University of Cambridge
Mathematical Tripos

Part III – **Algebraic Topology**

Based on Lectures by *I. Smith*
Notes taken by Zihan Yan

Michaelmas 2020

These notes may not reflect the full format and content that are actually lectured. I usually modify the notes after the lectures and sometimes my own thinking or interpretation might be blended in. Any mistake or typo should surely be mine. Be cautious if you are using this for self-study or revision.

Course Information

Algebraic Topology permeates modern pure mathematics and theoretical physics. This course will focus on (co)homology, with an emphasis on applications to the topology of manifolds. We will cover singular and cellular (co)homology; degrees of maps and cup-products; cohomology with compact supports and Poincaré duality; and Thom isomorphism and the Euler class. The course will not specifically assume any knowledge of algebraic topology, but will go quite fast in order to reach more interesting material, so some previous exposure to chain complexes (e.g. simplicial homology) would certainly be helpful.

Pre-requisites

Basic topology: topological spaces, compactness and connectedness, at the level of Sutherland's book. Some knowledge of the fundamental group would be helpful though not a requirement. Hatcher's book and Bott and Tu's book are especially recommended for accompanying the course, but there are many other suitable texts.

Contents

0 Introduction

Lecture 1
No-Revise

Algebraic Topology concerns the *connectivity* properties of topological spaces.

Definition 0.1. A space X is *path-connected* if for $p, q \in X$, $\exists \gamma : [0,1] \to X$ continuous with $\gamma(0) = p, \gamma(1) = q$.

[Need figure 1 here.]

Example 0.2. $\mathbb{R}$ is path-connected; $\mathbb{R}\backslash\{0\}$ is not.

Corollary 0.3 (The intermediate value theorem). *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $x < y$ satisfy $f(x) < 0, f(y) > 0$ then f takes the value 0 on $[x, y]$.*

Proof. Otherwise, $f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$ disconnects $[x, y]$, #. □

Definition 0.4. Let X, Y be topological spaces. We say maps $f_0, f_1 : Y \to X$ are *homotopic* if $\exists F : Y \times [0,1] \to X$ continuous such that

$$F|_{Y\times\{0\}} = f_0, \qquad F|_{Y\times\{1\}} = f_1$$

We write $f_0 \simeq f_1$ (or $f_0 \underset{F}{\simeq} f_1$).

[Need figure 2 here.]

Exercise 0.5. (On example sheet 1) $\simeq$ is an equivalence relation on the set of maps from Y to X.

Note. X is *path-connected* iff every two maps $\{\text{point}\} \to X$ are homotopic.

Definition 0.6. X is *simply-connected* if every two maps $S^1 \to X$ are homotopic.

Note. We often denote

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}, \qquad S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$$

Example 0.7. $\mathbb{R}^2$ is simply connected; $\mathbb{R}^2\backslash\{0\}$ is not.

From complex analysis we know $\gamma : S^1 \to \mathbb{R}^2\backslash\{0\}$ has a *winding number* or *degree* $\deg(\gamma) \in \mathbb{Z}$, for which

1. If $\gamma_n(t) = e^{2\pi i n t}$ then $\deg(\gamma_n) = n$;
2. $\deg(\gamma_1) = \deg(\gamma_2)$ if $\gamma_1 \simeq \gamma_2$.

[Need figure 3 here.]

For *differentiable* γ,

$$\deg(\gamma) = \int_\gamma \frac{\mathrm{d}z}{z}.$$

Corollary 0.8 (Fundamental theorem of algebra). *Every non-constant complex polynomial has a root.*

Proof. Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be non-constant and WLOG monic. Suppose $f(z) \neq 0, \forall z \in \mathbb{C}$, let $\gamma_R(t) := f\left(Re^{2\pi i t}\right)$ so that $\gamma_R : S^1 \to \mathbb{R}^2 \backslash \{0\}$. We know that

$$\gamma_0 \text{ is constant} \quad \Rightarrow \quad \deg(\gamma_0) = 0 \quad \Rightarrow \quad \deg(\gamma_R) = 0, \quad \forall R$$

However, if we take $R \gg \sum_i |a_i|$, let $f_s(z) = z^n + s\left(a_1 z^{n-1} + \cdots + a_n\right)$ with $0 \leq s \leq 1$. On the circle $|z| = R$, $f_s(z) \neq 0, \forall s$.

Therefore, if $\gamma_{R,s}(t) := f_s\left(Re^{2\pi i t}\right)$ then we have $\gamma_{R,1} = \gamma_R$ and $\gamma_{R,0} : t \mapsto R^n e^{2\pi i n t}$.

Clearly, we have

$$\deg(\gamma_{R,1}) = 0 \neq n = \deg(\gamma_{R,0})$$

as non-constant property suggests $n \neq 0$. This is a #. □

Definition 0.9. X is *k-connected* if every two maps $S^i \to X$ are homotopic whenever $i \leq k$.

Example 0.10. $\mathbb{R}^n$ is $(n-1)$-connected; $\mathbb{R}^n \backslash \{0\}$ is not. Maps $S^{n-1} \to \mathbb{R}^n \backslash \{0\}$ have a homotopy-invariant degree $\in \mathbb{Z}$ and deg(inclusion) $= 1$, deg(constant) $= 0$. (We'll prove it later.)

Corollary 0.11 (Brouwer's theorem). *For closed unit ball $\bar{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, any map $f : \bar{B}^n \to \bar{B}^n$ has a fixed point.*

Proof. Suppose f has no fixed point. Let $\gamma_R(v) := Rv - f(Rv)$ where $0 \leq R \leq 1$ and $v \in S^{n-1} = \partial \bar{B}^n$. Our assumption suggests γ_R takes values in $\mathbb{R}^n \backslash \{0\}$.

According to homotopy invariance, as γ_0 is constant, we have $\deg(\gamma_0) = 0$ hence $\deg(\gamma_1) = 0$.

Let $\gamma_{1,s}(v) := v - sf(v)$ for $0 \leq s \leq 1$. Then $\gamma_{1,1} = \gamma_1$ and $\text{image}(\gamma_{1,s}) \subseteq \mathbb{R} \backslash \{0\}$ as $\|v\| = 1, \|sf(v)\| = |s|\|f(v)\| < 1$ if $s < 1$.

Therefore, we have $\deg(\gamma_{1,0}) = \deg(\gamma_{1,1}) = 0$ by homotopy invariance. However, the inclusion $\gamma_{1,0}$ should have degree 1, thus #. □

Definition 0.12. $f : X \to Y$ is a *homotopy-equivalence* if $\exists g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y, g \circ f \simeq \mathrm{id}_X$. (We call g a "homotopy inverse" for f.)

Note. The homotopy equivalence can be shown as an equivalence relation on spaces.

Example 0.13. If X, Y are homeomorphic they are trivially homotopy equivalent: simply by taking $g = f^{-1}$.

Example 0.14. $\mathbb{R}^n \backslash \{0\} \simeq S^{n-1}$.

Let

$$f : \mathbb{R}^n \backslash \{0\} \to S^{n-1}, \quad v \overset{f}{\mapsto} \frac{v}{\|v\|}$$

$$g : S^{n-1} \hookrightarrow \mathbb{R}^n \backslash \{0\} \text{ by inclusion}$$

Then

$$f \circ g = \mathrm{id}_{S^{n-1}}, \qquad g \circ f \underset{F}{\simeq} \mathrm{id}_{\mathbb{R}^n \backslash \{0\}}$$

via the homotopy

$$F(t, v) = tv + (1 - t)\frac{v}{\|v\|}$$

[Need figure 4 here.]

Example 0.15. $\{0\} \overset{\sim}{\hookrightarrow} \mathbb{R}^n$ is a homotopy equivalence. (Check!)

Definition 0.16. If a space $X \simeq \{\text{point}\}$ we say X is *contractible.*

Talking about all these, we emphasise that

$$\boxed{\text{Algebraic topology is the study of topological spaces up to homotopy equivalence.}}$$

The main idea is that: homeomorphism is too delicate as a relation, but homotopy equivalence keeps track of "essential" topological information. More precisely, we assign

$$\{\text{Spaces}\} \to \{\text{Groups}\}$$
$$\{\text{Maps of spaces}\} \to \{\text{Homomorphisms of groups}\}$$

so we get algebraic invariants. (They are defined for *all* spaces, but have more structure and use/interest for "nicer" spaces.)

The classical first attempt of algebraic topology would be *homotopy theory.* One can *concatenate* loops: [Need figure 5 here.] for

$$\gamma * \tau(t) = \begin{cases} \gamma(2t), & t \leq \frac{1}{2} \\ \tau(1 - 2t), & t \geq \frac{1}{2} \end{cases}$$

which leads to

$$\{\text{Maps } S^1 \xrightarrow{\gamma} X\}/\simeq \quad \longrightarrow \quad \pi_1(X, x_0)$$

where γ fixes $\gamma(0) = x_0 \in X$ and the homotopies preserve x_0; [Need figure 6 here.] π_1 is called the *fundamental group* on which the group operation is the concatenation $(\gamma, \tau) \mapsto \gamma * \tau$.

Similarly, for higher dimensions [Need figure 7 here.]

giving

$$\pi_n(X, x_0) = \{\text{based maps}\}/\simeq$$

called the *n-th homotopy group* of X.

The issue is that these homotopy groups are very hard to compute. E.g. $\pi_n(S^2, x_0)$ is not known $\forall n$.

There is even *no* simply connected manifold (a space X locally homeomorphic to $\mathbb{R}^n$) of dimension > 0 with $\pi_n(X)$ known $\forall n$!

Therefore, we will do something else: *(co)homology*.

It is algebraically harder to set up, yet the computational gain is worth it. Please note that computing cohomology of "harder" spaces (e.g. $\mathrm{Diff}(X), \mathrm{Emb}(X, Y), \ldots$) is still very hard.

Some general remarks:

- Algebraic topology is all about being able to *compute*. It is important to do lots of examples;
- Our "nice spaces" are *manifolds* and indeed *smooth manifolds* — some of these will overlap with the course *Differential Geometry* which will be useful.

1 Homology

1.1 Chain & Cochain Complexes

Lecture 2
No-Revise

We will define invariants of spaces in two stages:

1. Associate to X a (co)chain complex;
2. Take the (co)homology of that complex.

Definition 1.1. A *chain complex* (C_*, ∂) is a sequence of abelian groupd and homomorphisms

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots$$

such that $\partial_{n-1} \circ \partial_n = 0, \forall n$. We write this as "$\partial^2 = 0$".

The *homology group* $H(C_*, \partial)$ is the graded group

$$H_n(C_*) := \ker(\partial_n)/\mathrm{im}(\partial_{n+1}).$$

Note. We may call ∂ the "differential" or "boundary map".

Definition 1.2. A *cochain complex* is a sequence of abelian groups and homomorphisms (C^*, ∂)

$$\cdots \to C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \xrightarrow{\partial^{n+1}} C^{n+2} \to \cdots$$

such that $\partial^n \circ \partial^{n-1} = 0, \forall n$ ("$\partial^2 = 0$" again).

The *cohomology groups* $H(C^*, \partial)$ are

$$H^n(C^*) := \ker(\partial^n)/\mathrm{im}(\partial^{n-1}).$$

Note. Here we introduce some terminologies.

- Elements of $\ker \partial : C_n \to C_{n-1}$ are called *cycles*;
- Elements of $\mathrm{im}\, \partial : C_{n+1} \to C_n$ are called *boundaries*;
- Elements of $\ker \partial : C^n \to C^{n+1}$ are called *cocycles*.

Exercise 1.3. Try to define *coboundary*.

Note. For convenience (we are lazy!), we write all ∂_i and ∂^i as ∂ (or occasionally ∂, ∂^*).

Definition 1.4. The elements of $H_*(C_*)$ are *homology classes* and those of $H^*(C^*)$ are *cohomology classes*.

Definition 1.5. A *chain map* between chain complexes (C_*, ∂) and (D_*, ∂) is a sequence of homomorphisms $f_n : C_n \to D_n$ such that $\forall n$ the following commutes:

$$\begin{tikzcd} \cdots \arrow[r] & C_n \arrow[r, "\partial"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_n \arrow[r, "\partial"] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

i.e. $f_{n-1} \circ \partial_n^{C_*} = \partial_n^{D_*} \circ f_n$.

Exercise 1.6. Define a *cochain map* of cochain complexes.

Lemma 1.7. *A chain map $f_* : C_* \to D_*$ induces homomorphisms*

$$(f_n)_\star : H_n(C_*) \to H_n(D_*)$$

for each n.

Proof. Let $[a] \in H_n(C_*)$, so a is represented by a cycle $\alpha \in C_n$. Use the commutativity of the diagram in the above definition, we have

$$\partial(f_n(\alpha)) = f_{n-1}(\underbrace{\partial\alpha}_{0}) = 0$$

so $f_n(\alpha)$ is a cycle.

Define

$$f_\star[a] := [f_n(\alpha)] \in H_n(D_*).$$

We made a choice of representing cycle α. But if $[a]$ is represented by α and α', then

$$\alpha - \alpha' \in \operatorname{im}(\partial_{n+1} : C_{n+1} \to C_n).$$

Say $\alpha - \alpha' = \partial\tau$, then

$$f_n(\alpha) - f_n(\alpha') = f_n(\alpha - \alpha') = f_n(\partial\tau) = \partial(f_{n+1}(\tau)).$$

Thus

$$[f_n(\alpha)] = [f_n(\alpha') + \partial f_{n+1}(\tau)] = [f_n(\alpha')]$$

as $[\operatorname{im}(\partial)] = 0$ in H_*.

So $f_\star$ is well-defined, and easy to see it's a homomorphism. $\square$

EXERCISE 1.8. If C_*, D_*, E_* are chain complexes and $f : C_* \to D_*$, $g : D_* \to E_*$ are chain maps then $\{g_n \circ f_n : C_n \to E_n\}_n$ defines a chain map and

$$(\dagger) \begin{cases} (g \circ f)_\star = g_\star \circ f_\star, \\ (\mathrm{id})_\star = \mathrm{id} \;\text{ on } H(C_*) \text{ themselves.} \end{cases}$$

Discussing all these, our goal is associating to space X (co)chain complexes $C_*(X), C_*(X)$ such that a map $f : X \to Y$ yields (co)chain maps

$$C_*(X) \xrightarrow{f_*} C_*(Y)$$

and

$$C^*(Y) \xrightarrow{f^*} C^*(X).$$

NOTE. (†) means that we can say we have a *functor* **Top** → **Groups**, $X \mapsto H_\star(X)$ where the categories are

Top = (spaces, continuous maps) and **Groups** = (abelian groups, homomorphisms).

Our complexes C_*, C^* will have the benefit that they are "intrinsic" but will be huge and unwieldy. We will

1. Prove structure theorems to help compute these;
2. Find "smaller" complexes later for nice spaces (e.g. CW-complexes).

DEFINITION 1.9. The *standard simplex* is defined as

$$\Delta^n := \left\{ (t_0, \cdots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0, \sum_i t_i = 1 \right\}.$$

The i-th *face* of Δ^n is

$$\Delta^n_i := \{\mathbf{t} \in \Delta^n : t_i = 0\}.$$

NOTE. There exists canonical homeomorphism $\Delta^{n-1} \xrightarrow{\delta_i} \Delta^n_i$ such that

$$(t_0, \cdots, t_{n-1}) \mapsto (t_0, \cdots, t_{i-1}, 0, t_{i+1}, t_{n-1}).$$

DEFINITION 1.10. If X is a space, a *singular n-simplex in X* is a map $\sigma : \Delta^n \to X$. The *singular chain complex* $(C_*(X), \partial)$ has

$$C_n(X) := \left\{ \sum_{i=1}^{N} n_i \sigma_i : N < \infty, n_i \in \mathbb{Z}, \sigma_i : \Delta^n \to X \right\},$$

the free abelian groups on the singular n-simplices in X, and

$$\partial : C_n(X) \to C_{n-1}(X), \quad \sigma \mapsto \sum_{i=0}^{n} (-1)^i (\sigma \circ \delta_i)$$

which extends linearly.

Note. The $n+1$ ordered points $\{v_i\}_{0\leq i\leq n} \subseteq \mathbb{R}^{n+1}$ determine an n-simplex if $\{v_i - v_0 : 1 \leq i \leq n\}$ are linearly independent. Take their convex hull and set

$$\sigma : \Delta^n \to \mathbb{R}^{n+1}, \quad \mathbf{t} \mapsto \sum_{i=0}^{n} t_i v_i.$$

We orient the edges as $v_i \to v_j$ if $i < j$. Write $\underbrace{[v_0 \cdots v_n]}_{\sigma}$ for this n-simplex, then

$$\partial\sigma := \sum_{i=0}^{n} (-1)^i \sigma\Big|_{[v_0\cdots\hat{v}_i\cdots v_n]}$$

where the hat means omission.

[Need figure 8 here.]

Lemma 1.11. $\partial^2 = 0$.

Proof.

$$\partial(\partial\sigma) = \sum_{j<i} (-1)^i (-1)^j \sigma\Big|_{[v_0\cdots\hat{v}_j\cdots\hat{v}_i\cdots v_n]} + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma\Big|_{[v_0\cdots\hat{v}_i\cdots\hat{v}_j\cdots v_n]}$$

Exchange the i and j in the second term, the two terms cancel. □

Definition 1.12. The *singular homology* is defined as $H_*(X) = H_*(x;\mathbb{Z}) := H(C_*(X), \partial)$.

Note. This is *trivially* a homeomorphism invariant of X since we only used the notion of continuous map to X to define it.

Example 1.13. [Need figure 9 here.]

We take four 1-simplices to form a closed path as shown.

The annulus to the left has

$$\partial(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) = 0$$

and to the right, the connected space also has

$$\partial(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) = 0.$$

But what we can confirm is that

$$\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \in \operatorname{im}(\partial)$$

e.g. $\partial(\tau) = [v_0 v_1] - [v_0 v_2] + \underbrace{[v_1 v_2]}_{\gamma_3}$. This means that the homology class defined by such $\sum_i \gamma_i$ is 0. However, for σ it should be non-zero — the singular homology probes 'holes' in a space.

Definition 1.14. The *sigular cochain complex* $C^*(X)$ has

$$C^n(X) := \mathrm{Hom}(C_n(X), \mathbb{Z})$$

$$\partial^* : C^n(X) \to C^{n+1}(X), \qquad (\partial^*\psi)(\sigma) := \psi(\partial\sigma), \quad \sigma \in C_{n+1}(X)$$

Then

$$\partial^*(\partial^*\psi)(\sigma) = \partial^*(\psi(\partial\sigma)) = \psi(\partial(\partial\sigma)) = 0.$$

i.e. $(\partial^*)^2 = 0$: this is indeed a cochain complex.

Note. $H^*(X;\mathbb{Z}) := H^*(C^*(X), \partial^*)$ is *singular cohomology* of X.

$H^*(X;\mathbb{Z}) \not\cong \mathrm{Hom}(H_*(X), \mathbb{Z})$ in general.

We bothered to define such cochain complex and cohomology because later we will show cohomology is a ring (which has better algebraic properties) while homology is not.

Note (Rough idea). We have several *heuristic* ideas (don't take too seriously!):

- $\partial^2 = 0$ means "the boundary of the boundary vanishes";
- $H_i(X)$ will probe "i-dimensional holes/regions" in X;
- $H^i(X)$ will be a rule associating an integer to an i-dimensional region of X.

Lecture 3
No-Revise

Remark 1.15. Let $f : X \to Y$ be continuous. If $\sigma : \Delta^n \to X$ then $f \circ \sigma : \Delta^n \to Y$, meaning that f induces homomorphisms

$$f_n : C_n(X) \to C_n(Y)$$

and by $f \circ (\sigma \circ \delta_i) = (f \circ \sigma) \circ \delta_i$, we have

$$f \circ \left(\sigma\big|_{i\text{-th face}}\right) = (f \circ \sigma)\big|_{i\text{-th face}}$$

Thus $f_* : C_*(X) \to C_*(Y)$ is a chain map

$$\begin{array}{ccc} C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) \end{array}$$

also giving homomorphisms

$$f_\star : H_\star(X) \to H_\star(Y).$$

Exercise 1.16. Show that

$$(f \circ g)_* = f_* \circ g_*, \qquad (\mathrm{id})_* = \mathrm{id}\,.$$

Note. $f : X \to Y$ also induces

$$f^* : C^*(Y) \to C^*(X), \quad (f^*\psi)(\sigma) := \psi(f \circ \sigma)$$

and

$$f^\star : H^\star(Y) \to H^\star(X)$$

where $\sigma : \Delta^n \to X$ and $\psi : C_n(Y) \to \mathbb{Z}$.

Note that in the language of category theory, we say "cohomology is a contravariant functor".

1.2 Homology of the Circle

So far, what can we compute?

Lemma 1.17. *Let X be a point. Then the singular homology is*

$$H_i(\{pt\}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. For each $n \geq 0$, there exists a unique n-simplex in X, $\sigma_n : \Delta^n \to \{\mathrm{pt}\}$, the constant map. So $C_*(\{\mathrm{pt}\})$ is

$$\begin{tikzcd}
\cdots \arrow[r] & C_3 \arrow[r] \arrow[d, equal] & C_2 \arrow[r] \arrow[d, equal] & C_1 \arrow[r] \arrow[d, equal] & C_0 \arrow[d, equal] \\
\cdots \arrow[r, "+1"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "+1"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z}
\end{tikzcd}$$

We find

$$\partial\sigma_1 = \partial(\bullet\!\!\longrightarrow\!\!\bullet) = \bullet - \bullet = 0$$

and

$$\partial\sigma_2 = \sigma_2 \circ \delta_0 - \sigma_2 \circ \delta_1 + \sigma_2 \circ \delta_2 = \sigma_1$$

Thus by induction,

$$\partial\sigma_n = \begin{cases} \sigma_{n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Then if we see how these relate kernels and images at different n, our result is clear. $\square$

Exercise 1.18. Check the cohomology

$$H^i(\{\mathrm{pt}\}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

There is basically only one other computation we can do from the definitions.

Lemma 1.19. *If $X = \bigsqcup_{\alpha \in I} X_\alpha$ is a disjoint union of path-components, then*

$$H_i(X) \cong \bigoplus_{\alpha \in I} H_i(X_\alpha), \quad \forall i.$$

Proof. Any continuous map $\sigma : \Delta^i \to X$ has image in one X_α and then all the faces of σ lie in the *same* X_α. So

$$C_*(X) = \bigoplus_\alpha C_*(X_\alpha)$$

compatibly with the differential. □

Lemma 1.20. *If X is path-connected (and non-empty), $H_0(X) \cong \mathbb{Z}$.*

(An aside note: We sometimes write $\pi_0(X)$ for the *set* of path-components of X.)

Proof. Define

$$\varepsilon : C_0(X) \to \mathbb{Z}, \quad \sum_{\text{finite}} n_i \sigma_i \mapsto \sum_i n_i$$

called the *augmentation*, where $\sigma_i : \{\text{pt}\} \to X$ are 0-simplices in X. As $X \neq \varnothing$, ε is surjective.

If $\tau : \Delta^1 \to X$,

[Need figure 10-1 here.]

we have

$$\varepsilon(\partial \tau) = \varepsilon(v_1 - v_0) = 0$$

so

$$\operatorname{im}(\partial : C_1 \to C_0) \subseteq \ker(\varepsilon)$$

i.e. ε defines $H_0(X) \to \mathbb{Z}$.

So far we didn't use path-connectivity. But suppose $\sum_i n_i \sigma_i \in \ker(\varepsilon)$. Fix a base point $p \in X$ and for every i we pick

$$\tau_i : \Delta^1 \cong [0,1] \to X, \quad \begin{cases} \tau_i(1) = \sigma_i(\Delta^0) \\ \tau_i(0) = p \end{cases}$$

[Need figure 10-2 here.]

and we have

$$\partial\left(\sum_i n_i \tau_i\right) = \sum_i n_i \sigma_i - \cancelto{0}{\left(\sum_i n_i\right)} p$$

as $\sum_i n_i \sigma_i \in \ker(\varepsilon)$, so

$$\ker(\varepsilon) \subseteq \operatorname{im}(\partial)$$

and we identify

$$\varepsilon : C_0(X)/\operatorname{im}(\partial) =: H_0(X) \xrightarrow{\cong} \mathbb{Z}.$$

□

Informal Picture Recall $\sigma : \Delta^1 \cong [0,1] \to X =$ Annulus has $\partial\sigma = \sigma(1) - \sigma(0) = 0$, so σ defines $[\sigma] \in H_1(X)$.

[Need figure 11-1 here.]

We would hope this is non-zero, as we can't "see" a way to fill in σ with 2-simplices. (Contrast with the case where $\tau \in \mathrm{im}(\partial)$.)

[Need figure 11-2 here.]

However, $C_i(X)$ is uncountably generated for all i, and is very hard to control.

The question is: how do we rule out *all* configurations of 2-simplices? Or, are there any other representations for $[\sigma] \in H_1(X)$?

Informal Conjecture In the realm of "nice" spaces, there is *nothing else* you can compute from the definition!

(Co)homology is rendered useful by a collection of structural theorems: we will state these and see how to use them and then return to prove them later.

Theorem 1.21 (Homotopy invariance). *If $f : X \to Y$ and $g : X \to Y$ are homotopic, then*

$$f_\star = g_\star : H_\star(X) \to H_\star(Y) \quad \textit{and} \quad f^\star = g^\star : H^\star(Y) \to H^\star(X).$$

Corollary 1.22. *If $X \simeq Y$ (homotopy-equivalent), then*

$$H_\star(X) \cong H_\star(Y) \quad \textit{and} \quad H^\star(X) \cong H^\star(Y).$$

Proof. There exists $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$ thus $(f_\star)^{-1} = g_\star$ are isomorphisms. □

Thus (co)homology is insensitive to "inessential" deformations of a space.

Corollary 1.23.

$$H_*(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \textit{otherwise} \end{cases}$$

for every n. And similar for H^*.

But we still cannot compute very much...

Definition 1.24. An *exact sequence* is a (co)chain complex with vanishing (co)homology:

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

such that $\ker(\partial_n) = \mathrm{im}(\partial_{n+1}), \forall n$.

Note. There are some additional terminologies/facts:

- Given homomorphisms
$$A \xrightarrow{f} B \xrightarrow{g} C$$
say this is *exact at* B if $\ker g = \operatorname{im} f$;
- If
$$0 \to A \xrightarrow{f} B \to 0$$
is exact, we have $A \cong B$;
- A *short exact sequence* is one of shape
$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$
which says f is injective and g is surjective.

THEOREM 1.25 (Mayer-Vietoris). *If $X = A \cup B$ with A, B open, there are "Mayer-Vietoris boundary homomorphisms"*
$$\partial_{MV} : H_{i+1}(X) \to H_i(A \cap B)$$
yielding a "long" exact sequence (LES):
$$\cdots \to H_{i+1}(X) \xrightarrow{\partial_{MV}} H_i(A \cap B) \xrightarrow{(i_{A\star}, i_{B\star})} H_i(A) \oplus H_i(B) \xrightarrow{j_{A\star} - j_{B\star}} H_i(X) \to \cdots$$
where $i_{A\star}, j_{A\star}$, etc. are induced from certain inclusion maps in the commutative diagram below
$$\begin{tikzcd} A \cap B \arrow[r, hook, "i_A"] \arrow[d, hook, "i_B"] & A \arrow[d, hook, "j_A"] \\ B \arrow[r, hook, "j_B"] & X \end{tikzcd}$$
and ∂_{MV} is defined algebraically, not associated to a map of spaces.

REMARK 1.26. Suppose $\sigma \in C_{i+1}(X)$ is a *cycle* and $\sigma = \alpha + \beta$, $\alpha \in C_{i+1}(A), \beta \in C_{i+1}(B)$ for *chains* α, β (i.e. $\partial\alpha, \partial\beta$ need not vanish). Then $\partial\alpha = -\partial\beta$ and
$$\partial_{\mathrm{MV}}[\sigma] = [\partial\alpha]$$
as $\partial\alpha = -\partial\beta$ gives $\partial\alpha \in C_i(A \cap B)$.

[Need figure 12 here.]

ADDENDUM. *The MV sequence is* natural. *If $X = A \cup B$ and $Y = C \cup D$ and $f : X \to Y$ has $f(A) \subseteq C$ and $f(B) \subseteq D$ (under maps of pairs), then there are homomorphisms of exact exact sequences*
$$\begin{tikzcd} \cdots \arrow[r] & H_{i+1}(X) \arrow[r, "\partial^X_{MV}"] \arrow[d, "f_\star"] & H_i(A \cap B) \arrow[r] \arrow[d, "f_\star"] & H_i(A) \oplus H_i(B) \arrow[r] \arrow[d, "f_\star"] & H_i(X) \arrow[r] \arrow[d, "f_\star"] & \cdots \\ \cdots \arrow[r] & H_{i+1}(Y) \arrow[r, "\partial^Y_{MV}"] & H_i(C \cap D) \arrow[r] & H_i(C) \oplus H_i(D) \arrow[r] & H_i(Y) \arrow[r] & \cdots \end{tikzcd}$$

such that all squares commute.

REMARK 1.27. There is a MV sequence in cohomology, also natural:

$$\partial^*_{\mathrm{MV}} : H^i(A \cap B) \to H^{i+1}(X)$$

such that

$$\cdots \to H^i(X) \xrightarrow{(j_A^\star, j_B^\star)} H^i(A) \oplus H^i(B) \xrightarrow{i_A^\star - i_B^\star} H^i(A \cap B) \xrightarrow{\partial^*_{\mathrm{MV}}} H^{i+1}(X) \to \cdots$$

is *exact.*

Finally, we get something to compute: the homology of the circle.

PROPOSITION 1.28.

$$H_i(S^1) \cong \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & \textit{otherwise} \end{cases}$$

and

$$H^i(S^1) \cong \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Choose A and B such that $S^1 = X = A \cup B$ and A, B are open intervals, $A \cap B$ is the union of 2 disjoint open intervals, shown as in the figure below

[Need figure 13 here.]

So, we have $A \simeq \{\mathrm{pt}\} \simeq B$ and $A \cap B \simeq \underbrace{\{p\} \sqcup \{q\}}_{S^0, \text{0-sphere}}$.

Recall that

$$H_*(\mathbb{R}) \cong \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

and $\mathbb{R} \simeq \{\mathrm{pt}\}$. By homotopy invariance, we know $H_*(A), H_*(B), H_*(A \cap B)$.

If we take the MV sequence for $i \geq 2$, we have

$$\cancelto{0}{H_i(A) \oplus H_i(B)} \to H_i(S^1) \to \cancelto{0}{H_{i-1}(A \cap B)} \quad \Rightarrow \quad H_i(S^1) = 0.$$

For $i = 1$, we have

$$\begin{tikzcd}
\cancelto{0}{H_1(A \cap B)} \arrow[r] & \cancelto{0}{H_1(A) \oplus H_1(B)} \arrow[r] & H_1(S^1) \arrow[dll, bend left] \\
H_0(A \cap B) \arrow[r, "(i_A, i_B)"] \arrow[d, equal, "\cong"] & H_0(A) \oplus H_0(B) \arrow[r, "(j_A - j_B)"] \arrow[d, equal, "\cong"] & H_0(S^1) \arrow[d, equal, "\cong"] \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[r, dashed, "\alpha"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, dashed, "\beta"] & \mathbb{Z}
\end{tikzcd}$$

Recall that for some path-connected space Z, $H_0(Z)$ is a free abelian group on $\pi_0(Z)$ (the set of path-components), and indeed this is generated by

$$\sigma : \{\text{pt}\} \to Z$$

for any choice of point in each component.

So from

$$H_0(A \cap B) = \mathbb{Z}\langle p\rangle \oplus \mathbb{Z}\langle q\rangle \xrightarrow{(i_A, i_B)} \mathbb{Z} \oplus \mathbb{Z}$$

we identify the map

$$\alpha : (a, b) \mapsto (a + b, a + b)$$

and from

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j_A - j_B} H_0(S^1)$$

we have

$$\beta : (u, v) \mapsto u - v.$$

The exactness of the sequence gives

$$H_1(S^1) \cong \ker(\alpha) \cong \mathbb{Z}$$

which is generated by

$$(1, -1) \equiv (p, -q) \in H_0(A) \oplus H_0(B).$$

□