University of Cambridge
Mathematical Tripos

Part III – **Algebraic Topology**

Based on Lectures by *I. Smith*
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These notes may not reflect the full format and content that are actually lectured. I usually modify the notes heavily after the lectures and sometimes my own thinking or interpretation might be blended in. Any mistake or typo should surely be mine. Be cautious if you are using this for self-study or revision.

Course Information

Algebraic Topology permeates modern pure mathematics and theoretical physics. This course will focus on (co)homology, with an emphasis on applications to the topology of manifolds. We will cover singular and cellular (co)homology; degrees of maps and cup-products; cohomology with compact supports and Poincaré duality; and Thom isomorphism and the Euler class. The course will not specifically assume any knowledge of algebraic topology, but will go quite fast in order to reach more interesting material, so some previous exposure to chain complexes (e.g. simplicial homology) would certainly be helpful.

Pre-requisites

Basic topology: topological spaces, compactness and connectedness, at the level of Sutherland's book. Some knowledge of the fundamental group would be helpful though not a requirement. Hatcher's book and Bott and Tu's book are especially recommended for accompanying the course, but there are many other suitable texts.

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0 Introduction

Lecture 1
No-Revise

Algebraic Topology concerns the *connectivity* properties of topological spaces.

Definition 0.1. A space X is *path-connected* if for $p, q \in X$, $\exists \gamma : [0,1] \to X$ continuous with $\gamma(0) = p, \gamma(1) = q$.

Example. $\mathbb{R}$ is path-connected; $\mathbb{R}\backslash\{0\}$ is not.

Corollary 0.0.1 (The intermediate value theorem). *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $x < y$ satisfy $f(x) < 0, f(y) > 0$ then f takes the value 0 on $[x, y]$.*

Proof. Otherwise, $f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$ disconnects $[x, y]$, #. □

Definition 0.2. Let X, Y be topological spaces. We say maps $f_0, f_1 : Y \to X$ are *homotopic* if $\exists F : Y \times [0,1] \to X$ continuous such that

$$F|_{Y\times\{0\}} = f_0, \qquad F|_{Y\times\{1\}} = f_1$$

We write $f_0 \simeq f_1$ (or $f_0 \simeq_F f_1$).

[Need a figure of square here.]

Exercise. (On example sheet 1) $\simeq$ is an equivalence relation on the set of maps from Y to X.

Note. X is *path-connected* iff every two maps $\{\text{point}\} \to X$ are homotopic.

Definition 0.3. X is *simply-connected* if every two maps $S^1 \to X$ are homotopic.

Note. We often denote

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}, \qquad S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$$

Example. $\mathbb{R}^2$ is simply connected; $\mathbb{R}^2\backslash\{0\}$ is not.