

UNIVERSITY OF CAMBRIDGE
MATHEMATICAL TRIPOS

Part III – **Differential Geometry**

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These notes may not reflect the full format and content that are actually lectured. I usually modify the notes heavily after the lectures and sometimes my own thinking or interpretation might be blended in. Any mistake or typo should surely be mine. Be cautious if you are using this for self-study or revision.

COURSE INFORMATION

Differential geometry is the study of manifolds — spaces built from smoothly gluing together open sets in Euclidean space — and structures that live on or in them. The goal of this course is to introduce the main ideas on both the abstract conceptual (‘coordinate-free’) level and the concrete computational (‘in coordinates’) level, and to develop fluency in passing between them. This will lay the foundation for future study in geometry and topology, and provide the language for modern theoretical physics. Throughout the emphasis will be on building up geometric intuition. Topics will include:

- Manifolds, tangent and cotangent spaces, smooth maps and their derivatives. Tangent and cotangent bundles, tensors. Vector fields, flows, the Lie derivative.
- Differential forms, the exterior derivative, de Rham cohomology. Orientability. Integration and Stokes’s theorem. Frobenius integrability.
- Lie groups and algebras. Principal bundles, connections (from multiple perspectives), curvature. Associated bundles, reduction of the structure group, vector bundles.
- Riemannian metrics, the Levi-Civita connection, geodesics and the exponential map. The Riemann tensor and its symmetries and contractions. The Hodge star, the Laplacian, statement of the Hodge decomposition.

PRE-REQUISITES

Familiarity with point set topology (including compactness), multi-variable calculus (including the inverse function theorem), and linear algebra (including dual spaces and bilinear forms) is essential. No previous exposure to geometry will be assumed.

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1 MANIFOLDS AND SMOOTH MAPS

Rec 1
No-Revise

1.1 Manifolds

A manifold is a space which locally looks like \mathbb{R}^n .

DEFINITION 1.1. A *topological n -manifold* is a topological space X such that for every point p in X there exists an open neighbourhood U of p in X , an open set V in \mathbb{R}^n , and a homeomorphism $\varphi : U \xrightarrow{\sim} V$.

We also require X to be

- *Hausdorff*: given distinct points p_1 and p_2 in X there exist disjoint open neighbourhoods U_1 and U_2 of p_1 and p_2 respectively.
- *second-countable*: there exists a countable collection of open sets which form a basis for the topology, i.e. every open set is a union of sets in the collection.

EXAMPLE. \mathbb{R}^n is a topological n -manifold:

- For every p take $U = V = \mathbb{R}^n$ and $\varphi = \text{id}_{\mathbb{R}^n}$.
- Hausdorffness is obvious (e.g. since \mathbb{R}^n is metrisable).
- A countable basis for the topology is given by open balls of rational radius with rational centre.

REMARK. 1. Hausdorff and second-countable are important but are not restrictive in practice.

2. They're automatic for embedded submanifolds of \mathbb{R}^n .

3. They're equivalent to ' X is metrisable and has countably many components'.

Terminology:

- Each map φ is a *chart* (about p).
- The set U is a *coordinate patch*.
- If x_1, \dots, x_n are the standard coordinates on \mathbb{R}^n then

$$x_1 \circ \varphi, \dots, x_n \circ \varphi$$

are *local coordinates on U* or *local coordinates about p* . Usually we'll just call these x_1, \dots, x_n or similar.

- The inverse of a chart is called a *parametrisation*. (It's easier to remember which direction a parametrisation goes than a chart!)

EXAMPLE. If X is a topological n -manifold, so is any open $W \subset X$:

- If $p \in W$ and $\varphi : U \xrightarrow{\sim} V$ is a chart about p in X then

$$\varphi|_{U \cap W} : W \cap U \xrightarrow{\sim} \varphi(U \cap W)$$

is a chart about p in W .

- Hausdorffness and second-countability are inherited from X .

More terminology:

Given overlapping charts $\varphi : U_1 \rightarrow V_1$ and $\varphi_2 : U_2 \rightarrow V_2$, the corresponding local coordinates x_1, \dots, x_n and y_1, \dots, y_n are related by the *transition map*

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2).$$

This is a map between open subsets of \mathbb{R}^n . Such a map is *smooth* if each component has all partial derivatives of all orders, i.e. if when we express each y_i as a function of x_1, \dots, x_n using $\varphi_2 \circ \varphi_1^{-1}$

$$\frac{\partial^k y_i}{\partial x_{j_1} \cdots \partial x_{j_k}}$$

exists for all $k \geq 1$ and all j_1, \dots, j_k .

We want a notion of smoothness for functions on manifolds.

A function $f : W \rightarrow \mathbb{R}$ on an open subset $W \subset X$ may be written locally on a coordinate patch as a function $f(x_1, \dots, x_n)$ of the local coordinates. PRELIMINARY DEFINITION. f is *smooth* if and only if this local expression has all partial derivatives of all orders. PROBLEM. On overlaps between coordinate patches this depends on the choice of local coordinates.

A natural solution is to require all transition maps to be smooth. Then smoothness in one chart implies smoothness in other charts on overlaps, by the chain rule.

DEFINITION 1.2. • An *atlas* for a topological n -manifold X is a collection of charts

$$\{\varphi_\alpha : U_\alpha \xrightarrow{\sim} V_\alpha\}_{\alpha \in \mathcal{A}}$$

that covers X , i.e. such that $\bigcup_\alpha U_\alpha = X$.

- An atlas is *smooth* if every transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ is smooth.
- Given an atlas \mathfrak{A} and open $W \subset X$, a function $f : W \rightarrow \mathbb{R}$ is *smooth with respect to \mathfrak{A}* if $f \circ \varphi_\alpha^{-1}$ is smooth for all α , i.e. if all local coordinate expressions $f(x_1, \dots, x_n)$ are smooth.

LEMMA 1.1. If \mathfrak{A} is smooth then f is smooth if and only if for all p in W there exists U_α containing p such that $f \circ \varphi_\alpha^{-1}$ is smooth, i.e. if $f(x_1, \dots, x_n)$ is smooth for some local coordinates x_1, \dots, x_n about p .

COROLLARY 1.1.1. *Given a smooth atlas \mathfrak{A} all local coordinate functions are smooth with respect to the atlas.*

We'll think of two smooth atlases as being the same if they have the same smooth functions.

DEFINITION 1.3. • Two smooth atlases are *smoothly equivalent* if and only if their union is smooth (this is an equivalence relation).

- A *smooth structure* of X is an equivalence class of smooth atlases under this relation.
- A *smooth n -manifold* is a topological n -manifold equipped with a choice of smooth structure. We'll abbreviate it to ' n -manifold' or even just 'manifold'.

LEMMA 1.2. *If \mathfrak{A} and \mathfrak{B} are smoothly equivalent then $f : W \rightarrow \mathbb{R}$ is smooth with respect to \mathfrak{A} if and only if it's smooth with respect to \mathfrak{B} .*

DEFINITION 1.4. Given a smooth n -manifold X , a function $F : W \rightarrow \mathbb{R}$ is *smooth* if and only if it's smooth with respect to some (or, equivalently, all) smooth atlas(es) representing the smooth structure.

EXAMPLE. \mathbb{R}^n is naturally an n -manifold via the atlas

$$\{\text{id} : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n\}$$

EXAMPLE. If X is an n -manifold, then any open $W \subset X$ inherits the structure of an n -manifold, by restricting charts on X to W .

EXAMPLE. If X is an n -manifold and Y an m -manifold then $X \times Y$ is naturally an $(m + n)$ -manifold, by equipping it with the product topology and the smooth structure induced by products of charts on X and Y .

- REMARK.**
1. Being a topological n -manifold is a *property*.
 2. Being a smooth n -manifold is a property (being a topological n -manifold and admitting a smooth structure) *plus* a choice of smooth structure.
 3. When $n = 1, 2$, or 3 , every topological n -manifold admits an essentially unique smooth structure.
 4. For $n \geq 4$ a topological n -manifold may admit no smooth structure (e.g. the E_8 manifold) or many essentially different smooth structures (e.g. exotic 7-spheres, or exotic \mathbb{R}^4). But these results are hard.

DEFINITION 1.5. The integer n is the *dimension* of X , denoted $\dim X$.

- REMARK.
1. We'll show that a (non=empty!) smooth manifold has a unique dimension.
 2. A topological manifold also has a unique dimension but this requires algebraic topology to prove. It's at least as hard as showing \mathbb{R}^m and \mathbb{R}^n are not homeomorphic for $m \neq n$.
 3. A manifold of negative dimension is empty.

Conventions:

- Whenever we talk about an atlas on a manifold, it will always implicitly be a representative of the smooth structure.
- If we construct a new chart then we'll say that it's *compatible (with the smooth structure)* if it can be added to an atlas representing the smooth structure whilst preserving smoothness.
- If we say 'take a chart satisfying...', or 'we may assume our chart satisfies...', or similar, we mean that either our atlas already contains such a chart, or we may add the chart to our atlas (i.e. the chart is compatible). Adding charts in this way makes no real difference.

EXAMPLE. We may want a chart about p contained in a given open neighbourhood W . To do this we can take an arbitrary chart $\varphi : U \xrightarrow{\sim} V$ about p and then choose the chart

$$\varphi|_{U \cap W} : U \cap W \xrightarrow{\sim} \varphi(U \cap W),$$

adding it to the atlas first if necessary.

- Likewise 'take local coordinates satisfying...' or similar, means choose a chart whose associated coordinates have these properties, or add such a chart to the atlas if non exists.

EXAMPLE. Given a point p in a manifold X we may always choose local coordinates x_1, \dots, x_n about p in which p is given by $\mathbf{x} = 0$: take any chart $\varphi : U \xrightarrow{\sim} V$ about p and add the chart

$$\varphi - \varphi(p) : U \xrightarrow{\sim} \{\mathbf{v} - \varphi(p) : \mathbf{v} \in V\}$$

to the atlas if it's not already there.

Some people avoid this by working with the *maximal atlas*, meaning the union of all atlases representing the smooth structure. But this obscures the fact that it's only the equivalence class that matters.

EXAMPLE. The n -sphere, S^n , is the n -manifold whose underlying topological space is

$$\{\mathbf{y} = (y_0, \dots, y_n) \in \mathbb{R}^{n+1} : \|\mathbf{y}\|^2 = 1\}$$

with the subspace topology, and whose smooth structure is defined by the following atlas. There are two charts $\varphi_{\pm} : U_{\pm} \xrightarrow{\sim} \mathbb{R}^n$, where $U_{\pm} = S^n \setminus \{(\pm 1, 0, \dots, 0)\}$ and φ_{\pm} is stereographic projection

$$\varphi_{\pm}(y_0, \dots, y_n) = \frac{1}{1 \mp y_0}(y_1, \dots, y_n).$$

The local coordinates \mathbf{x}^{\pm} associated to φ_{\pm} satisfy $x_i^{\pm} = y_i/(1 \mp y_0)$.

The *height function* $y_0 : S^n \rightarrow \mathbb{R}$ is smooth, since it is given by

$$y_0 = \pm \frac{\|\mathbf{x}^{\pm}\|^2 - 1}{\|\mathbf{x}^{\pm}\|^2 + 1} \quad \text{on } U_{\pm}$$

REMARK. This may seem asymmetric because we singled out two points to project from, but charts obtained by stereographic projection from any other point are compatible. We'll see later that S^n is a *submanifold* of \mathbb{R}^{n+1} and its smooth structure is inherited from \mathbb{R}^{n+1} .

1.2 Manifolds from Sets

Rec 2 A set can be made into a manifold by identifying subsets with subsets of \mathbb{R}^n .
No-Revise A smooth n -manifold X is a set equipped with:

- A topology satisfying various conditions;
- An (equivalence class) of smooth atlas.

The atlas presents X as a union of sets U_{α} , each identified with an open set $V_{\alpha} \subset \mathbb{R}^n$ by a homeomorphism $\varphi_{\alpha} : U_{\alpha} \xrightarrow{\sim} V_{\alpha}$.

It knows the topology on X : a subset $W \subset X$ is open $\Leftrightarrow W \cap U_{\alpha}$ is open in U_{α} for all $\alpha \Leftrightarrow \varphi_{\alpha}(W \cap U_{\alpha})$ is open in V_{α} for all α .

So we can describe X by giving the underlying set, the subset U_{α} , and identifications $\varphi_{\alpha} : U_{\alpha} \xrightarrow{\sim} V_{\alpha}$ which match up smoothly.

EXAMPLE. We can make the set $\mathbb{C} \cup \{\infty\}$ into a manifold by covering it with $U_0 = \mathbb{C}$ and $U_{\infty} = \mathbb{C}^* \cup \{\infty\}$ and defining

- $\varphi_0 : U_0 \xrightarrow{\sim} \mathbb{C} \cong \mathbb{R}^2$ by $\text{id}_{\mathbb{C}}$;
- $\varphi_{\infty} : U_{\infty} \xrightarrow{\sim} \mathbb{C} \cong \mathbb{R}^2$ by $z \mapsto 1/z$ on \mathbb{C}^* and $\infty \mapsto 0$.

The transition function $\mathbb{C}^* \rightarrow \mathbb{C}^*$ is $z \mapsto 1/z$ which is smooth.

Now check for Hausdorff property: given points $p_1 \neq p_2$, either

- They're both contained in (WLOG) U_0 and $\varphi_0(p_1), \varphi_0(p_2)$ are separated by disjoint open sets in $\varphi_0(U_0)$;
- Or they're $0, \infty$, separated by $\varphi_0^{-1}(\text{unit ball})$ and $\varphi_\infty^{-1}(\text{unit ball})$.

For second-countability: take $\varphi_0^{-1}(\text{rational balls})$ and $\varphi_\infty^{-1}(\text{rational balls})$.

Alternative perspective:

- There's no need to talk about the underlying set;
- Instead we could start with open sets $V_\alpha \subset \mathbb{R}^n$ and specify how to glue them together smoothly on open subsets;
- The first step is then to construct the underlying set, by taking the disjoint union of the V_α and quotienting by the equivalence relation generated by the gluing instructions.

This is 'building a manifold by gluing open sets in \mathbb{R}^n '.

But it's cumbersome, and one often starts with a nice description of the underlying set anyway, so we shall take it as given.

Suppose we're given:

- A set X ;
- A collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of subsets covering X ;
- For each α an open set $V_\alpha \subset \mathbb{R}^n$ and a bijection $\varphi_\alpha : U_\alpha \rightarrow V_\alpha$.

Suppose also that for all α and β in \mathcal{A} the set $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in V_α (or, equivalently, open in \mathbb{R}^n), and that

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$$

is smooth.

DEFINITION 1.6. Call the data above a *smooth pseudo-atlas*, and each φ_α a *pseudo-chart*. (Non-standard definition.)

Declare a subset $W \subset X$ to be open if and only if $\varphi_\alpha(W \cap U_\alpha)$ is open in V_α for all α .

LEMMA 1.3. This defines a topology on X .

Proof. Easy to check. □

PROPOSITION 1.4. *Apart from the possible failure of Hausdorff and second countable, the resulting topological space X is a topological n -manifold and the pseudo-atlas $\{\varphi_\alpha : U_\alpha \rightarrow V_\alpha\}_{\alpha \in \mathcal{A}}$ forms a smooth atlas.*

Proof. We just need to check that the U_α are open and that the pseudo-charts φ_α are homeomorphisms with respect to the topology we have defined on X . So take an arbitrary α and a subset $W \subset U_\alpha$. We need to show that W is open in X if and only if $\varphi_\alpha(W)$ is open in V_α .

To show $W \subset U_\alpha$ is open $\Leftrightarrow \varphi_\alpha(W)$ is open in V_α .

\Rightarrow : Clear.

\Leftarrow : Suppose $\varphi_\alpha(W)$ is open. Required to prove that for all β the set $\varphi_\beta(W \cap U_\beta)$ is open in V_β . For all β we have

$$\begin{aligned}\varphi_\beta(W \cap U_\beta) &= \varphi_\beta \circ \varphi_\alpha^{-1}(\varphi_\alpha(W \cap U_\beta)) \\ &= (\varphi_\alpha \circ \varphi_\beta^{-1})^{-1}(\varphi_\alpha(W) \cap \varphi_\alpha(U_\alpha \cap U_\beta)).\end{aligned}$$

We're assuming $\varphi_\alpha(W)$ is open in V_α and our hypotheses mean

- $\varphi_\alpha(U_\alpha \cap U_\beta)$ is also open;
- $\varphi_\alpha \circ \varphi_\beta^{-1}$ is smooth and hence continuous.

Thus $\varphi_\beta(W \cap U_\beta)$ is indeed open. □

Say two smooth pseudo-atlases are equivalent if their union is also a smooth pseudo-atlas.

LEMMA 1.5. *Equivalent smooth pseudo-atlases define the same topology and smooth structure on X .*

Sketch Proof. Reduce to the case where one pseudo-atlas contains the other. Then check by hand. □

There's no easy general method for checking whether the topology induced by a pseudo-atlas is Hausdorff. One sufficient condition is that for all p_1, p_2 in X some pseudo-chart U_α covers both points.

Second-countability is much easier: it's enough for X to be covered by countably many of the pseudo-charts.

1.3 Projective Spaces and Grassmannians

Rec 3 Projective spaces and Grassmannians, parametrising subspaces of a fixed vector space
No-Revise are all manifolds.