Homework#8

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5.1. (a) Evaluate T^2, for testing H_0: $\boldsymbol{\mu}' = [7, \quad 11]$, using the data

$$\mathbf{X} = \begin{bmatrix} 2 & 12 \\ 8 & 9 \\ 6 & 9 \\ 8 & 10 \end{bmatrix}$$

(b) Specify the distribution of T^2 for the situation in (a).

(c) Using (a) and (b), test H_0 at the $\alpha = .05$ level. What conclusion do you reach?

a) We have $\overline{\boldsymbol{X}}' = [6, 10]$, $n = 4, p = 2$,

$$\boldsymbol{S} = \frac{1}{n-1}\boldsymbol{X}'\boldsymbol{H}\boldsymbol{X} = \frac{1}{3}\begin{bmatrix} 2 & 8 & 6 & 8 \\ 12 & 9 & 9 & 10 \end{bmatrix}\left[I_4 - \frac{1}{4}(1_4 1_4')\right]\begin{bmatrix} 2 & 12 \\ 8 & 9 \\ 6 & 9 \\ 8 & 10 \end{bmatrix} =$$

$$\frac{1}{12}\begin{bmatrix} 2 & 8 & 6 & 8 \\ 12 & 9 & 9 & 10 \end{bmatrix}\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 12 \\ 8 & 9 \\ 6 & 9 \\ 8 & 10 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 2 & 8 & 6 & 8 \\ 12 & 9 & 9 & 10 \end{bmatrix}\begin{bmatrix} -16 & 8 \\ 8 & -4 \\ 0 & -4 \\ 8 & 0 \end{bmatrix}$$

$$\frac{1}{12}\begin{bmatrix} 96 & -40 \\ -40 & 24 \end{bmatrix} = \begin{bmatrix} 8 & -\frac{10}{3} \\ -\frac{10}{3} & 2 \end{bmatrix}$$

$$\overline{\boldsymbol{X}} - \boldsymbol{\mu}_0 = \begin{bmatrix} 6 \\ 10 \end{bmatrix} - \begin{bmatrix} 7 \\ 11 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\boldsymbol{T}^2 = n(\overline{\boldsymbol{X}} - \boldsymbol{\mu}_0)'\boldsymbol{S}^{-1}(\overline{\boldsymbol{X}} - \boldsymbol{\mu}_0) = \frac{4}{3}[-1, -1]\begin{bmatrix} \frac{9}{22} & \frac{15}{22} \\ \frac{15}{22} & \frac{18}{11} \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} =$$

$$4[-1,-1]\begin{bmatrix}-\frac{12}{11}\\-\frac{51}{22}\end{bmatrix}=\frac{150}{11}$$

b) We have that $\boldsymbol{T^2} = n(\overline{\boldsymbol{X}}-\boldsymbol{\mu}_0)'\boldsymbol{S^{-1}}(\overline{\boldsymbol{X}}-\boldsymbol{\mu}_0)\sim \frac{(n-1)p}{n-p}\boldsymbol{F}_{p,n-p}$,

then in our case $\boldsymbol{T^2}\sim\boldsymbol{3F_{2,2}}$

c) From b) we have that $\boldsymbol{T^2}\sim\boldsymbol{3F_{2,2}}$, at level $\alpha = 0.05$, is $\boldsymbol{3F_{2,2}(0.05)} = \boldsymbol{3(19)}$, so

$$\boldsymbol{T^2} = 13.6 < \ \boldsymbol{3F_{2,2}(0.05)} = 57$$

Thus, we do not reject H_0 at level $\alpha = 0.05$

5.13. Determine the approximate distribution of $-n\ln(|\hat{\boldsymbol{\Sigma}}|/|\hat{\boldsymbol{\Sigma}}_0|)$ for the sweat data in Table 5.1. (See Result 5.2.)

We have that $-2\ln(\Lambda) = -2\ln\left(\frac{\max_{\theta\in\Theta_0}L(\theta)}{\max_{\theta\in\Theta}L(\theta)}\right)\sim\chi^2_{v-v_0}$, where Λ is the likelihood ratio,

$v - v_0 = dimension(\Theta) - \text{dimension}(\Theta_0)$ and $v = p + \frac{p(p+1)}{2}$, $v_0 = \frac{p(p+1)}{2}$.

Then, $-n\ln\left(\frac{|\hat\Sigma|}{|\hat\Sigma_0|}\right) = -\frac{2n}{2}\ln\left(\frac{|\hat\Sigma|}{|\hat\Sigma_0|}\right) = -2\ln\left(\left(\frac{|\hat\Sigma|}{|\hat\Sigma_0|}\right)^{\frac{n}{2}}\right) = -2\ln(\Lambda)$,

which will have $-2\ln\left(\left(\frac{|\hat\Sigma|}{|\hat\Sigma_0|}\right)^{\frac{n}{2}}\right)\sim\chi^2_3$.

As X is an 20x3 matrix so $v = 3 + \frac{3(3+1)}{2}$, and $v_0 = \frac{3(3+1)}{2}$ so $v - v_0 = 3$.

5.15. Let X_{ji} and X_{jk} be the ith and kth components, respectively, of $\mathbf{X}_j$.

(a) Show that $\mu_i = E(X_{ji}) = p_i$ and $\sigma_{ii} = \text{Var}(X_{ji}) = p_i(1-p_i), i = 1,2,\ldots,p$.

(b) Show that $\sigma_{ik} = \text{Cov}(X_{ji}, X_{jk}) = -p_ip_k, i \neq k$. Why must this covariance necessarily be negative?

a)

$$E(X_{ji}) = (1)p_i + (0)(1-p_i) = p_i$$
$$Var(X_{ji}) = (1-p_i)^2 + (0-p_i)^2(1-p_i) = p_i(1-p_i)$$

b)

$$Cov(X_{ji},X_{jk}) = E(X_{ji}X_{jk}) - E(X_{ji})E(X_{jk}) = 0 - p_ip_k = -p_ip_k$$

5.29 Refer to the car body data in Exercise 5.28. These are all measured as deviations from target value so it is appropriate to test the null hypothesis that the mean vector is zero. Using the first 30 cases, test $H_0: \boldsymbol{\mu} = \mathbf{0}$ at $\alpha = .05$

mu0	ybar
0	-0.506333
0	-0.207
0	-0.062
0	-0.031667
0	0.698
0	-0.065

s					
0.0626033	0.0615852	0.0473834	0.0082822	0.0197386	0.0031397
0.0615852	0.0924493	0.0267717	-0.000843	0.0227648	0.0154914
0.0473834	0.0267717	0.1446166	0.0078448	0.0210993	-0.004907
0.0082822	-0.000843	0.0078448	0.1086489	0.0220724	0.0065569
0.0197386	0.0227648	0.0210993	0.0220724	0.3428441	0.0145828
0.0031397	0.0154914	-0.004907	0.0065569	0.0145828	0.0366052

t2	f	df1	df2	p
374.72269	51.685888	6	24	1.497E-12

From the table we get that the p-value = $1.497 \times 10^{-12} < \alpha = 0.05$. Thus, we reject the null hypothesis at level α.