THE ISOPERIMETRIC PROBLEM

Dissertation submitted at the University of Leicester
in partial fulfilment of the requirements for
the degree of Bachelor of Science of Mathematics

by

Steven Cheung
Department of Mathematics
University of Leicester

May 2024

Contents

Declaration

All sentences or passages quoted in this project dissertation from other people's work have been specifically acknowledged by clear cross referencing to author, work and page(s). I understand that failure to do this amounts to plagiarism and will be considered grounds for failure in this module and the degree examination as a whole.

Name: Steven Cheung

Signed: (add the signature from adobe pdf editor)

Date:

Abstract

In general, we want the maximum area whose boundary has a specific length.

For the 2-dimensional case.

For the 3-dimensional.

For the n-dimensional.

Manifolds?

Introduction

Historical Notes

Isoperimeter, isos which is ancient Greek for equal and perimetron for perimeter. The isoperimetric perimetric problem, even though not properly formalized, was already thought about all throughout the history.

Book V of Pappus of Alexandria's Mathematical collections [1] began with a preface titled "On the Sagacity of Bees", rather than the history of mathematicians past or their accomplishments to follow. This was written near the end of the 3rd century $A.D.$. Observing, Pappus credited the bees with "a certain geometrical forethought" (Thomas 593 [2]) for their nearly faultless hexagonal comb structre. He wrote

> "Bees know just this fact which is useful to them, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material in constructing each"
>
> (Thomas 593 [2])

Beyond the efficiency of bees, Pappus prefaced, "We" he continued

> "... will investigate a somewhat wider problem, namely that, of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always great and the greates of them all is the circle having it's perimeter equal to them"
>
> (Thomas 593 [2])

Pappus then started working on the isoperimetric problem, which included numerous smaller problems within it. The main objective of the problem is to determine which of the planes and figures with the same perimeter has the largest area amongst them.

Although Pappus addressed the problem in the collections, it has been a topic of interest for centuries before. Appearing in both mathematical and literary materials and captivating the minds of mathematicians.

The isoperimetric issue demonstrates ancient mathmaticians' perceptiveness and the consistency of mathematical endeavor over time, even in the modern age. The isoperimetric problem have been used by poets and historians, despite its mathematical nature. Most famously, Virgil made use of the concept in his Roman epic, The Aeneid. To quote Virgil

> "At last they landed, where from far your eyes May view the turrets of new Carthage rise; There bought a space of ground, which Byrsa call'd, From the bull's hide they first inclos'd and wall'd.""
>
> (Book I of Aeneid [3])

Virgil's version has it that Dido, daughter of the king of Tyre, fled home after her brother killed her husband. Dido ended up on the north coast of Africa, where she bargained to buy as much land as she could enclose with an oxhide. Thus, she cut the hide into thin strips, presumably met and solved enclosing the largest area with a given perimeter - the isoperimetric problem. Dido may have been clever enough and chose an area by the coast to exploit the shore as part of the perimeter. But this mostly spoils the purity of posed problem. Kline concludes [4]. The Aeneid was written between 29 and 19$B.C.$.

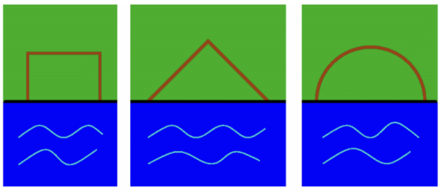


Figure 1: Representations of areas bounded by common shapes of the same perimeter. The semicircle, answer to Dido's problem which contains the greatest area. (image and caption from [5])

In the 3rd century *A.D.*, Roman historian Marcus Junianus Justinus compliled an account of Carthaginian folklore that the legendary founding of Carthage by Dido, called Elissa by the Greeks (the mythological origin of the city):

> "Then [Elissa] bout some land, just as much as could be covered by a cow's hide, where she could give some recreation to her men... She next gave orders for the hide to be cut into very fine strips, and in this way she took possession of a great area than she had apparently bargained for"
>
> (Book XVIII 157 [6])

So since ancient Greece, around 100*B.C.*, they wanted to measure the size of islands by timing how long it takes to circumnavigate the entire island. Proclus, a classical mathematician, mocked geometers for "measuring the size of a city from the lengths of its walls". To the common person of antiquity, two shapes with equal perimeter may have different areas. Interestingly, some individuals exploited this misconception to defraud others of land. Considerably more amusing, these con artists were viewed as liberal which demonstrates how unnatural the idea of shapes with a similar edge having different regions was to the old Greeks.

Geoffrey of Monmouth's Historia Regum Britanniae (History of the Kings of England), a 12th century *A.D.* chronicle of Arthurian legends, mentions the isoperimetric problem. In this story, Hengist, a German duke, appeals to King Vortigern for land in exchange for his military service:

> "'Grant', said Vortigern, 'unto thy servant but so much only as may be compassed round about by a single thong within the land thou hast given me, that so I ma build me a high place therein whereunto if need be I may betake me'...Straightaway...Hengist took a bull's hide, and wrought the same into a single thong throughout. He then compassed round with his thong a stony place that he had right cunningly chosen, and within the space thus meted out did begin to build a castle that was afterwards called in British , Kaercorrei, but in Saxon, Thongceaster, the which in Latin's speech is called Castrum corrigae'"
>
> (Monmouth 105-106 [7])

Thus, the poets and historians who chronicled the exploits of these mythological characters as well as the figures themselves found special meaning in the isoperimetric problem. Despite its extensive implications, the notion of isoperimetry was "naturally greek". The Greeks have pretty much solved it, by their standards. Zenodorus was an ancient Greek mathematician from around 200*B.C.* to 120*B.C.*. And have mostly proved that a circle has greater area than any polygon with the same perimeter. Majority of his work was lost. However, fortunately, parts of his work survived through references by Pappus and Theon of Alexandria.

Theon of Alexandria then develops this idea, with a summary of the proofs present by Zenodorus in "On Isoperimetric Figures". According to Theon, Zenodorus did not initiate his disussion of isoperimetry with the circle. Rather he stated that "Of all rectilinear figures having an equal perimeter - I mean equilateral and equiangular figures - the greatest is that which has the most angles" (Thomas 388-389 [2]). In more modern language, the proposition is stated as follows: "Given two regular n-gons with the same perimeter, one with $n = n_1$ and the other with $n = n_2 > n_1$ then the regualr n_2-gon has the larger area" (Nahin 47 [8]). Following this, Zenodorus was able to arrive at the proposition that "if a circle have an equal primter with an equilateral and equiangular rectilinear figure, the circle shall be the great" (Thomas 391 [2]). As Heath notes in his "History of Greek Mathematics", Zenodorus chose to base his proof of this proposition on the theorem already established by Archimedes that "the area of a circle is eaual to the right-angled triange with perpendicular side equal to the radius and base equal to the perimeter of the circle" (Heath 209 [9]). Frome here, Zenodorus proceeded on the basis of two preliminary lemmas: first that "if there be two triangels on the same base and with the same perimeter one bein isosceles and the other scalene, the isosceles triangle has the greater area" (Heath 209 [9]); second that "given two isosceles triangles not similar to one another, if [one constructs] on the same bases two triangles similar to one another such that the sme of the areas of the similar triangles is great than the sum of the ares of the non-similar triangles" (Heath 210 [9]). Both commentors seem to hint that it will be covered in subsequent chapters, but as Heath bemoans "in the text as we have it the promise is not fulfilled" (Heath 212 [9]) (this entire paragraph is lifted from [1])

In the ancient world, the problem of isoperimetry was associated with the work of Zenodorus and his commentator Pappus. It was the work of a Swiss mathematician Jacob Steiner (1796-1863) who tackled the isoperimetric theorem in the modern word.

Indeed, the problem of isoperimetry in the niniteenth century emerged at an important juncture in mathematical thought. Mathematicians working in all fields of inquiry struggled over the use of analytic (i.e. calculus) or synthentic (i.e pure geometry) methods in solving problems. [1] Nahin notes that Steiner's 1842 geometrical proof of the isoperimetric theorem is still regarded as a "model of mathematical ingenuity" despite subsequent discoveries of defects in the synthetic approach. The following propositions must be understood to be logically equivalent in order for Steiner's proof of the isoperimetric theorem to hold:

> A. "Of all closed curves in a plane with equal perimeters, the cicle bounds the largest area"
> [and]
> B. "Of all closed curves in a plane with equal areas, the cricle has the smallest perimeter"
>
> (Nahim 55 [8])

Steiner thought he had demonstrated that the circle was the answer to the isoperimetric problem. As later researchers, especially the German mathematician Peter Dirichlet (1805-1859), remarked, Steiner had made an underlying assumption not explicitly addressed in his proof, namely that a solution existed (Nahin 59 [8]).

Other mathematicians attempted to tackle the isoperimetric problem from the analytic or calculus-based perspective. And to no avail.

Problems posed by the ancients not only speeded the progression towards more rigorous, complete systems of mathematics, but also prompted later innovators to develop new systems to deal with these early questions. The isoperimetric problem thus demonstrates an important continuity in mathematical thought. From Zenodorus to Pappus and from Steiner to the mathematicians of the twenty-first century, isoperimetry has transcended its origins in ancient geometry to become a building block of more modern analytic systems of mathematics [1]. Below is a table summary:

Name	Time Period	What they did?
Pappus	written 3rd Century $A.D.$	started working about the isoperimetric problem from bee's hexagonal comb structure
Dido (The Aeneid)	$29B.C.$-$19B.C.$	enclose as much land with oxhide
Zenodorus	$200B.C.$-$120B.C.$	more angles means more area
Ancient Greece	$100B.C.$	circumnavigate land
Arthurian Legends	12th Century $A.D.$	exchanged land for military service
Steiner	1842	first proof (existence)
Peter Dirichlet	1805-1859	noticed flaw with Steiner's proof

Important Preliminaries

We will take for granted the Jordan Curve Theorem

Theorem 1 (Jordan Curve Theorem)**.** *A simple closed curve in the plane divides the plane into two regions, one compact and one non-compact, and in the common boundary of both regions.*

The Jordan Curve Theorem is just a standard, but highly non-trivial, result of the topology of $\mathbb{R}^2$, that any simple closed curve in the plane has an 'interior' and 'exterior': more precisely, the complement of the image of γ (i.e. the set of two points $\mathbb{R}^2$ that are <u>not</u> in the image of γ) is the disjoint union of two subsets of $\mathbb{R}^2$, denoted by $int(\gamma)$ and $ext(\gamma)$, with the following properties:

1. $int(\gamma)$ is bounded, i.e. it is contained inside the circle of sufficiently large radius.
2. $ext(\gamma)$ is unbounded
3. Both of the regions $int(\gamma)$ and $ext(\gamma)$ are connected, i.e. they have the property that any two points in the same region can be joined by a curve contained entirely in the region (but any curve joining a point of $int(\gamma)$) to a point of $ext(\gamma)$ must cross the curve γ)

Note. *When we talk of the region bounded by a simple closed curve in the plane, we always mean the compact region*

Definition 1. *A closed curve, is a curve that changes direction but does not cross itself whilst changing direction.*

Definition 2. *A simple curve, is a curve that changes direction but does not cross itself whilst changing direction.*

Definition 3. *A simple closed curve in $\mathbb{R}^2$, is a closed curve in $\mathbb{R}^2$ that has no self-intersections*

The two definitions, above, are vital into understanding the main theorem. Since the isoperimetric inequality is a global result, we borrow concepts from topology, such as:

Definition 4. *A function is bounded if $\exists M \in \mathbb{R}$ such that $|f(x)| \leq M$.*

Definition 5. *Let X be a topological space and $A \subset X$. An open cover for A is a family $\{U_\lambda\}_{\lambda\in I}$ of open subsets of X such that*

$$A \subset \cup_{\lambda\in I} U_\lambda$$

An open cover is called finite if $\|I\| < \infty$. If $\{U_\lambda\}_{\lambda\in I}$ is an open cover for A and $J \subset I$ is such that $A \subset \cup_{\lambda\in J} U_\lambda$, then $\{U_\lambda\}_{\lambda\in J}$ is called a subcover of $\{U_\lambda\}_{\lambda\in I}$.

Definition 6. *A subset $A \subset X$ of a topological space called compact if every open cover of A has a finite subcover. A space is called compact space if it is a compact subset of itself.*

Definition 7. *(Heine-Borel Theorem) A set in $\mathbb{R}^n$ is said to be compact if it is closed and bounded.*

The Isoperimetric Theorems for 2D, 3D and nD Cases

1.1 2 Dimensional Case (Plane)

Theorem 2. *Let C be a simple closed curve in the plane with length L and bounding a region of area A . Then $L^2 \leq 4\pi A$ with equality if and only if C is a circle.*

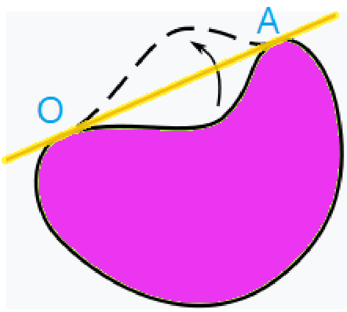
The circle therefore bounds the biggest area among all simple closed curves in the plane with a given length.

1.1.1 Unpacking Steiner's proof

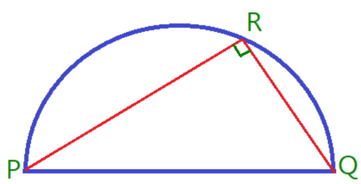
The proof that I will be unpacking and taking a closer look at will be from the book (reference the book here), and is credited by them to Jakob Sternier. Packed and concise proof from [10].
(todo: add my own insights and increase the length of the proof below)

Proof. We will begin the proof by assuming the solution exists. So there exists a simple closed curve C of a specified length L, enclosing a region with the maximum possible area.

1. We claim that the curve C must by convex, meaning that any straight line connecting two points (O and A) on C should entirely lie within the region enclosed by C. We justify this assertion thus, suppose that the curve is not convex then we could draw a segment OA connecting the two points of C, with the entire segment (except for its endpoints) situated outside of C. By mirroring the relevant arc of C between O and A along this line, a new curve of an identical length emerges, encapsulating a greater area, as illustrated in the figure below. Therefore, C is already a convex curve.



2. Select two points, P and Q, to partition our solution curve C into arcs of identical length. Consequently, the line segment PQ must divide the region enclosed by C into two sections of equal area. Failure to do so would imply that the portion with a larger area could be mirrored across PQ, resulting in another curve of the same length but with greater area.



3. Next we will need to prove that the region bounded by the line PQ and the arc with length L/2 and endpoints P,Q bounds a region of maximum area and we will prove that the region is a semi-circle. Next we will need to prove that the region bounded by the line PQ and the arc with length L/2 and endpoints P,Q bounds a region of maximum area and we will prove that the region is a semi-circle. For the sake of contradiction assume that the angle PRQ is not a right angle. we can treat the arc as hinged at R and either open or close it to make the angle at R a right angle. This adjustment does not alter the arc's length, consequently expanding the total area encapsulated by the arc PRQ and the line PQ. However, this contradicts the maximization achieved by the original arc, which had already optimized the enclosed total area. Therefore, the inscribed angle at R must be a right angle, confirming that the arc is indeed a semi-circle. Now since we have proven that it is a semi circle, we can use this logic and expand it to the whole curve to conclude that it is a circle.

4. Thus far, we have established that when considering a simple closed curve C with a designated length L, enclosing a region with maximum area, the curve in question must be a circle. Moving forward, we will demonstrate the existence of a maximizer and present this as the comprehensive Isoperimetric problem. In order to achieve this, we will approximate the curve C using a polygonal curve. Use Lemma X (need to prove that among all 2n-sided polygons with the same length L , the regular 2n-gon has the largest area. Now, let C be a smooth, regular simple closed curve in the plane, with length L and enclosing a region of area A. $\forall \epsilon > 0$ we can inscribe in C a polygon P_ϵ whose length L_ϵ $|L_\epsilon - L| < \epsilon$ and the area A_ϵ which is $|A_\epsilon - A| < \epsilon$. Using Lemma Y, now as $\epsilon \to 0$ and we obtain the desired inequality. Since the circle of the same length L encloses a region of area A_0 complying with $L^2 = 4\pi A_0$. Therefore, we ascertain that a circle achieves the maximum enclosed area among all smooth, regular, simple closed curves of the identical length. This confirms the existence of the maximizer. Hence the proof is complete.

□

1.2 3 Dimensional Case (Sphere)

1.3 n Dimensional Case ($\mathbb{R}^n$)

Manifolds

Bibliography

[1] Jennifer Wiegert. The sagacity of circles: a history of the isoperimetric problem. *Loci, July*, 2010.

[2] Ivor Bulmer-Thomas. *Selections illustrating the history of Greek mathematics*, volume 2. Harvard University Press, 1941.

[3] Virgil. *The Aeneid*. Trans. Robert Fitzgerald. New York: Random House, 1981.

[4] Morris Kline. *Mathematics for the Nonmathematician.* Courier Corporation, 1985.

[5] S Demjanenko. The isoperimetric inequality: A history of the problem, proofs and applications, 2008.

[6] JC Yardley and R Develin. Justin, epitome of the philippic history of pompeius trogus, translated by jc yardley, with introduction and explanatory notes by r. *Develin, Seattle*, 1994.

[7] Sebastian Evans et al. *Manmouth, Geoffrey of. Histories of the Kings of Britain.* Number 577. JM Dent & sons, Limited, 1920.

[8] Paul J Nahin. *When Least is Best: how mathematicians discovered many clever ways to make things as small (or as large) as possible*, volume 114. Princeton University Press, 2021.

[9] Thomas Little Heath. *A history of Greek mathematics*, volume 1. Cambridge University Press, 2013.

[10] Herman Gluck. 5. the isoperimetric problem. *Math 501 - Differential Geometry*, 2012.