University of Cambridge
Mathematical Tripos

Part III – **Quantum Field Theory**

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These notes may not reflect the full format and content that are actually lectured. I usually modify the notes after the lectures and sometimes my own thinking or interpretation might be blended in. Any mistake or typo should surely be mine. Be cautious if you are using this for self-study or revision.

Course Information

Quantum Field Theory is the marriage of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, spinor (Dirac), and vector (gauge) fields. The relativistic invariance and symmetry properties of these fields are discussed using the language of Lagrangians and Noether's theorem.

The quantisation of the basic non-interacting free fields is firstly developed using the Hamiltonian and canonical methods in terms of operators which create and annihilate particles and anti-particles. The associated Fock space of quantum physical states is explained together with ideas about how particles propagate in spacetime and their statistics.

Interactions betweeen fields are examined next, using the interaction picture, Dyson's formula and Wick's theorem. A 'short version' of these techniques is introduced: Feynman diagrams. Decay rates and interaction cross-sections are introduced, along with the associated kinematics and Mandelstam variables.

Spinors and the Dirac equation are explored in detail, along with parity and γ^5. Fermionic quantisation is developed, along with Feynman rules and Feynman propagators for fermions.

Finally, quantum electrodynamics (QED) is developed. A connection between the field strength tensor and Maxwell's equations is carefully made, before gauge symmetry is introduced. Lorentz gauge is used as an example, before quantisation of the electromagnetic field and the Gupta-Bleuler condition. The interactions between photons and charged matter is governed by the principal of minimal coupling. Finally, an example QED cross-section calculation is performed.

Pre-requisites

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics.

Introduction

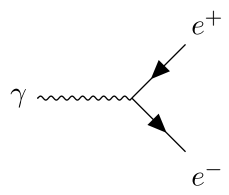
Why do we need QFT?

If we consider the description of the interaction between two charged particles, for an accurate prediction, we need

- *Maxwell's theory* using a field theoretic perspective (using electric field $\mathbf{E}(\mathbf{x}, t)$ and magnetic field $\mathbf{B}(\mathbf{x}, t)$), which embodies *locality* and encodes *Lorentz invariance*;
- And *quantum mechanics* as the charged particles are too small to be considered classically. This encodes the *"discrete"* and *probabilistic* nature.

The reconcile of special relativity and quantum mechanics leads to *Quantum Field Theory*, where new phenomena emerge:

- Particle creation;



with $\quad E_\gamma \geq 2m_e c^2$

- Bose/Fermi statistics;
- And more...

It is typical to consider experiments where some initial state $|i\rangle$ transitions to $|f\rangle$.

[Need figure 1 here.]

The challenge we are facing is to calculate the probability

$$P_{i\to f} = |\mathcal{A}_{i\to f}|^2$$

where $A_{i\to f} \in \mathbb{C}$ is the amplitude of such transition.

The properties of such probability are

- Unitarity: $\sum_f P_{i\to f} = 1$;
- Lorentz covariance.

What is QFT?

Instead of QM, we start from classical field theory.

For a classical field $\phi(\mathbf{x}, t)$, we can use *canonical quantisation* analogous to that in particle QM to obtain a quantum field operator $\hat{\phi}(\mathbf{x}, t)$. We will mainly follow this route in this course.

Quantum field operators are very complicated things — they are operator-valued functions in space and time.

There are some key facts in QFT:

- Eigenstates of *Hamiltonian* $\hat{H}[\hat{\phi}, \hat{\pi}]$ describes *multi-particle states.* (We can obtain the eigenstates at least in a free field theory);
- We can calculate
$$\mathcal{A}_{i\to f} = \langle f|e^{\mathrm{i}\hat{H}T}|i\rangle \,.$$

The remarkable feature of field theories is that there are very few which they can be consistently calculated. Thus we are left to limited, confined choices, which is an attractive character QFTs. Among these, *gauge theories* are a special type, using which we can successfully describe our universe to some extent.

Outline

1. Classical Field Theory
2. Free QFT
3. Interacting QFT
4. Fermions
5. QED

Contents

0 Preliminaries

Lecture 1
No-Revise

0.1 Units

In SI units, we know the dimensions of several fundamental constants are

$$\begin{aligned} c &\sim LT^{-1} \\ \hbar &\sim L^2 M T^{-1} \\ G &\sim L^3 M^{-1} T^{-2} \end{aligned}$$

In this course, we choose *natural units* such that

$$\hbar = c = 1$$

Note that we are eliminating several dimensions identified in SI units, thus all quantities scale with some power of *mass* or *energy*.

$$X \sim M^\delta$$

where we call δ the dimension of the quantity and write

$$[X] = \delta$$

Example 0.1.

$$[E] = +1 \qquad [L] = -1$$

0.2 Relativity

Unless otherwise stated, we use Einstein summation convention throughout the course.

We work in Minkowski spacetime $\mathbb{R}^{3,1}$ with metric tensor

$$\eta_{\mu\nu} = \mathrm{diag}\{+1, -1, -1, -1\}$$

with the above conventional signature.

We denote the coordinates in spacetime as

$$x^\mu = (t, \mathbf{x})$$

and using the metric, we have

$$x_\mu = \eta_{\mu\nu} x^\nu$$

known as "lowering" the indices.

Similarly, for "raising" the indices we use the inverse metric tensor $\eta^{\mu\nu}$ defined by

$$\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho$$

1 Classical Field Theory

1.1 Lorentz Covariant Fields

Firstly, let's recall that the Minkowski metric is invariant under Lorentz transformations. And we consider the Lorentz transformation of coordinates:

$$x^\mu \mapsto (x')^\mu = \Lambda^\mu{}_\nu x^\nu \tag{1.1.1}$$

where $\Lambda^\mu{}_\nu$ is a 4×4 matrix encoding the Lorentz transformation. To find the properties of such Λ, we write

$$\Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau \eta^{\sigma\tau} = \eta^{\mu\nu} \tag{1.1.2}$$

Also, we exclude the time-reversal transformations by imposing

$$\det \Lambda = +1$$

The conditions above fix *proper orthochronous* Lorentz transformations. These have 6 degrees of freedom: 3 rotations + 3 boosts.

Lorentz transformations form a group under composition. It is actually a Lie group, called *Lorentz group*, denoted as $\mathrm{SO}(3,1)$.

Now we bring out the main characters — *fields.*

Definition 1.1. A *scalar field* is a function $\phi(x) = \phi(t, \mathbf{x})$

$$\phi : \underbrace{\mathbb{R}^{3,1}}_{\text{spacetime}} \to \underbrace{\mathbb{R}}_{\text{field space}}$$

such that it transforms as

$$\phi(x) \to \phi'(x) := \phi(\Lambda^{-1} \cdot x) \tag{1.1.3}$$

under (active) Lorentz transformation.

Note. By the group property of Lorentz transformations, we can write

$$\left(\Lambda^{-1} \cdot x\right)^\mu = \left(\Lambda^{-1}\right)^\mu{}_\nu x^\nu$$

where

$$\left(\Lambda^{-1}\right)^\mu{}_\nu \Lambda^\nu{}_\rho = \delta^\mu_\rho$$

Often we denote $\left(\Lambda^{-1}\right)^\mu{}_\nu$ as $\Lambda_\nu{}^\mu$.

Now consider the spacetime derivatives of some scalar field ϕ:

$$\partial_\mu \phi(x) := \frac{\partial \phi(x)}{\partial x^\mu}$$

We find it transforms as

$$\partial_\mu \phi(x) \to \Lambda_\mu{}^\nu \partial_\nu \phi(\Lambda^{-1} \cdot x)$$

We can raise the index as

$$\partial^\mu \phi(x) = \eta^{\mu\nu} \partial_\nu \phi(x)$$

EXERCISE 1.2. Show $\partial^\mu \phi(x)$ transforms as a *4-vector field* such that

$$\partial^\mu \phi(x) \to \Lambda^\mu{}_\nu \partial^\nu \phi(\Lambda^{-1} \cdot x)$$

The consequence is that

$$\partial_\mu \phi(x) \partial^\mu \phi(x) = \eta^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x)$$

transforms as a scalar field. That is

$$\partial_\mu \phi \partial^\mu \phi(x) \to \partial_\mu \phi \partial^\mu \phi(\Lambda^{-1} \cdot x)$$

1.2 Lagrangian Formulation

Lecture 2
No-Revise

1.2.1 Review of Lagrangian Formulation

Here we consider a non-relativistic particle with mass m and moving in a potential $V(q)$. Here we use $q = q(t)$ to denote its position at time t. Then its Lagrangian is

$$L(t) = L\left(q(t), \dot{q}(t)\right) = \frac{1}{2} \dot{q}^2 - V(q) \tag{1.2.1}$$

The *action* in time interval $[t_i, t_f]$ of this Lagrangian L is defined to be

$$S[q] := \int_{t_i}^{t_f} \mathrm{d}t\, L\left(q(t), \dot{q}(t)\right) \tag{1.2.2}$$

The equation(s) of motion can be obtained by the *principle of least action*. The plan is

- We vary the path of the particle as $q(t) \to q(t) + \delta q(t)$ and fix the end points by $\delta q(t_i) = \delta(q_f) = 0$.
- The equations of motion, known as *Euler-Lagrange equations* are obtained by imposing the condition that the action is stationary, i.e.

$$\delta S = 0.$$

The above principle can be easily and clearly generalised in later contexts.

For this specific example, we get the Euler-Lagrange equation

$$\ddot{q} = -\frac{\partial V}{\partial q}. \tag{1.2.3}$$

1.2.2 Scalar Field Theories

To construct such a scalar field theory of physical significance, we require

- It should have *Lorentz invariant* action;

- It has *locality*, i.e. no coupling between fields (and their spacetime derivatives) at different points;
- It has *at most two time derivatives.*

From now on, for convenience, we use x to denote some point $(\mathbf{x}, t)$ in spacetime.

DEFINITION 1.3. The *Lagrangian* of a scalar field $\phi(x)$ is some time-dependent functional of ϕ and its derivatives, in the form of

$$L(t) = L[\phi, \partial_\mu \phi] = \int \mathrm{d}^3x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \tag{1.2.4}$$

where $\mathcal{L}$ is called the *lagrangian density*, which itself is a function of ϕ and its derivatives.

NOTE. The proper 'Lagrangian' is in fact barely used in the context. Instead, the 'Lagrangian density' appears far more often. Thus, we usually abuse the terminology and refer to 'Lagrangian density' just as 'Lagrangian'.

DEFINITION 1.4. The *action* of a scalar field $\phi(x)$ for some time interval $[t_i, t_f]$ is a functional of the field and its derivatives in the form of

$$S_{t_i,t_f}[\phi, \partial_\mu \phi] = \int_{t_i}^{t_f} \mathrm{d}t \, L[\phi, \partial_\mu \phi] = \int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}^3x \, \mathcal{L}. \tag{1.2.5}$$

Practically, we often choose infinite time interval $t_i \to -\infty, t_f \to +\infty$, thus

$$S[\phi, \partial_\mu \phi] = \int_{\mathbb{R}^{3,1}} \mathrm{d}^4x \, \mathcal{L}(\phi(x), \partial_\mu \phi(x)). \tag{1.2.6}$$

Under Lorentz transformation

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

we require that

$$\mathcal{L}(x) = \mathcal{L}(\phi(x), \partial_\mu \phi(x))$$

transforms as a scalar field, i.e.

$$\mathcal{L}(x) \to \mathcal{L}(\Lambda^{-1} \cdot x)$$

Thus, changing variables by $y^\mu = \Lambda_\nu{}^\mu x^\nu$ and noting $\det \Lambda = +1$ we have

$$S \to S' = \int \mathrm{d}^4x \, \mathcal{L}(\Lambda^{-1} \cdot x) = \int \mathrm{d}^4y \, \mathcal{L}(y) = S,$$

i.e. the action is invariant under Lorentz transformation.

Recall our requirements for any physical scalar field, the most general Lagrangian of some scalar field ϕ is of the form

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \tag{1.2.7}$$

for some scalar function $V(\phi)$.

NOTE. We don't need to consider terms like $\phi\partial_\mu\partial^\mu\phi$, as they are the same as $\partial_\mu\phi\partial^\mu\phi$ up to some surface term (which has no contribution to the action).

Now we generalise the principle of least action to classical (scalar) field theories.

POSTULATE 1 (Principle of least action). *The equations of motion (i.e. Euler-Lagrange equations) of a scalar field $\phi(x)$ should make the action $S[\phi, \partial_\mu\phi]$ stationary, i.e. $\delta S = 0$, under variations $\phi(x) \to \phi(x) + \delta\phi(x)$ subject to the boundary condition $\delta\phi(x) = \delta\phi(t, \mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ or $t \to \pm\infty$.*

To get the equation(s) of motion of ϕ, we vary the action and set it to zero, by

$$\begin{aligned}\delta S &= \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right] \\ &= \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\delta\phi + \underbrace{\int \mathrm{d}^4x\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)}_{=\int_{\partial\mathbb{R}^{3,1}}\mathrm{d}S_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi=0}.\end{aligned}$$

Thus, by setting $\delta S = 0$, we have, for any variation $\delta\phi$, the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0. \tag{1.2.8}$$

For our 'ansatz'

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \tag{1.2.9}$$

we have

$$\frac{\partial\mathcal{L}}{\partial\phi} = -V'(\phi) := \frac{\mathrm{d}V}{\mathrm{d}\phi} \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi.$$

so that the Euler-Lagrange equation is

$$\partial_\mu\partial^\mu\phi + V'(\phi) = 0. \tag{1.2.10}$$

Let's see a special case of particular importance.

EXAMPLE 1.5. Klein-Gordon field theory has potential term

$$V(\phi) = \frac{1}{2}m^2\phi^2$$

and the equation of motion is called *Klein-Gordon equation*

$$\boxed{\partial_\mu\partial^\mu\phi + m^2\phi = 0}. \tag{1.2.11}$$

If we write the derivatives in terms of space and time explicitly, we have

$$\partial_\mu\partial^\mu = \frac{\partial^2}{\partial t^2} - \sum_i \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial t^2} - \nabla^2$$

and the Klein-Gordon equation is actually a wave equation. It is a linear partial differential equation and has wavelike solutions such as

$$\phi \sim e^{\mathrm{i}p\cdot x} = e^{\mathrm{i}\omega t - \mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

with dispersion relation

$$\omega_{\mathbf{k}} = \sqrt{|\mathbf{k}|^2 + m^2}.$$

These solutions can be superposed, giving certain wave packets. (Think about Fourier transform.)