$$

where we denoted the *critical temperature* $T_c = Jq$.

[Need figure 2 here.]

At high T, we have $m = 0$ as the minimum. For low T, specifically $T < T_c$, we have the minima

$$m = \pm m_0 = \pm\sqrt{\frac{3(T_c - T)}{T}}$$

and note that this holds only when close to T_c where m is small.

If we plot the phase diagram [Need figure 3 here.], we find m turns on abruptly as we lower T. Here, m is continuous, so this is called a *continuous phase transition* (also known as a 2nd order phase transition).

There are two phases:

- $m = 0$ is the *disordered phase*;

- $m \neq 0$ is the *ordered phase*.

$f(m)$ is invariant under a $\mathbb{Z}_2$ symmetry $m \to -m$, which is inherited from the $S_i \to -S_i$ symmetry of the Ising model when $B = 0$.

For $T < T_c$, the system chooses one of the two states $m = \pm m_0$, i.e. the $\mathbb{Z}_2$ symmetry is *spontaneously broken*.

Consider the heat capacity at constant $B = 0$ (this is a response function):

$$C = \frac{\partial \langle E \rangle}{\partial T} = \beta^2 \frac{\partial^2}{\partial \beta^2} \ln Z$$

Some algebra gives

$$f(m_{\min}) = \begin{cases} 0 & T > T_c \\ -\dfrac{3}{4}\dfrac{(T_c - T)^2}{T} & T < T_c \end{cases}$$

Using $\ln Z \approx -\beta N f(m_{\min})$ gives

$$c = \frac{C}{N} \to \begin{cases} 0 & T \to T_c^+ \\ 3/2 & T \to T_c^- \end{cases}$$

This means c jumps discontinuously.

[Need figure 4 here.]

1.2.2 $B \neq 0$

$$f(m) \approx -Bm + \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4 + \cdots$$

For high T, we have a unique minimum. For low T, we have two minima: one meta-stable, one true minimum.

[Need figure 5 here.]

As we vary T, the magnetisation varies smoothly

$$m \to \frac{B}{T} \quad \text{as} \quad T \to \infty$$

and

$$m \to \pm 1 \quad \text{as} \quad T \to 0$$

But now the sign of the magnetic field determines the sign of m. Thus no phase transition as we vary T.

[Need figure 6 here.]

If instead we fix T (low T) and vary B, as B passes through 0, m jumps discontinuously from $m = -m_0$ to $m = +m_0$ — this is a *first-order phase transition*.

[Need figure 7 here.]

The phase diagram is shown below.

[Need figure 8 here.]

What happens close to the critical point? If we are right at the critical point,

$$f(m) \approx -Bm + \frac{1}{12}T_c m^4 + \cdots$$

which gives

$$m \sim B^{1/3} \quad (B > 0)$$

Consider magnetic susceptibility $\chi = \frac{\partial m}{\partial B}\big|_T$, which is another response function. Then for $T > T_c$,

$$f(m) \approx -Bm + \frac{1}{2}(T - T_c)m^2 + \cdots \quad \text{and} \quad m \approx \frac{B}{T - T_c}$$

so

$$\chi \sim \frac{1}{T - T_c} \quad \text{as} \quad T \to T_c^+$$

For $T < T_c$, similarly,

$$\chi \sim \frac{1}{|T - T_c|} \quad \text{as} \quad T \to T_c^-$$

1.2.3 Validity of Mean Field Theory

Are these results correct?

- $d = 1$: No. MFT is completely wrong! No phase transitions;

- $d = 2, 3$: Basic structure of phase diagram is right, details near T_c are wrong;

- $d \geq 4$: MFT gives right answers.

Similar story for other systems:

- MFT fails completely for $d \leq d_l$, the *lower critical dimension*;

- MFT always works for $d \geq d_u$, the *upper critical dimension*;

- What about $d_l < d < d_u$? Often these d are interesting.

In Ising model, $d_l = 1$ and $d_u = 4$.

1.2.4 Critical Exponents

We've seen that various quantities scale as power laws as we approach the critical point.

$$m \sim (T_c - T)^\beta \qquad \beta = 1/2 \quad (T < T_c)$$
$$c \sim c_\pm |T - T_c|^{-\alpha} \qquad \alpha = 0 \text{ (discontinuity)}$$
$$\chi \sim |T - T_c|^{-\gamma} \qquad \gamma = 1$$
$$m \sim B^{1/\delta} \qquad \delta = 3$$

Compare MFT results with true values for Ising model in $d = 2, 3$, we have

	MFT	$d = 2$	$d = 3$
α	0 (disc.)	0 (log div.)	0.1101...
β	1/2	1/8	0.3264...
γ	1	7/4	1.2371...
δ	3	15	4.7898...

1.2.5 Universality

Liquid-gas transition shows many similarities to the Ising model. [Need figure 9 here.]

Again, a line of first-order phase transitions ending at a critical point.

At fixed p, $v = V/N$ jumps discontinuously at liquid-gas first-order phase transition. Suggests v is order parameter analogous to m in Ising model.

Using an equation of state (e.g. van der Waals equation), at the critical point:

$$v_\text{gas} - v_\text{liquid} \sim (T_c - T)^\beta \qquad \beta = 1/2 \quad (T < T_c)$$
$$v_\text{gas} - v_\text{liquid} \sim (p - p_c)^{1/\delta} \qquad \delta = 3 \quad (p \to p_c^+)$$
$$\text{Compressibility, } \kappa = -\frac{1}{v}\frac{\partial v}{\partial p}\bigg|_T \sim |T - T_c|^{-\gamma} \qquad \gamma = 1$$
$$\text{Heat capacity, } C_V \sim C_\pm |T - T_c|^{-\alpha} \qquad \alpha = 0$$

None of these agree with experiment.

All liquid-gas transition show the same critical exponents, and these agree with $d = 3$ Ising.

Universality: critical point governs behaviour of many different physical systems. Why?!

1.2.6 Ising Model as Lattice Gas

Consider the same d-dimensional lattice, but now with particles hopping between sites.

The number of particles on site is n_i, and we allow no more than one particle on each site, i.e. either $n_i = 0$ (empty) or $n_i = 1$ (filled). Then we have

$$E = -4J \sum_{\langle ij \rangle} n_i n_j - \mu \sum_i n_i$$

where $J > 0$ is acting as an attractive force and μ is the chemical potential.

This is equivalent to the Ising model if we identify $S_i = 2n_i - 1$.

1.3 Landau-Ginzburg Theory

Rec 1-3
No-Revise

The Landau-Ginzberg (LG) approach is an extension of Landau's idea that, treated correctly, gives the correct physics.

We now allow m to vary in space: $m \to m(\mathbf{x})$. It is now a field.

Start with a lattice with N sites, and divide into boxes each with N' sites such that $1 \ll N' \ll N$.

[Need figure 10 here.]

For each box

$$m(\mathbf{x}) = \frac{1}{N'} \sum_i S_i$$

where $\mathbf{x}$ is the centre of the box and the sum is only over the sites in box.

This is a *coarse graining*. We require

- N' is large enough so $m \in [-1, +1]$ is essentially continuous;

- Box side length $a \ll$ other relevant scales so $\mathbf{x}$ can be treated as continuous. But we must remember that m cannot vary on scales $< a$.

Write the partition function as

$$Z = \sum_{m(\mathbf{x})} \sum_{\{S_i\}|m(\mathbf{x})} e^{-\beta E[S_i]} := \sum_{m(\mathbf{x})} e^{-\beta F[m(\mathbf{x})]}$$

where the summation is over all $\{S_i\}$ such that coarse graining gives $m(\mathbf{x})$ and we defined the free energy $F[m(\mathbf{x})]$ to be a functional (that takes a function $m(\mathbf{x})$ and gives a number).

Now sum is over all possible fields $m(\mathbf{x})$.

We write Z as a *functional integral*, also known as a *path integral*.

$$Z = \int \mathcal{D}m(\mathbf{x}) e^{-\beta F[m(\mathbf{x})]}$$

The probability of field configuration $m(\mathbf{x})$ is

$$P[m(\mathbf{x})] = \frac{e^{-\beta F[m(\mathbf{x})]}}{Z}.$$

1.3.1 Free Energy Functional

What is the free energy functional $F[m(\mathbf{x})]$?

It must have the following properties:

- **Locality**: in the Ising model spins only affect their neighbours

$$F[m(\mathbf{x})] = \int \mathrm{d}^d\mathbf{x}\, f[m(\mathbf{x})]$$

  where $f[m(\mathbf{x})]$ depends only on $m(\mathbf{x}), \boldsymbol{\nabla}m(\mathbf{x}), \nabla^2 m(\mathbf{x}), \cdots$;

- **Translation and rotation symmetry**: inherited from discrete versions of these symmetries on the lattice;

- A $\mathbb{Z}_2$ **symmetry** when $B = 0$: $m(\mathbf{x}) \to -m(\mathbf{x})$ and when $B \neq 0$ there's a symmetry $B \to -B, m(\mathbf{x}) \to -m(\mathbf{x})$;

- **Analyticity**: we are interested in the critical point when m is small. Assume $f[m(\mathbf{x})]$ has a Taylor series expansion;

- We are interested in $m(\mathbf{x})$ that are slowly varying in space, e.g. ∇m terms are more important than $a\nabla^2 m$ terms, etc.

Most general free energy when $B = 0$ writes

$$F[m(\mathbf{x})] = \int \mathrm{d}^d\mathbf{x} \left[\frac{1}{2}\alpha_2(T)m^2 + \frac{1}{4}\alpha_4(T)m^4 + \frac{1}{2}\gamma(T)(\nabla m)^2 + \cdots \right].$$

Note that we only have even powers in m as odd powers break the $\mathbb{Z}_2$ symmetry. Similar for ∇m terms as odd ones kills the rotational symmetry off.

When $B \neq 0$ we can have Bm, Bm^3, etc. terms.

It's hard to compute coefficients $\alpha_2(T), \alpha_4(T), \gamma(T)$ from first principles. From MFT we expect

$$\alpha_2(T) \sim (T - T_c),$$
$$\alpha_4(T) \sim \frac{1}{3}T.$$

But we only need

$$\alpha_4(T), \gamma(T) > 0 \quad \text{and} \quad \alpha_2(T) \text{ changes sign at } T = T_c.$$

1.3.2 Saddle Point

For the path integral

$$Z = \int \mathcal{D}m(\mathbf{x})e^{-\beta F[m(\mathbf{x})]},$$

our first guess is to assume integral is dominated by the saddle point.

Consider $m(\mathbf{x}) + \delta m(\mathbf{x})$, we get

$$\delta F = \int \mathrm{d}^d\mathbf{x} \left[\alpha_2 m \, \delta m + \alpha_4 m^3 \, \delta m + \gamma(\boldsymbol{\nabla}m) \cdot (\boldsymbol{\nabla}\delta m) + \mathcal{O}((\delta m)^2)\right]$$

$$= \int \mathrm{d}^d\mathbf{x} \left[\alpha_2 m + \alpha_4 m^3 - \gamma\nabla^2 m\right](\delta m) + \cdots \quad \text{[integrated by parts]}$$

This has minimum when

$$\gamma\nabla^2 m = \alpha_2 m + \alpha_4 m^3.$$

The simplest solution is $m(\mathbf{x}) = m = \text{constant}$ which reproduces results of Landau theory

$$T > T_c \Rightarrow \alpha_2 > 0 \Rightarrow m = 0$$

$$T < T_v \Rightarrow \alpha_2 < 0 \Rightarrow m = \pm m_0 = \pm\sqrt{\frac{-\alpha_2}{\alpha_4}} = \pm\sqrt{\frac{3(T_c - T)}{T}}.$$

1.3.3 Domain Walls

For $T < T_c$, there are two ground states $m \pm m_0$.

Consider state where part of space has $m = +m_0$ and the other part has $m = -m_0$.

Two different '*domains*' separated by a '*domain wall*'.

[Need figure 11 here.]

Write $m(\mathbf{x}) = m(x)$ since we here only vary the x-coordinate. We get

$$\gamma\frac{\mathrm{d}^2 m}{\mathrm{d}x^2} = \alpha_2 m + \alpha_4 m^3$$

with boundary conditions $m(x) \to \pm m_0$ as $x \to \pm\infty$.

This has solution

$$m = m_0 \tanh\left(\frac{x - x_0}{W}\right)$$

where x_0 is the position of the domain wall and $W = \sqrt{-\gamma/\alpha_2}$ is the width of the wall.

[Need figure 12 here.]

If linear size of system is L, the excess cost in free energy is

$$F_{\text{wall}} \sim L^{d-1} \int \mathrm{d}x \, \gamma \left(\frac{\mathrm{d}m}{\mathrm{d}x}\right)^2 \sim L^{d-1} W \gamma \left(\frac{m_0}{W}\right)^2.$$

where we regard L^{d-1} as the 'area' of the domain wall. Plug W in, we have

$$F_{\text{wall}} \sim L^{d-1}\sqrt{\frac{-\gamma\alpha_2^3}{\alpha_4^2}}.$$

1.3.4 Lower Critical Dimension

It turns out that domain walls are responsible for the lack of phase transition for $d = 1$.

Suppose the system is in the ordered phase, $\alpha_2(T) < 0$, and consider an interval of length L.

Suppose at $x = -L/2$ we fix $m = +m_0$, we want to calculate the probability of $m = +m_0$ at the other end $x = +L/2$.

[Need figure 13 here.]

A domain wall flips the state, and

$$P[\text{wall at } x = x_0] = \frac{e^{-\beta F_{\text{wall}}}}{Z}$$

and

$$P[\text{wall anywhere}] \sim \frac{L}{W} \frac{e^{-\beta F_{\text{wall}}}}{Z}$$

Suppose we have n domain walls, if n is even, it eventually gets back to $+m_0$; for n is odd, it ends up at $-m_0$.

$$P[n \text{ walls}] = \frac{e^{-n\beta F_{\text{wall}}}}{Z} \frac{1}{W^n} \int_{-L/2}^{L/2} \mathrm{d}x_1 \int_{x_1}^{L/2} \mathrm{d}x_2 \cdots \int_{x_{n-1}}^{L/2} \mathrm{d}x_n = \frac{1}{Zn!} \left(\frac{Le^{-\beta F_{\text{wall}}}}{W} \right)^n$$

Now we can answer the question

$$P[m \to +m_0] = \frac{1}{Z} \sum_{n \text{ even}} \frac{1}{n!} \left(\frac{Le^{-\beta F_{\text{wall}}}}{W} \right)^n = \frac{1}{Z} \cosh\left(\frac{Le^{-\beta F_{\text{wall}}}}{W} \right)$$

and

$$P[m \to -m_0] = \frac{1}{Z} \sum_{n \text{ odd}} \frac{1}{n!} \left(\frac{Le^{-\beta F_{\text{wall}}}}{W} \right)^n = \frac{1}{Z} \sinh\left(\frac{Le^{-\beta F_{\text{wall}}}}{W} \right)$$

As $L \to \infty$, $P[m \to +m_0] = P[m \to -m_0]$, i.e. all memory of the starting point gets washed away.

Fluctuations of domain walls are unsuppressed because of entropy — they proliferate.

Thus there is no ordered phase and no spontaneous symmetry breaking in $d = 1$.

This suggests (although we haven't proved yet) the lower critical dimension for Ising model is $d_l = 1$.

2   PATH INTEGRALS

We want to compute

$$Z = \int \mathcal{D}m(\mathbf{x})e^{-\beta F[m(\mathbf{x})]}.$$

To be more general, we change notation $m(\mathbf{x}) \to \phi(\mathbf{x})$ which can be used to denote a generic field.

For L-G theory, at $B = 0$ we have

$$F[\phi(\mathbf{x})] = \int \mathrm{d}^d\mathbf{x} \left[\frac{1}{2}\gamma(T)(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}\alpha_2(T)\phi^2 + \frac{1}{4}\alpha_4(T)\phi^4 + \cdots\right].$$

Path integrals are:

- Easy to do if $F[\phi(\mathbf{x})]$ is quadratic in ϕ;

- Possible if higher order terms are small (try perturbation);

- Very difficult to compute otherwise.

We start by neglecting higher order terms.

For $T > T_c$, by denoting $\mu^2 = \alpha_2(T) > 0$, we have

$$F[\phi(\mathbf{x})] = \frac{1}{2} \int \mathrm{d}^d\mathbf{x}\, [\gamma(\boldsymbol{\nabla}\phi)^2 + \mu^2\phi^2]. \tag{2.1}$$

NOTE. When close to the critical point, $\mu^2 \approx 0$ and so we cannot neglect higher order terms (see later).

When $T < T_c$ the saddle point is at $\langle\phi\rangle = \pm m_0$ and we write

$$\tilde{\phi}(\mathbf{x}) = \phi(\mathbf{x}) - \langle\phi\rangle$$

giving

$$F[\tilde{\phi}(\mathbf{x})] = F[m_0] + \frac{1}{2} \int \mathrm{d}^d\mathbf{x} \left[\gamma(\boldsymbol{\nabla}\tilde{\phi})^2 + \alpha_2'\tilde{\phi}^2\right]$$

where $\alpha_2' = \alpha_2 + 3m_0^2\alpha_4 = -2\alpha_2 > 0$.

This is the same form as (2.1) ($T > T_c$) but with $\phi \to \tilde{\phi}$ and $\mu^2 \to \alpha_2'$.

2.1   Thermodynamic Free Energy

We have already seen some contributions to F_{thermo}:

$$F_0(T) \text{ (e.g. } -T\ln 2) \quad \text{and} \quad F(m_0) \text{ when } T < T_c.$$

Now we compute further contributions from fluctuations of $\phi(\mathbf{x})$. Work in Fourier space:

$$\phi_{\mathbf{k}} = \int \mathrm{d}^d\mathbf{x}\, e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{x}).$$

$\phi(\mathbf{x})$ is real means

$$\phi_{\mathbf{k}}^* = \phi_{-\mathbf{k}}$$

and we refer to $\mathbf{k}$ as *momenta*.

Because we started with a spatial lattice, we must have

$$\phi_{\mathbf{k}} = 0 \quad \forall |\mathbf{k}| > \Lambda$$

where Λ is the *UV cutoff*. For us, $\Lambda = \pi/a$.

In a finite system with volume $V = L^d$, momenta are quantised

$$\mathbf{k} = \frac{2\pi}{L}\mathbf{n} \quad \mathbf{n} \in \mathbb{Z}^d.$$

Then

$$\phi(\mathbf{x}) = \frac{1}{V}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\phi_{\mathbf{k}}.$$

As $V \to \infty$, this becomes

$$\phi(\mathbf{x}) = \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{x}}\phi_{\mathbf{k}}.$$

Substitute into F, we have

$$\begin{aligned}
F[\phi_{\mathbf{k}}] &= \frac{1}{2}\int \frac{\mathrm{d}^d\mathbf{k}_1}{(2\pi)^d}\int \frac{\mathrm{d}^d\mathbf{k}_2}{(2\pi)^d}\int \mathrm{d}^d\mathbf{x}\,(-\gamma\mathbf{k}_1\cdot\mathbf{k}_2 + \mu^2)\phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} \\
&= \frac{1}{2}\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}(-\gamma(-\mathbf{k})\cdot\mathbf{k} + \mu^2)\phi_{\mathbf{k}}\phi_{-\mathbf{k}} \\
&= \frac{1}{2}\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}(\gamma k^2 + \mu^2)|\phi_{\mathbf{k}}|^2 \\
&= \frac{1}{2V}\sum_{\mathbf{k}}(\gamma k^2 + \mu^2)|\phi_{\mathbf{k}}|^2.
\end{aligned}$$

The measure is

$$\int \mathcal{D}\phi(\mathbf{x}) = \prod_{\mathbf{k}}\left[N\int \mathrm{d}\phi_{\mathbf{k}}\,\mathrm{d}\phi_{\mathbf{k}}^*\right].$$

NOTE. We'll drop N because it doesn't play any role in the quantities we're interested in.

Then the partition function is

$$\begin{aligned}
Z &= \left(\prod_{\mathbf{k}}\int \mathrm{d}\phi_{\mathbf{k}}\,\mathrm{d}\phi_{\mathbf{k}}^*\right)\exp\left[-\frac{\beta}{2V}\sum_{\mathbf{k}}(\gamma k^2 + \mu^2)|\phi_{\mathbf{k}}|^2\right] \\
&= \prod_{\mathbf{k}}\left[\int \mathrm{d}\phi_{\mathbf{k}}\,\mathrm{d}\phi_{\mathbf{k}}^*\exp\left(-\frac{\beta}{2V}(\gamma k^2 + \mu^2)|\phi_{\mathbf{k}}|^2\right)\right].
\end{aligned}$$

This is a Gaussian integral for each $\mathbf{k}$

$$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2a}} = \sqrt{2\pi a}.$$

So

$$Z = \prod_{\mathbf{k}} \sqrt{\frac{2\pi V}{\beta(\gamma k^2 + \mu^2)}} = \prod_{\mathbf{k}} \sqrt{\frac{2\pi VT}{\gamma k^2 + \mu^2}} := e^{-\beta F_{\text{thermo}}}.$$

thus

$$\frac{F_{\text{thermo}}}{V} = -\frac{T}{V} \ln Z = -\frac{T}{2V} \sum_{\mathbf{k}} \ln\left(\frac{2\pi VT}{\gamma k^2 + \mu^2}\right) = -\frac{T}{2} \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \ln\left(\frac{2\pi VT}{\gamma k^2 + \mu^2}\right).$$

We now compute the heat capacity per unit volume $c = C/V$.

Take γ constant and $\mu^2 \sim (T - T_c)$, for $T > T_c$

$$c = \frac{\beta^2}{V} \frac{\partial^2}{\partial\beta^2} \ln Z$$

$$= \frac{1}{2}\left(T^2 \frac{\partial^2}{\partial T^2} + 2T \frac{\partial}{\partial T}\right) \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \ln\left(\frac{2\pi VT}{\gamma k^2 + \mu^2}\right)$$

$$= \frac{1}{2} \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \left[\underbrace{1}_{(1)} - \underbrace{\frac{2T}{(\gamma k^2 + \mu^2)}}_{(2)} + \underbrace{\frac{T^2}{(\gamma k^2 + \mu^2)^2}}_{(3)}\right]$$

where (1) is $\frac{1}{2}k$ per d.o.f. (from equipartition) from $\ln T$ bit; (2), (3) are from T dependence of μ. The behaviour of (2) and (3) terms depends on whether the integral converges as $\Lambda \to \infty$ — which depends on d.

The measure $\mathrm{d}^d \mathbf{k}$ is proportional to $k^{d-1} \, \mathrm{d}k$ in spherical coordinates. Then we can consider

$$\int_0^\Lambda \mathrm{d}k \, \frac{k^{d-1}}{(\gamma k^2 + \mu^2)} \sim \begin{cases} \Lambda^{d-2} + \cdots & d > 2 \\ \ln\Lambda + \cdots & d = 2 \\ \mu^{-1} + \cdots & d = 1 \end{cases}$$

and

$$\int_0^\Lambda \mathrm{d}k \, \frac{k^{d-1}}{(\gamma k^2 + \mu^2)^2} \sim \begin{cases} \Lambda^{d-4} + \cdots & d > 4 \\ \ln\Lambda + \cdots & d = 4 \\ \mu^{d-4} + \cdots & d < 4 \end{cases}$$

For $d \geq 4$, the leading contribution to c is independent of T;

For $d < 4$, $c \sim \mu^{d-4} \sim |T - T_c|^{-\alpha}$ with $\alpha = 2 - \frac{d}{2}$, i.e. change of critical exponent (recall MFT gives $\alpha = 0$).

But we shouldn't trust this near the critical point — we neglected the ϕ^4 term.

2.2   Correlation Functions

We know that

$$\langle\phi(\mathbf{x})\rangle = \begin{cases} 0 & T > T_c \\ \pm m_0 & T < T_c \end{cases}$$

Now look at the correlation function $\langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle$ which characterises how the magnetisation at $\mathbf{x}$ is correlated with that at $\mathbf{y}$.

The connected correlation function is defined by

$$\langle \phi(\mathbf{x})\phi(\mathbf{y})\rangle_c = \langle \phi(\mathbf{x})\phi(\mathbf{y})\rangle - \langle \phi \rangle^2 \,.$$

Now let's include spatially varying $B(\mathbf{x})$. The free energy is now

$$F = \int \mathrm{d}^d\mathbf{x} \left[\frac{1}{2}\gamma(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}\mu^2\phi^2 - B(\mathbf{x})\phi(\mathbf{x}) \right]$$

and the partition function

$$Z[B(\mathbf{x})] = \int \mathcal{D}\phi \ e^{-\beta F}$$

is a functional of $B(\mathbf{x})$, which contains lots of information.

Consider

$$\frac{1}{\beta}\frac{\delta \ln Z}{\delta B(\mathbf{x})} = \frac{1}{\beta Z}\frac{\delta Z}{\delta B(\mathbf{x})} = -\frac{1}{Z}\int \mathcal{D}\phi \ \phi(\mathbf{x})e^{-\beta F} = -\langle \phi(\mathbf{x})\rangle_B$$

which is the expectation in presence of B

Consider

$$\frac{1}{\beta^2}\frac{\delta^2 \ln Z}{\delta B(\mathbf{x})\delta B(\mathbf{y})} = \frac{1}{\beta^2 Z}\frac{\delta Z}{\delta B(\mathbf{x})\delta B(\mathbf{y})} - \frac{1}{\beta^2 Z^2}\frac{\delta Z}{\delta B(\mathbf{x})}\frac{\delta Z}{\delta B(\mathbf{y})}$$

$$= \langle \phi(\mathbf{x})\phi(\mathbf{y})\rangle_B - \langle \phi(\mathbf{x})\rangle_B \langle \phi(\mathbf{y})\rangle_B$$

which, when evaluated at $B = 0$, is what we want:

$$\frac{1}{\beta^2}\frac{\delta^2 \ln Z}{\delta B(\mathbf{x})\delta B(\mathbf{y})}\bigg|_{B=0} = \langle \phi(\mathbf{x})\phi(\mathbf{y})\rangle - \langle \phi \rangle^2 = \langle \phi(\mathbf{x})\phi(\mathbf{y})\rangle_c \,. \tag{2.2}$$

We just need to compute $Z[B(\mathbf{x})]$.

In Fourier space

$$F[\phi_{\mathbf{k}}] = \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \left[\frac{1}{2}(\gamma k^2 + \mu^2)\phi_{\mathbf{k}}\phi_{-\mathbf{k}} - B_{-\mathbf{k}}\phi_{\mathbf{k}} \right]$$

where $B_{\pm\mathbf{k}}$ are Fourier modes of $B(\mathbf{x})$.

Complete the square

$$\hat{\phi}_{\mathbf{k}} = \phi_{\mathbf{k}} - \frac{B_{\mathbf{k}}}{\gamma k^2 + \mu^2}$$

we have

$$F[\hat{\phi}_{\mathbf{k}}] = \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} \left[\frac{1}{2}(\gamma k^2 + \mu^2)\left|\hat{\phi}_{\mathbf{k}}\right|^2 - \frac{1}{2}\frac{|B_{\mathbf{k}}|^2}{\gamma k^2 + \mu^2} \right]$$

and so

$$Z[B(\mathbf{x})] = \prod_{\mathbf{k}} \int \mathrm{d}\hat{\phi}_{\mathbf{k}}\,\mathrm{d}\hat{\phi}_{\mathbf{k}}^*\, e^{-\beta F[\hat{\phi}]}$$

$$= e^{-\beta F_{\text{thermo}}} \exp\left[\frac{\beta}{2}\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{|B_{\mathbf{k}}|^2}{\gamma k^2 + \mu^2} \right]$$

$$= e^{-\beta F_{\text{thermo}}} \exp\left[\frac{\beta}{2}\int \mathrm{d}^d\mathbf{x}\,\mathrm{d}^d\mathbf{y}\, B(\mathbf{x})G(\mathbf{x} - \mathbf{y})B(\mathbf{y}) \right]$$

where

$$G(\mathbf{x} - \mathbf{y}) = \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\gamma k^2 + \mu^2}.$$

From (2.2) we have

$$\langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle - \langle \phi \rangle^2 = \frac{1}{\beta} G(\mathbf{x} - \mathbf{y}).$$

Now we just need to compute this.

Correlation function depends on $\xi^2 = \gamma/\mu^2$, where ξ is called the *correlation length*.

Actually, $G(\mathbf{x}) = G(r)$ where $r = |\mathbf{x}|$, and we can write

$$G(r) = \frac{1}{\gamma} \int \frac{\mathrm{d}^d \mathbf{k}}{(2\pi)^d} \frac{e^{-i\mathbf{k}\cdot\mathbf{x}}}{k^2 + 1/\xi^2}.$$

We claim that (proof is exercise on example sheet 2):

$$G(r) \sim \begin{cases} \dfrac{1}{r^{d-2}} & r \ll \xi \\[2mm] \dfrac{e^{-r/\xi}}{r^{(d-1)/2}} & r \gg \xi \end{cases}$$

[Need figure 14, power law + exponential here.]

Roughly, fluctuations occur over sizes $\sim \xi$ or less.

$\mu^2 \sim (T - T_c)$ suggests that $\xi \to \infty$ at the critical point — this means fluctuations on all length scales (c.f. critical opalescence). We want to understand this.

The correlator suggests two more critical exponents

$$\xi \sim |T - T_c|^{-\nu} \quad \text{with} \quad \nu = 1/2$$

and, at the critical point, $\xi \to \infty$,

$$\langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle \sim \frac{1}{r^{d-2+\eta}} \quad \text{with} \quad \eta = 0.$$

These are correct for $d \geq 4$, but not for lower d.

It turns out that, for both Ising and liquid/gas transition,

	MFT	$d = 2$	$d = 3$
η	0	1/4	0.0363...
ν	1/2	1	0.6300...

Something else to understand!

2.2.1 Aside: the Upper Critical Dimension

Recall that, in the ordered phase $(T < T_c)$, $\langle \phi(\mathbf{x}) \rangle = \pm m_0$.

Mean field results are not obviously wrong provided $\langle \phi^2 \rangle \ll \langle \phi \rangle^2$.

Consider

$$R = \frac{\int_0^\xi \mathrm{d}^d \mathbf{x} \langle \phi(\mathbf{x})\phi(\mathbf{0}) \rangle}{\int_0^\xi \mathrm{d}^d \mathbf{x}\, m_0^2} \sim \frac{1}{m_0^2 \xi^d} \int_0^\xi \mathrm{d}r \frac{r^{d-1}}{r^{d-2}} \sim \frac{\xi^{2-d}}{m_0^2}.$$

But $m_0 \sim |T - T_c|^{1/2}$ and $\xi \sim |T - T_c|^{-1/2}$, we have

$$R \sim |T - T_c|^{(d-4)/2}.$$

We see that $R \to \infty$ as $T \to T_c$ if $d < d_c = 4$ — this means the MFT results are definitely wrong.

For $d > 4$, $R \to 0$ as $T \to T_c$, so the MFT results have at least a chance of being correct!

(For $d = 4$ there is a log divergence — need to work harder.)