Risk-Averse Portfolio Management Using GARCH-Based Forecasts of Time-Varying Returns and Volatility

Time Series Analysis (11320STAT521000) Final Project

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1 Introduction

The problem of portfolio optimization have been revolutionized by Markowitz's mean-variance model in 1952 [1] titled simply Portfolio Selection[1]. However the portfolio optimization proposed assume perfect knowledge of expected returns and portfolio covariance matrix, which is impossible to have considering the unpredictable nature of the financial market. Many work have been proposed to address this issue. For example, robust(or ditributionally robust) stochastic framework uses worst case scenarios in which either investor considers the worst expected return, or the most volatile price dynamics for a specific time period to optimize their portfolio. Optimal stochastic framework aims to either minimize portfolio variance, maximize expected returns, or considers both via risk averse reformulation of the Markowitz mean-variance model. There are numerous methods that have been put forward to solve the optimization problem mentioned above. Layman practitioners often uses historical data to estimate expected returns and price volatilities to assume future parameters. But considering the uncertainty essence of the market, this method rarely works satisfactorily.

In this work, we will attempt to use volatility and expected returns forecast via GARCH model to solve the a risk averse[2] optimization problems; i.) risk averse formulation of Markowitz's mean variance portfolio model. Then we will backtest the result using portfolio of ten assets with varying degree of characteristics. We then compare the performance difference of two methods using historical data methods as a baseline comparison.

We carefully design this portfolio to achieve diversification across multiple asset classes and macroeconomic risk hedging. It includes assets from technology, finance, energy, manufacturing, and digital assets. This diverse set of assets ensures emphasizing the methods we utilized works with many portfolio of assets. To add to that, this portfolio will show us the behavior of the solution againts differing kind of assets with different degree of returns volatility.

The portfolio is structured across multiple sectors—technology (AAPL, MSFT, META), finance (V, JPM, C), energy/commodities (UEC, ET), manufacturing (GM), and digital assets (BTC)—to balance growth, value, and income generation. Growth exposure focuses on innovative tech leaders, while value and income are achieved through stable financials (JPM, C, GM) and high-yield assets (ET). Additionally, the allocation provides macroeconomic hedging, with rate-sensitive financials, inflation-resistant commodities, and defensive equities (AAPL, V) mitigating risks across different economic conditions.

[1] There are no universal agreed upon "best" formulation of mean-variance model, but almost all have the same objectives of either minimizing portfolio variance or maximizing expected portfolio return.

[2] Risk averse in the sense that investors wants to mitigate risks to the degree desired by their risk tolerance.

2 EDA

Data Description Using the ten stocks listed above, we obtain price data for the period from January 3, 2022, to April 26, 2024. We partition the dataset into training data (observations before January 1, 2024) and testing data (observations on or after January 1, 2024).

	AAPL	MSFT	META	V	JPM	C	BTC.USD	UEC	GM	ET
2022-01-03	178.6457	325.0381	336.9520	215.6605	146.9958	55.07719	46458.12	3.70	59.47835	6.530941
2022-01-04	176.3783	319.4646	334.9514	216.6637	152.5684	55.50489	45897.57	3.81	63.92197	6.688404
2022-01-05	171.6867	307.2010	322.6494	214.2678	149.7791	54.85898	43569.00	3.86	61.00493	6.605924
2022-01-06	168.8207	304.7735	330.9005	214.0243	151.3704	56.65707	43160.93	3.68	61.13134	6.808375
2022-01-07	168.9876	304.9289	330.2336	211.3070	152.8702	57.41645	41557.90	3.88	60.54793	6.928347
2022-01-10	169.0072	305.1522	326.5311	206.4470	153.0165	57.63466	41821.26	3.76	59.38111	6.838368

Figure 1: Stock Data first 6 rows

Source Real stock price data retrieved via the Yahoo Finance API.

Remark In this EDA section, without loss of context and information, we will use one stock namely AAPL as an example. Other plots however will be provided under Appendix (subsection 7.1). Reader will observe that the prices across these 10 assets exhibits similar pattern of long term trends. This pattern emerges as a consequence of a large scale economic factor happening in the real world[2].

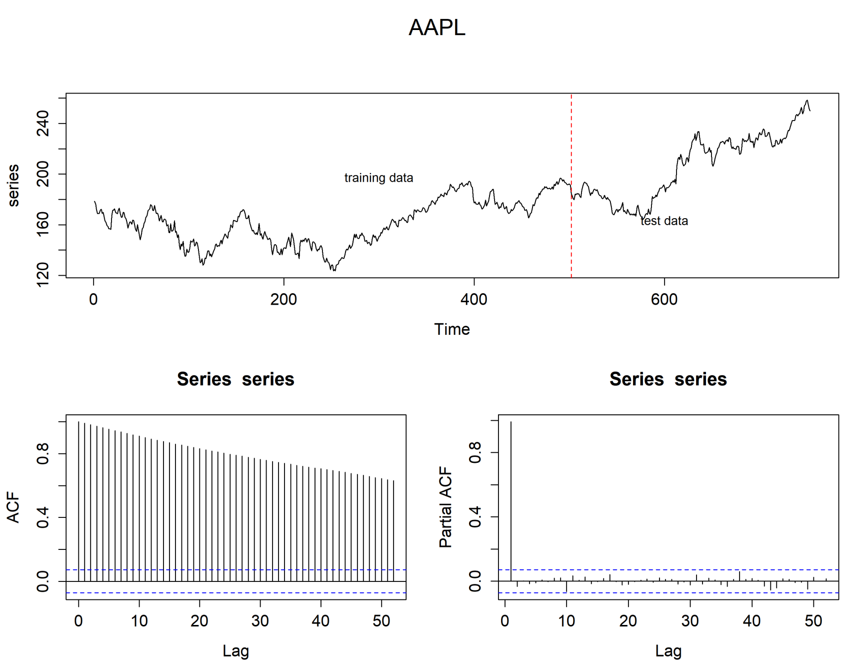


Figure 2: Apple Inc.

- **Price Trend and Trajectory** AAPL's price shows a sustained upward trend over the sample. After an initial consolidation, prices climbed to successive highs. Despite occasional pullbacks and brief volatility spikes, the long-run trajectory remains firmly positive.
- **ACF** Raw prices exhibit significant positive autocorrelation through lag 50, with a slow, long-tail decay—indicative of near–unit-root behavior.
- **PACF** Only lags 1 exceeds significance bounds; higher lags drop off rapidly. While an AR(1) could capture level-series dynamics, the near–unit-root nature suggests modeling returns instead of levels.
- **Volatility Clustering** AAPL's return volatility is clearly time-varying, with bouts of high-volatility "clusters" followed by calmer periods. Notably, in the months after January 1, 2024, volatility surged before settling into a more stable range. This confirms that conditional variance is not constant and that a GARCH-type model is warranted.

In our preliminary EDA, the ACF of AAPL's price series decays slowly and the PACF remains significant only at lags 1, which is a hallmark of near–unit-root behavior. In other words, the price level itself is nonstationary, and directly applying ARMA or GARCH models to these levels would violate the stationarity assumption.

By taking first-differences of logarithmic prices (i.e., log returns), we effectively remove this trend and unit-root component, yielding a series that is approximately stationary and therefore suitable for autocorrelation analysis and heteroskedasticity testing.

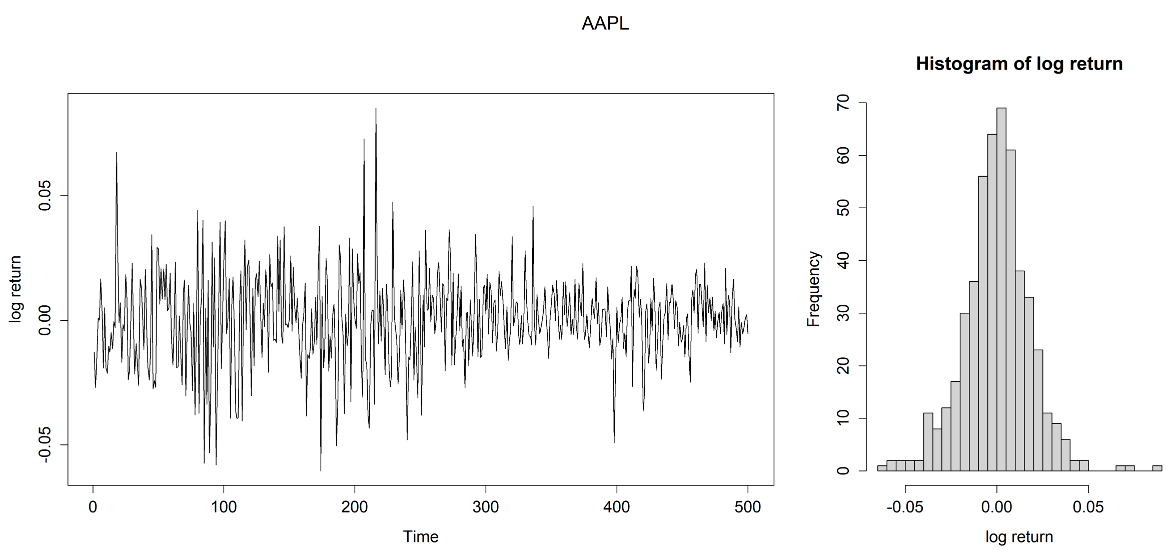


Figure 3: Log Return of Apple Inc.

- **Mean-Reverting Behavior** The log-return series fluctuates around zero without any obvious drift or long-term trend. Both positive and negative shocks appear to alternate without persistent, systematic upward or downward movement. This suggests that the log-return process is plausibly stationary in mean.

- **Volatility Clustering** We observe periods in which the magnitude of the log returns is relatively large, followed by intervals of comparatively low amplitude. In particular, there are several episodes where large spikes (both positive and negative) occur in quick succession. This pattern—consecutive large shocks followed by calmer stretches—is a classic indication of conditional heteroskedasticity.
- **Approximately Symmetric but Slightly Fat-Tailed Distribution** The histogram of log returns is centered near zero and appears fairly symmetric. However, extreme observations (both positive and negative) occur more frequently than under a pure Gaussian distribution, indicating some "fat-tail" characteristics. Whether the series better follows a normal or a Student's t distribution will be determined by subsequent residual diagnostics and distribution-fitting results.

In the log return EDA, the series oscillates around zero with no clear trend and shows volatility clustering; its histogram is roughly symmetric but slightly fat-tailed as expected for a financial data. These observations are qualitative and cannot confirm the absence of a unit root. Therefore, we perform the Augmented Dickey–Fuller (null: unit root) and KPSS (null: stationarity) tests—only if ADF rejects and KPSS fails to reject can we assume covariance stationarity and proceed with ARMA/GARCH models.

3 Modelling

Using Apple Inc. (AAPL) as an example, below is the Model selection of AAPL. (Model selection for the remaining nine stocks is provided in the appendix).

Table 1: ADF Test Results for Log Return Series

Model Type	**Lag 0**	**Lag 1**	**Lag 2**	**Lag 3**	**Lag 4**	**Lag 5**
Type 1: no drift, no trend	-22.31	-16.91	-13.49	-11.71	-9.71	-8.86
Type 2: with drift, no trend	-22.29	-16.89	-13.48	-11.70	-9.70	-8.86
Type 3: with drift and trend	-22.33	-16.93	-13.52	-11.75	-9.75	-8.92

Note: All p-values are ≤ 0.01 under the alternative hypothesis of stationarity.

Table 2: Unit Root Test (Lag=5)

Model Type	**KPSS Statistic**	**p.value**
Type 1: no drift, no trend	0.137	≥ 0.10
Type 2: with drift, no trend	0.183	≥ 0.10
Type 3: with drift & trend	0.0411	≥ 0.10

From the results of both the ADF and KPSS tests, we conclude that the AAPL series is stationary.

- **Returns Series (ACF/PACF):** The autocorrelation function (ACF) and partial autocorrelation function (PACF) of the raw returns show no significant spikes beyond the 95% confidence bounds at any lag. This indicates that the AAPL return series behaves essentially like white noise, with no strong linear dependence or ARMA structure in the mean equation.
- **Squared Returns Series (Volatility Clustering):** The ACF and PACF of the squared returns exhibit pronounced positive autocorrelations at several short lags (e.g., lags 1–10). These significant autocorrelations confirm the presence of volatility clustering, meaning that large (small) returns tend to be followed by large (small) returns in absolute value, hence indicating ARCH effect presence in the series. Below, we will give our result of ARCH test.

We apply Engle's Lagrange Multiplier (ARCH) test [3] to examine whether the residuals obtained by assuming ARCH model exhibit time-varying heteroskedasticity, known as ARCH effects. The null hypothesis assumes constant variance (no ARCH effects), while the alternative suggests the presence of time-varying volatility.

Lag	1	2	3	4	5	6	7	8	9	10	11	12
Statistic	7.256	7.286	8.540	15.885	24.372	24.622	24.741	24.936	50.978	51.567	54.076	54.391
P_Value	0.007	0.026	0.036	0.003	0.000	0.000	0.001	0.002	0.000	0.000	0.000	0.000

Table 3: ARCH Test Results by Lag

Reader may readily observe that the p-value shows significant value for all lags which stipulated the necessities of ARCH/GARCH model.

3.1 Model Selection

We will use method to select ARMA order in an ARMA-GARCH model inspired this article written by (Siaw et al., 2017) [7]

In this work, we will consider ARMA(p,q)-(DCC-)GARCH(1,1) model, one of multivariate alternative of a GARCH family model proposed by Engle as a natural extension of GARCH [5]. Our methodology of selecting ARMA order uses AIC[4] as a information criterion. As a sanity test, we first consider the model under ARMA(0,0) applied on AAPL log return series.

Parameter	**Standard Errors**				**Robust Standard Errors**			
	Estimate	**Std. Error**	**t value**	**Pr($\|t\|$)**	**Estimate**	**Std. Error**	**t value**	**Pr($\|t\|$)**
μ	0.000941	0.000766	1.22873	0.21918	0.000941	0.001967	0.47838	0.63238
ω	0.000002	0.000005	0.40886	0.68264	0.000002	0.000040	0.05302	0.95771
α_1	0.044585	0.027274	1.63472	0.10211	0.044585	0.180848	0.24654	0.80527
β_1	0.947283	0.030562	30.99595	0.00000	0.947283	0.210354	4.50328	0.00001

Information Criterion	**Value**
Akaike	-5.2871
Bayes (BIC)	-5.2534
Shibata	-5.2872
Hannan–Quinn	-5.2739

Table 4: Summary of ARMA(0,0)+GARCH(1,1)

- Since $\beta_1 \approx 0.9473$ is highly significant, the conditional variance of the series exhibits strong persistence—meaning volatility clustering is pronounced and today's volatility is largely driven by yesterday's volatility.
- In contrast, $\alpha_1 \approx 0.0446$ is relatively small and not significant under robust standard errors. This indicates that "last period's shock" has only a limited direct effect on today's variance. In other words, using an ARCH term to capture sudden large shocks driving next-period volatility is not strongly supported by this data.
- The constant term ω is nearly zero and not significant, emphasizing that the conditional variance is driven primarily by the autoregressive component β_1 rather than a fixed baseline variance.
- The mean term μ is not statistically significant, implying that there is no obvious drift in the daily return series over the sample period. Even with μ included in the model, one can effectively treat the mean as zero and consider the process as "zero-mean + GARCH(1,1) noise."
- Overall, the **ARMA(0,0) + GARCH(1,1)** specification (i.e., only a constant mean and a highly persistent GARCH(1,1) variance equation) appears sufficient to capture the series' characteristics:

- The mean equation does not require any autoregressive or moving average structure.
- The variance equation relies primarily on the GARCH term to reflect volatility persistence, while the ARCH term contributes little.

We now find the best ARMA order with maximum p and q (the AR order and MA order respectively) both set at 7 via Akaike information criterion. In our observation, during fitting, Akaike information criterion maintains balance of model complexitites and conservativeness.

Table 5: Model Selection Criteria for ARMA Models

Model	Asset	Akaike	Bayes	Shibata	HQ
ARMA(7,7)	AAPL	-5.389034	-5.245737	-5.391246	-5.332804
ARMA(7,2)	MSFT	-5.153228	-5.052077	-5.154344	-5.113537
ARMA(7,5)	META	-4.066253	-3.939815	-4.067984	-4.016639
ARMA(4,5)	V	-5.714279	-5.613129	-5.715396	-5.674588
ARMA(4,6)	JPM	-5.555521	-5.445942	-5.556828	-5.512523
ARMA(5,4)	C	-5.268432	-5.167282	-5.269549	-5.228741
ARMA(3,4)	BTC.USD	-3.918000	-3.833708	-3.918779	-3.884924
ARMA(6,4)	UEC	-3.443777	-3.334197	-3.445084	-3.400778
ARMA(4,7)	GM	-4.631907	-4.513898	-4.633419	-4.585601
ARMA(6,6)	ET	-5.601636	-5.475198	-5.603368	-5.552022

The coefficients for the models together with DCC-GARCH with its respective p-values can be found under Appendix

3.2 Model Diagnosis

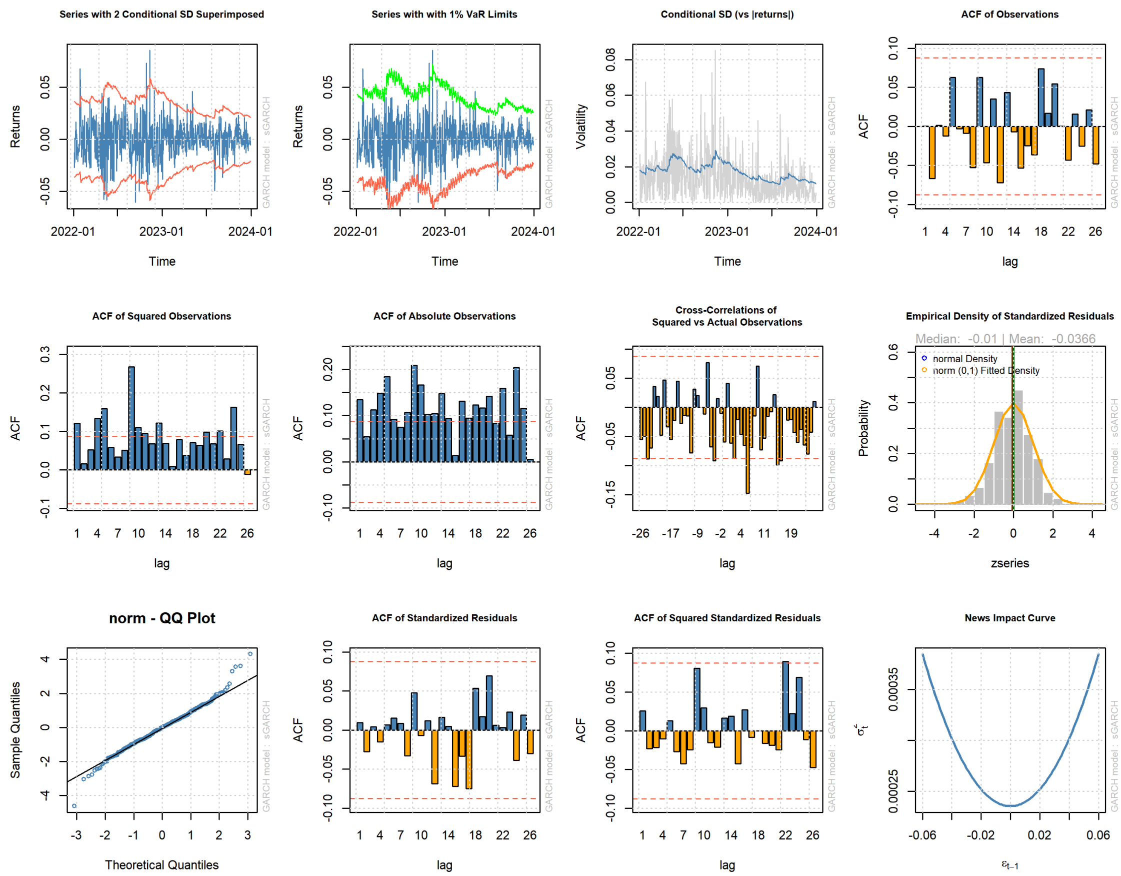


Figure 4: Model Diagnosis

- **Volatility Envelopes (Returns vs. $\pm 2\sigma$)**Most returns stay within the ± 2 conditional standard deviation band, indicating reasonable volatility forecasts.
- **Tail-Risk Check (Returns vs. 1% VaR)**Only a few observations breach the 1% VaR lower bound, suggesting adequate tail-risk control with occasional extreme events.
- **Conditional Standard Deviation** The estimated $\sqrt{h_t}$ spikes during high-volatility periods (e.g., 2022 turmoil) and falls in calmer times, matching expected behavior.
- **ACFs**
  - **Returns ACF** No significant lags, which indicates that returns behave like white noise.
  - **Squared/Absolute Returns ACF** Several significant lags, which confirms volatility clustering.

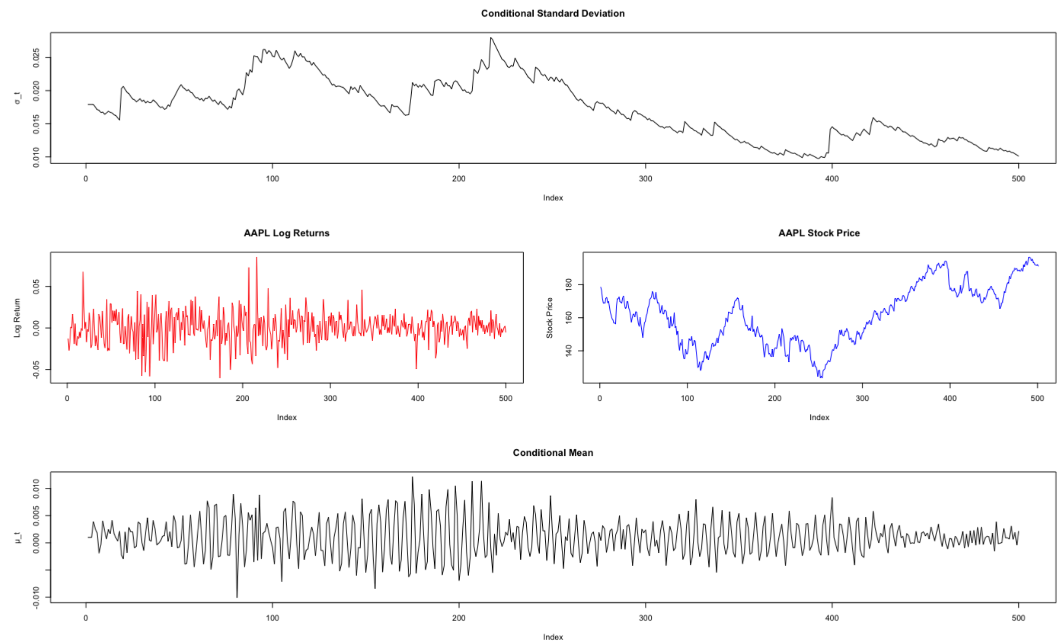


Figure 5: This plot is a fitted vs data under ARMA(3,4)-GARCH(1,1) in AAPL series; i.) the first row plot is the conditional standard deviation, ii) The second row plot is a log return and stock price of AAPL stock, iii) The third row is the conditional mean. Observe that when the log return and stock price is very volatile, and then followed by a calm price movements, the conditional standard deviation is very high followed by a very high value, and when it decreases, it decreases slowly and very smooth, followed by a low value, resulting in a smooth curve. This shows the ARCH effect in action. The conditional mean corresponds to the volatiliy of the log return, volatile at the start, but is less jumpy over time.

  - **Cross-Correlations** Positive lags between past returns and current squared returns indicate ARCH effects.

- **Standardized Residuals**
  - **Distribution (Histogram & QQ-Plot)** Roughly normal but slightly fat-tailed.
  - **ACF of Residuals** No significant autocorrelation, implying the mean equation is well specified.
  - **ACF of Squared Residuals** No remaining ARCH, showing the volatility equation captures most clustering.

- **News Impact Curve** Displays a symmetric U-shape, meaning positive and negative shocks have similar effects on future volatility (no obvious leverage effect).

We conclude that GARCH(1,1) adequately captures mean and volatility dynamics.

3.3 GARCH Model Forecast

Table 6: 10-Step GARCH(1,1) Forecast Starting at 2023–12–29

Horizon	t+1	t+2	t+3	t+4	t+5	t+6	t+7	t+8	t+9	t+10
Series	–0.0010460	0.0009306	–0.0001998	0.0024697	0.0011931	0.0020308	–0.0002463	0.0004694	–0.0002064	0.0017332
Sigma	0.01045	0.01050	0.01055	0.01060	0.01065	0.01070	0.01075	0.01080	0.01084	0.01089

Above is a multi-period forecast starting from the last training observation (2023-12-29), with no rolling updates (Roll Steps: 0) and no out-of-sample validation (Out of Sample: 0). The results show that the conditional mean returns for the next 10 trading days are all very close to zero (ranging from about –0.10% to +0.20%), indicating that the model does not anticipate any significant price direction. At the same time, the conditional standard deviation gradually rises from approximately 1.045% to 1.089%, suggesting a slight increase in implied volatility over the next few days, but overall volatility remains low (around 1%). Therefore, we can infer that the GARCH(1,1) model expects AAPL to remain relatively stable over the short term, with only a modest uptick in volatility.

3.4 DCC-GARCH

Given an N-dimensional return vector $\mathbf{r}_t = (r_{1t}, \ldots, r_{Nt})^\top$, the DCC-GARCH model decomposes its conditional covariance matrix $\mathbf{H}_t$ into a diagonal volatility-matrix D_t and a dynamic correlation-matrix R_t:

$$\mathbf{H}_t \;=\; D_t\, R_t\, D_t, \quad D_t \;=\; \operatorname{diag}\big(\sqrt{h_{1t}}, \ldots, \sqrt{h_{Nt}}\big).$$

Here each h_{it} is the conditional variance of asset i, estimated from a univariate GARCH(1,1):

$$r_{it} \;=\; \mu_{it} + \varepsilon_{it}, \quad \varepsilon_{it} = \sqrt{h_{it}}\, z_{it}, \quad h_{it} = \omega_i + \alpha_i\, \varepsilon_{i,t-1}^2 + \beta_i\, h_{i,t-1}.$$

Define the standardized residual $z_{it} = \varepsilon_{it}/\sqrt{h_{it}}$. After estimating each GARCH model, collect the N-vector of standardized residuals $\widehat{\mathbf{z}}_t = (\widehat{z}_{1t}, \ldots, \widehat{z}_{Nt})^\top$.

Next compute the long-run (unconditional) correlation matrix

$$S \;=\; \frac{1}{T} \sum_{t=1}^{T} \widehat{\mathbf{z}}_t\, \widehat{\mathbf{z}}_t^\top,$$

and initialize

$$Q_1 = S.$$

For $t = 2, \dots, T$, update the *intermediate* matrix Q_t via

$$Q_t \;=\; (1 - a - b)\, S \;+\; a\, \widehat{\mathbf{z}}_{t-1} \widehat{\mathbf{z}}_{t-1}^{\top} \;+\; b\, Q_{t-1}, \quad a \geq 0,\; b \geq 0,\; a + b < 1.$$

Then obtain the dynamic correlation matrix by normalizing Q_t:

$$R_t \;=\; \operatorname{diag}(Q_t)^{-\frac{1}{2}}\, Q_t\, \operatorname{diag}(Q_t)^{-\frac{1}{2}}, \quad \text{so that each } R_{ii,t} = 1.$$

Finally, the full conditional covariance is

$$\mathbf{H}_t = D_t\, R_t\, D_t.$$

Two-Stage Estimation

1. **Stage 1 (Univariate GARCH).** Independently fit GARCH(1,1) to each series $\{r_{it}\}$ to obtain
$$\{\widehat{h}_{it}\}_{t=1}^{T}, \quad \widehat{z}_{it} = \widehat{\varepsilon}_{it} / \sqrt{\widehat{h}_{it}}.$$

2. **Stage 2 (DCC parameters).** Using the standardized residuals $\{\widehat{\mathbf{z}}_t\}$, maximize the "DCC likelihood" in order to estimate the scalars a, b:
$$\mathcal{L}_{\text{DCC}}(a, b) \;=\; -\tfrac{1}{2} \sum_{t=1}^{T} \Big[\ln \det(R_t) + \widehat{\mathbf{z}}_t^{\top} R_t^{-1} \widehat{\mathbf{z}}_t \Big].$$

After obtaining $\widehat{a}, \widehat{b}$, generate $\{R_t\}$ via the recursion above. Then at each time t,

$$\widehat{\mathbf{H}}_t \;=\; \big[\operatorname{diag}(\sqrt{\widehat{h}_{1t}}, \dots, \sqrt{\widehat{h}_{Nt}})\big]\; R_t\; \big[\operatorname{diag}(\sqrt{\widehat{h}_{1t}}, \dots, \sqrt{\widehat{h}_{Nt}})\big].$$

4 Portfolio Optimization

We assume several assumptions on our implementation of portfolio optimization problem:

- **Perfect liquidity:** All assets can be bought or sold in any quantity without affecting their market prices.
- **No frictional costs:** There are no transaction costs, taxes, or other fees associated with trading.
- **Instantaneous execution:** All trades are executed immediately at the desired prices without delay.

- **Insignificant changes to the market:** The portfolio's trades are too small to influence market dynamics or asset prices.

4.1 Preliminary Observation via Efficient Frontier

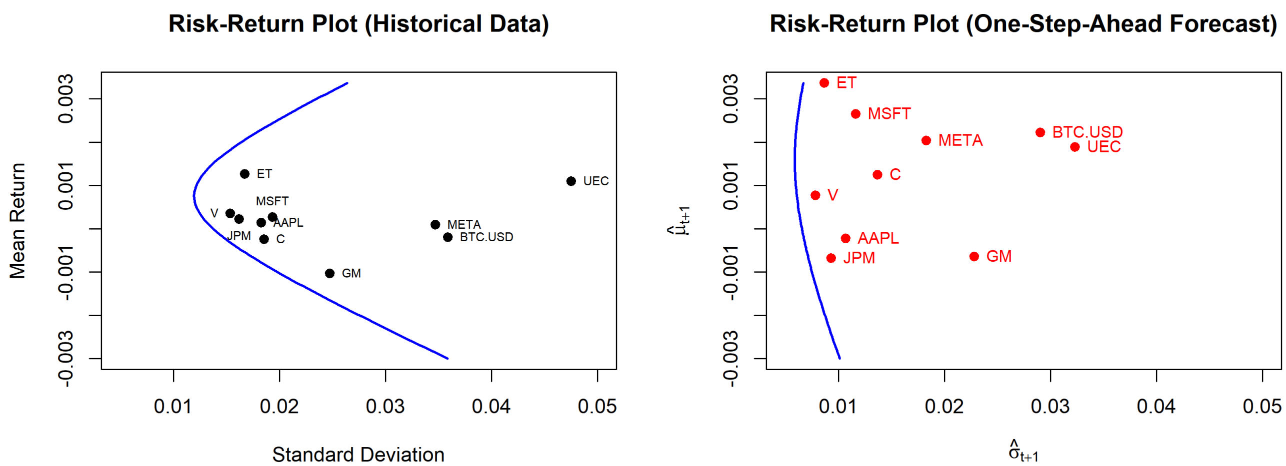


Figure 6: Risk-Return Plots

- **Historical vs. Forecast Frontier:** The historical frontier reflects realized risk-return tradeoffs using sample means and covariances. The forecast frontier uses ARMA–GARCH projections to estimate near-term risk and return. It often lies closer to the origin at low returns but rises more steeply at higher return levels.

- **Top Assets:** "ET" and "UEC" align closely with the historical frontier, suggesting strong risk-adjusted performance. Forecasts favor "ET," "MSFT," and "META" for high projected return and low volatility. "GM" and "JPM" underperform across both views.

- **Blended Strategy:** Use the historical frontier for core allocation and adjust tactically with the forecast frontier—underweight assets falling outside it and overweight those near or within it for short-term gains.

Note. Both frontiers use the same optimization method but differ in inputs—realized vs. forecast. Combining both mitigates overreliance on past data or forecasts. Ensure forecast accuracy and manage risks when rebalancing.

4.2 Risk Averse Markowitz Mean-Variance Problem

The Risk Averse Markowitz mean-variance problem is defined as:

$$\begin{aligned} \max_{\mathbf{w}} \quad & \mathbf{w}^\top \boldsymbol{\mu} - \lambda\, \mathbf{w}^\top \Sigma \mathbf{w} \\ \text{s.t} \quad & \mathbf{1}^T \mathbf{w} = 1 \\ & |w_i| \leq 0.3 \end{aligned}$$

Where:

- $\mathbf{w} \in \mathbb{R}^n$ is the vector of portfolio weights (decision variable),
- $\boldsymbol{\mu} \in \mathbb{R}^n$ is the vector of expected returns,
- $\Sigma \in \mathbb{R}^{n \times n}$ is the covariance matrix,
- $\lambda > 0$ is the risk-aversion parameter.

We deliberately limit $|w_i| \leq 0.3$ to control the movement of our portfolio and reducing the variance of our portfolio.

We consider Markowitz mean–variance portfolio optimization under two information sets: one using the historical-sample covariance matrix and mean returns, and the other using one-step-ahead forecasts of returns and covariance matrix. Hence, under one-step ahead forecast using DCC GARCH, we can denote our conditinal expected return vector and conditional covariance matrix as $\mu_{\mathbf{t+1}}$ and $\mathbf{\Sigma_{t+1}}$

4.3 Rolling Window Portfolio Optimization Using GARCH and Historical Estimates

We conduct a rolling-window portfolio optimization over a test period using both historical and GARCH-based estimates of return and risk. At each step in the test period, it first calculates the historical mean and covariance of returns using all data available up to that point. These estimates are used to construct one portfolio via mean-variance optimization. Simultaneously, the function fits a Dynamic Conditional Correlation (DCC) GARCH model to the historical returns, forecasts the next-day covariance matrix, and estimates expected returns using a moving average of the most recent window. This information is then used to generate an alternative optimized portfolio based on conditional forecasts.

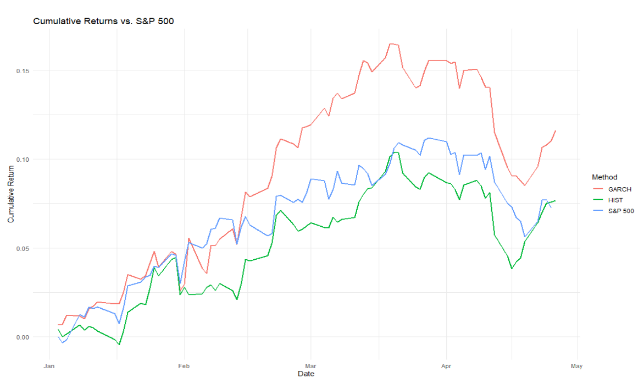
For both the historical and GARCH-based approaches, the function solves a convex optimization problem that maximizes expected return penalized by risk, subject to budget and weight constraints (allowing mild short-selling). It stores the optimal weights, associated portfolio variances (objectives), and realized out-of-sample returns across the rolling period. Finally, the function compiles and returns these results in structured data frames, facilitating further analysis or comparison between the historical and model-based strategies.

4.4 Empirical Result

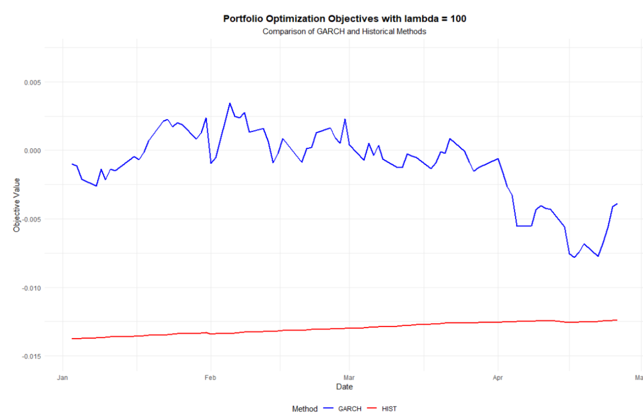
Cumulative Return Comparison

sharpe ratio assuming zero risk free rate:

garch 0.1701974 hist 0.1412727



(a) Comparison of GARCH and Historical method with $\lambda = 100$ and windows = 10



(b) Comparison of Cumulative Return

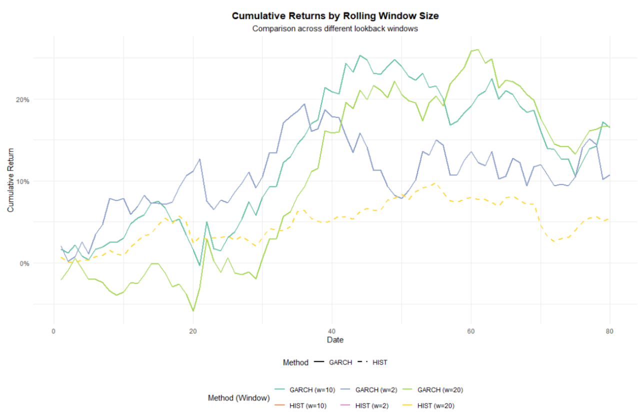
Figure 7: Side-by-side comparison of objective value and cumulative performance

1. **Cumulative Returns and Sharpe Ratio:** Under the same parameter settings $\lambda = 10$ and window length $= 10$, the minimum-variance portfolio constructed using GARCH-estimated covariance (blue) significantly outperforms both the portfolio based on rolling historical covariance (green) and the S&P 500 index (red) during the backtest period. In particular, assuming a zero risk-free rate, the Sharpe ratio of the GARCH-based portfolio is approximately 0.171, whereas the historical method yields only about 0.141.

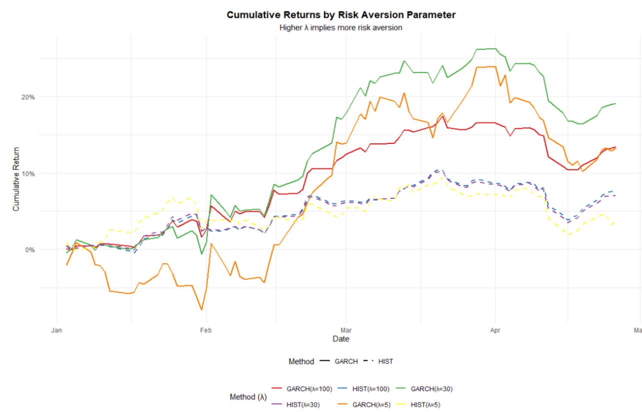
2. **Risk Forecasting and Portfolio Variance:** At each rebalance date, the GARCH model updates its volatility forecasts using the most recent market data: as soon as the market experiences heightened volatility, the predicted covariance matrix Σ_{GARCH} spikes, causing the minimized objective to increase accordingly. This forces the optimization to adopt a more conservative allocation during turbulent periods. When volatility subsides, Σ_{GARCH}

declines and the objective value returns to a lower level. In contrast, the rolling historical covariance method relies solely on the past 10 days of sample data and cannot react promptly to sudden market moves, failing to capture real-time increases in risk.

Parameters Changes Comparison



(a) Comparison of cumulative returns under the same λ but different rolling window sizes w.



(b) Comparison of cumulative returns under the same window $w = 10$ but different risk-aversion parameters λ.

Figure 8: Side-by-side comparison of objective value and cumulative performance

Summary

- **(a):**
  - *Short window* ($w = 5$): Reacts fastest to market changes, adjusting positions quickly, but exhibits more pronounced oscillations in cumulative return.
  - *Medium window* ($w = 10$): In this study, it strikes a balance between "reaction speed" and "estimation stability," resulting in a smoother return path and the highest ultimate return.
  - *Long window* ($w = 20$): Although the return curve is very smooth, it cannot capture short-term market moves promptly, missing out on momentum and producing the weakest overall performance.
- **(b):**
  - *Lower* λ ($\lambda = 5$): Prefers higher risk, willing to tolerate volatility. This yields higher short-term gains but also larger drawdowns—characteristic of a "high-risk, high-return" strategy.

– *Higher* λ ($\lambda = 20$): Prefers lower risk and adopts a more conservative allocation. The cumulative return is smooth but misses out on significant market opportunities.

– *Intermediate value* ($\lambda = 10$): During the backtest period, it achieves a good compromise between return and volatility. It follows the market rally in February–March and quickly reduces risk when volatility spikes, leading to the best risk-adjusted performance.

Overall conclusion

1. The dynamic risk forecasting and real-time adjustment inherent in GARCH (or DCC–GARCH) frameworks significantly outperform methods relying solely on historical sample covariance, under a wide range of parameter settings.

5 Future Works

Dynamic λ_t

Based on above problem and resulting plot, we can readily see that lambda affects the variance of our portfolio. It would be interesting to see a model construction using dynamic λ_t based on the forecast of the variance-covariance matrix to reduce the variance of our portfolio return.

Assumptions Relaxation

The relaxation of the assumptions mentioned above in section 4 is highly desired since real-world financial markets rarely satisfy such idealized conditions. Incorporating transaction costs, liquidity constraints, and market impact leads to more realistic and robust portfolio strategies. For example, Hedge funds usually buy assets in significant volume, enough to change the market price[6][8]. Furthermore, Limit Order Book (LOB)[9] concept might help to adress the issue of market delay.

6 Reference

References

[1] Harry Markowitz. “Portfolio Selection”. In: *The Journal of Finance* 7.1 (1952), pp. 77–91. ISSN: 00221082, 15406261. URL: http://www.jstor.org/stable/2975974 (visited on 06/01/2025).

[2] Barry Bosworth, Saul Hymans, and Franco Modigliani. “The Stock Market and the Economy”. In: *Brookings Papers on Economic Activity* 1975.2 (1975), pp. 257–300. ISSN: 00072303, 15334465. URL: http://www.jstor.org/stable/2534104 (visited on 06/01/2025).

[3] Robert F. Engle. “Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation”. In: *Econometrica* 50.4 (1982), pp. 987–1007. ISSN: 00129682, 14680262. URL: http://www.jstor.org/stable/1912773 (visited on 06/01/2025).

[4] Hirotogu Akaike. “Information Theory and an Extension of the Maximum Likelihood Principle”. In: *Selected Papers of Hirotugu Akaike*. Ed. by Emanuel Parzen, Kunio Tanabe, and Genshiro Kitagawa. New York, NY: Springer New York, 1998, pp. 199–213. ISBN: 978-1-4612-1694-0. DOI: 10.1007/978-1-4612-1694-0_15. URL: https://doi.org/10.1007/978-1-4612-1694-0_15.

[5] Robert Engle and. “Dynamic Conditional Correlation”. In: *Journal of Business & Economic Statistics* 20.3 (2002), pp. 339–350. DOI: 10.1198/073500102288618487.

[6] Nidal Rashid Sabri. “The impact of trading volume on stock price volatility in the Arab economy”. In: *Journal of derivatives & Hedge funds* 14 (2008), pp. 285–298.

[7] Richmond Siaw, Eric Ofosu-Hene, and Evans Tee. “Investment Portfolio Optimization with GARCH Models”. In: *Elk Asia Pacific Journal of Finance and Risk Management* 8.2 (2017). Available at SSRN: https://ssrn.com/abstract=2987932. URL: https://ssrn.com/abstract=2987932 (visited on 06/02/2025).

[8] Charles Cao et al. “Hedge fund holdings and stock market efficiency”. In: *The Review of Asset Pricing Studies* 8.1 (2018), pp. 77–116.

[9] B. Ross Barmish et al. “A Jump Start to Stock Trading Research for the Uninitiated Control Scientist: A Tutorial”. In: *2024 IEEE 63rd Conference on Decision and Control (CDC)*. 2024, pp. 7441–7457. DOI: 10.1109/CDC56724.2024.10886117.

7 Appendix

7.1 EDA of Stock Price

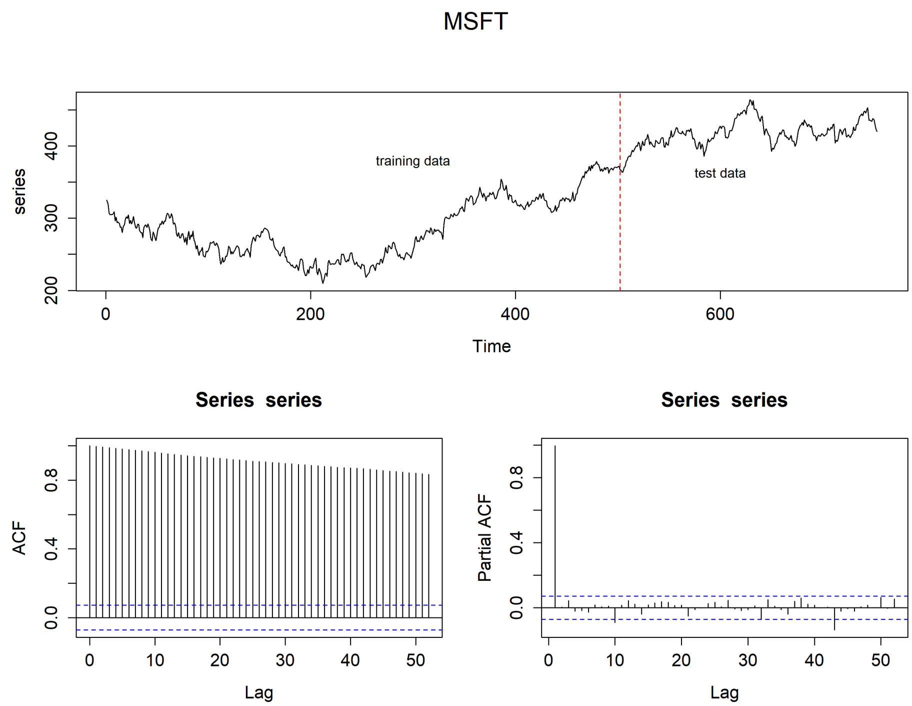


Figure 9: Microsoft Corporation

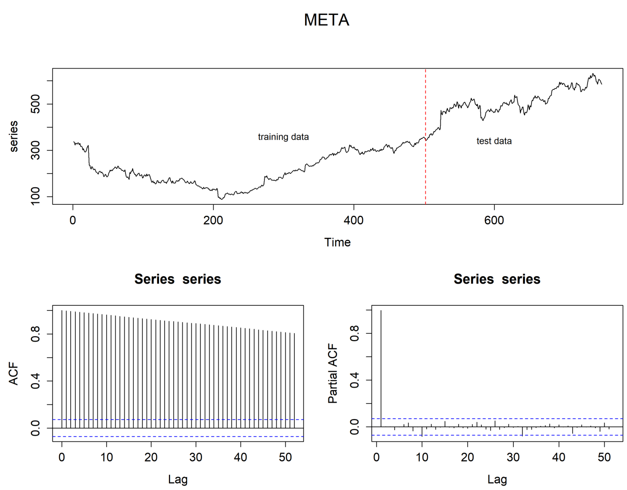


Figure 10: Meta Platforms, Inc.

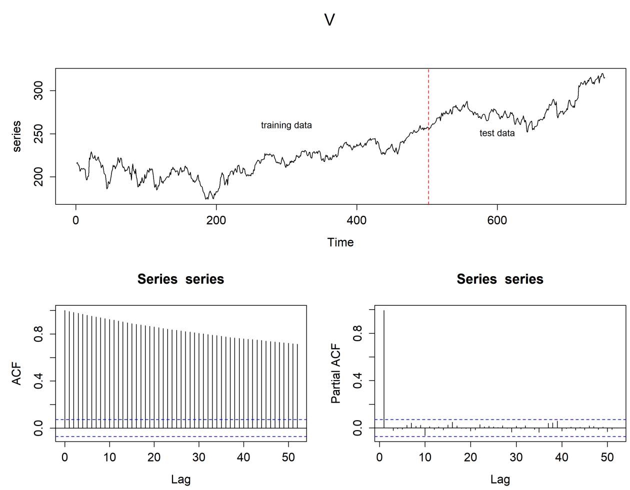


Figure 11: Visa Inc.

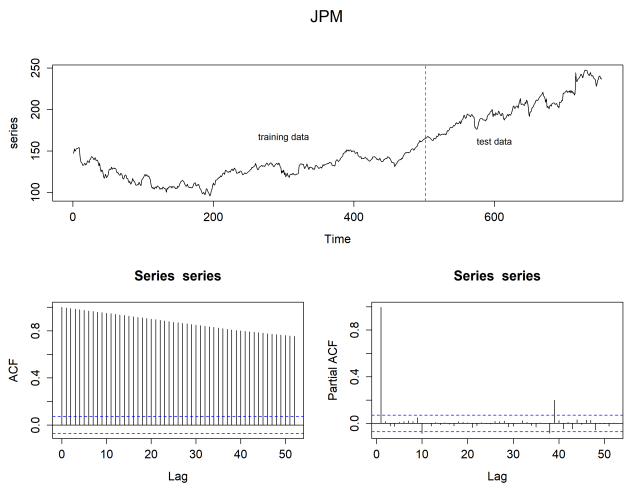


Figure 12: JPMorgan Chase & Co.

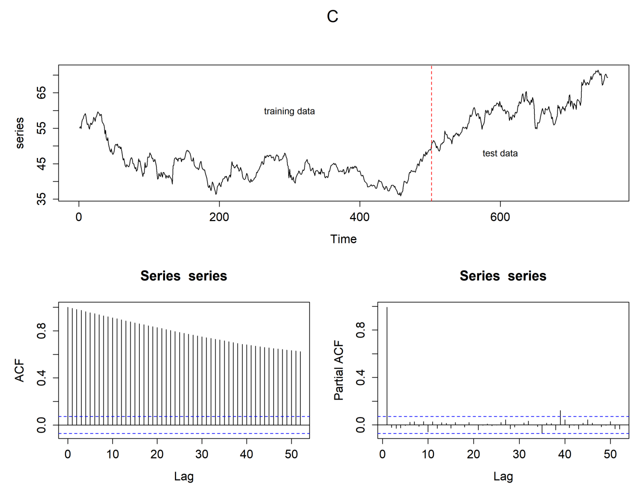


Figure 13: Citigroup Inc.

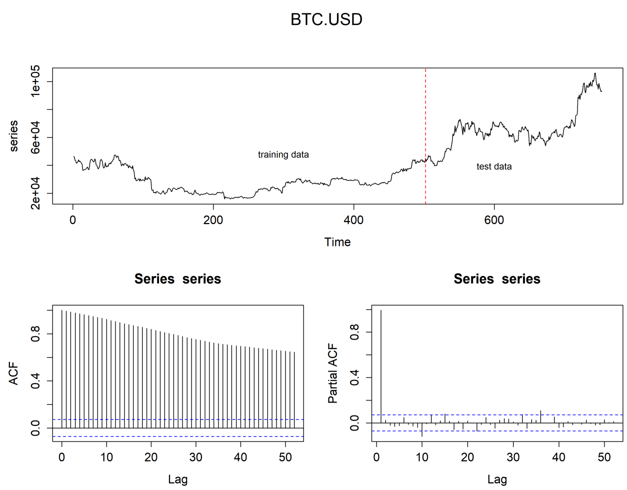


Figure 14: Bitcoin

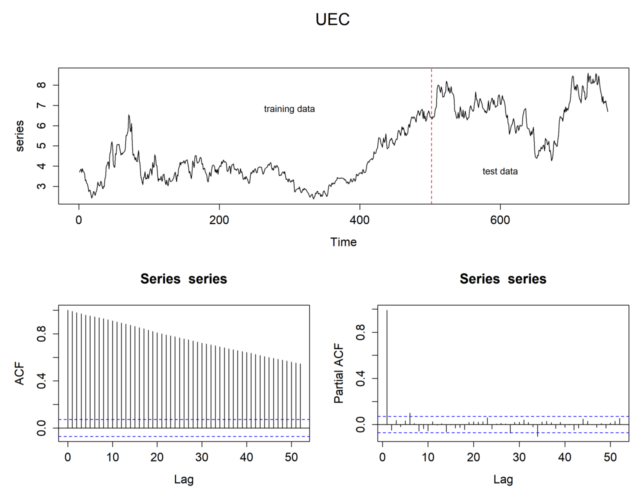


Figure 15: Uranium Energy Corp.

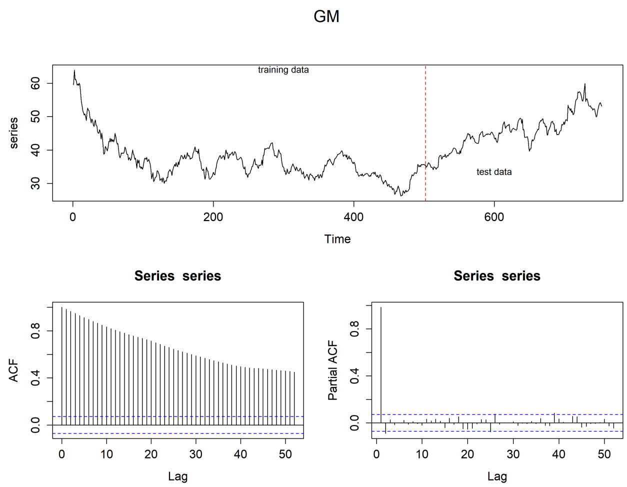


Figure 16: General Motors Company

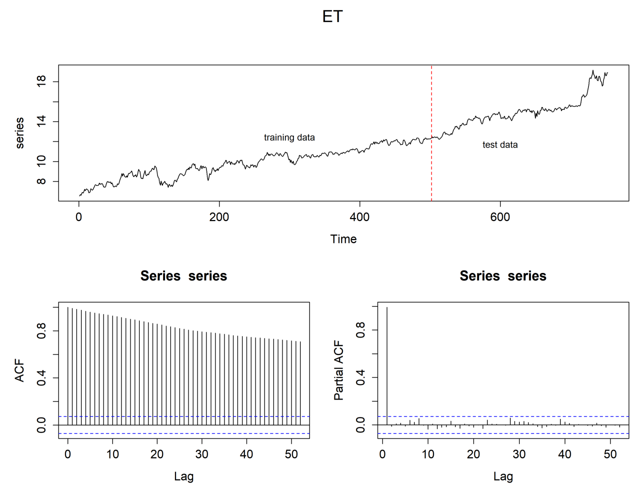


Figure 17: Energy Transfer LP

7.2 EDA of Log Return

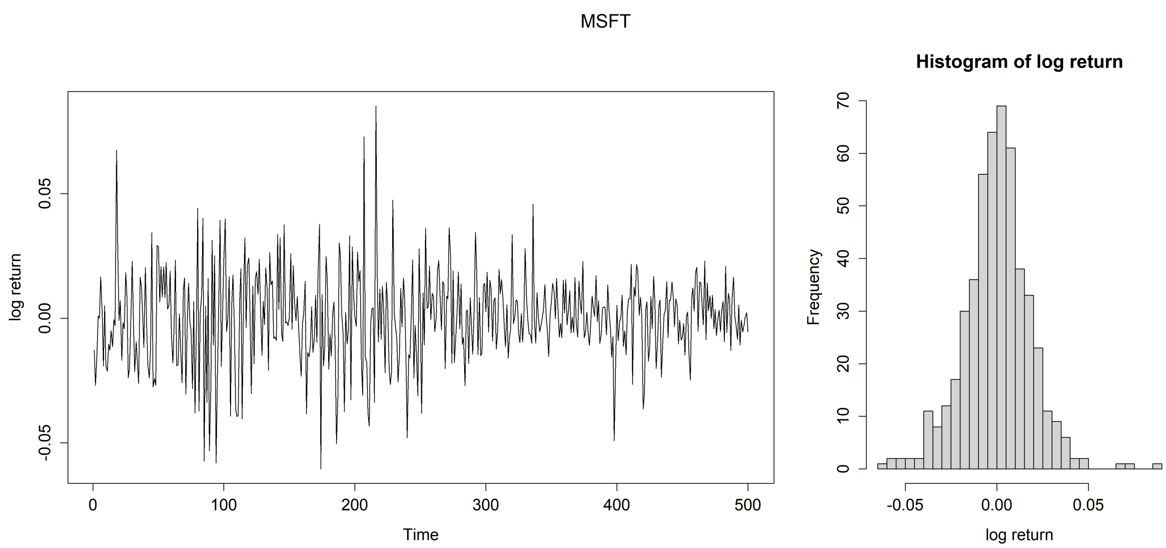


Figure 18: Log Return of Apple Inc.

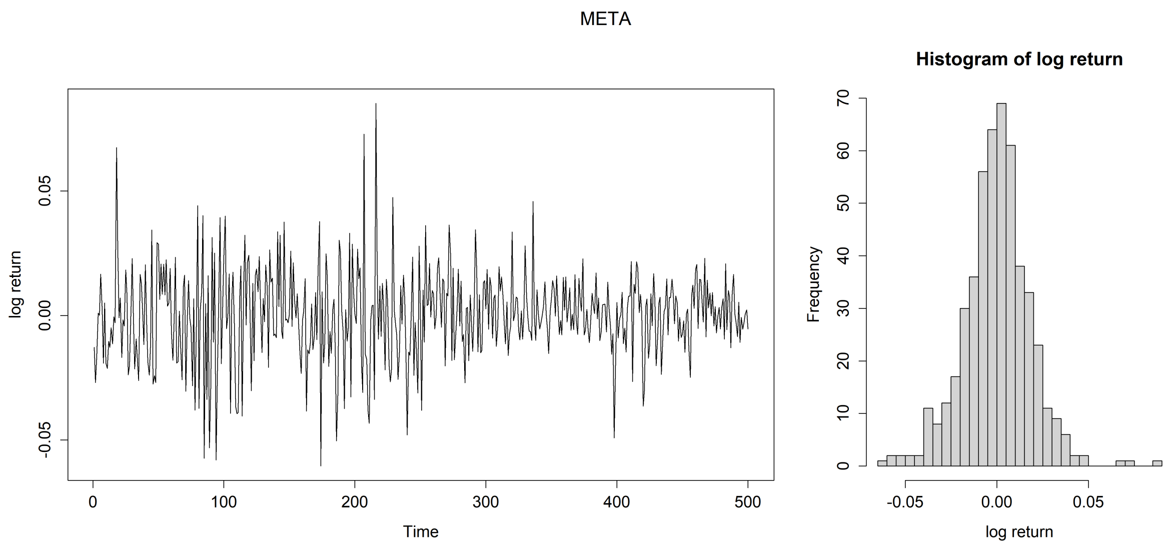


Figure 19: Log Return of Meta Platforms, Inc.

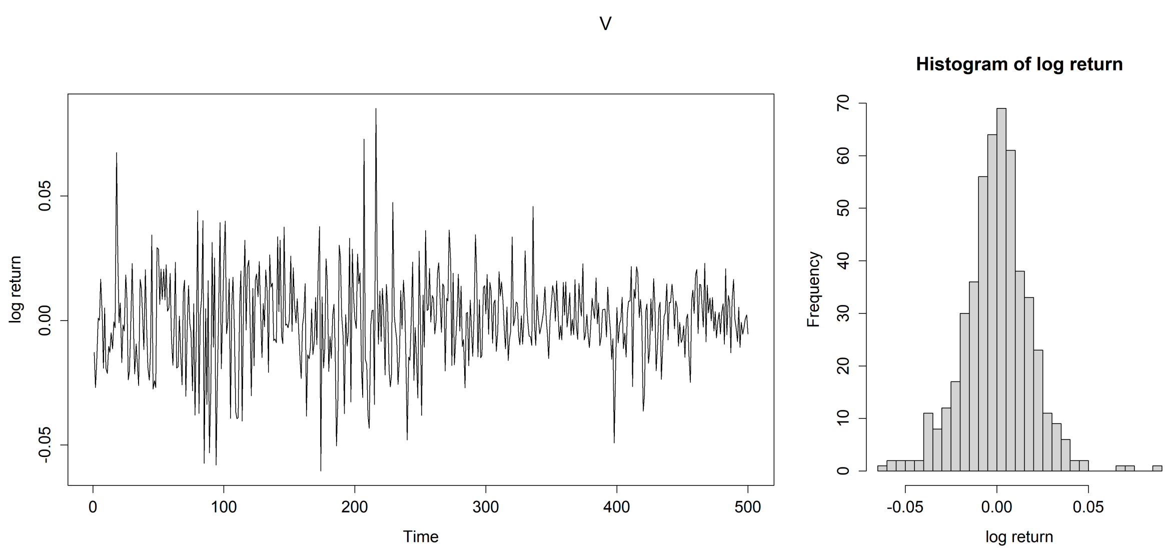


Figure 20: Log Return of Visa Inc.

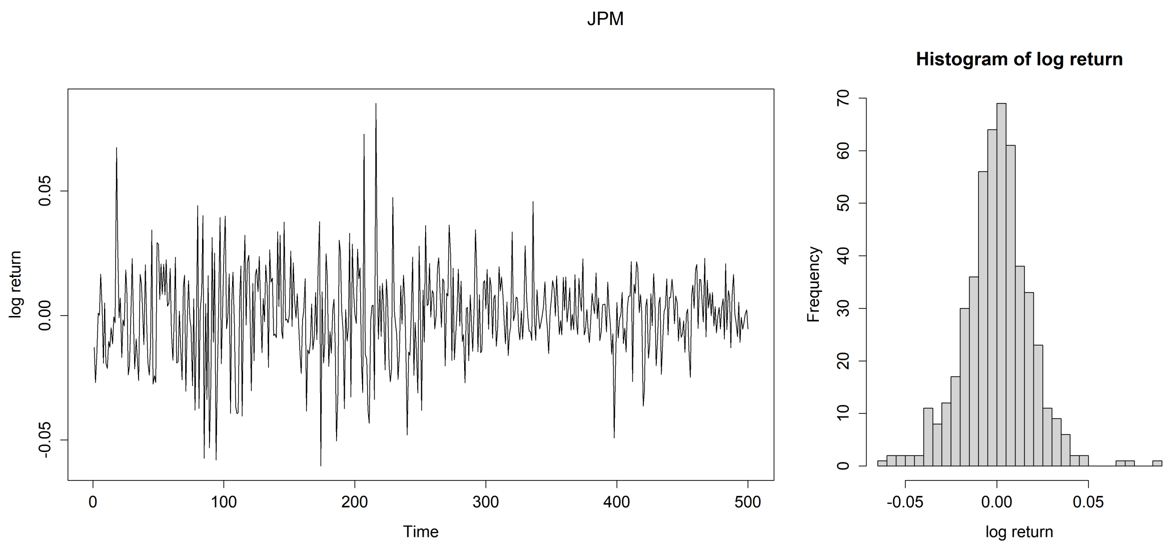


Figure 21: Log Return of JPMorgan Chase & Co.

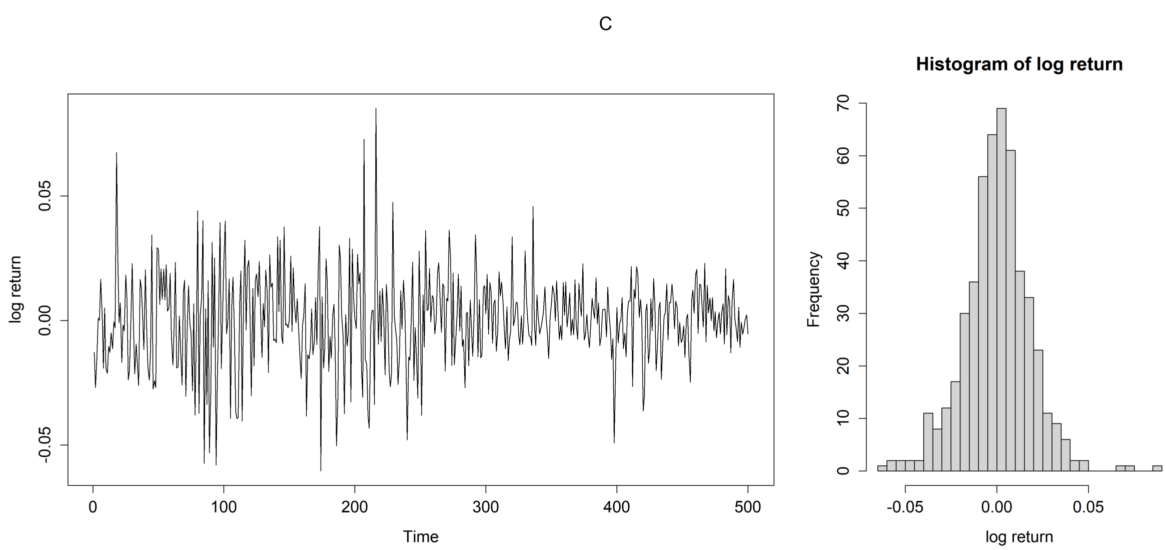


Figure 22: Log Return of Citigroup Inc.

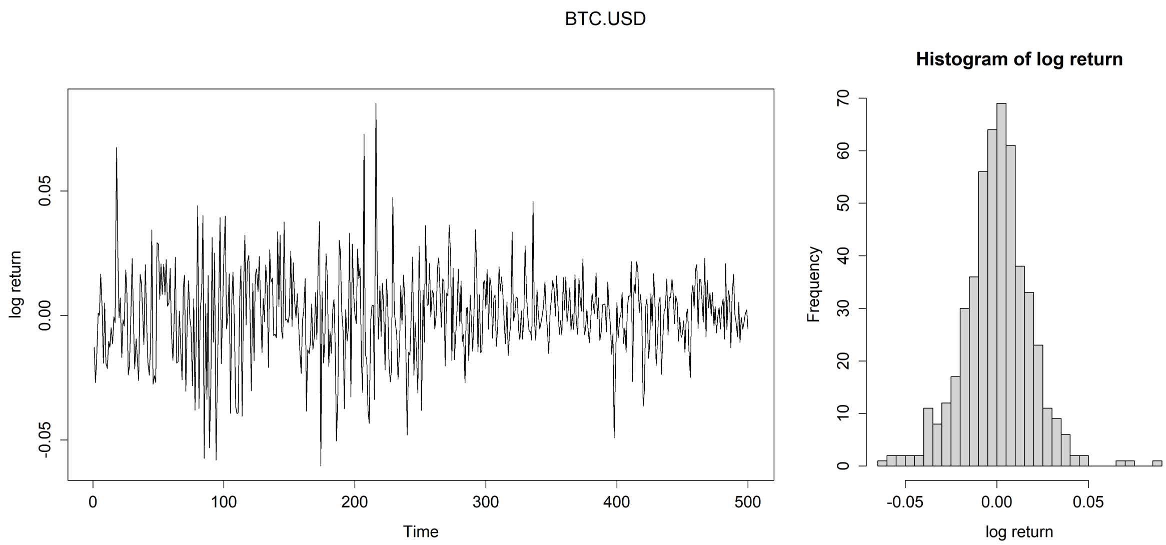


Figure 23: Log Return of Bitcoin

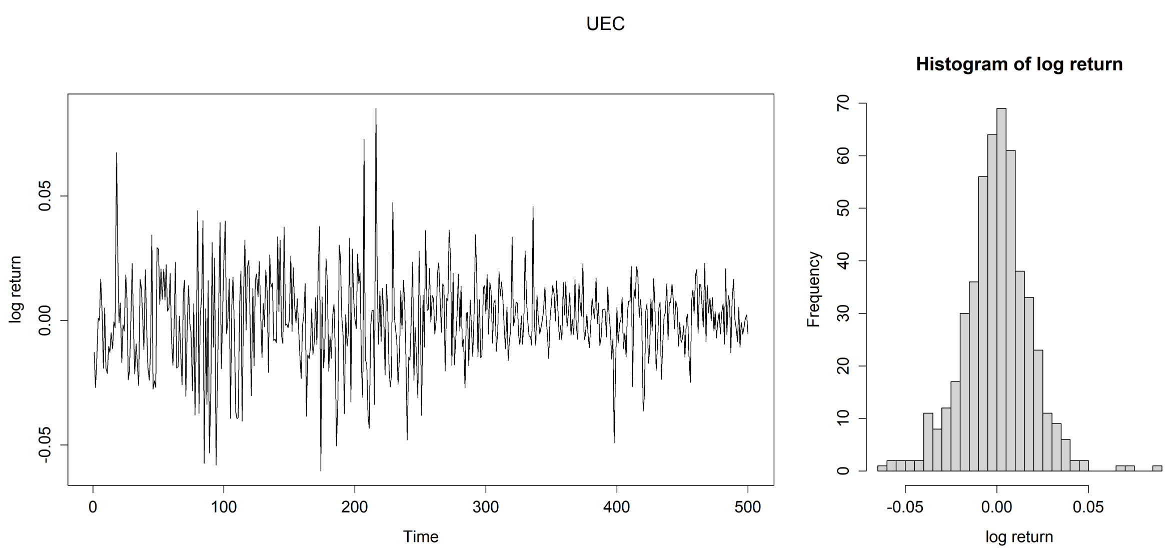


Figure 24: Log Return of Uranium Energy Corp.

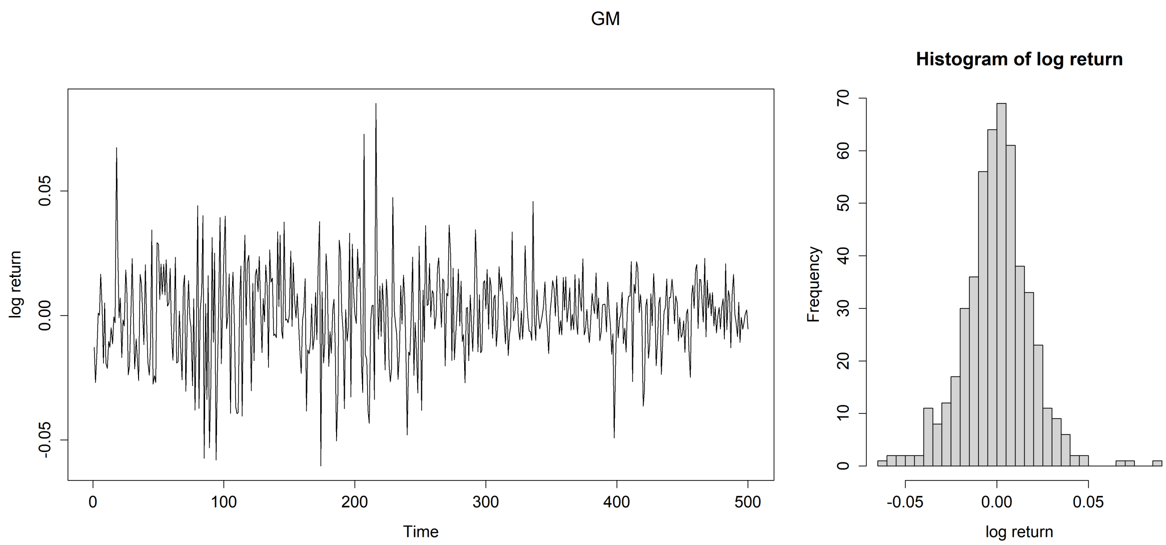


Figure 25: Log Return of General Motors Company

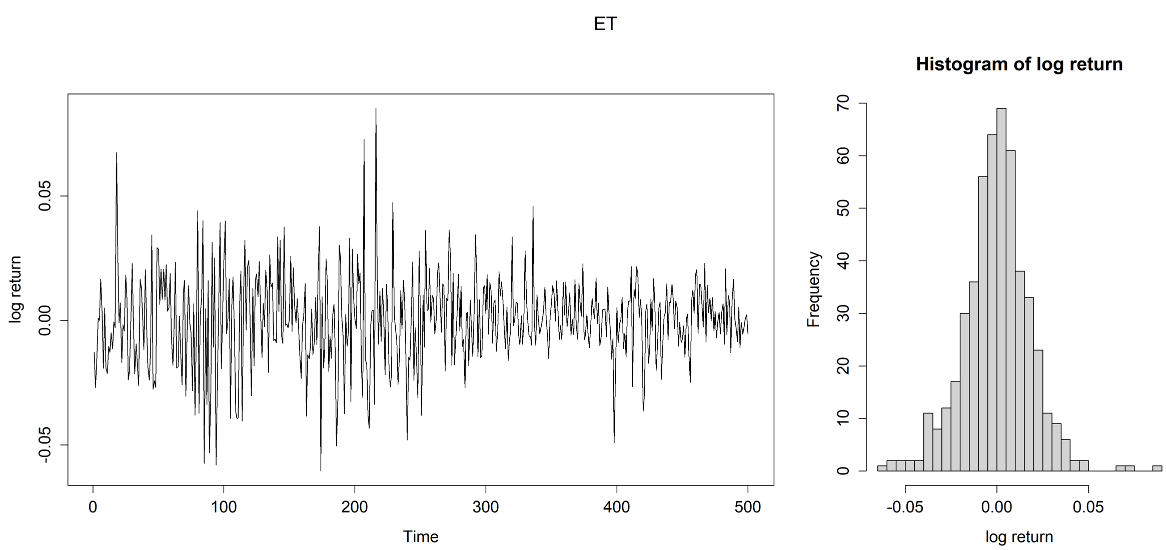


Figure 26: Log Return of Energy Transfer LP

7.3 ARMA-GARCH coefficients

Series	Parameter	Estimate	Std. Error	t value	Pr(> $\|t\|$)
JPM	μ	0.001530	0.000102	15.07	0.000000
	ar1	−0.766316	0.001732	−442.45	0.000000
	ar2	−1.374795	0.001615	−851.45	0.000000
	ar3	−0.737189	0.001760	−418.90	0.000000
	ar4	−0.938967	0.002521	−372.59	0.000000
	ma1	0.666324	0.000604	1103.20	0.000000
	ma2	1.509345	0.003404	443.42	0.000000
	ma3	0.760585	0.000759	1002.60	0.000000
	ma4	1.104062	0.001194	924.94	0.000000
	ma5	−0.051266	0.000164	−312.90	0.000000
	ma6	0.133993	0.000110	1220.60	0.000000
C	μ	0.000607	0.001014	0.60	0.549171
	ar1	−0.573656	0.160434	−3.58	0.000349
	ar2	0.221855	0.049298	4.50	0.000007
	ar3	−0.798085	0.038890	−20.52	0.000000
	ar4	−0.740008	0.118617	−6.24	0.000000
	ar5	0.123869	0.086057	1.44	0.150043
	ma1	0.627580	0.151300	4.15	0.000034
	ma2	−0.147268	0.063325	−2.33	0.020040
	ma3	0.846118	0.038474	21.99	0.000000
	ma4	0.822368	0.119564	6.88	0.000000
BTC.USD	μ	0.003788	0.000255	14.88	0.000000
	ar1	0.104019	0.000327	318.49	0.000000
	ar2	0.206698	0.000387	534.56	0.000000
	ar3	−0.999080	0.002754	−362.70	0.000000
	ma1	−0.022853	0.000245	−93.34	0.000000
	ma2	−0.234053	0.000474	−493.36	0.000000
	ma3	1.029462	0.002374	433.60	0.000000
	ma4	0.113189	0.000348	325.10	0.000000
UEC	μ	0.001660	0.004617	0.36	0.719152
	ar1	0.054650	1.382220	0.04	0.968462
	ar2	1.162361	0.868167	1.34	0.180614
	ar3	0.065796	0.147354	0.45	0.655224
	ar4	−0.593722	0.121289	−4.90	0.000001
	ar5	0.081457	0.101697	0.80	0.423145
	ar6	0.116337	0.131149	0.89	0.375047
	ma1	−0.021274	1.271682	−0.02	0.986653
	ma2	−1.322113	0.750792	−1.76	0.078246
	ma3	−0.149846	0.366841	−0.41	0.682923
	ma4	0.647287	0.062463	10.36	0.000000
GM	μ	0.000439	0.001915	0.23	0.818535
	ar1	0.396549	0.349200	1.14	0.256128
	ar2	−0.298851	0.280634	−1.06	0.286914
	ar3	0.591192	0.162306	3.64	0.000270
	ar4	0.060486	0.287282	0.21	0.833242
	ma1	−0.376675	0.359984	−1.05	0.295392
	ma2	0.329730	0.295642	1.12	0.264720
	ma3	−0.587734	0.182085	−3.23	0.001247
	ma4	0.012289	0.319281	0.04	0.969298
	ma5	−0.166576	0.107439	−1.55	0.121039
	ma6	0.176234	0.117769	1.50	0.134539
	ma7	−0.027832	0.111134	−0.25	0.802247

Table 10: Coefficient of ARMA + DCC–GARCH (ARMA)

Series	Parameter	Estimate	Std. Error	t value	Pr(> $\lvert t \rvert$)
ET	μ	0.001367	0.000009	146.96	0.000000
	ar1	1.053422	0.002371	444.29	0.000000
	ar2	−1.096402	0.002563	−427.80	0.000000
	ar3	−0.467431	0.001961	−238.32	0.000000
	ar4	0.987185	0.003372	292.80	0.000000
	ar5	−1.005353	0.002445	−411.20	0.000000
	ar6	0.380528	0.000299	1270.08	0.000000
	ma1	−1.051154	0.002465	−426.51	0.000000
	ma2	1.169908	0.002609	448.39	0.000000
	ma3	0.460835	0.001159	397.62	0.000000
	ma4	−1.131491	0.002424	−466.80	0.000000
	ma5	1.100485	0.002431	452.74	0.000000
	ma6	−0.516764	0.001066	−484.80	0.000000

Table 11: Coefficient of ARMA + DCC–GARCH (ARMA)

Series	Parameter	Estimate	Std. Error	t value	Pr($>\|t\|$)
AAPL	ω	0.000018	0.000092	0.199	0.842045
	α_1	0.072687	0.213980	0.340	0.734089
	β_1	0.782199	0.917892	0.852	0.394120
MSFT	ω	0.000000	0.000001	0.000	1.000000
	α_1	0.001637	0.000343	4.765	0.000002
	β_1	0.996521	0.000497	2003.10	0.000000
META	ω	0.000000	0.000004	0.000	1.000000
	α_1	0.000030	0.000363	0.082	0.934599
	β_1	0.996712	0.000019	52035.00	0.000000
V	ω	0.000000	0.000001	0.031	0.975393
	α_1	0.000317	0.003369	0.094	0.925010
	β_1	0.998440	0.002806	355.81	0.000000
JPM	ω	0.000001	0.000015	0.065	0.948265
	α_1	0.036292	0.152936	0.237	0.812424
	β_1	0.953038	0.158273	6.021	0.000000
C	ω	0.000053	0.000018	2.879	0.003986
	α_1	0.173472	0.100538	1.725	0.084449
	β_1	0.607689	0.084106	7.225	0.000000
BTC.USD	ω	0.000000	0.000006	0.000	1.000000
	α_1	0.000000	0.000287	0.000	0.999912
	β_1	0.998999	0.000008	123500.00	0.000000
UEC	ω	0.000001	0.000042	0.018	0.985983
	α_1	0.000000	0.006024	0.000	1.000000
	β_1	0.999000	0.014490	68.94	0.000000
GM	ω	0.000011	0.000000	32.57	0.000000
	α_1	0.020693	0.001371	15.09	0.000000
	β_1	0.951785	0.005178	183.83	0.000000
ET	ω	0.000026	0.000082	0.317	0.751097
	α_1	0.257218	1.008857	0.255	0.798754
	β_1	0.485489	0.411715	1.179	0.238323

Table 12: Coefficient of ARMA + DCC–GARCH (GARCH)

Parameter	Estimate	Std. Error	t value	Pr($> t$)
$dcca_1$	0.000000	0.000003	0.0002	0.999806
$dccb_1$	0.910263	0.051133	17.8020	0.000000

Table 13: Coefficient of ARMA + DCC–GARCH (DCC)