University of Cambridge
Mathematical Tripos

Part III – **Symmetries, Fields and Particles**

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These notes may not reflect the full format and content that are actually lectured. I usually modify the notes after the lectures and sometimes my own thinking or interpretation might be blended in. Any mistake or typo should surely be mine. Be cautious if you are using this for self-study or revision.

Course Information

Lie groups and Lie algebras are important in the construction of quantum field theories which describe interactions between known particles. Gauge theories, which describe many of the interactions in the Standard Model, rely on them. After some other preliminaries, we introduce representations in terms of square matrices. The group of rotations in three-dimensional space SO(3) is covered, along with SU(2) and the connection to angular momentum. Relativistic symmetries are discussed: in particular, the Lorentz and Poincaré groups and quantum fields. Lie groups and Lie algebras are covered in more generality, focusing on SU(3) as a useful example. An overview of the results of the Cartan classification of simple Lie algebras is included. Finally, gauge theory is introduced.

Pre-requisites

Linear algebra including direct sums and tensor products of vector spaces. Special relativity and quantum theory, including orbital angular momentum theory and Pauli spin matrices.

Contents

0 Introduction

0.1 Symmetries

Lecture 1
No-Revise

Definition 0.1. A *group* G is a set $G = \{g_1, g_2, \dots\}$ with

1. A composition rule (binary operation) $*$ such that $g * g' \in G, \forall g, g' \in G$, which we shall write as gg';
2. A unique identity e such that $eg = ge = g, \forall g \in G$;
3. Associativity: $(gg')g'' = g(g'g'') := gg'g'', \forall g, g', g'' \in G$;
4. A unique inverse $\forall g \in G, \exists g^{-1}$ such that $gg^{-1} = g^{-1}g = e$.

If the binary operation is commutative, we say that G is *abelian*.

Example 0.2. Group $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ with group operation being addition modulo n and identity $e = 0$.

Cyclic group $C_n = \{e^{2\pi i r/n} \in \mathbb{C} : r = 0, 1, \dots, n-1\}$, certain complex numbers of modulus 1, under multiplication.

$\mathbb{Z}_n$ and C_n are clearly abelian. In fact, $C_n \cong \mathbb{Z}_n$, i.e. they're *isomorphic*, that is to say there exists a one-to-one correspondence between the elements consistent with group composition rules.

Example 0.3. Symmetry groups such as the dihedral group D_3 [Need figure 1 here.] containing reflections along axes and rotations by $120°, 240°, 360°$.

Example 0.4. *Lie groups* are the generalisation to continuous symmetries, e.g. rotations by $\theta \in \mathbb{R}$ of a circle ("SO(2)"). Lie groups are essential to the description of particles and their interactions.

To identify the connection between symmetries and groups, we first make the following definition.

Definition 0.5. A *symmetry* is a transformation that leaves physical properties (e.g. energy, scattering probability, etc.) unchanged. They have properties:

- Symmetries can be composed: gg' := act first with g', then with g;
- Doing nothing is a symmetry, e, the identity;
- A symmetry transformation g can be reversed by g^{-1}, which is itself a symmetry.

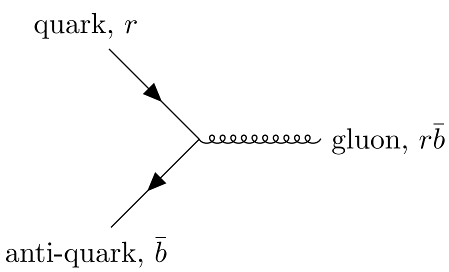
From above, it is clear that the set of all symmetries forms a group. Symmetry often greatly simplifies analysis. It leads to conservation rules and constrains interactions.

0.1.1 Internal Symmetries

Internal symmetries are properties of particles or fields themselves.

Example 0.6 (Colour states of a quark). Quarks come in three otherwise identical copies — called 'colours' (red, green and blue). One can continuously rotate the colours into each other, resulting in a symmetry.

One can rotate the colour differently at different points of spacetime. In fact, one finds that one has to add a force-carrying particle to make the whole theory invariant under the symmetry. This is the gluon, which carries a colour and an anti-colour. Below is a Feynman diagram representing the fusion of a quark and an anti-quark



Anti-quarks carry anti-colour $\{\bar{r}.\bar{g},\bar{b}\}$. The group structure implies that colour is conserved by interactions (i.e. $r\bar{b} \to r\bar{b}$ in $q\bar{q} \to g$).

When the theory is left invariant by a symmetry transformation that's the same across whole spacetime, it's called a *global symmetry*.

The theory of quarks, anti-quarks and gluons is called *Quantum Chromodynamics* (QCD), which is a part of the Standard Model of particle physics.

Since the colour rotations may differ at different points $(\mathbf{x}, t)$ in spacetime, it is called a *local* or *gauge* symmetry.

0.1.2 External Symmetries

External symmetries involve spacetime coordinates.

Example 0.7.

- Translation in $(\mathbf{x}, t)$;
- Lorentz transformation: boosts/rotations;

Conserved quantities come from the group structure: e.g. energy, momentum, angular momentum, etc. The *Poincaré group* consists of all these symmetries: 3 boosts, 3 rotations and 4 translations.

Group theory has also been used in cases where the symmetries are approximate but not exact, to explain the spectrum of hadrons, for instance.

0.2 Particles

0.2.1 Force-carriers

Force-carriers are particles with spin $1(\hbar)$ (convention $\hbar \to 1$, see QFT course).

Example 0.8.

- g, gluon carries colour force;
- γ, photons carry the electromagnetic force.
- $W^{\pm}, Z^0$ boson carry electroweak force that mediates radioactive decay.

Note. Bosons are integer spin particles. Fermions are half-integer spin particles.

For spin 2, we have graviton, the force carrier of gravity. It is not seen yet because gravity is so weak.

Force carriers belonging to a good symmetry are *massless.* Those corresponding to one where the vacuum "spontaneously" breaks an underlying symmetry may be massive (i.e. $W^{\pm}, Z^0$ bosons, the symmetry is broken by the Higgs mechanism).

0.2.2 Matter Particles

Matter particles are of spin $\frac{1}{2}$.

Example 0.9.

- Up quarks, electric charge $Q = +2/3$ (choosing units where $e = 1$);
- Down quarks, $Q = -1/3$;
- Neutrinos, $Q = 0$;
- Electrons, $Q = -1$.

They all have anti-particles, with opposite sign charge or anti-colour.

Matter particles additionally come in 3 families, each are heavier than the last but otherwise with the same colour and charge.

Family	$Q = +2/3$	$Q = -1/3$	$Q = -1$	$Q = 0$
1	up u	down d	electron e	e-neutrino ν_e
2	charm c	strange s	muon μ	μ-neutrino ν_μ
3	top t	bottom b	tauon τ	τ-neutrino ν_τ

Anti-particles are denoted with a bar above, e.g. $\bar{\nu}_e, \bar{u}$, etc.

The Standard Model explains many of these features with a QFT possessing a particular group structure of symmetries. Each particle has its own field which fills the spacetime. Quantum excitations of the fields are observed in experiments.

1 Groups

Lecture 2
No-Revise

1.1 Basics

Recall the definition of a group from last section. Here, we introduce some more definitions and facts about groups.

Definition 1.1. A discrete group G with n elements has *order* $|G| = n$.

Definition 1.2. For any group G, a *subgroup* $H \subset G$ is naturally defined as a set of elements belonging to G which is also a group itself. A *proper subgroup* is when $H \neq G$, and is denoted $H < G$.

Definition 1.3. For any subgroup H, we may define an equivalence relation between $g_i, g_i' : g_i \sim g_i' \Leftrightarrow g_i = g_i' h$ for $h \in H$. Each equivalence class defines a *coset* and has $|H|$ elements. The cosets form a *coset space* G/H such that $G/H \simeq G/\sim$ and $\dim G/H = |G|/|H|$. In general G/H isn't a group.

Theorem 1.4 (Lagrange's theorem). *For any subgroup $H \subset G$, $|H|$ divides $|G|$.*

Definition 1.5. The *index* of subgroup H in G is the number of cosets in G/H, denoted $G : H = |G|/|H|$.

Definition 1.6. A *normal*, or *invariant* subgroup is a subgroup $H \subset G$ such that

$$gHg^{-1} = H \quad \forall g \in G.$$

This is denoted $H \triangleleft G$ (or $G \triangleright H$).

Proposition 1.7. *For a normal subgroup H of G, G/H becomes a group.*

Proof Sketch. For $g_i' = g_i h_i, g_j' = g_j h_j$ with $h_i, h_j \in H$, then $g_i' g_j' = g_i g_j h$ for some $h \in H$. (Try to complete the proof.) □

Corollary 1.8. *For an abelian group, all subgroups are normal subgroups.*

Definition 1.9. A group is *simple* if the only normal subgroups are G and the trivial subgroup formed by the identity e itself.

Definition 1.10. The *centre* of a group G, denoted $\mathcal{Z}(G)$, is the set of all elements which commute with all elements of G. It is an abelian, normal subgroup.

DEFINITION 1.11. For two groups G_1 and G_2, we may define a *direct product group* $G_1 \times G_2$ formed by pairs of elements $\{(g_1, g_2)\}$ belonging to (G_1, G_2), defined by the rules

- $(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2')$;
- $(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1})$;
- $e = (e_1, e_2)$.

So long as it is clear which elements belong to G_1 and which to G_2, we may write the elements of $G_1 \times G_2$ as $g_1 g_2$ or $g_2 g_1$.

COROLLARY 1.12. *For finite groups* $|G_1 \times G_2| = |G_1||G_2|$.

1.2 Cyclic, Dihedral and Permutation Groups

1.2.1 Cyclic Groups

Clearly $\mathbb{Z}_n$ is abelian. For p prime, $\mathbb{Z}_p$ has no subgroups. p has no divisors, hence $\mathbb{Z}_p$ is simple.

PROPOSITION 1.13. *If $n = pq$ then $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are normal subgroups of $\mathbb{Z}_n$ and $\mathbb{Z}_{pq}/\mathbb{Z}_p \simeq \mathbb{Z}_q$. If p, q are co-prime (i.e. no common factors), then $\mathbb{Z}_{pq} \simeq \mathbb{Z}_p \times \mathbb{Z}_q$.*

1.2.2 Dihedral Groups

DEFINITION 1.14. The *dihedral group* D_n of order $2n$, is the symmetry groups of a regular n-sided polygon, formed by rotations a through angles $2\pi r/n, r = 0, \cdots, n-1$, together with reflections$b$. In general,

$$D_n = \{a^r, a^r b : (r = 0, \cdots, n-1; a^0 = a^n = e; b^2 = e; ab = ba^{r-1})\}.$$

For any r, we have $(a^r b)^2 = e$.

For $n > 2$, the group is non-abelian since $ab \neq ba$.

It is an easy observation that

$$D_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$$

1.2.3 Permutation (or Symmetric) Groups

DEFINITION 1.15. The *permutation group* S_n acting on n objects, is the group of all permutations of these n objects. It has $|S_n| = n!$.

EXAMPLE 1.16. $S_3 \simeq D_3$: symmetry group of equilateral triangle under permutations of vertices.

Definition 1.17. A_n, the *alternating group*, is a normal subgroup of S_n formed by the even permutations. $|A_n| = n!/2$.

Corollary 1.18. *For $n \geq 5$, A_n is simple.*

Proposition 1.19.

$$S_n/A_n \simeq \mathbb{Z}_2.$$

Also, note $A_3 \simeq \mathbb{Z}_3$.

The elements of the permutation group can be decomposed into *cycles*. Acting on $\{1, 2, \cdots, n\}$,

- 2-cycle (ij) with $i \neq j$, swaps i and j;
- 3-cycle (ijk) with $i \neq j \neq k$, takes $i \to j \to k \to i$;
- ...

Thus we generalise to the following.

Definition 1.20. A *p-cycle* $(i_1 i_2 \cdots i_p)$ for all i_j different with $p \leq n, 1 \leq i_j \leq n$ generates cyclic permutations of $\{i_1, \cdots, i_p\}$.

Note that $(i_1 i_2 \cdots i_p)^p = e$.

For any one of the $\binom{n}{p}$ choices of $\{i_j\}$, there exists $(p-1)!$ choices for the p-cycle involving $\{i_j\}$, since any p-cycle is invariant under cyclic permutations.

For distinct i, j, k, l

$$(ij)(kl) = (kl)(ij) \quad \text{and} \quad (ij)(jk) = (ijk).$$

Proposition 1.21. *The action of some $g \in S_n$ can be decomposed into cycles.*

Proof. Consider an arbitrary $i \in \{1, 2, \cdots, n\}$ and act g on i by $g^r i, r = 1, 2, \cdots$ For some minimal p, we have $g^p i = i$. The action of g then generates a p-cycle $(i_1 \cdots i_p)$.

Now we pick $j \in \{1, 2, \cdots, n\} \backslash \{i_1, \cdots, i_p\}$ acting repeatedly with g generates a new q-cycle $(j_1 \cdots j_q)$ for some q and $j_1 = j$. Continuing, any element of $\{1, 2, \cdots, n\}$ belongs to some cycle. $\square$

If $gk = k$, then the element belongs to the 1-cycle (k).

We may denote g as $g_{(i_1 \cdots i_p)(j_1 \cdots j_q) \cdots}$, $e = e_{(1)(2) \cdots (n)}$ and $g^{-1} = g_{(i_p \cdots i_1)(j_q \cdots j_1) \cdots}$.

If h corresponds to a permutation σ where $\sigma\{1, 2, \cdots, n\} = \{\sigma(1), \sigma(2), \cdots, \sigma(n)\}$, then

$$h g_{(i_1 \cdots i_p)(j_1 \cdots j_q) \cdots} h^{-1} = g_{(\sigma(i_1) \cdots \sigma(i_p))(\sigma(j_1) \cdots \sigma(j_q)) \cdots}.$$

1.3 Orbit Stabiliser Theorem

Now we introduce *orbit stabiliser theorem*, which applies when a group G acts an a space $X = \{x\}$ such that $\forall g \in G, x \to gx$.

> DEFINITION 1.22. For any particular $x \in X$, the *stabiliser group* or *little group* G_x is defined by those elements of G which leave x invariant, i.e.
>
> $$G_x = \{h : h \in G, hx = x\}.$$
>
> a subgroup of G.

> DEFINITION 1.23. The *orbit* of x is the set of points in X obtained by the action of G,
>
> $$O_x = \{x' : x' = gx, \forall g \in G\}.$$

> THEOREM 1.24 (Orbit stabiliser theorem). *O_x can be identified with G/G_x. For a finite group G, $\dim O_x = |G|/|G_x| \in \mathbb{Z}$ by Lagrange's theorem. For $x' \in O_x$, $G_{x'} \simeq G_x$. In general, the space X can be decomposed into orbits under the action of G.*

HINT. $G_{x'} \simeq G_x$ since $hx = x$ and $x' = gx$, giving $h'x' = x'$ for $h' = ghg^{-1}$.