University of Cambridge
Mathematical Tripos

# Part III – **Symmetries, Fields and Particles**

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*These notes may not reflect the full format and content that are actually lectured. I usually modify the notes after the lectures and sometimes my own thinking or interpretation might be blended in. Any mistake or typo should surely be mine. Be cautious if you are using this for self-study or revision.*

## Course Information

Lie groups and Lie algebras are important in the construction of quantum field theories which describe interactions between known particles. Gauge theories, which describe many of the interactions in the Standard Model, rely on them. After some other preliminaries, we introduce representations in terms of square matrices. The group of rotations in three-dimensional space SO(3) is covered, along with SU(2) and the connection to angular momentum. Relativistic symmetries are discussed: in particular, the Lorentz and Poincaré groups and quantum fields. Lie groups and Lie algebras are covered in more generality, focusing on SU(3) as a useful example. An overview of the results of the Cartan classification of simple Lie algebras is included. Finally, gauge theory is introduced.

## Pre-requisites

Linear algebra including direct sums and tensor products of vector spaces. Special relativity and quantum theory, including orbital angular momentum theory and Pauli spin matrices.

# Contents

# 0 Introduction

## 0.1 Symmetries

Lecture 1
No-Revise

Definition 0.1. A *group* $G$ is a set $G = \{g_1, g_2, \dots\}$ with

1. A composition rule (binary operation) $*$ such that $g * g' \in G, \forall g, g' \in G$, which we shall write as $gg'$;
2. A unique identity $e$ such that $eg = ge = g, \forall g \in G$;
3. Associativity: $(gg')g'' = g(g'g'') := gg'g'', \forall g, g', g'' \in G$;
4. A unique inverse $\forall g \in G, \exists g^{-1}$ such that $gg^{-1} = g^{-1}g = e$.

If the binary operation is commutative, we say that $G$ is *abelian*.

Example 0.2. Group $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ with group operation being addition modulo $n$ and identity $e = 0$.

Cyclic group $C_n = \{e^{2\pi \mathrm{i} r/n} \in \mathbb{C} : r = 0, 1, \dots, n-1\}$, certain complex numbers of modulus 1, under multiplication.

$\mathbb{Z}_n$ and $C_n$ are clearly abelian. In fact, $C_n \cong \mathbb{Z}_n$, i.e. they're *isomorphic*, that is to say there exists a one-to-one correspondence between the elements consistent with group composition rules.

Example 0.3. Symmetry groups such as the dihedral group $D_3$ [Need figure 1 here.] containing reflections along axes and rotations by $120°, 240°, 360°$.

Example 0.4. *Lie groups* are the generalisation to continuous symmetries, e.g. rotations by $\theta \in \mathbb{R}$ of a circle ("SO(2)"). Lie groups are essential to the description of particles and their interactions.

To identify the connection between symmetries and groups, we first make the following definition.

Definition 0.5. A *symmetry* is a transformation that leaves physical properties (e.g. energy, scattering probability, etc.) unchanged. They have properties:

- Symmetries can be composed: $gg'$ := act first with $g'$, then with $g$;
- Doing nothing is a symmetry, $e$, the identity;
- A symmetry transformation $g$ can be reversed by $g^{-1}$, which is itself a symmetry.

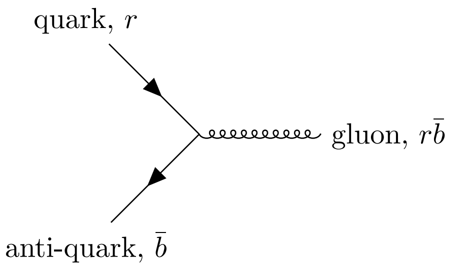
From above, it is clear that the set of all symmetries forms a group. Symmetry often greatly simplifies analysis. It leads to conservation rules and constrains interactions.

### 0.1.1 Internal Symmetries

*Internal symmetries* are properties of particles or fields themselves.

EXAMPLE 0.6 (Colour states of a quark). Quarks come in three otherwise identical copies — called 'colours' (red, green and blue). One can continuously rotate the colours into each other, resulting in a symmetry.

One can rotate the colour differently at different points of spacetime. In fact, one finds that one has to add a force-carrying particle to make the whole theory invariant under the symmetry. This is the gluon, which carries a colour and an anti-colour. Below is a Feynman diagram representing the fusion of a quark and an anti-quark




Anti-quarks carry anti-colour $\{\bar{r}.\bar{g},\bar{b}\}$. The group structure implies that colour is conserved by interactions (i.e. $r\bar{b} \to r\bar{b}$ in $q\bar{q} \to g$).

When the theory is left invariant by a symmetry transformation that's the same across whole spacetime, it's called a *global symmetry*.

The theory of quarks, anti-quarks and gluons is called *Quantum Chromodynamics* (QCD), which is a part of the Standard Model of particle physics.

Since the colour rotations may differ at different points $(\mathbf{x}, t)$ in spacetime, it is called a *local* or *gauge* symmetry.

### 0.1.2 External Symmetries

*External symmetries* involve spacetime coordinates.

EXAMPLE 0.7.

- Translation in $(\mathbf{x}, t)$;
- Lorentz transformation: boosts/rotations;

Conserved quantities come from the group structure: e.g. energy, momentum, angular momentum, etc. The *Poincaré group* consists of all these symmetries: 3 boosts, 3 rotations and 4 translations.

Group theory has also been used in cases where the symmetries are approximate but not exact, to explain the spectrum of hadrons, for instance.

## 0.2 Particles

### 0.2.1 Force-carriers

Force-carriers are particles with spin $1(\hbar)$ (convention $\hbar \to 1$, see QFT course).

EXAMPLE 0.8.

- $g$, gluon carries colour force;
- $\gamma$, photons carry the electromagnetic force.
- $W^{\pm}, Z^0$ boson carry electroweak force that mediates radioactive decay.

NOTE. Bosons are integer spin particles. Fermions are half-integer spin particles.

For spin 2, we have graviton, the force carrier of gravity. It is not seen yet because gravity is so weak.

Force carriers belonging to a good symmetry are *massless*. Those corresponding to one where the vacuum "spontaneously" breaks an underlying symmetry may be massive (i.e. $W^{\pm}, Z^0$ bosons, the symmetry is broken by the Higgs mechanism).

### 0.2.2 Matter Particles

Matter particles are of spin $\frac{1}{2}$.

EXAMPLE 0.9.

- Up quarks, electric charge $Q = +2/3$ (choosing units where $e = 1$);
- Down quarks, $Q = -1/3$;
- Neutrinos, $Q = 0$;
- Electrons, $Q = -1$.

They all have anti-particles, with opposite sign charge or anti-colour.

Matter particles additionally come in 3 families, each are heavier than the last but otherwise with the same colour and charge.

| Family | $Q = +2/3$ | $Q = -1/3$ | $Q = -1$ | $Q = 0$ |
|---|---|---|---|---|
| 1 | up $u$ | down $d$ | electron $e$ | $e$-neutrino $\nu_e$ |
| 2 | charm $c$ | strange $s$ | muon $\mu$ | $\mu$-neutrino $\nu_\mu$ |
| 3 | top $t$ | bottom $b$ | tauon $\tau$ | $\tau$-neutrino $\nu_\tau$ |

Anti-particles are denoted with a bar above, e.g. $\bar{\nu}_e, \bar{u}$, etc.

The Standard Model explains many of these features with a QFT possessing a particular group structure of symmetries. Each particle has its own field which fills the spacetime. Quantum excitations of the fields are observed in experiments.

# 1 Groups: Basics

Lecture 2
No-Revise

## 1.1 Basic Concepts

Recall the definition of a group from last section. Here, we introduce some more definitions and facts about groups.

Definition 1.1. A discrete group $G$ with $n$ elements has *order* $|G| = n$.

Definition 1.2. For any group $G$, a *subgroup* $H \subset G$ is naturally defined as a set of elements belonging to $G$ which is also a group itself. A *proper subgroup* is when $H \neq G$, and is denoted $H < G$.

Definition 1.3. For any subgroup $H$, we may define an equivalence relation between $g_i, g_i' : g_i \sim g_i' \Leftrightarrow g_i = g_i' h$ for $h \in H$. Each equivalence class defines a *coset* and has $|H|$ elements. The cosets form a *coset space* $G/H$ such that $G/H \simeq G/\sim$ and $\dim G/H = |G|/|H|$. In general $G/H$ isn't a group.

Theorem 1.4 (Lagrange's theorem). *For any subgroup $H \subset G$, $|H|$ divides $|G|$.*

Definition 1.5. The *index* of subgroup $H$ in $G$ is the number of cosets in $G/H$, denoted $G : H = |G|/|H|$.

Definition 1.6. A *normal*, or *invariant* subgroup is a subgroup $H \subset G$ such that

$$gHg^{-1} = H \quad \forall g \in G.$$

This is denoted $H \triangleleft G$ (or $G \triangleright H$).

Proposition 1.7. *For a normal subgroup $H$ of $G$, $G/H$ becomes a group.*

*Proof Sketch.* For $g_i' = g_i h_i, g_j' = g_j h_j$ with $h_i, h_j \in H$, then $g_i' g_j' = g_i g_j h$ for some $h \in H$. (Try to complete the proof.) □

Corollary 1.8. *For an abelian group, all subgroups are normal subgroups.*

Definition 1.9. A group is *simple* if the only normal subgroups are $G$ and the trivial subgroup formed by the identity $e$ itself.

Definition 1.10. The *centre* of a group $G$, denoted $\mathcal{Z}(G)$, is the set of all elements which commute with all elements of $G$. It is an abelian, normal subgroup.

Definition 1.11. For two groups $G_1$ and $G_2$, we may define a *direct product group* $G_1 \times G_2$ formed by pairs of elements $\{(g_1, g_2)\}$ belonging to $(G_1, G_2)$, defined by the rules

- $(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2')$;
- $(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1})$;

- $e = (e_1, e_2)$.

So long as it is clear which elements belong to $G_1$ and which to $G_2$, we may write the elements of $G_1 \times G_2$ as $g_1 g_2$ or $g_2 g_1$.

Corollary 1.12. *For finite groups* $|G_1 \times G_2| = |G_1||G_2|$.

## 1.2 Cyclic, Dihedral and Permutation Groups

### 1.2.1 Cyclic Groups

Clearly $\mathbb{Z}_n$ is abelian. For $p$ prime, $\mathbb{Z}_p$ has no subgroups. $p$ has no divisors, hence $\mathbb{Z}_p$ is simple.

Proposition 1.13. *If* $n = pq$ *then* $\mathbb{Z}_p$ *and* $\mathbb{Z}_q$ *are normal subgroups of* $\mathbb{Z}_n$ *and* $\mathbb{Z}_{pq}/\mathbb{Z}_p \simeq \mathbb{Z}_q$. *If* $p, q$ *are co-prime (i.e. no common factors), then* $\mathbb{Z}_{pq} \simeq \mathbb{Z}_p \times \mathbb{Z}_q$.

### 1.2.2 Dihedral Groups

Definition 1.14. The *dihedral group* $D_n$ of order $2n$, is the symmetry groups of a regular $n$-sided polygon, formed by rotations $a$ through angles $2\pi r/n, r = 0, \cdots, n-1$, together with reflections$b$. In general,

$$D_n = \{a^r, a^r b : (r = 0, \cdots, n-1; a^0 = a^n = e; b^2 = e; ab = ba^{r-1})\}.$$

For any $r$, we have $(a^r b)^2 = e$.

For $n > 2$, the group is non-abelian since $ab \neq ba$.

It is an easy observation that

$$D_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

### 1.2.3 Permutation (or Symmetric) Groups

Definition 1.15. The *permutation group* $S_n$ acting on $n$ objects, is the group of all permutations of these $n$ objects. It has $|S_n| = n!$.

Example 1.16. $S_3 \simeq D_3$: symmetry group of equilateral triangle under permutations of vertices.

Definition 1.17. $A_n$, the *alternating group*, is a normal subgroup of $S_n$ formed by the even permutations. $|A_n| = n!/2$.

Corollary 1.18. *For* $n \geq 5$, $A_n$ *is simple.*

Proposition 1.19.
$$S_n/A_n \simeq \mathbb{Z}_2.$$

Also, note $A_3 \simeq \mathbb{Z}_3$.

The elements of the permutation group can be decomposed into *cycles*. Acting on $\{1, 2, \cdots, n\}$,

- 2-cycle $(ij)$ with $i \neq j$, swaps $i$ and $j$;
- 3-cycle $(ijk)$ with $i \neq j \neq k$, takes $i \to j \to k \to i$;
- ...

Thus we generalise to the following.

Definition 1.20. A *p-cycle* $(i_1 i_2 \cdots i_p)$ for all $i_j$ different with $p \leq n, 1 \leq i_j \leq n$ generates cyclic permutations of $\{i_1, \cdots, i_p\}$.

Note that $(i_1 i_2 \cdots i_p)^p = e$.

For any one of the $\binom{n}{p}$ choices of $\{i_j\}$, there exists $(p-1)!$ choices for the $p$-cycle involving $\{i_j\}$, since any $p$-cycle is invariant under cyclic permutations.

For distinct $i, j, k, l$
$$(ij)(kl) = (kl)(ij) \quad \text{and} \quad (ij)(jk) = (ijk).$$

Proposition 1.21. *The action of some $g \in S_n$ can be decomposed into cycles.*

*Proof.* Consider an arbitrary $i \in \{1, 2, \cdots, n\}$ and act $g$ on $i$ by $g^r i, r = 1, 2, \cdots$ For some minimal $p$, we have $g^p i = i$. The action of $g$ then generates a $p$-cycle $(i_1 \cdots i_p)$.

Now we pick $j \in \{1, 2, \cdots, n\} \backslash \{i_1, \cdots, i_p\}$ acting repeatedly with $g$ generates a new $q$-cycle $(j_1 \cdots j_q)$ for some $q$ and $j_1 = j$. Continuing, any element of $\{1, 2, \cdots, n\}$ belongs to some cycle. □

If $gk = k$, then the element belongs to the 1-cycle $(k)$.

We may denote $g$ as $g_{(i_1 \cdots i_p)(j_1 \cdots j_q)\cdots}$, $e = e_{(1)(2)\cdots(n)}$ and $g^{-1} = g_{(i_p \cdots i_1)(j_q \cdots j_1)\cdots}$.

If $h$ corresponds to a permutation $\sigma$ where $\sigma\{1, 2, \cdots, n\} = \{\sigma(1), \sigma(2), \cdots, \sigma(n)\}$, then
$$h g_{(i_1 \cdots i_p)(j_1 \cdots j_q)\cdots} h^{-1} = g_{(\sigma(i_1) \cdots \sigma(i_p))(\sigma(j_1) \cdots \sigma(j_q))\cdots}.$$

## 1.3 Orbit Stabiliser Theorem

Now we introduce *orbit stabiliser theorem*, which applies when a group $G$ acts an a space $X = \{x\}$ such that $\forall g \in G, x \to gx$.

DEFINITION 1.22. For any particular $x \in X$, the *stabiliser group* or *little group* $G_x$ is defined by those elements of $G$ which leave $x$ invariant, i.e.

$$G_x = \{h : h \in G, hx = x\}.$$

a subgroup of $G$.

DEFINITION 1.23. The *orbit* of $x$ is the set of points in $X$ obtained by the action of $G$,

$$O_x = \{x' : x' = gx, \forall g \in G\}.$$

THEOREM 1.24 (Orbit stabiliser theorem). *$O_x$ can be identified with $G/G_x$. For a finite group $G$, $\dim O_x = |G|/|G_x| \in \mathbb{Z}$ by Lagrange's theorem. For $x' \in O_x$, $G_{x'} \simeq G_x$. In general, the space $X$ can be decomposed into orbits under the action of $G$.*

HINT. $G_{x'} \simeq G_x$ since $hx = x$ and $x' = gx$, giving $h'x' = x'$ for $h' = ghg^{-1}$.

### 1.4 Automorphisms and Semi-Direct Product

Lecture 3
No-Revise

DEFINITION 1.25. An *automorphism* of a group $G = \{g_i\}$ is defined as a mapping between elements $g_i \to \phi(g_i)$ such that the product rule is preserved, i.e.

$$\phi(g_i)\phi(g_j) = \phi(g_i g_j), \quad g_i, g_j \in G$$

and $G_\phi = \{\phi(g_i)\} \simeq G$. It must have $\phi(e) = e$ and $\phi(g^{-1}) = \phi(g)^{-1}$.

For any fixed $g \in G$ we may define an *inner automorphism* by

$$\phi_g(g_i) = g g_i g^{-1}.$$

Automorphisms not of this form are called *outer automorphisms*.

PROPOSITION 1.26. *The set of all automorphisms forms a group* $\operatorname{Aut} G$*, which should include $G/\mathcal{Z}(G)$ as a normal subgroup.*

PROPOSITION 1.27. *For any abelian group, there are no non-trivial inner automorphisms, but there can be outer ones.*

EXAMPLE 1.28. For $\mathbb{Z}_3$, take $\{e, a, a^2\} \to \{e, a^2, a\}$. In this case, $\operatorname{Aut} \mathbb{Z}_3 = \mathbb{Z}_2$ and $\mathbb{Z}_3/\mathcal{Z}(\mathbb{Z}_3) = \{e\}$, the trivial one-element group.

LEMMA 1.29. *If $H \subset \operatorname{Aut} G$ such that for any $h \in H$ and any $g \in G$, we have*

$$g \xrightarrow{h} \phi_h(g)$$

*with*

$$\phi_h(g_1)\phi_h(g_2) = \phi_h(g_1 g_2)$$

*and*

$$\phi_{h_1}(\phi_{h_2}(g)) = \phi_{h_1 h_2}(g)$$

*also* $\phi_h(e) = e, \phi_e(g) = g, \phi_{h^{-1}}(g) = \phi_h^{-1}(g)$.

DEFINITION 1.30. The above proposition allows us to define a new group called the *semi-direct product* of $H$ with $G$, denoted as $H \ltimes G$. As with the direct product, this is defined in terms of pairs of elements $(h, g)$ belonging to $(H, G)$ but with a less trivial product rule

$$\begin{aligned}(h, g)(h', g') &= (hh', g\phi_h(g')),\\ (h, g)^{-1} &= (h^{-1}, \phi_{h^{-1}}(g)).\end{aligned}$$

COROLLARY 1.31. *The above product rule of* $H \ltimes G$ *implies*

$$(h, e)(e, g)(h, e)^{-1} = (e, \phi_h(g)).$$

It's often convenient to write the elements of $H \ltimes G$ as $(h, g) \to hg := \phi_h(g)h$ as an abbreviation.

PROPOSITION 1.32. *$G$ is a normal subgroup of $H \ltimes G$.*

*Proof sketch.*

$$(h, g)(e, g')(h, g)^{-1} = (e, g\phi_h(g')g^{-1})$$

for any $g, g' \in G$ and $h \in H$ so that

$$H \simeq (H \ltimes G/G).$$

□

EXAMPLE 1.33. $D_n \simeq \mathbb{Z}_2 \ltimes \mathbb{Z}_n$, where

$$\mathbb{Z}_2 = \{e, b : b^2 = e\}$$

and

$$\mathbb{Z}_n = \{a^r : r = 0, \cdots, n-1, a^n = e\}$$

and we define for any $g = a^r \in \mathbb{Z}_n$,

$$\phi_b(g) = g^{-1} = bgb^{-1}.$$

## 1.5 Conjugacy Classes

DEFINITION 1.34. If $g_j = gg_jg^{-1}$ for some $g \in G$, then $g_i$ is *conjugate* to $g_j$, written as $g_i \sim g_j$. The equivalence relation $\sim$ divides $G$ into *conjugacy classes*

$$\mathcal{C}_r = \{g_i : g_i \sim g_i' = gg_ig^{-1}, g \in G\}.$$

The identity $e$ is in a conjugacy class by itself. For an abelian group, all elements have their own conjugacy class.

Elements of a conjugacy class have similar properties, e.g. $g_i^n = e$ for the same $n$ for all $g_i \in \mathcal{C}_r$.

EXAMPLE 1.35. For $S_3 = \{e, a, a^2, b, ab, a^2b\}$ with $b = (1\ 2), a = (1\ 2\ 3)$, there exist 3 conjugacy classes

$$\{e\}, \quad \{a, a^2\}, \quad \{b, ab, a^2b\}.$$

NOTE. If we have $a^n = e$, then $ga^ng^{-1} = e$ so that $(gag^{-1})^n = e$, too.

### 1.6 Normaliser, Centraliser, Commutator

DEFINITION 1.36. For a subgroup $H \subset G$, the elements $g \in G$ such that $ghg^{-1} \in H, \forall h \in H$ (we write this as $gHg^{-1} = H$) form a subgroup of $G$ which contains $H$ itself, called the *normaliser* of $H$ in $G$, denoted as $N_G(H)$.

COROLLARY 1.37. *Clearly,* $H \triangleleft N_G(H)$.

DEFINITION 1.38. The subgroup of $G$ formed by elements such that $ghg^{-1} = h, \forall h \in H$ forms the *centraliser* $C_G(H)$.

COROLLARY 1.39. *Necessarily,* $C_G(H) \subset N_G(H)$.

DEFINITION 1.40. For any $g \in G, h \in G$,

$$[g, h] := g^{-1}h^{-1}gh$$

is the *commutator* of $g$ and $h$.

We say $g$ is *abelian*, if

$$[g, h] = e, \quad \forall h \in G.$$

More generally, if $[g, h] = e$, we say that $g$ and $h$ *commute.*

COROLLARY 1.41. *In general*

$$[g, h]^{-1} = [h, g]$$

*and for any* $g' \in G$,

$$g'[g, h]g'^{-1} = [g'gg'^{-1}, g'hg'^{-1}].$$

DEFINITION 1.42. The *commutator subgroup* or *derived subgroup* of $G$, denoted $G' = [G, G]$, is formed by arbitrary products of commutators.

Corollary 1.43. *From above,*

$$g[G,G]g^{-1} = [G,G] \quad \forall g \in G$$

*so* $[G,G]$ *is a normal subgroup.*

Corollary 1.44. *For any* $g_1, g_2 \in G$*, we have*

$$g_1 g_2 = g_2 g_1 [g_1, g_2]$$

*so* $G/[G,G]$ *is abelian.*

Definition 1.45. A group is *perfect* if $G = [G,G]$.

# 2 Matrix Groups and Representations

Any set of non-singular matrices which is closed under matrix multiplication forms a *matrix group*. We choose $e$ to be identity matrix, inverse to be the matrix inverse. Many groups are defined in terms of matrices.

Definition 2.1. The real *general linear group* $\mathrm{GL}(n,\mathbb{R})$ is the set of all real $n\times n$ non-singular matrices (i.e. $\det M\neq 0, \forall M\in \mathrm{GL}(n,\mathbb{R})$). Its real dimension is $n^2$.

Similarly, we can define the complex general linear group $\mathrm{GL}(n,\mathbb{C})$, which has real dimension $2n^2$.

Definition 2.2. The real *special linear group* $\mathrm{SL}(n,\mathbb{R})$ is the set of all real non-singular $n\times n$ matrices with $\det M = 1, \forall M\in \mathrm{SL}(n,\mathbb{R})$. It has real dimension $n^2-1$.

## 2.1 Continuous Matrix Groups of Interest

Definition 2.3. The *orthogonal group* $\mathrm{O}(n)$ is the group of real orthogonal $n\times n$ matrices such that

$$M^T M = I, \quad \forall M\in \mathrm{O}(n). \tag{2.1.1}$$

For *special orthogonal group* $\mathrm{SO}(n)$, we have the condition $\det = 1$ as well.

To find the (real) dimension of such groups, note that a general $n\times n$ real matrix has $n^2$ parameters. A symmetric one has $\frac{n}{2}(n+1)$ free entries. $M^TM$ is a symmetric matrix, so (2.1.1) imposes $\frac{n}{2}(n+1)$ constraints. So, $\mathrm{O}(n)$ has $\frac{1}{2}n(n-1)$ parameters. So does $\mathrm{SO}(n)$ as the defining relation already gives $|\det M| = 1$.

If $v, v'$ belong to the $n$-dimensional representation space (to be defined) of $\mathrm{O}(n)$ or $\mathrm{SO}(n)$, then the scalar product $v'^T v$ is invariant under

$$v\to Mv, \quad v'\to Mv'.$$

Definition 2.4. The *unitary group* $\mathrm{U}(n)$ is the group of complex unitary $n\times n$ matrices with

$$M^\dagger M = I, \quad \forall M\in \mathrm{U}(n). \tag{2.1.2}$$

A Hermitian complex matrix has $n^2$ real parameters. $M^\dagger M$ is Hermitian, then (2.1.2) contains $n^2$ constraints. Thus in $\mathrm{U}(n)$, we have $2n^2 - n^2 = n^2$ real parameters.

(2.1.2) implies $|\det M| = 1$, so imposing $\det M = 1$ provides one additional constraint, giving the *special unitary group*, $\mathrm{SU}(n)$, with $n^2-1$ real parameters.

The $\mathrm{U}(n)$ invariant product for $n$-dimensional complex vectors $v, v'$ is $v'^\dagger v$.

Example 2.5. $\mathrm{SO}(2)\simeq \mathrm{U}(1)$ since a general $\mathrm{SO}(2)$ matrix

$$\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

with $0 \leq \theta \leq 2\pi$ is in one-to-one correspondence with a general element of U(1): $e^{\mathrm{i}\theta}$, $0 \leq \theta \leq 2\pi$.

Topologically, $\mathrm{U}(1) \simeq S^1$.

For SU(2),

$$g = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

where $|\alpha|^2 + |\beta|^2 = 1$.

Write

$$\alpha = a + \mathrm{i}b, \quad \beta = c + \mathrm{i}d$$

we get

$$a^2 + b^2 + c^2 + d^2 = 1$$

i.e. the 3-sphere $S^3$.

Lecture 4
No-Revise

Definition 2.6. The real (complex) *symplectic groups* $\mathrm{Sp}(2n, \mathbb{R})$ ($\mathrm{Sp}(2n, \mathbb{C})$) consists of $2n \times 2n$ matrices $M$, satisfying

$$M^T J M = J \tag{2.1.3}$$

where

$$J = \begin{pmatrix} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{pmatrix}$$

is a $2n \times 2n$ matrix.

$M^T J M$ is antisymmetric, so (2.1.3) comprises $n(2n-1)$ conditions. Therefore, $\mathrm{Sp}(2n, \mathbb{R})$ has $n(2n+1)$ real parameters. (2.1.3) already imposes $\det M = 1$ so there exists no further restrictions on parameters. (To understand this, see Hugh Osborne's notes and the definition of the Pfaffian etc.)

Definition 2.7. The *antisymmetric invariant form* is defined as

$$\langle v', v \rangle = -\langle v, v' \rangle = v'^T J v.$$

Proposition 2.8. $\mathrm{SO}(n), \mathrm{SU}(n)$ *are* compact*, i.e. the natural parameters vary over a finite range.* $\mathrm{Sp}(2n, \mathbb{R})$ *isn't.*

Example 2.9. Consider $\mathrm{Sp}(2, \mathbb{R})$, it has elements

$$M = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix},$$

and $-\infty < \theta < \infty$. The parameters can tend to infinity, thus not compact.

DEFINITION 2.10. The *pseudo-orthogonal group* $\mathrm{O}(n, m)$ of $(n+m) \times (n+m)$ matrices is such that its elements $M$ satisfy

$$M^T g M = g$$

where

$$g = \begin{pmatrix} I_n & \\ & -I_m \end{pmatrix}$$

is an $(n+m) \times (n+m)$ matrix, with $I_n$ the $n \times n$ identity matrix and $I_m$ the $m \times m$ one.

This definition naturally tends to $\mathrm{SO}(n, m)$.

Similarly we can define $\mathrm{U}(n, m)$ and $\mathrm{SU}(n, m)$. Parameter counts are the same as for $\mathrm{O}(n+m)$ or $\mathrm{U}(n+m)$.

NOTE. $\mathrm{SO}(1, 1)$ is the same as $\mathrm{Sp}(2, \mathbb{R})$.

## 2.2 Representations

Representations play a crucial role in physics.

DEFINITION 2.11. For any group $G$, a *representation* is a set of non-singular square matrices $\{D(g)\}, \forall g \in G$ such that

- $D(g_1)D(g_2) = D(g_1 g_2)$;
- $D(e) = I$, the identity matrix;
- $D(g^{-1}) = D(g)^{-1}$.

If $D(g)$ are $n \times n$, the representation has *dimension* $n$.

For each matrix group, its definition provides a representation called the *fundamental representation*.

For complex matrices, the *conjugate representation* is defined to be $D(g)^*$.

NOTE. $(D(g)^{-1})^T$ also forms a representation.

DEFINITION 2.12. Two representations of the same dimension $D(g)$ and $D'(g)$ are said to be *equivalent* if

$$D'(g) = S D(g) S^{-1}, \quad \forall g \in G \tag{2.2.1}$$

where $S$ is an $n \times n$ invertible matrix.

DEFINITION 2.13. The $n$-dimensional vector space $V$ that some representation of dimension $n$ acts on, is called the *representation space*. For $v \in V$, we define a *group transformation* acting on it by

$$v \xrightarrow{g} v^g = D(g)v.$$

NOTE. Thus (2.2.1) corresponds to a change of basis of $V$.

DEFINITION 2.14. A representation is *reducible* if there exists a subspace $U \subset V, U \neq V$ such that

$$D(g)u \in U, \quad \forall u \in U.$$

Otherwise it is an *irreducible representation*, often called '*irrep*'.

For a reducible representation, we may define a representation of lower dimension by restricting to a invariant subspace. For example, with a suitable choice of basis

$$D(g) = \begin{pmatrix} \hat{D}(g) & B(g) \\ 0 & C(g) \end{pmatrix}, \quad \text{for } u = \begin{pmatrix} \hat{u} \\ 0 \end{pmatrix}$$

where $\hat{D}(g)$ form a representation of $G$.

DEFINITION 2.15. For the decomposition above, if $B(g) = 0, \forall g$, the representation is *completely reducible*.

COROLLARY 2.16. *For a completely reducible representation, the representation space $V$ decomposes into a direct sum of invariant spaces $U_r$ which are not further reducible.*

*Hence, there exists a matrix $S$ such that*

$$SD(g)S^{-1} = \begin{pmatrix} D_1(g) & & & \\ & D_2(g) & & \\ & & \ddots & \\ & & & D_k(g) \end{pmatrix}$$

*where $D_r(g)$ are irreps, and*

$$V = \bigoplus_{r=1}^{k} U_r.$$

Writing $R$ for representation given by matrices $D(g)$ and $R_s$ for irrep matrices $D_s(g)$, this is written as

$$R = R_1 \oplus \cdots \oplus R_k$$

A particular $R_s$ may appear more than once. A 1-dimensional trivial irrep is given by $D_0(g) = 1, \forall g \in G$.

LEMMA 2.17 (Schur's lemmas). *If $D_1(g)$ and $D_2(g)$ are two irreps, then*

1. $SD_1(g) = D_2(g)S, \forall g \quad \Rightarrow \quad D_1(g)$ *is equivalent to* $D_2(g)$ *or* $S = 0$;
2. $SD_1(g) = D_1(g)S, \forall g \quad \Rightarrow \quad S \propto I$.

For quantum applications, we are usually interested in *unitary representations*, where

$$D(g)^\dagger = D(g^{-1}) = D(g)^{-1}.$$

For such a representation, the usual scalar product on $V$ is invariant, since

$$v_1^\dagger v_2 = (v_1^g)^\dagger v_2^g, \quad v_1, v_2 \in V.$$

DEFINITION 2.18. For any representation $R$, the *character* is defined by

$$\chi_R(g) = \mathrm{tr}_R(D^{(R)}(g)).$$

COROLLARY 2.19. *Matrix traces are unchanged by cyclic permutations, so*

$$\chi_R(g' g g'^{-1}) = \chi_R(g).$$

*Therefore, the character depends on the conjugacy class of each element. Thus*

$$\chi(g_i) = \chi(\mathcal{C}_r)$$

*for any* $g_i \in \mathcal{C}_r$. *For* $n_{char}$ *different conjugacy classes in* $G$, $r = 1, \cdots, n_{char}$.

*The character is also unchanged for equivalent representations as*

$$D'(g) = SD(g)S^{-1}, \quad \forall g \in G.$$

DEFINITION 2.20. If $V_1, V_2$ are representation spaces for representations $R_1, R_2$ given by matrices $D_1(g), D_2(g)$ for a group $G$, we define a *tensor product representation* $R_1 \otimes R_2$ in terms of $D_1(g) \otimes D_2(g)$ acting on the tensor product space $V_1 \otimes V_2$ where

$$D(g)v = \sum_{r,s} a_{rs} D_1(g) v_{1,r} D_2(g) v_{2,s}$$

with $D(g) \in R_1 \otimes R_2$, $v \in V_1 \otimes V_2$ and $v_{1,r}$ the $r$-th component of the vector $v_1$, etc.

The dimension of tensor product space is $\dim V_1 \times \dim V_2$.

PROPOSITION 2.21. *In general, the tensor product of two representations* $R_1 \otimes R_2$ *is reducible and can be decomposed into irreps*

$$R_r \otimes R_s \simeq R_s \otimes R_r \simeq \bigoplus_t n_{rs,t} R_t \tag{2.2.2}$$

*where* $n_{rs,t} = 0, 1, 2, \cdots$.

*Even though for non-finite groups there exists infinitely many irreps, the direct sum* *is still finite dimensional.*

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DEFINITION 2.22. The *trace for product representation* is defined as

$$\mathrm{tr}_{V_r\otimes V_s}\left(D^{(R_r)}(g)\otimes D^{(R_s)}(g)\right) := \mathrm{tr}_{V_r}\left(D^{(R_r)}(g)\right)\mathrm{tr}_{V_s}\left(D^{(R_s)}(g)\right).$$

COROLLARY 2.23. *(2.2.2) suggests*

$$\chi_{R_r}(g)\chi_{R_s}(g) = \sum_t n_{rs,t}\chi_{R_t}(g).$$

(2.2.2) is equivalent to the decomposition of the associated representation spaces, with the same expansion for $V_r \otimes V_s$ into a direct sum of irreducible spaces $V_t$.

PROPOSITION 2.24. *If $R_r \otimes R_s$ contains the singlet representations, it's possible to construct a scalar product $\langle v, v'\rangle$ between vector $v \in V_r, v' \in V_s$, which is invariant under the group transformation rule, i.e.*

$$\left\langle D^{(R_r)}(g)v, D^{(R_s)}(g)v'\right\rangle = \left\langle v, v'\right\rangle.$$

## 2.3 Symmetries and Quantum Mechanics

In Quantum Mechanics, the state vector $|\psi\rangle$ lives in Hilbert space $\mathcal{H}$. Observables are probability $|\langle\phi|\psi\rangle|^2$: finding $|\phi\rangle$ after measurement of a state prepared in $|\psi\rangle$.

DEFINITION 2.25. A *symmetry transformation* $|\psi\rangle \to |\psi'\rangle$ is such that,

$$|\langle\phi|\psi\rangle|^2 = \left|\left\langle\phi'\middle|\psi'\right\rangle\right|^2 \quad \forall\,|\phi\rangle\,,|\psi\rangle \in \mathcal{H}.$$

The phase of each state is arbitrary.

Wigner used this to prove that there exists an operator $U$ such that

$$U\,|\psi\rangle = \left|\psi'\right\rangle$$

and $U$ is either *linear* or *anti-linear*, i.e.

$$\begin{aligned}
\text{linear} \quad &\Rightarrow \quad \left\langle\phi'\middle|\psi'\right\rangle = \langle\phi|\psi\rangle \quad \text{and} \quad U(a\,|\psi_1\rangle + b\,|\psi_2\rangle) = aU\,|\psi_1\rangle + bU\,|\psi_2\rangle\,;\\
\text{anti-linear} \quad &\Rightarrow \quad \left\langle\phi'\middle|\psi'\right\rangle = \langle\psi|\phi\rangle \quad \text{and} \quad U(a\,|\psi_1\rangle + b\,|\psi_2\rangle) = a^*U\,|\psi_1\rangle + b^*U\,|\psi_2\rangle\,.
\end{aligned}$$

In physics, the second cae is only relevant for $T$-reversal symmetries, so for now, we'll assume $U$ linear.

For a symmetry group $G = \{g\}$, we must have unitary operators $U(g)$ where $U(e) = 1$, $U(g^{-1}) = U(g)^{-1}$. The product rule is

$$U(g_i)U(g_j) = e^{\mathrm{i}\gamma(g_i,g_j)}U(g_ig_j)$$

where a phase $\gamma$ is allowed because of phase freedom of states. But it's mostly irrelevant for us. Take $\gamma = 0$ for now.

If $G$ is a symmetry for a physical system with Hamiltonian $H$, we require

$$U(g_i)HU(g_i)^{-1} = H, \quad \forall g_i \in G.$$

$H\left|\psi_r\right\rangle = E\left|\psi_r\right\rangle$ for $r = 1, \cdots, n$, then

$$H\left(U(g)\left|\psi_r\right\rangle\right) = E\left(U(g)\left|\psi_r\right\rangle\right).$$

Hence,

$$U(g)\left|\psi_r\right\rangle = \sum_{s=1}^{n}\left|\psi_s\right\rangle D_{sr}(g)$$

where $D_{sr}(g)$ matrices form an $n$-dimensional representation of $G$. For physical cases, these are irreps.

# 3 Rotations, SO(3) and SU(2)

Many physical systems have a symmetry with respect to 3-dimensional rotations. The fundamental property is that scalar products of vectors, and therefore their lengths, are invariant under a rotation.

Rotations correspond to orthogonal matrices acting on $v$, so they leave $v^T v$ invariant. For real-valued vectors $v$, the length $|v|$ is given by

$$|v|^2 = v^T v.$$

For a real orthogonal matrix, if $v$ is an eigenvector (which in general is complex-valued), then

$$Mv = \lambda v \quad \text{and} \quad Mv^* = \lambda^* v^*$$

i.e. if $\lambda \in \mathbb{C}$ is an eigenvalue, so is $\lambda^*$.

Thus

$$(Mv^*)^T Mv = |\lambda|^2 v^\dagger v = v^\dagger v$$

so $|\lambda|^2 = 1$.

## 3.1 3-Dimensional Rotations

Lemma 3.1. *For $R \in \mathrm{O}(3)$ (i.e. $R^T R = I$), the eigenvalues can only be $\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}$, so a general $R$ can be reduced via a real transformation $S$, to*

$$SRS^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

For $\det R = 1$, i.e. $R \in \mathrm{SO}(3)$, we have the $+1$ case, i.e.

$$\operatorname{tr} R = 2\cos\theta + 1.$$

Acting on a spatial vector $x$, the matrix induces a linear transformation

$$x \xrightarrow{R} x' = x^R$$

where $x' = Rx$.

For $\det R = -1$, the transformation involves a reflection.

Lemma 3.2. *A general $R \in \mathrm{SO}(3)$ has three parameters which may be taken to be a rotation angle $\theta$ and a unit vector $n$ (specified by other two angles) and is determined by*

$$Rn = n.$$

*$n$ is the axis of rotation.*

PROPOSITION 3.3. *In general,*

$$R_{ij} = \cos\theta\delta_{ij} + (1-\cos\theta)n_i n_j - \sin\theta\epsilon_{ijk}n_k. \tag{3.1.1}$$

*for* $i, j, k \in \{1, 2, 3\}$, $\delta_{ij}$ *the Kronecker delta and* $\epsilon_{ijk}$ *the 3-d Levi-Civita tensor.*

NOTE. Throughout we use the Einstein summation convention.

The parameters cover all rotations if $n \in S^2$, i.e. $n$ is anywhere on the unit sphere. Here, $0 \leq \theta \leq \pi$ and $(\pi, n) \simeq (\pi, -n)$.

For an infinitesimal rotation $R(\delta\theta, n)$,

$$\mathbf{x} \to \mathbf{x}' = \mathbf{x} + \delta\theta\mathbf{n} \times \mathbf{x} + \mathcal{O}(\delta\theta^2)$$

so that

$$\left|\mathbf{x}'\right|^2 = |\mathbf{x}|^2 + \mathcal{O}(\delta\theta^2).$$

## 3.2 Isomorphism of SO(3) and SU(2)/$\mathbb{Z}_2$

THEOREM 3.4.

$$\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$$

*where* $\mathbb{Z}_2 = \{I, -I\}$ *is the centre of* SU(2).

We will prove this later.

DEFINITION 3.5. The *Pauli matrices* are $2 \times 2$ matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

COROLLARY 3.6. *Pauli matrices satisfy*

$$\sigma_i\sigma_j = \delta_{ij}I + \mathrm{i}\epsilon_{ijk}\sigma_k. \tag{3.2.1}$$

*which is equivalent to*

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = (\mathbf{a}\cdot\mathbf{b})I + \mathrm{i}(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}.$$

Notice that the Pauli matrices are traceless Hermitian.

LEMMA 3.7. *(3.2.1) implies*

$$\mathrm{tr}(\sigma_i\sigma_j) = 2\delta_{ij}$$

*which means that any* $2 \times 2$ *matrix* $A$ *can be expressed in the form*

$$A = \frac{1}{2}\mathrm{tr}(A)I + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}A)\cdot\boldsymbol{\sigma}.$$

*i.e. the Pauli matrices form a complete set of traceless Hermitian* $2 \times 2$ *matrices.*

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*Proof of Theorem 3.4.* There exists a one-to-one correspondence between real 3-vectors and Hermitian traceless $2 \times 2$ matrices

$$\mathbf{x} \to \mathbf{x} \cdot \boldsymbol{\sigma} = (\mathbf{x} \cdot \boldsymbol{\sigma})^\dagger$$

and we have

$$\mathbf{x} = \frac{1}{2} \operatorname{tr}(\boldsymbol{\sigma} \mathbf{x} \cdot \boldsymbol{\sigma}).$$

Also

$$(\mathbf{x} \cdot \boldsymbol{\sigma})^2 = \mathbf{x}^2 I \tag{3.2.2}$$

(3.2.1) suggests the eigenvalues of $\mathbf{x} \cdot \boldsymbol{\sigma}$ are $\pm\sqrt{\mathbf{x}^2}$. Therefore,

$$\det(\mathbf{x} \cdot \boldsymbol{\sigma}) = -\mathbf{x}^2 \tag{3.2.3}$$

For any $A \in \mathrm{SU}(2)$, we can define a linear transformation $\mathbf{x} \to \mathbf{x}'$ by

$$\mathbf{x}' \cdot \boldsymbol{\sigma} = A \mathbf{x} \cdot \boldsymbol{\sigma} A^\dagger \tag{3.2.4}$$

to show

$$\mathbf{x}'^2 \overset{(3.2.3)}{=} -\det\big(\mathbf{x}' \cdot \boldsymbol{\sigma}\big) \overset{(3.2.4)}{=} -\det\Big(A \mathbf{x} \cdot \boldsymbol{\sigma} A^\dagger\Big) = -\det(\mathbf{x} \cdot \boldsymbol{\sigma}) \overset{(3.2.3)}{=} \mathbf{x}^2.$$

Thus the length is invariant. Therefore

$$x_i' = R_{ij} x_j$$

with $R$ an orthogonal matrix.

Since as $A \to I$, $R \to \delta_{ij}$. Therefore, we have

$$\det(R_{ij}) = +1.$$

(3.2.4) implies

$$\sigma_i R_{ij} x_j = A x_j \sigma_j A^\dagger \quad \text{or} \quad \sigma_i R_{ij} = A \sigma_j A^\dagger$$

which implies

$$R_{ij} = \frac{1}{2} \operatorname{tr}\Big(\sigma_i A \sigma_j A^\dagger\Big)$$

To find $A$ in terms of $R_{ij}$, note (3.2.1) suggests $\sigma_j \sigma_i \sigma_j = -\sigma_i$,

$$\sigma_j A^\dagger \sigma_j = 2(\operatorname{tr} A) I - A^\dagger.$$

We find

$$R_{jj} = |\operatorname{tr} A|^2 - 1 \tag{3.2.5}$$

and

$$\sigma_i R_{ij} \sigma_j = 2(\operatorname{tr} A^\dagger) A - I \tag{3.2.6}$$

For $A \in \mathrm{SU}(2)$, $\operatorname{tr} A = \operatorname{tr} A^\dagger \in \mathbb{R}$ (eigenvalues are $e^{\pm \mathrm{i}\alpha}$ suggesting $\operatorname{tr} A = 2\cos\alpha$), so (3.2.5), (3.2.6) give

$$A = \pm \frac{I + \sigma_i R_{ij} \sigma_j}{2\sqrt{1 + R_{jj}}}.$$

The arbitrary $\pm$ sign ensures $\pm A \to R_{ij}$.

For an infinitesimal rotation $\delta\theta$, (3.1.1) gives

$$R_{ij} = \delta_{ij} - \delta\theta \epsilon_{ijk} n_k.$$

Assuming $A \to +I$ as $\delta\theta \to 0$,

$$A = I - \frac{\mathrm{i}}{2} \delta\theta \mathbf{n} \cdot \boldsymbol{\sigma}.$$

Since $\det(1 + X) = 1 + \operatorname{tr} X$ to first order in $X$ for any matrix, the tracelessness of Pauli matrices is necessary to be compatible with $\det A = 1$.

For a finite rotation, we have

$$\operatorname{tr} R = 2\cos\theta + 1 = |\operatorname{tr} A|^2 - 1$$

and we can find $A$ by exponentiation

$$\exp M := I + \sum_{n=1}^{\infty} \frac{M^n}{n!}.$$

(for square matrix $M$) as

$$A(\theta, \mathbf{n}) = \exp\left(-\frac{\mathrm{i}}{2} \mathbf{n} \cdot \boldsymbol{\sigma} \theta\right) = \cos\frac{\theta}{2} I - \mathrm{i} \mathbf{n} \cdot \boldsymbol{\sigma} \sin\frac{\theta}{2}.$$

The parameters $(\theta, \mathbf{n})$ cover all SU(2) matrices for $n \in S^2$, $0 \le \theta \le 2\pi$ (i.e. *double* the range of SO(3)). □

## 3.3 Infinitesimal Generators and Rotations

Consider 2 infinitesimal transformation

$$R_1 = R(\delta\theta_1, \mathbf{n}_1) \quad \text{and} \quad R_2 = R(\delta\theta_2, \mathbf{n}_2).$$

The commutator is

$$R_c = R_2^{-1} R_1^{-1} R_2 R_1 = I + \mathcal{O}(\delta\theta_1, \delta\theta_2).$$

and

$$\begin{aligned} \mathbf{x} \to \mathbf{x} &+ \delta\theta_1 \delta\theta_2 (\mathbf{n}_2 \times (\mathbf{n}_1 \times \mathbf{x}) - \mathbf{n}_1 \times (\mathbf{n}_2 \times \mathbf{x})) \\ &= \mathbf{x} + \delta\theta_1 \delta\theta_2 (\mathbf{n}_2 \times \mathbf{n}_1) \times \mathbf{x} \end{aligned}$$

Acting on QM vector space, the corresponding unitary operators are assumed to be of the form

$$U[R(\delta\theta, \mathbf{n})] = 1 - \mathrm{i}\delta\theta \mathbf{n} \cdot \mathbf{J}$$

where $\mathbf{J}$ are the generators of the rotation group.

The inverse is

$$U[R(\delta\theta, \mathbf{n})]^{-1} = 1 + \mathrm{i}\delta\theta \mathbf{n} \cdot \mathbf{J}.$$

The condition for $U$ to be a unitary operator becomes $\mathbf{J} = \mathbf{J}^\dagger$.

$$\begin{aligned} U[R_c] &= 1 - \mathrm{i}\delta\theta_1\delta\theta_2(\mathbf{n}_2 \times \mathbf{n}_1) \cdot \mathbf{J} \\ &= U[R_2]^{-1}U[R_1]^{-1}U[R_2]U[R_1] \\ &= 1 - \delta\theta_1\delta\theta_2[\mathbf{n}_2 \cdot \mathbf{J}, \mathbf{n}_1 \cdot \mathbf{J}] \end{aligned}$$

hence

$$[\mathbf{n}_2 \cdot \mathbf{J}, \mathbf{n}_1 \cdot \mathbf{J}] = \mathrm{i}(\mathbf{n}_2 \times \mathbf{n}_1) \cdot \mathbf{J} \quad \Leftrightarrow \quad [J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k.$$

Acting on functions of $\mathbf{x}$,

$$\mathbf{J} \to \mathbf{L} = -\mathrm{i}\mathbf{x} \times \boldsymbol{\nabla},$$

so

$$(1 - \mathrm{i}\delta\theta \mathbf{n} \cdot \mathbf{L})f(\mathbf{x}) = f(\mathbf{x} - \delta\theta\mathbf{n} \times \mathbf{x})$$

corresponds to an infinitesimal rotation.

One can check $L_i$ the same commutation relations as $J_i$.

For finite rotations,

$$U[R(\theta, \mathbf{n})] = \exp(-\mathrm{i}\theta\mathbf{n} \cdot \mathbf{J}) = \lim_{N\to\infty}\left(1 - \frac{\mathrm{i}\theta}{N}\mathbf{n} \cdot \mathbf{J}\right)^N.$$

$\mathbf{J}$ is a vector under rotations, since from before

$$U[R]U[R(\delta\theta, \mathbf{n})]U[R]^{-1} = U[R(\delta\theta, R\mathbf{n})]$$

implying

$$U[R]J_iU[R]^{-1} = (R^{-1})_{ij}J_j.$$

For a physical system, the rotation group operator $\mathbf{J}$ is identified as the total angular momentum of the system and the commutation relations are the fundamental angular momentum commutation relations.

Rotational invariance implies conservation of angular momentum since

$$U[R]HU[R]^{-1} = H \quad \Leftrightarrow \quad [\mathbf{J}, H] = 0.$$

ensuring that degenerate energy states for each energy must belong to a representation space for a representation of the rotation group.

## 3.4 Representations of the Angular Momentum Commutation Relations

Here, we shall see how we can analyse $[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k$ to determine possible representation spaces $V$ on which the action of $\mathbf{J}$ is determined.

We can define operators
$$J_\pm := J_1 \pm \mathrm{i}J_2$$
so that
$$[J_3, J_\pm] = \pm J_\pm \quad \text{and} \quad [J_+, J_-] = 2J_3.$$

NOTE. $J_3 = J_3^\dagger$ but $J_+^\dagger = J_-$, i.e. $J_\pm$ are not Hermitian.

We define a basis for a representation of angular momentum in terms of eigenvalues of $J_3$. Let
$$J_3 \left|m\right\rangle = m \left|m\right\rangle$$
so
$$J_\pm \left|m\right\rangle \propto \left|m \pm 1\right\rangle \quad \text{or} \quad 0.$$

The possible sequence of $J_3$ eigenvalues is
$$\cdots, m-1, m, m+1, \cdots.$$

If the states $\left|m \pm 1\right\rangle$ are non-zero, we define
$$\begin{aligned} J_- \left|m\right\rangle &= \left|m-1\right\rangle \\ J_+ \left|m\right\rangle &= \lambda_m \left|m+1\right\rangle. \end{aligned}$$
which gives
$$J_+J_- \left|m\right\rangle = \lambda_{m-1} \left|m\right\rangle \quad \text{and} \quad J_-J_+ \left|m\right\rangle = \lambda_m \left|m\right\rangle.$$

$\lambda_{m-1} - \lambda_m = 2m$ if $\left|m \pm 1\right\rangle \neq 0$.

This can be solved for any $m$ by
$$\lambda_m = j(j+1) - m(m+1). \tag{3.4.1}$$

for some constant which we've written as $j(j+1)$.

For large enough $|m|$, $\lambda_m < 0$.