

UNIVERSITY OF CAMBRIDGE  
MATHEMATICAL TRIPOS

## Part III – **Statistical Field Theory**

Based on Lectures by *C. Thomas*  
Notes taken by ZIHAN YAN

Michaelmas 2020

*These notes may not reflect the full format and content that are actually lectured. I usually modify the notes after the lectures and sometimes my own thinking or interpretation might be blended in. Any mistake or typo should surely be mine. Be cautious if you are using this for self-study or revision.*

## COURSE INFORMATION

This course introduces the renormalization group, focusing on statistical systems such as spin models with further connections to quantum field theory.

After introducing the Ising Model, Landau's mean field theory is introduced and used to describe phase transitions. The extension to the Landau-Ginzburg theory reveals broader aspects of fluctuations whilst consolidating connections to quantum field theory. At second order phase transitions, also known as 'critical points', renormalisation group methods play a starring role. Ideas such as scaling, critical exponents and anomalous dimensions are developed and applied to a number of different systems.

## PRE-REQUISITES

Background knowledge of Statistical Mechanics at an undergraduate level is essential. This course complements the Quantum Field Theory and Advanced Quantum Field Theory courses.

## OVERVIEW

1. Phase Transitions (discontinuous, sudden change of properties)

2. Critical Points, e.g.

$$C \sim \frac{1}{|T - T_C|^{0.11008\dots}}$$

3. Universality:

- Symmetry
- Physics of different length scales, Renormalisation Group

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## 1 FROM SPIN TO FIELDS

Rec 1-1  
No-Revise

## 1.1 The Ising Model

Think about a simple model for a magnet. Consider a lattice in  $d$  spatial dimensions with  $N$  sites. On each site  $i = 1, \dots, N$  there is a discrete variable, called ‘spin’, which takes values:

$$\begin{aligned} S_i &= +1 = \uparrow \\ S_i &= -1 = \downarrow \end{aligned}$$

[Need figure Ising Lattice here.]

The system of spins  $\{S_i\}$  has energy

$$E = -B \sum_i S_i - J \sum_{\langle ij \rangle} S_i S_j$$

where  $\langle ij \rangle$  = nearest neighbour pairs,  $B$  is an external magnetic field:

$$\begin{aligned} B > 0 &\Rightarrow \text{spins prefer } \uparrow \\ B < 0 &\Rightarrow \text{spins prefer } \downarrow \end{aligned}$$

$$\begin{aligned} J > 0 &\Rightarrow \text{spins want to align} \\ J < 0 &\Rightarrow \text{spins want to anti-align} \end{aligned}$$

and we call  $\uparrow\uparrow$  ferromagnet and  $\downarrow\uparrow$  anti-ferromagnet.

In the scope of this course, we take  $J > 0$ .

For a finite temperature  $T$ , spins want to randomise to increase entropy.

For canonical ensemble, the probability of configuration  $\{S_i\}$  is

$$P[\{S_i\}] = \frac{e^{-\beta E[\{S_i\}]}}{Z}$$

where  $Z(T, J, B) = \sum_{\{S_i\}} e^{-\beta E[\{S_i\}]}$  is the *partition function*,  $\beta = 1/T$  and we choose units such that  $k_B = 1$ .

All physics can be extracted from  $Z$ .

## EXAMPLE 1.1.

- Thermodynamic free energy  $F_{\text{thermo}}(T, J, B) = \langle E \rangle - TS = -T \ln Z$ . For this we have

$$\begin{aligned} dF &= -S dT - p dV - M dB \\ -S &= \left. \frac{\partial F}{\partial T} \right|_{V, B} \quad -p = \left. \frac{\partial F}{\partial V} \right|_{T, B} \quad -M = \left. \frac{\partial F}{\partial B} \right|_{T, V} \end{aligned}$$

- Equilibrium magnetisation

$$[-1, +1] \ni m = \frac{1}{N} \sum_i \langle S_i \rangle = \frac{1}{N} \sum_{\{S_i\}} \frac{e^{-\beta E[S_i]}}{Z} \sum_i S_i = \frac{1}{N\beta} \frac{\partial \ln Z}{\partial B}$$

We want to understand how this changes as we vary  $T$  and  $B$  (equation of state).

All we need to do is compute  $Z$ . But this is very difficult and sometimes impossible. We need a different approach...

### Effective Free Energy

We rewrite  $Z$  as

$$Z = \sum_m \sum_{\{S_i\}|m} e^{-\beta E[S_i]} := \sum_m e^{-\beta F(m)}$$

where the expression  $\{S_i\}|m$  means we choose  $\{S_i\}$  such that  $m = \frac{1}{N} \sum_i S_i$ .

For large  $N$  ( $N \approx 10^{23}$ ),  $m$  is essentially continuous, we can write

$$Z = \frac{N}{2} \int_{-1}^{+1} dm e^{-\beta F(m)}$$

where  $N/2$  is there to make the normalisation correct.

It is easy to see that effective free energy  $F(m, T, J, B)$  contains more information than  $F_{\text{thermo}}$ .

We can determine the equilibrium value of  $m$  from it. Define

$$f(m) := \frac{F(m)}{N} \quad \text{such that} \quad Z = \frac{N}{2} \int_{-1}^{+1} dm e^{\beta N f(m)}$$

where  $N$  is large and  $\beta f(m)$  is of order 1.

The above integral is dominated by the minimum value of  $f(m)$  where  $m = m_{\min}$  at  $\left. \frac{\partial f}{\partial m} \right|_{m=m_{\min}} = 0$ . This is known as the saddle point/steepest descent technique. With this approximation, we have

$$Z \approx e^{-\beta N f(m_{\min})} \quad \text{so} \quad F_{\text{thermo}} \approx F(m_{\min})$$

We want to compute  $f(m)$  but this is also hard.

### Mean Field ‘Approximation’

For each  $\{S_i\}$  s.t.  $\sum_i S_i = Nm$ , we guess that the energy is approximately

$$\begin{aligned} E &\approx -B \sum_i m - J \sum_{\langle ij \rangle} m^2 \\ &= -BNm - \frac{1}{2} NJqm^2 \end{aligned}$$

where the factor of half comes from the fact that we are summing over pairs but not sites, and  $q$  is the number of nearest neighbours. (E.g.  $q = 2$  for  $d = 1$ ,  $q = 4$  for  $d = 2$  square lattice,  $q = 6$  for  $d = 3$  simple cubic lattice.)

Now we just need to count the number of configurations. If there are  $N_\uparrow$  up spins and  $N_\downarrow = N - N_\uparrow$  down spins, we have

$$m = \frac{N_\uparrow - N_\downarrow}{N} = \frac{2N_\uparrow - N}{N}$$

and the number of configurations is

$$\Omega = \frac{N!}{N_\uparrow!(N - N_\uparrow)!}$$

Using Stirling's formula

$$\ln \Omega \approx N \ln N - N_\uparrow \ln N_\uparrow - (N - N_\uparrow) \ln(N - N_\uparrow) - N + N_\uparrow + N - N_\uparrow + \dots$$

so that

$$\frac{1}{N} \ln \Omega = \ln 2 - \frac{1}{2}(1 + m) \ln(1 + m) - \frac{1}{2}(1 - m) \ln(1 - m)$$

Therefore, in the mean field approximation, we have

$$\begin{aligned} e_{-\beta N f(m)} &= \sum_{\{S_i\}|m} e^{-\beta E[S_i]} \\ &\approx \Omega(m) e^{-\beta E(m)} \\ -\beta N f(m) &\approx \ln \Omega(m) - \beta E(m) \end{aligned}$$

Then we have

$$f(m) \approx -Bm - \frac{1}{2}Jqm^2 - \frac{1}{\beta} \left[ \ln 2 - \frac{1}{2}(1 + m) \ln(1 + m) - \frac{1}{2}(1 - m) \ln(1 - m) \right]$$

Then to get the minimum value of  $m$ ,

$$\frac{\partial f}{\partial m} = 0 \quad \Rightarrow \quad \beta(B + Jqm) = \frac{1}{2} \ln \left( \frac{1 + m}{1 - m} \right)$$

and rearrange, we get

$$m = \tanh[\beta(B + Jqm)]$$

We can think about this as if

$$B \rightarrow B_{\text{eff}} = B + Jqm$$

as the background magnetic field seen from some site can be seen as modified by the magnetisation of neighbouring sites.



## 1.2 Landau Theory of Phase Transitions

Rec 1-2-1  
No-Revise

A phase transition occurs when some quantity changes discontinuously. For us this *order parameter* is  $m$ .

Landau theory is a simple yet effective way to understand phase transitions qualitatively. Based on free energy and symmetry.

Recall the free energy for the Ising model in the mean field approximation

$$f(m) \approx -Bm - \frac{1}{2}Jqm^2 - \frac{1}{\beta} \left[ \ln 2 - \frac{1}{2}(1+m) \ln(1+m) - \frac{1}{2}(1-m) \ln(1-m) \right]$$

Taylor expand for small  $m$  and discarding the constant term (as it is not important), we get

$$f(m) \approx -Bm + \frac{1}{2} \left( \frac{1}{\beta} - Jq \right) m^2 + \frac{1}{12} \frac{1}{\beta} m^4 + \dots$$

The equilibrium value of  $m$  is  $m_{\min}$ . Landau theory investigates how  $m_{\min}$  changes as we vary the parameters.

### 1.2.1 $B = 0$

$$f(m) \approx \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4 + \dots$$

where we denoted the *critical temperature*  $T_c = Jq$ .

[Need figure 2 here.]

At high  $T$ , we have  $m = 0$  as the minimum. For low  $T$ , specifically  $T < T_c$ , we have the minima

$$m = \pm m_0 = \pm \sqrt{\frac{3(T_c - T)}{T}}$$

and note that this holds only when close to  $T_c$  where  $m$  is small.

If we plot the phase diagram [Need figure 3 here.], we find  $m$  turns on abruptly as we lower  $T$ . Here,  $m$  is continuous, so this is called a *continuous phase transition* (also known as a 2<sup>nd</sup> order phase transition).

There are two phases:

- $m = 0$  is the *disordered phase*;
- $m \neq 0$  is the *ordered phase*.

$f(m)$  is invariant under a  $\mathbb{Z}_2$  symmetry  $m \rightarrow -m$ , which is inherited from the  $S_i \rightarrow -S_i$  symmetry of the Ising model when  $B = 0$ .

For  $T < T_c$ , the system chooses one of the two states  $m = \pm m_0$ , i.e. the  $\mathbb{Z}_2$  symmetry is *spontaneously broken*.

Consider the heat capacity at constant  $B = 0$  (this is a response function):

$$C = \frac{\partial \langle E \rangle}{\partial T} = \beta^2 \frac{\partial^2}{\partial \beta^2} \ln Z$$

Some algebra gives

$$f(m_{\min}) = \begin{cases} 0 & T > T_c \\ -\frac{3}{4} \frac{(T_c - T)^2}{T} & T < T_c \end{cases}$$

Using  $\ln Z \approx -\beta N f(m_{\min})$  gives

$$c = \frac{C}{N} \rightarrow \begin{cases} 0 & T \rightarrow T_c^+ \\ 3/2 & T \rightarrow T_c^- \end{cases}$$

This means  $c$  jumps discontinuously.

[Need figure 4 here.]

### 1.2.2 $B \neq 0$

$$f(m) \approx -Bm + \frac{1}{2}(T - T_c)m^2 + \frac{1}{12}Tm^4 + \dots$$

For high  $T$ , we have a unique minimum. For low  $T$ , we have two minima: one meta-stable, one true minimum.

[Need figure 5 here.]

As we vary  $T$ , the magnetisation varies smoothly

$$m \rightarrow \frac{B}{T} \quad \text{as } T \rightarrow \infty$$

and

$$m \rightarrow \pm 1 \quad \text{as } T \rightarrow 0$$

But now the sign of the magnetic field determines the sign of  $m$ . Thus no phase transition as we vary  $T$ .

[Need figure 6 here.]

If instead we fix  $T$  (low  $T$ ) and vary  $B$ , as  $B$  passes through 0,  $m$  jumps discontinuously from  $m = -m_0$  to  $m = +m_0$  — this is a *first-order phase transition*.

[Need figure 7 here.]

The phase diagram is shown below.

[Need figure 8 here.]

What happens close to the critical point? If we are right at the critical point,

$$f(m) \approx -Bm + \frac{1}{12}T_cm^4 + \dots$$

which gives

$$m \sim B^{1/3} \quad (B > 0)$$

Consider magnetic susceptibility  $\chi = \frac{\partial m}{\partial B}|_T$ , which is another response function.

Then for  $T > T_c$ ,

$$f(m) \approx -Bm + \frac{1}{2}(T - T_c)m^2 + \dots \quad \text{and} \quad m \approx \frac{B}{T - T_c}$$

so

$$\chi \sim \frac{1}{T - T_c} \quad \text{as} \quad T \rightarrow T_c^+$$

For  $T < T_c$ , similarly,

$$\chi \sim \frac{1}{|T - T_c|} \quad \text{as} \quad T \rightarrow T_c^-$$

### 1.2.3 Validity of Mean Field Theory

Rec 1-2-2  
No-Revise

Are these results correct?

- $d = 1$ : No. MFT is completely wrong! No phase transitions;
- $d = 2, 3$ : Basic structure of phase diagram is right, details near  $T_c$  are wrong;
- $d \geq 4$ : MFT gives right answers.

Similar story for other systems:

- MFT fails completely for  $d \leq d_l$ , the *lower critical dimension*;
- MFT always works for  $d \geq d_u$ , the *upper critical dimension*;
- What about  $d_l < d < d_u$ ? Often these  $d$  are interesting.

In Ising model,  $d_l = 1$  and  $d_u = 4$ .

### 1.2.4 Critical Exponents

We've seen that various quantities scale as power laws as we approach the critical point.

$$\begin{aligned} m &\sim (T_c - T)^\beta & \beta = 1/2 \quad (T < T_c) \\ c &\sim c_\pm |T - T_c|^{-\alpha} & \alpha = 0 \text{ (discontinuity)} \\ \chi &\sim |T - T_c|^{-\gamma} & \gamma = 1 \\ m &\sim B^{1/\delta} & \delta = 3 \end{aligned}$$

Compare MFT results with true values for Ising model in  $d = 2, 3$ , we have

	MFT	$d = 2$	$d = 3$
$\alpha$	0 (disc.)	0 (log div.)	0.1101...
$\beta$	1/2	1/8	0.3264...
$\gamma$	1	7/4	1.2371...
$\delta$	3	15	4.7898...

### 1.2.5 Universality

Liquid-gas transition shows many similarities to the Ising model. [\[Need figure 9 here.\]](#)

Again, a line of first-order phase transitions ending at a critical point.

At fixed  $p$ ,  $v = V/N$  jumps discontinuously at liquid-gas first-order phase transition. Suggests  $v$  is order parameter analogous to  $m$  in Ising model.

Using an equation of state (e.g. van der Waals equation), at the critical point:

$$\begin{aligned} v_{\text{gas}} - v_{\text{liquid}} &\sim (T_c - T)^\beta & \beta = 1/2 \quad (T < T_c) \\ v_{\text{gas}} - v_{\text{liquid}} &\sim (p - p_c)^{1/\delta} & \delta = 3 \quad (p \rightarrow p_c^+) \\ \text{Compressibility, } \kappa = -\frac{1}{v} \frac{\partial v}{\partial p} \Big|_T &\sim |T - T_c|^{-\gamma} & \gamma = 1 \\ \text{Heat capacity, } C_V &\sim C_\pm |T - T_c|^{-\alpha} & \alpha = 0 \end{aligned}$$

None of these agree with experiment.

All liquid-gas transition show the same critical exponents, and these agree with  $d = 3$  Ising.

*Universality*: critical point governs behaviour of many different physical systems. Why?!

### 1.2.6 Ising Model as Lattice Gas

Consider the same  $d$ -dimensional lattice, but now with particles hopping between sites.

The number of particles on site is  $n_i$ , and we allow no more than one particle on each site, i.e. either  $n_i = 0$  (empty) or  $n_i = 1$  (filled). Then we have

$$E = -4J \sum_{\langle ij \rangle} n_i n_j - \mu \sum_i n_i$$

where  $J > 0$  is acting as an attractive force and  $\mu$  is the chemical potential.

This is equivalent to the Ising model if we identify  $S_i = 2n_i - 1$ .